Self-similar analytical model of the plasma expansion in a magnetic field

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Abstract

The study of hot plasma expansion in a magnetic field is of interest for many astrophysical applications. In order to observe this process in laboratory, an experiment is proposed in which an ultrashort laser pulse produces a high-temperature plasma by irradiation of a small target. In this Letter an analytical model is proposed for an expanding plasma cloud in an external dipole or homogeneous magnetic field. The model is based on the self-similar solution of a similar problem which deals with sudden expansion of spherical plasma into a vacuum without ambient magnetic field. The expansion characteristics of the plasma and deceleration caused by the magnetic field are examined analytically. The results obtained can be used in treating experimental and simulation data, and many phenomena of astrophysical and laboratory significance.

Keywords: Plasma expansion, Self-similar model, Magnetic field

1. Introduction

The problem of sudden expansion of hot plasma into a vacuum in the presence of an external magnetic field has been intensively studied in the mid-1960s in connection with the high-altitude nuclear explosions. It has also been discussed in the analysis of many astrophysical and laboratory applications (see, e.g., Refs. [1, 2] and references therein). Such kind of processes arise during the dynamics of solar flares and flow of the solar wind around the Earth's magnetosphere, in active experiments with plasma clouds in space, and in the course of interpreting a number of astrophysical observations [1–6]. Researches on this problem are of considerable interest in connection with the experiments on controlled thermonuclear fusion [7] (a review [1] summarizes research in this area over the past four decades).

The property of expanding plasma to push magnetic field out is a sequence of magnetic flux conservation. Even if a magnetic field and, magnetic flux are nonzero initially inside the plasma, then internal field tends to zero fast, providing magnetic flux in plasma to be constant. Plasma is shielded from the penetration of the large external field by means of surface currents circulating inside the thin layer on the plasma boundary. Ponderomotive forces resulting from interaction of these currents with the magnetic field would act on the plasma surface as if there were magnetic pressure applied from outside. Thus after some period of accelerated motion, plasma gets decelerated as a result of this external force acting inward. The plasma has been considered as a highly conducting media with zero magnetic field inside. From the point of view of electrodynamics it is similar to the expansion of a superconductor in a magnetic field. An exact analytic solution for a uniformly expanding, highly conducting plasma sphere in an external uniform and constant magnetic field has been obtained in [8]. The non-relativistic limit of this theory has been used by Raizer [9] to analyse the energy balance (energy emission and transformation) during the plasma expansion. The similar problem has been considered in Ref. [4] within one-dimensional geometry for a plasma layer. In our previous papers [10] and [11] we obtained an exact analytic solution for the uniform relativistic expansion of the highly conducting plasma sphere or cylinder in the presence of a dipole or homogeneous magnetic field, respectively. In the present Letter we study the expansion of the spherical plasma cloud in the presence of a dipole or homogeneous magnetic field taking into account the thermal effects. For this geometry we found an analytical solution which can be used in analysing the recent experimental and simulation data (see, e.g., Refs. [1, 2] and references therein).

2. Theoretical model

Usually the motion of the expanding plasma boundary is approximated as the motion with constant velocity (uniform expansion). In the present study a quantitative analysis of plasma dynamics is developed on the basis of one-dimensional spherical radial model. Within the scope of this analysis the initial stage of plasma acceleration, later stage of deceleration and the process of stopping at the point of maximum expansion are examined.

Consider the magnetic dipole $\mathbf{p}$ and a plasma spherical cloud with radius $a(t)$ located at the origin of the coordinate system. The dipole is placed in the position $\mathbf{r}_0$ from the center of the plasma cloud ($a(t) < r_0$). The orientation of the dipole is given by the angle θ_p between the vectors $\mathbf{p}$ and $\mathbf{r}_0$. We denote the strength of the magnetic field of the dipole by $\mathbf{H}_0(\mathbf{r})$. The energy, which is transferred from plasma to electromagnetic field

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is the mechanical work performed by the plasma on the external magnetic pressure $H_0^2(\mathbf{r})/8\pi$. Taking into account this effect, the equation of balance of plasma energy is as follows:

$$\frac{4\pi}{\gamma-1}\int_0^{a(t)} pr^2dr+2\pi\int_0^{a(t)}\rho v^2r^2dr+\int_\Omega \frac{H_0^2(\mathbf{r})}{8\pi}d\mathbf{r}=W_0,\quad (1)$$

where Ω is the volume of the spherical shell $a_0 \leqslant r \leqslant a(t)$, $a_0 = a(0)$ ($a(t) \geqslant a_0$) and W_0 are the initial radius and energy of the plasma. Also $\gamma = C_p/C_V > 1$, v, p and ρ are the adiabatic index, the velocity, the pressure and the mass density of the plasma, respectively. C_p and C_V are the heat capacities at constant pressure and constant volume, respectively. When the plasma cloud is introduced into a background magnetic field, the plasma expands and excludes the background magnetic field to form a magnetic cavity. The magnetic energy of the dipole in the excluded volume is represented by the last term in Eq. (1). Initial plasma velocity is supposed to be $v(r,0) = v_m(r/a_0)$ at $r \leqslant a_0$ and $v(r,0) = 0$ at $r > a_0$, where v_m is the initial velocity of the plasma boundary ($v_m = \dot{a}(0)$).

The obtained energy balance equation can be effectively used if profiles of velocity $v(r,t)$, pressure $p(r,t)$, and mass density $\rho(r,t)$ are known functions of the plasma radius $a(t)$. We will take these dependences from the solution of a similar problem which deals with sudden expansion of spherical plasma into a vacuum without ambient magnetic field. The simplest class of solutions available in this case are so-called self-similar solutions. They are realized under the specified initial conditions. We will set the initial conditions with a parabolic distribution of pressure and mass density, which describe hot and dense initial plasma state with sharp boundary localized at $r = a_0$. The self-similar solutions are characterized by a velocity distribution linearly dependent on r. At $r \leqslant a(t)$

$$v(r,t) = r\frac{\dot{a}(t)}{a(t)}, \quad (2)$$

where unknown $a(t)$ is the radius of sharp plasma boundary while $\dot{a}(t)$ is the velocity of the boundary. The specification of the mass density profile at $r \leqslant a(t)$ is given by

$$\rho(r,t) = \frac{\Gamma\left(\frac{5}{2}+q\right)}{\pi^{3/2}\Gamma(1+q)}\frac{M}{a^3(t)}\left[1-\frac{r^2}{a^2(t)}\right]^q \quad (3)$$

and Eq. (2) for velocity, automatically satisfies the continuity equation for an arbitrary function $a(t)$ and for an arbitrary parameter q. Here $M =$ const is the total mass of plasma cloud and $\Gamma(z)$ is the Euler function. Substitution of ρ and p into the entropy equation gives at $r \leqslant a(t)$ the following solution for the pressure

$$p(r,t) = p_{\max}\left[\frac{a_0}{a(t)}\right]^{3\gamma}\left[1-\frac{r^2}{a^2(t)}\right]^s, \quad (4)$$

where s is an arbitrary parameter and $p_{\max}$ is the thermal pressure at the center of the spherical plasma cloud at $t = 0$. In addition the quantities v, ρ and p vanish at $r > a(t)$, $v(r,t) = \rho(r,t) = p(r,t) = 0$. Substituting above expressions into the fluid equation of motion yields a second-order differential equation governing the motion of the plasma boundary $a(t)$. The problem considered is not isentropic in general except the case when $s = q\gamma$. In the latter case of the isentropic expansion the equation of state is given by $p\rho^{-\gamma} =$ const. Throughout in this paper we will assume that $q \geqslant 0$ and $s \geqslant 0$.

Equations (2)-(4) are an exact solution in the case of expansion into a vacuum without magnetic field. However, Eq. (4) does not satisfy the boundary condition, $p(a(t),t) = H_0^2/8\pi$, which is imposed in the case of expansion into an ambient magnetic field. On the other hand if the magnetic pressure is smaller than the plasma pressure, $p_{\rm mag}/p_{\max} \ll 1$, the difference between the exact solution in the magnetic field and free expansion model is small and is localized in a narrow area near the surface of the cloud. These deviations are additionally reduced due to integration in the equation of energy balance. Estimating accuracy of the free expansion model, one should take into account that the long stage of plasma deceleration corresponds to a high expansion ratio, $a(t)/a_0 \gg 1$. Average plasma pressure drops significantly and plays no role in energy balance equation (1) during this stage. In accordance with the above boundary condition local pressure near the plasma edge must be equal to the magnetic pressure outside. It causes deviation from the profile equation (3) and accumulation of plasma in this area. This is confirmed independently by the numerical simulations [12]. In the limiting case when all plasma is localized near the front, one can expect an increase of the kinetic energy and longer stage of plasma deceleration as compared with the free expansion model.

In the case of dipole magnetic field the volume integral in the last term of Eq. (1) has been evaluated in Ref. [10]. The result reads

$$\int_\Omega \frac{H_0^2(\mathbf{r})}{8\pi}d\mathbf{r} = \frac{p^2}{32r_0^3}\left[Q(\eta x(t)) - Q(\eta)\right], \quad (5)$$

where $\eta = a_0/r_0 < 1$, $x(t) = a(t)/a_0$ (note that $a(t)/r_0 = \eta x(t) < 1$), and

$$Q(\eta) = \frac{1}{(1-\eta^2)^3}\left[\eta\left(1-\eta^4\right)\left(3\cos^2\theta_p - 1\right)\right. \quad (6)$$
$$\left. +8\eta^3\left(1+\cos^2\theta_p\right)\right] - \frac{3\cos^2\theta_p-1}{2}\ln\frac{1+\eta}{1-\eta}.$$

Substituting Eqs. (2)-(4) into (1) and integrating over r yields first-order differential equation for $a(t)$, which already satisfies initial condition $\dot{a}(0) = v_m$,

$$\dot{x}^2(\tau) + \frac{\beta}{x^{3(\gamma-1)}} + \alpha\left[Q(\eta x(\tau)) - Q(\eta)\right] = 1. \quad (7)$$

Here two dimensionless quantities are introduced

$$\alpha = \frac{p^2}{32W_0r_0^3}, \quad \beta = \frac{\pi^{3/2}p_{\max}a_0^3}{(\gamma-1)W_0}\frac{\Gamma(1+s)}{\Gamma(\frac{5}{2}+s)} < 1 \quad (8)$$

which determine the magnetic and the thermal energies, respectively, in terms of the total initial energy W_0. The latter is easily obtained from Eq. (1) and reads

$$W_0 = \frac{3Mv_m^2}{2(5+2q)} + \frac{\pi^{3/2}p_{\max}a_0^3}{\gamma-1}\frac{\Gamma(1+s)}{\Gamma(\frac{5}{2}+s)}. \quad (9)$$

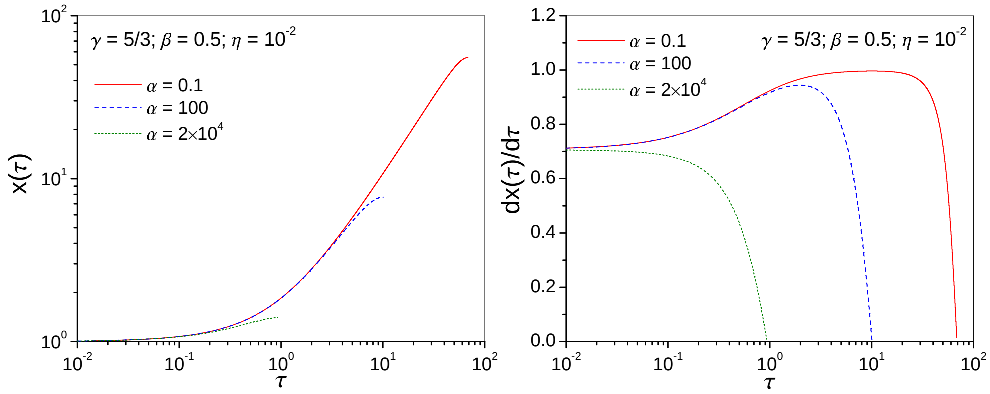

Figure 1: (Color online.) The dynamics of the plasma cloud expanding in a dipole magnetic field. Shown are the scaled radius $a(t)/a_0$ (left panel) and the velocity $\dot{a}(t)/u_m$ (right panel) of the plasma boundary vs time (in units of t_0) at $\gamma = 5/3$, $\beta = 0.5$, $\eta = 10^{-2}$, $\theta_p = 0$ and for $\alpha = 0.1$ (solid lines), $\alpha = 100$ (dashed lines) and $\alpha = 2 \times 10^4$ (dotted lines).

From Eqs. (8) and (9) it is seen that $\beta < 1$. New dimensionless variables are introduced as follows: $x(\tau) = a(t)/a_0$, $\tau = t/t_0$, $t_0 = a_0/u_m$, where $u_m = [2(5+2q)W_0/3M]^{1/2}$ is the velocity of plasma expansion, achieved asymptotically at $t \to \infty$ in the case of expansion into a vacuum without magnetic field (i.e. at $\alpha = 0$).

The total energy of the plasma cloud at time t is obtained from Eq. (7)

$$W(t) = W_0 - \frac{p^2}{32r_0^3}\left[Q(\eta x(t)) - Q(\eta)\right]. \qquad (10)$$

Note that the function $Q(\eta)$ monotonically increases with the argument and the plasma cloud energy decreases with time.

Consider also the case of uniform magnetic field when $\mathbf{H}_0 =$ const. In this case the volume integral in Eq. (5) is replaced by $(H_0^2/6)(a^3(t) - a_0^3)$ and the differential equation (7) for the plasma boundary reads

$$\dot{x}^2(\tau) + \frac{\beta}{x^{3(\gamma-1)}} + \sigma[x^3(\tau) - 1] = 1, \qquad (11)$$

where $\sigma = W_{\rm mag}/W_0$, $W_{\rm mag} = (4\pi a_0^3/3)p_{\rm mag}$ is the initial magnetic energy in the plasma volume, and $p_{\rm mag} = H_0^2/8\pi$ is the magnetic field pressure.

Equations (7) and (11) coincide with the equation of the one-dimensional motion of the point-like particle in the potential $U(x)$ which is determined by second and third terms of Eqs. (7) and (11). The distance x_s of the plasma cloud motion up to the full stop (the stopping length) at the turning point is determined by $U(x_s) = 1$. In particular, it is easier to obtain the stopping length in the case of homogeneous magnetic field and at vanishing thermal pressure ($\beta = 0$). Then from Eq. (11) one obtains the equation of motion

$$\dot{x}^2 = 1 + \sigma - \sigma x^3. \qquad (12)$$

It is seen that in this case the stopping length is given by $x_s = (1+1/\sigma)^{1/3}$. The solution of Eq. (12) can be represented in the form

$$t = \frac{t_0}{\sqrt{\sigma+1}}\left[x(t)\mathcal{F}\left(\frac{x^3(t)}{x_s^3}\right) - \mathcal{F}\left(\frac{1}{x_s^3}\right)\right], \qquad (13)$$

where $\mathcal{F}(z) = F\left(\frac{1}{3}, \frac{1}{2}; \frac{4}{3}; z\right)$ and the latter is the hypergeometric function. Substituting in Eq. (13) $x(t) = x_s$ we obtain the corresponding stopping time as a function of the magnetic field and the plasma kinetic energy

$$t_s = \frac{t_0}{\sqrt{\sigma+1}}\left[C\left(\frac{\sigma+1}{\sigma}\right)^{1/3} - \mathcal{F}\left(\frac{\sigma}{\sigma+1}\right)\right]. \qquad (14)$$

Here $C = \mathcal{F}(1) = \sqrt{\pi}\Gamma\left(\frac{4}{3}\right)/\Gamma\left(\frac{5}{6}\right) \simeq 1.4$ is a constant. At vanishing ($\sigma \ll 1$) and very strong ($\sigma \gg 1$) magnetic fields the stopping time becomes $t_s \simeq CR_m/v_m = C(a_0/v_m)\sigma^{-1/3}$, $t_s \simeq 2R_m^3/3v_m a_0^2 = (2a_0/3v_m)\sigma^{-1}$, respectively, where the radius $R_m = (6W_0/H_0^2)^{1/3}$ is obtained by equating the initial kinetic energy W_0 of an initially spherical plasma cloud to the energy of the magnetic field that it pushes out in expanding to the radius R_m. It is worth mentioning that in the case of weak magnetic field, $\sigma \ll 1$, and at vanishing thermal pressure ($\beta = 0$) the stopping time does not depend on the initial plasma radius, $t_s \sim (M/v_m p_{\rm mag})^{1/3}$.

We now turn to the general equations determined by Eqs. (7) and (11). At the initial stage of plasma expansion ($t \ll t_0$) from these equations we obtain

$$x(t) \simeq 1 + \frac{v_m t}{a_0} + \frac{3}{4}h\left(\frac{t}{t_0}\right)^2, \qquad (15)$$

where $h = \beta(\gamma-1) - \kappa$, $\kappa = \frac{\alpha}{3}\eta Q'(\eta)$ and $\kappa = \sigma$ for the dipole and homogeneous magnetic fields, respectively. Here the prime indicates the derivative with respect to the argument. Thus at the initial stage the plasma cloud may get accelerated or decelerated depending on the sign of the quantity h (in other words on the relation between thermal and magnetic pressures). For instance, in the homogeneous magnetic field the acceleration occurs when $p_{\max} > p_c$, where

$$p_c = \frac{4}{3\sqrt{\pi}}\frac{\Gamma\left(\frac{5}{2}+s\right)}{\Gamma(1+s)}p_{\rm mag} \qquad (16)$$

(i.e. at $h > 0$) and continues until $x(t)$ reaches some value $x_c > 1$ given by $x_c = (p_{\max}/p_c)^{1/3\gamma}$. The time interval $0 \leqslant t < t_c$

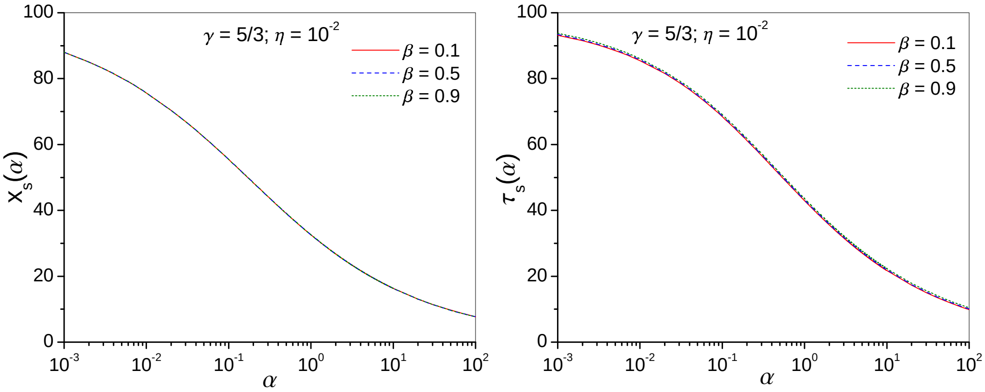

Figure 2: (Color online.) The normalized stopping length $a_s(\alpha)/a_0$ (left panel) and the stopping time $t_s(\alpha)/t_0$ (right panel) of the plasma cloud expanding in a dipole magnetic field vs the normalized dipole magnetic field α at $\gamma = 5/3$, $\eta = 10^{-2}$, $\theta_p = 0$ and for $\beta = 0.1$ (solid lines), $\beta = 0.5$ (dashed lines) and $\beta = 0.9$ (dotted lines).

of the acceleration is determined from the equation of motion (11). The critical radius x_c and time t_c correspond to the beginning of plasma deceleration. Further plasma motion at $t > t_c$ is an expansion with slowing-down velocity. It ends up at the turning point which corresponds to the maximum of expansion, $U(x_s) = 1$. However, in the opposite case of the low thermal pressure with $p_{\max} < p_c$ the plasma systematically get decelerated in the whole time interval of its dynamics.

A characteristic stopping time of plasma motion up to the full stop at the turning point is given by the integral of the Eqs. (7) and (11)

$$\tau_s = \int_1^{x_s} \frac{dy}{\sqrt{1 - U(y)}} \simeq 2\sqrt{\frac{x_s - 1}{U'(x_s)}}. \tag{17}$$

Calculating time t_s needed for plasma to reach this point, one can simplify the integrand taking into account that the main contribution comes from the vicinity of upper limit of integration. This approximation is expressed by the second part of Eq. (17). In the case of weak and homogeneous magnetic field this yields universal expressions, $t_s \sim (M/u_m p_{\text{mag}})^{1/3}$ and $a_s \sim u_m t_s$. It is worth mentioning that in the case of weak magnetic field the stopping time and length do not depend on the initial plasma radius but depend on the thermal pressure (or temperature) (cf. these relations with those obtained above). At very strong magnetic fields, $a_s \simeq a_0 + (1/2)v_m t_s$ and $t_s \sim M v_m / a_0^2 p_{\text{mag}}$, and the stopping characteristics of the plasma essentially depend on the initial radius but are now independent on the thermal pressure. The similar estimates can be found for the dipole magnetic field. However, we note that the latter case significantly differs from the homogeneous field situation considered above. Since in the vicinity of the dipole the magnetic field is arbitrary large the stopping length cannot naturally exceed r_0 for any thermal energy of the plasma ($x(t) < 1/\eta$ in Eq. (7)). For a weak magnetic field this simply yields $a_s \simeq r_0$ and $t_s \simeq r_0/u_m$.

As an example in Fig. 1 we show the results of model calculations for the normalized radius $a(t)/a_0$ (left panel) and the velocity $\dot{a}(t)/u_m$ (right panel) as a function of time (in units of t_0) at $\gamma = 5/3$, $\beta = 0.5$, $\eta = 10^{-2}$, $\theta_p = 0$ and for different values of the parameter α. In this figure the dimensionless strengths α of the dipole magnetic field are chosen such that the coefficient h in Eq. (15) is positive, $h > 0$, for solid and dashed lines and negative, $h < 0$, for dotted lines. From the right panel of Fig. 1 it is seen that at $h > 0$ (solid and dashed lines) there is a short initial period of acceleration, $0 \leqslant t \lesssim t_0$, when the plasma boundary is accelerated according to Eq. (15). During this period (which is only weakly sensitive to the magnetic field strength) the dimensionless radius $a(t)/a_0$ increases up to $2-3$, and at $t_0 \lesssim t < t_c$ almost all initial total energy W_0 is transferred into kinetic energy of free radial expansion at constant velocity $\sim u_m$. As expected (see above) the time t_c is reduced with increasing the strength of the magnetic field and the free expansion period is shorter for larger α. The further increasing the strength of the magnetic field (Fig. 1, dotted line) results in a plasma dynamics with systematically slowing-down velocity.

For the same set of the parameters γ, η and θ_p in Fig. 2 it is shown the normalized stopping length (left panel) and the stopping time (right panel) of the plasma cloud as a function of the dimensionless strength α of the dipole magnetic field for some values of the normalized plasma thermal pressure β. It is seen that the stopping length and time decrease with the strength of the magnetic field and practically are not sensitive to the variation of the plasma thermal pressure.

Note that at otherwise unchanged parameters the strength of the dipole magnetic field is maximal at the orientation $\theta_p = 0$ and monotonically decreases with θ_p. For instance, the strength $H_0(0)$ of the dipole magnetic field at the center of the plasma cloud is reduced by a factor of 2 by varying the dipole orientation from $\theta_p = 0$ to $\theta_p = \pi/2$. Therefore the effect of the magnetic field shown in Figs. 1 and 2 is weakened at the orientation $\theta_p = \pi/2$ of the dipole. In particular, this results in a larger stopping lengths and times than those shown in Fig. 2.

3. Conclusion

An analytical self-similar solution of the radial expansion of a spherical plasma cloud in the presence of a dipole or homogeneous magnetic field has been obtained. The analysis of the plasma expansion into ambient magnetic field shows that there are processes of acceleration, retardation and stopping at the point of maximum expansion that are very distinct and separated in space and time. The scaling laws obtained are, in general, the functions of two dimensionless parameters, α (or σ for constant magnetic field) and β, which can be varied by means of the choice of the external magnetic field, the thermal pressure and the initial energy of the plasma. It allows to test the different regimes of plasma dynamics in a wide range of external conditions.

We expect our theoretical findings to be useful in experimental investigations as well as in numerical simulations of the plasma expansion into an ambient magnetic field (either uniform or nonuniform). One of the improvements of our present model will be the derivation of the dynamical equation for the plasma surface deformation. In this case it is evident that the problem is not isotropic with respect to the center of the plasma cloud ($\mathbf{r} = 0$) and a full three-dimensional analysis is required. A study of this and other aspects will be reported elsewhere.

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