Positron annihilation in the MuPs system

Alexei M. Frolov[*]

Department of Chemistry

University of Western Ontario,

London, Ontario N6H 5B7, Canada

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Abstract

The life-time of the four-body atomic system MuPs ($\mu^+ e_2^- e^+$ or muonium-positronium) against positron annihilation has been evaluated as $\tau = \frac{1}{\Gamma} \approx 4.07635 \cdot 10^{-10}$ sec. Various annihilation rates for MuPs are determined to a good numerical accuracy, e.g., $\Gamma_{2\gamma} \approx 2.446549 \cdot 10^9$ sec^{-1}, $\Gamma_{3\gamma} \approx 6.62815 \cdot 10^6$ sec^{-1}, $\Gamma_{4\gamma} \approx 3.61689 \cdot 10^4$ sec^{-1}, $\Gamma_{5\gamma} \approx 6.32989 \cdot 10^1$ sec^{-1}. The hyperfine structure splitting for the ground state in the MuPs system has also been evaluated as $\Delta = 23.074$ MHz.

[*]E–mail address: afrolov@uwo.ca

In our earlier study [1] we have considered the bound states in the positronium hydrides ∞HPs, TPs, DPs, ^{1}HPs and MuPs ($\mu^+ e_2^- e^+$). In this work we report the improved results obtained recently for four-body muonium-positronium system MuPs. In these calculations we could not improve the total (and binding) energies of the MuPs system, but some expectation values of interparticle delta-functions have been changed noticeably (in contrast with the positronium hydrides!). These changes in delta-functions lead to variations in the computed rates of positron annihilation. The improved annihilation rates can directly be compared with the experimental values, e.g., with the life-time of MuPs against positron annihilation.

In general, for the four-body MuPs system ($\mu^+ e_2^- e^+$) one can find many interesting physical problems to investigate (see, e.g., [1] and references therein). In this study we restrict ourselves to the discussions of the positron annihilation and hyperfine structure splitting. The muonium-positronium conversion and other interesting problems [1] are not considered here.

The Hamiltonian of the four-body $\mu^+ e_2^- e^+$ system is written in the form (in atomic units $\hbar = 1, m_e = 1, e = 1$):

$$H = -\frac{1}{2m_\mu}\Delta_1 - \frac{1}{2}\Delta_2 - \frac{1}{2}\Delta_3 - \frac{1}{2}\Delta_4 + \frac{1}{r_{12}} - \frac{1}{r_{13}} - \frac{1}{r_{14}} - \frac{1}{r_{23}} - \frac{1}{r_{24}} + \frac{1}{r_{34}} \quad (1)$$

where the notation 1 designates the positively charged muon μ^+, the notation 2 (or +) means the positron, while 3 (or -) and 4 (or -) stand for electrons. This system of notations will be used everywhere below in our study. By solving the corresponding Schrödinger equation $H\Psi = E\Psi$ one can determine the wave function of all bound states in the MuPs system. In reality, there is only one bound (ground) state in the muonium-positronium MuPs. The total energy of this ground state is $\approx$ -0.78631730(15) $a.u.$ [1]. The known accurate wave functions Ψ allows one to obtain the expectation values of many operators, including the electron-positron and muon-positron delta-functions, respectively. In turn, such delta-functions are needed to evaluate the positron annihilation rate(s) in the MuPs system and predict the hyperfine structure splitting.

Annihilation of electron-positron pairs in the MuPs system is a very interesting process which can be observed experimentally. The largest annihilation rates correspond to the two- and three-photon annihilation. For the two-photon annihilation in [1] we have obtained the

following formula

$$\Gamma_{2\gamma}(\text{MuPs}) = 2\pi\alpha^4 ca_0^{-1}\Big[1 - \frac{\alpha}{\pi}\Big(5 - \frac{\pi^2}{4}\Big)\Big]\langle\delta(\mathbf{r}_{+-})\rangle = 100.3456053781 \cdot 10^9\langle\delta_{+-}\rangle \ sec^{-1} \quad . \ (2)$$

where $\langle\delta_{+-}\rangle$ is the expectation value of the electron-positron delta-function determined for the ground state in the MuPs system. The formula for the three-photon annihilation rate $\Gamma_{3\gamma}(\text{MuPs})$ takes the form

$$\Gamma_{3\gamma}(\text{MuPs}) = 2\frac{4(\pi^2 - 9)}{3}\alpha^5 ca_0^{-1}\langle\delta(\mathbf{r}_{+-})\rangle = 2.718545954 \cdot 10^8\langle\delta_{+-}\rangle \ sec^{-1} \quad\quad (3)$$

In these formulas and everywhere below α is the fine structure constant, c is the speed of light in vacuum and a_0 is the Bohr radius. Below, the numerical values of these constants have been taken from [2] and [3].

The rates of the four- and five-photon annihilations of the electron-positron pairs in the MuPs system are related with the $\Gamma_{2\gamma}(\text{MuPs})$ and $\Gamma_{3\gamma}(\text{MuPs})$, respectively, by the following approximate relations [4]

$$\Gamma_{4\gamma}(\text{MuPs}) \approx 0.274\Big(\frac{\alpha}{\pi}\Big)^2\Gamma_{2\gamma}(\text{MuPs}) \approx 1.478364 \cdot 10^{-6} \cdot \Gamma_{2\gamma}(\text{MuPs}) \quad\quad (4)$$

and

$$\Gamma_{5\gamma}(\text{MuPs}) \approx 0.177\Big(\frac{\alpha}{\pi}\Big)^2\Gamma_{3\gamma}(\text{MuPs}) \approx 9.550018 \cdot 10^{-7} \cdot \Gamma_{3\gamma}(\text{MuPs}) \quad\quad (5)$$

To a very good accuracy one can evaluate the total annihilation rate of the MuPs system by the following sum $\Gamma \approx \Gamma_{2\gamma} + \Gamma_{3\gamma} + \Gamma_{4\gamma} + \Gamma_{5\gamma} \approx \Gamma_{2\gamma} + \Gamma_{3\gamma} \approx 1006.174599735 \cdot 10^8\langle\delta_{+-}\rangle$ $sec^{-1} \approx 2.45317693 \cdot 10^9 \ sec^{-1}$. In other words, the knowledge of accurate values of the $\Gamma_{2\gamma}$ and $\Gamma_{3\gamma}$ annihilation rates is sufficient to predict the total life-time of the MuPs system against positron annihilation $\tau = \frac{1}{\Gamma} \approx 4.07635 \cdot 10^{-10} \ sec$.

In addition to the two-, three-, four- and five-photon annihilations of the electron-positron pair the one-photon and zero-photon annihilations may also play an important role in some applications. In general, these rates are very small. Note that these rates have been discussed (and evaluated) in our earlier studies (see, e.g., [1] and references therein). In particular, the closed formula for the zero-photon annihilation rate $\Gamma_{0\gamma}$ takes the form

$$\Gamma_{0\gamma} = \xi\frac{147\sqrt{3}\pi^3}{2} \cdot \alpha^{12}(ca_0^{-1}) \cdot \langle\delta_{\mu^+ + --}\rangle = 5.0991890 \cdot 10^{-4} \cdot \xi \cdot \langle\delta_{\mu^+ + --}\rangle \quad sec^{-1} \quad\quad (6)$$

where $\langle\delta_{\mu^+ + --}\rangle$ is the expectation value of the four-particle delta-function in the ground state of muonium-positronium (MuPs). The numerical value of $\langle\delta_{\mu^+ + --}\rangle$ is the probability to find

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all four particles at one volume with the radius αa_0. The unknown (dimensionless) factor ξ has the numerical value close to unity. The expectation value of the four-particle delta-function determined in our calculations is $\approx 1.785222 \cdot 10^{-4}$ (in $a.u.$). From here one finds that $\Gamma_{0\gamma}(\text{MuPs}) \approx 9.10318(10) \cdot 10^{-8}\xi\ sec^{-1}$. For approximate evaluations we can assume that the factor ξ equals unity. In this case one finds that $\Gamma_{0\gamma}(\text{MuPs}) \approx 9.10318(10) \cdot 10^{-8}\ sec^{-1}$.

The one-photon annihilation of the electron-positron pair in MuPs can be considered as a regular two-photon annihilation, but one of the two emitted photons is absorbed either by the remaining electron e^-, or by the muon μ^+. In the first case one can observe the emission of the fast electron. The probability of this process is given by the formula

$$\Gamma_{1\gamma}^{(1)} = \frac{64\pi^2}{27} \cdot \alpha^8 (ca_0^{-1}) \cdot \langle \delta_{+--} \rangle = 1.066420947 \cdot 10^3 \cdot \langle \delta_{+--} \rangle \quad sec^{-1}, \tag{7}$$

where $\langle \delta_{+--} \rangle = \langle \delta(\mathbf{r}_{+-})\delta(\mathbf{r}_{--}) \rangle$ is the expectation value of the triple electron-positron delta-function determined for the ground state of the MuPs system. Its numerical value is the probability to find all three corresponding particles at one spatial point with spatial radius $\alpha a_0 \approx \frac{a_0}{137}$. Our best numerical treatment to-date for the $\langle \delta_{+--} \rangle$ value gives $\approx 3.631815 \cdot 10^{-4}$, and therefore, $\Gamma_{1\gamma}^{(1)} \approx 3.87063(10) \cdot 10^{-1}\ sec^{-1}$ for the ground state in the MuPs system.

Analysis of the second one-photon annihilation in MuPs is significantly more complicated (see discussion in [1]). To evaluate the corresponding annihilation rate $\Gamma_{1\gamma}^{(2)}$ one needs undertake an extensive QED consideration [5]. Such an analysis is also required to determine the accurate value of the factor ξ in Eq.(6). Here we shall not discuss these problems.

Now, let us discuss the hyperfine structure splitting in the MuPs system. The hyperfine structure splitting in the MuPs system is written in the form [1]

$$a = 14229.1255 \cdot \langle \delta_{\mu^+ e^+} \rangle \tag{8}$$

To obtain this formula we have used the following values for the muon mass m_μ and for the factors g_+ and g_μ [2], [3]:

$$m_\mu = 206.768264 m_e \quad , \quad g_+ = -2.0023193043718 \quad g_\mu = -2.0023318396 \tag{9}$$

where m_e is the electron/positron mass at rest. By using our improved expectation value for the muon-positron delta-function $\langle \delta_{\mu^+ e^+} \rangle \approx 1.6216075 \cdot 10^{-3}\ a.u.$, one finds that the constant in Eq.(8) is $a \approx 23.074\ MHz$. This is the energy difference between the state with $J = 0$ and $J = 1$, where the notation J stands the total spin of the muon-positron pair.

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Thus, we have considered the positron annihilation in the MuPs system ($\mu^+ e_2^- e^+$ or muonium-positronium). The two-, three-, four- and five-photon annihilation rates of the MuPs system are determined to high numerical accuracy (see Table I). The rate of zero-photon annihilation $\Gamma_{0\gamma}(\text{MuPs})$ and first one-photon annihilation rate $\Gamma_{1\gamma}^{(1)}(\text{MuPs})$ have also been estimated (see Table I). To evaluate zero-photon annihilation rate $\Gamma_{0\gamma}(\text{MuPs})$ to better accuracy and obtain the explicit formula for the second one-photon annihilation rate $\Gamma_{1\gamma}^{(1)}(\text{MuPs})$ one needs to perform an extensive QED analysis. We also determine the hyperfine structure splitting between singlet $J = 0$ and triplet $J = 1$ spin states of the muon-positron pair in MuPs. By using our expectation value for the $\delta_{\mu^+ +}$ delta-function (see Table I) we have found that the hyperfine splitting in the ground state of the MuPs system is $\approx 23.074(10)\ MHz$.

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TABLE I: Annihilation rates $\Gamma_{n\gamma}$ (in sec^{-1}) of the n-photon annihilation ($n = 2, 3, 4, 5, 1, 0$) of the electron-positron pair in the four-body MuPs system.

$\langle \delta_{+-} \rangle$	$\Gamma_{2\gamma}$	$\Gamma_{3\gamma}$	$\Gamma_{4\gamma}$	$\Gamma_{5\gamma}$
$2.4381225 \cdot 10^{-2}$	$2.446549 \cdot 10^{9}$	$6.62815 \cdot 10^{6}$	$3.61689 \cdot 10^{4}$	$6.32989 \cdot 10^{1}$
$\langle \delta_{+--} \rangle$	$\Gamma_{1\gamma}^{(1)}$	$\langle \delta_{\mu^{+}+--} \rangle$	$\Gamma_{0\gamma}$	$\langle \delta_{\mu^{+}+} \rangle$
$3.679653 \cdot 10^{-4}$	$3.92406 \cdot 10^{-1}$	$1.7755 \cdot 10^{-4}$	$9.0537 \cdot 10^{-8}$	$1.621618 \cdot 10^{-3}$