

Scaling and intermittency in incoherent α –shear dynamo

Dhrubaditya Mitra^{1*} and Axel Brandenburg^{1,2†}

¹*NORDITA, AlbaNova University Center, Roslagstullsbacken 23, SE-10691 Stockholm, Sweden*

²*Department of Astronomy, AlbaNova University Center, Stockholm University, SE-10691 Stockholm, Sweden*

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ABSTRACT

We consider mean-field dynamo models with fluctuating α effect, both with and without shear. The α effect is chosen to be Gaussian white noise with zero mean and given covariance. We show analytically that the mean magnetic field does not grow, but, in an infinitely large domain, the mean-squared magnetic field shows exponential growth of the fastest growing mode at a rate proportional to the shear rate, which agrees with earlier numerical results of Yousef et al. (2008) and recent analytical treatment by Heinemann et al. (2011) who use a method different from ours. In the absence of shear, an incoherent α^2 dynamo may also be possible. We further show by explicit calculation of the growth rate of third and fourth order moments of the magnetic field that the probability density function of the mean magnetic field generated by this dynamo is non-Gaussian.

Key words: Dynamo – magnetic fields – MHD – turbulence

1 INTRODUCTION

The dynamo mechanism that generates large-scale magnetic fields in astrophysical objects is a topic of active research. Almost all astrophysical objects, e.g., the galaxy, or the sun, shows presence of large-scale shear (differential rotation). It is now well established that this shear is an essential ingredient to the dynamo mechanism. This view is also supported by direct numerical simulations (DNS). In particular, DNS of convective flows have been able to generate large-scale dynamos predominantly in the presence of shear (Käpylä et al. 2008; Hughes & Proctor 2009), while non-shearing large-scale dynamos are only possible at very high rotation rates (Käpylä et al. 2009). The other essential ingredient to the large-scale dynamo mechanism is helicity of the flow which is often described by the α effect. Indeed most dynamos, including the early model by Parker (1955), are the result of an α effect combined with shear. Of these two ingredients, shear is typically constant over the time scale of generation of the dynamo. For the case of the solar dynamo, shear or differential rotation are constrained by helioseismology. By contrast, measuring the α effect is a non-trivial exercise. Whenever it has been obtained from DNS studies, it was found to have large fluctuations in space and time; see e.g., BRRK for the PDF of different components of α and Cattaneo & Hughes (1996) for the fluctuating time series of the total electromotive force, which is however different from the α effect. What effect do these fluctuations have on the properties

of the α –shear dynamo? In particular, can a fluctuating α effect about a *zero* mean drive a large-scale dynamo in conjunction with shear? Recent DNS studies (Brandenburg 2005a; Yousef et al. 2008; Brandenburg et al. 2008, hereafter referred to as BRRK) suggest that the answer to this question is yes. Yousef et al. (2008) have further provided compelling evidence that the growth rate of the large-scale magnetic energy in such dynamos scales linearly with the shear rate, and the wavenumber of the fastest growing mode scales as the square root of the shear rate. It has been proposed that such a dynamo can also emerge by an alternate mechanism (which has nothing to do with fluctuations of α effect) involving the interaction between shear and mean current density. This mechanism is called the *shear–current* effect (Rogachevskii & Kleeorin 2003). However recent DNS studies of BRRK as well as those of Brandenburg (2005b) have not found evidence in support of it, and analytical works (Rädler & Stepanov 2006; Rüdiger & Kitchatinov 2006; Sridhar & Subramanian 2009) have doubted its existence. Furthermore, the shear–current effects does not produce the observed scaling, namely, growth-rate scales with the shear rate squared, and the wavenumber of the fastest growing mode scales linearly with the shear rate; see section 4.2 of BRRK.

It behooves us then to consider the interaction between fluctuating α effect and shear as a possible dynamo mechanism. Kraichnan (1976) was first to propose that fluctuations of kinetic helicity can give rise to negative turbulent diffusivities, thereby giving rise to a dynamo that is effective only in small scales (Moffatt 1978). However, more relevant in the present context is the work of

* E-mail: dhruba.mitra@gmail.com

† E-mail: brandenb@nordita.org

Vishniac & Brandenburg (1997), who investigated, in addition to a one-dimensional numerical model, a simple zero-dimensional analytical mean-field model in a one-mode truncation with shear and fluctuating α effect, which they referred to as incoherent α - Ω dynamos. By dividing the poloidal field component by the toroidal one, they turned the mean-field dynamo equations with multiplicative noise into one with additive noise of Langevin type. They drew an analogy between the behaviour of the mean magnetic field and Brownian motion in the sense that the mean field does not grow although the mean-square field grows with a growth rate that scales with shear rate to the 2/3 power.

Over the last decade, the problem has been studied using a variety of different approaches (Sokolov 1997; Silant'ev 2000; Fedotov et al. 2006 2006; Proctor 2007; Kleorin & Rogachevskii 2008; Sur & Subramanian 2009). Models with fluctuating α effect which depends on both space and time can be divided into two categories, one in which α is inhomogeneous (Silant'ev 2000; Proctor 2007; Kleorin & Rogachevskii 2008) and the other in which it is a constant in space. In this paper we shall confine ourselves to the second class of models. For the second category, using a multiscale expansion, Proctor (2007) found a quadratic dependence between growth rate and shear rate. This result is contradicted by Kleorin & Rogachevskii (2008) who found that the mean field averaged over α does not show a dynamo, actually the fluctuating α effect adds up to the diagonal components of the turbulent magnetic diffusivity tensor. Heinemann et al. (2011) have recently proposed a simple analytically tractable model, in which they consider the first and second moments (calculated over the distribution of α) of the horizontally averaged mean field. The first moment shows the absence of dynamo effect, but the second moment shows exponential growth with the same scaling behaviour observed by Yousef et al. (2008). Motivated by their study, we solve here a similar model using a technique known as *Gaussian integration by parts* to obtain the same results. We further show that, even in the absence of shear, it may be possible for incoherent α^2 dynamos to operate. For the model of Vishniac & Brandenburg (1997) we can also calculate the higher (third and fourth) order moments of the magnetic field to demonstrate that the probability distribution function (PDF) of the mean magnetic field generated by an incoherent α -shear dynamo is non-Gaussian.

2 MEAN-FIELD MODEL

Our mean-field model is designed to describe the simulations of Yousef et al. (2008) and BRRK. In particular, there is a large-scale velocity given by $\bar{\mathbf{U}} = Sx\mathbf{e}_y$. The turbulence is generated by an isotropic external random force with zero net helicity. The mean fields are defined by averaging over two coordinate directions x and y . By assuming shearing-periodic boundary conditions, as done in the simulations, this averaging obeys the Reynolds rules. By virtue of the divergence-less properties of the mean magnetic field, v.i.z., $\partial_j \bar{B}_j = 0$, \bar{B}_z can be set to zero. The resultant mean-field equations then have the following form (BRRK)

$$\partial_t \bar{B}_x = -\alpha_{yx} \partial_z \bar{B}_x - \alpha_{yy} \partial_z \bar{B}_y - \eta_{yx} \partial_z^2 \bar{B}_y + \eta_{yy} \partial_z^2 \bar{B}_x, \quad (1)$$

$$\partial_t \bar{B}_y = S \bar{B}_x + \alpha_{xx} \partial_z \bar{B}_x + \alpha_{xy} \partial_z \bar{B}_y - \eta_{xy} \partial_z^2 \bar{B}_x + \eta_{xx} \partial_z^2 \bar{B}_y. \quad (2)$$

Here, \bar{B}_x and \bar{B}_y are the x and y components of the mean (averaged over x and y coordinate directions) magnetic field, η_{ij}^\dagger are the four relevant components of the turbulent magnetic diffusivity tensor and α_{ij} are the four relevant components of the α tensor. The shear-current effect works via a non-zero η_{yx} , provided its sign is the same as that of S . In this paper we do not consider the possibility of the shear-current effect hence we set $\eta_{yx} = 0$. Here we have ignored the molecular diffusivity, as we are interested in the limit of very high magnetic Reynolds numbers.

The fluctuating α effect is modelled by choosing each component of the α tensor be an independent Gaussian random number with zero mean and the following covariance (no summation over repeated indices is assumed):

$$\langle \alpha_{ij}(t) \alpha_{kl}(t') \rangle = D_{ij} \delta_{ik} \delta_{jl} \delta(t - t'). \quad (3)$$

We also assume that α_{ij} are constant in space. Note that DNS studies of BRRK have shown that the coefficients of turbulent diffusivity also show fluctuations in time, but we have ignored that in this paper. To make our notation clear, let us clearly distinguish between two different kinds of averaging we need to perform. The mean fields themselves are constructed by Reynolds averaging, which in our case is horizontal averaging, and is denoted by $\bar{\cdot}$. As we are dealing with mean field models, the quantities appearing in our equations have already been Reynolds averaged. But, as the α_{ij} in our mean field equation are stochastic, we study the properties of our model by writing down evolution equations for different moments of the mean magnetic field averaged over the statistics of α_{ij} . This is denoted by the symbol $\langle \cdot \rangle$. To give an example, the first moment (mean over statistics of α) of the x component of the mean magnetic field is denoted by $\langle \bar{B}_x \rangle$ and the second moment (mean square) is denoted by $\langle \bar{B}_x^2 \rangle$.

3 RESULTS

3.1 First and second moments

Let us begin by averaging both sides of Equation (1) over α_{ij} . Two terms are non-trivial to average, v.i.z., $\langle \alpha_{yx} \partial_z \bar{B}_x \rangle$ and $\langle \alpha_{yy} \partial_z \bar{B}_y \rangle$. As the α effect is Gaussian with the covariance given in Equation (3), we have

$$\langle \alpha_{yx} \partial_z \bar{B}_x \rangle = D_{yx} \left\langle \frac{\delta \bar{B}_x}{\delta \alpha_{yx}} \right\rangle = D_{yx} \langle -\partial_z \bar{B}_x \rangle, \quad (4)$$

where the operator $\delta(\cdot)/\delta \alpha_{yx}$ denotes a functional derivative with respect to α_{yx} ; see Appendix A for a detailed derivation. It is convenient to do this in Fourier space. The Fourier transforms are given by

$$\hat{B}_x = \int \bar{B}_x e^{-ikz} dk, \quad \hat{B}_y = \int \bar{B}_y e^{-ikz} dk. \quad (5)$$

Proceeding with the other terms in a similar fashion as in Equation (4), we obtain

$$\partial_t \mathbf{C}^1 = \mathbf{N}_1 \mathbf{C}^1 \quad (6)$$

with $\mathbf{C}^1 = (\langle \hat{B}_x \rangle, \langle \hat{B}_y \rangle)$ and

$$\mathbf{N}_1 = \begin{bmatrix} -k^2(\eta_{yy} + D_{yx}) & k^2 \eta_{yx} \\ S + k^2 \eta_{xy} & -k^2(\eta_{xx} + D_{xy}) \end{bmatrix}. \quad (7)$$

If there is no shear-current effect, i.e., $\eta_{yx} = 0$, there is no dynamo. The fluctuations of the α effect actually enhance the diagonal components of the turbulent magnetic diffusivity. This result agrees with Heinemann et al. (2011) who used a different model and found that the mean magnetic field does not grow. Vishniac & Brandenburg (1997) also found the same for their simplified zero-dimensional model.

Although the mean magnetic field does not grow, the mean-squared magnetic field can still show growth. To study this we now write a set of equations for the time evolution of the second moment (covariance) of the mean magnetic field averaged over α_{ij} . We emphasize that we are not considering here the covariance of the actual magnetic field that would include also the small-scale magnetic fluctuations, which are relevant to the small-scale dynamo (Kazantsev 1968). The covariance of the actual magnetic field was later also considered by Hoyng (1987) in connection with α - Ω dynamos. Following Heinemann et al. (2011) we define a covariance vector,

$$\mathbf{C}^2 \equiv (\langle \hat{B}_x \hat{B}_x^* \rangle, \langle \hat{B}_y \hat{B}_y^* \rangle, \langle \hat{B}_x \hat{B}_y^* \rangle + \langle \hat{B}_x^* \hat{B}_y \rangle). \quad (8)$$

The evolution equation for the covariance vector is given by

$$\partial_t \mathbf{C}^2 = \mathbf{N}_2 \mathbf{C}^2, \quad (9)$$

where \mathbf{N}_2 is given by

$$\begin{bmatrix} -2k^2\eta_{yy} & 2k^2D_{yy} & k^2\eta_{yx} \\ 2k^2D_{xx} & -2k^2\eta_{xx} & S + k^2\eta_{xy} \\ 2(S + k^2\eta_{xy}) & 2k^2\eta_{yx} & -k^2(D_{yx} + D_{xy} + 2\eta_{xx}) \end{bmatrix}.$$

The only non-trivial terms in the derivation of the above equation are the terms which are products of components of α effect and two components of the magnetic field. We evaluate them by using the same technique used to obtain Equation (4); see Appendix A for details.

The characteristic equation of the matrix \mathbf{N}_2 is a third order equation, the solutions of which give the three solutions for the growth rate 2γ . For simplicity let us also choose $\eta_{xx} = \eta_{yy} \equiv \eta_t$. In other words, we take the turbulent magnetic diffusivity tensor to be diagonal. We further note that D_{yx} and D_{xy} contribute only in enhancing the turbulent magnetic diffusivity of C_3 , so we can safely ignore them compared to η_{xx} . With these simplifying assumptions the equation for the growth rate reduces to

$$\xi^3 - 4k^2D_{yy} [S^2 + k^2D_{xx}\xi] = 0, \quad (10)$$

where $\xi = 2(k^2\eta_t + \gamma)$. For large enough S we can always ignore the second term inside the parenthesis of Equation (10). This gives the three roots of γ as

$$\gamma = -k^2\eta_t + \left(\frac{1}{2}k^2D_{yy}S^2\right)^{1/3} (1, \omega, \omega^2), \quad (11)$$

where $(1, \omega, \omega^2)$ are the three cube roots of unity, of which ω and ω^2 have negative real parts. The same dispersion relation is obtained by Heinemann et al. (2011). The wavenumber of the fastest growing mode, k^{peak} , is given by

$$k^{\text{peak}} = |S|^{1/2} \left(\frac{D_{yy}}{54\eta_t^3}\right)^{1/4}. \quad (12)$$

The growth rate of the fastest growing mode is given by

$$\gamma = \frac{2^{1/3}}{6} \left(1 - \frac{2^{1/6}}{\sqrt{3}}\right) \left(\frac{D_{yy}}{\eta_t}\right)^{1/2} |S|. \quad (13)$$

This is the same scaling numerically obtained by Yousef et al. (2008).

3.2 Incoherent α^2 dynamo

Let us now consider a different case where shear is zero. In that case, Equation (10) becomes a quadratic equation in ξ ,

$$\xi^2 - 4k^4D_{xx}D_{yy} = 0, \quad (14)$$

with solutions,

$$\gamma = k^2 \left(-\eta_t \pm \sqrt{D_{xx}D_{yy}}\right). \quad (15)$$

Hence, it may be possible for fluctuations of α to drive a large-scale dynamo (in the mean-square sense) even in the absence of velocity shear.

To summarise there are two possible dynamo mechanisms in our dynamo model. In both of them the magnetic field grows in the mean-square sense. The first one is an incoherent α -shear dynamo. For large enough shear this is the fastest growing mode. However, this dynamo has no oscillating modes because the modes for which γ have a non-zero imaginary part have negative real part. An incoherent α^2 dynamo mechanism also exists in this model. The condition for excitation of a fluctuating α -shear dynamo is

$$\frac{(k^2D_{yy}S^2/2)^{1/3}}{k^2\eta_t} > 1, \quad (16)$$

and the condition for excitation of a fluctuation α^2 dynamo is

$$\frac{\sqrt{D_{xx}D_{yy}}}{\eta_t} > 1. \quad (17)$$

The condition that a fluctuating α^2 dynamo is preferred compared to an α -shear one is

$$\frac{4k^8D_{xx}^3D_{yy}}{S^4} > 1, \quad (18)$$

or $\sqrt{2}k^2D_{xx}/|S| > 1$ for $D_{xx} = D_{yy}$.

To compare with DNS we need to use some estimates of η_t , D_{xx} and D_{yy} . We use $\eta_t = u_{\text{rms}}/3k_f$, as obtained by Sur et al. (2008) without shear, and a slightly larger value by BRRK in the presence of shear. We further use $D_{xx} = D_{yy} = u_{\text{rms}}^2/9$. For this choice of parameters the incoherent α^2 dynamo does not grow. Here, u_{rms} is the mean-squared velocity and k_f corresponds to the characteristic Fourier mode of the forcing if the turbulence has been maintained by an external force, as done by Yousef et al. (2008) or BRRK. For turbulence maintained by convection, k_f should be replaced by the Fourier mode corresponding to the integral scale of the turbulence. Typically, mean-field theory applies for modes with $k < k_f$. Lengths are measured in units of $1/k_f$ and velocity is measured in the unit of u_{rms} . This makes $1/u_{\text{rms}}k_f$ the unit of time. The two dispersion relations are plotted in Fig. 1 for different values of S .

3.3 Scaling in a simpler zero-dimensional model

The essential physics of Equations (1) and (2) can be captured by an even simpler mean-field model in a one-mode truncation, but with fluctuating α effect, introduced by

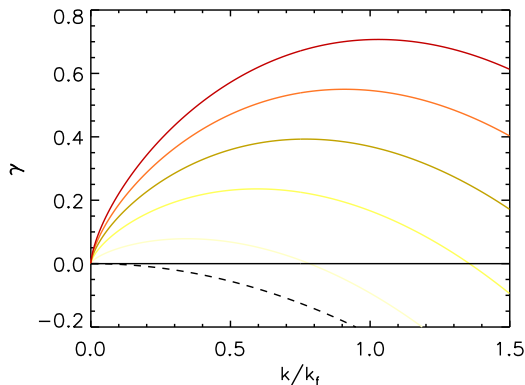


Figure 1. Sketch of dispersion relations, Equation (10) for different values of velocity shear S (continuous lines with difference colours from bottom to top $S = 0.5, 1.5, 2.5, 3.5$), and Equation (15) (broken line). Velocity shear, and γ has dimensions of inverse time and are measured in the units of $u_{\text{rms}}k_f$.

Vishniac & Brandenburg (1997). Their model, rewritten in our notations and dropping all k factors,

$$\partial_t \bar{B}_x = \alpha \bar{B}_y - \eta_t \bar{B}_x, \quad (19)$$

$$\partial_t \bar{B}_y = -S \bar{B}_x - \eta_t \bar{B}_y, \quad (20)$$

can be analysed in exactly similar ways. By construction, this model does not have fluctuating α^2 effect, and α is a Gaussian random variable with zero mean and covariance

$$\langle \alpha(t)\alpha(t') \rangle = D\delta(t-t'). \quad (21)$$

For this model we adopt a more general framework and define the growth-rate of the p -th order moment of the magnetic field (the first order is the mean and the second order is the covariance) to be $p\gamma_p$. Explicit calculations, shown in Appendix B, give,

$$\gamma_1 = -\eta_t, \quad \gamma_2 = -\eta_t + \left(\frac{4DS^2}{8}\right)^{1/3} \sim S^{2/3}. \quad (22)$$

We show in the Appendix B that this two-third scaling with shear rate in this zero-dimensional model is equivalent to $\gamma \sim S$ scaling in for Equations (1) and (2).

3.4 Possibility of intermittency

We note that Equations (1) and (2) can be considered as coupled ordinary differential equations of the Langevin type but with multiplicative noise. We have taken the probability distribution function (PDF) of the noise to be Gaussian. But, by virtue of multiplicative noise, the PDF of the magnetic field may be non-Gaussian. We have already calculated the first and second moments of this PDF. To probe non-Gaussianity we need to calculate the higher order moments. For Equations (1) and (2) this is a formidable problem. But it is far simpler for the model of Vishniac & Brandenburg (1997). Sokolov (1997) has already argued that the statistics of the magnetic field in the model of Vishniac & Brandenburg (1997) is intermittent; see also Sur & Subramanian (2009). In Appendix B we show that the growth rate for the third and fourth order moment are given by

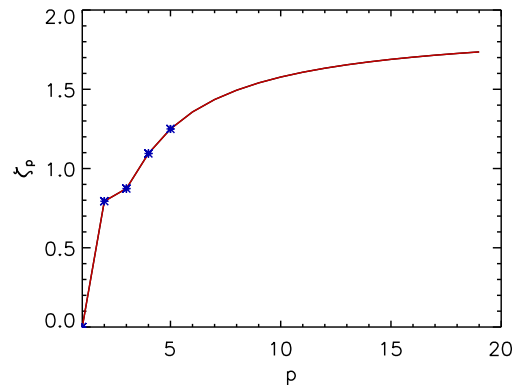


Figure 2. ζ_p versus p as obtained from Equation (24). If the magnetic field had obeyed Gaussian statistics, ζ_p versus p would have been constant.

$$\gamma_3 = -\eta_t + \left(\frac{18DS^2}{27}\right)^{1/3}, \quad \gamma_4 = -\eta_t + \left(\frac{84DS^2}{64}\right)^{1/3}. \quad (23)$$

Clearly, γ_p has the same scaling dependence on D , and S independent of p , but nevertheless they are different, i.e., the PDF is non-Gaussian. This non-Gaussianity is best described by plotting

$$\zeta_p = \frac{\gamma_p + \eta_t}{(DS^2)^{1/3}} \quad (24)$$

versus p in Fig. 2.

Let us now conjecture that as $p \rightarrow \infty$, ζ_p remains finite. Remembering that $\zeta_1 = 0$, the general form would then be

$$\zeta_p = \left(\frac{(p-1)(a_0 + a_1p + a_2p^2)}{p^3}\right)^{1/3}. \quad (25)$$

Substituting the form back in Equations (22) and (23) we find $a_0 = 36$, $a_1 = -30$, and $a_2 = 7$. This formula is also plotted in Fig. 2.

4 CONCLUSIONS

In this paper we have analytically solved a mean-field dynamo model with fluctuating α effect to find self-excited solutions. We have studied the growth rate of different moments (calculated over the statistics of α) of the magnetic field. There are three crucial aspects in which our results, the DNS of Yousef et al. (2008), and the analytical results by Heinemann et al. (2011) agree: (a) There is no dynamo for the first moment of the magnetic field, (b) the second moment (mean-square) of the magnetic field shows dynamo action, and (c) the fastest growing mode has a growth rate $\gamma \sim S$ at Fourier mode $k^{\text{peak}} \sim \sqrt{S}$. We have further shown that these aspects of our results can even be reproduced by a simpler zero-dimensional mean field model due to Vishniac & Brandenburg (1997). For this simpler model we have also calculated the growth rate for third and fourth order moments and have explicitly demonstrated non-Gaussian nature of the PDF of the magnetic field. Given that the incoherent α -shear dynamo (often with an additional coherent part) is the most common dynamo mechanism our results provide a qualitative reasoning of why

the large-scale magnetic field in the universe are intermittent. However note also that we have merely shown that the growth rate of the different moments of the magnetic field are different. The eventual nature of the PDF of the magnetic field will also be influenced by the saturation of this dynamo which is outside the realm of this paper.

The most restrictive assumption in our model is the assumption of the white-in-time nature of the α effect. This assumption however allows us to obtain closed equations for all the moments of the magnetic field. Numerical solutions of fluctuating α -shear dynamos often uses an α effect which has finite correlation time, typically telegraphic noise (Sur & Subramanian 2009). It is not immediately obvious how our results would be applied to such models. We speculate that our results, in particular the scaling behaviour, would still apply, with

$$D \approx \alpha_{\text{rms}}^2 \tau_{\text{cor}}, \quad (26)$$

where α_{rms} is the root-mean-square value of α and τ_{cor} is the characteristic correlation time of α . It is interesting to note that, even under the restrictive assumption of the white-in-time nature of the α effect, we obtain the same scaling behaviour as the DNS studies. The assumption of the Gaussian nature of α is well supported by numerical evidence; see Fig. 10 of BRRK. We have also assumed that different components of α_{ij} are independent of each other. This assumption should be checked from future DNS studies. The formalism described in this paper is capable of dealing with the more general case where the different components are indeed correlated.

Let us stress here that our results are also valid for the usual large-scale mean-field dynamo. Kolokolov, Lebedev and Sizov (2010) have recently applied similar techniques to study small-scale kinematic dynamos in a smooth delta-correlated velocity field in the presence of shear to find

$$\gamma_n = \frac{3}{2^{5/3}} n^{4/3} D^{1/3} S^{2/3} \sim \lambda n^{4/3}, \quad (27)$$

where λ is the expression for the largest Lyapunov exponent describing the divergence of two initially close fluid particles, This Lyapunov exponent was earlier obtained for such flows by Turitsyn (2007). Interestingly, this is exactly the same scaling with shear as in Vishniac & Brandenburg (1997). In the absence of shear the small-scale dynamo can still operate (Chertkov, Kolokolov and Vergassola 1997) with $\gamma_n \sim n^2$.

Proctor (2007) have also considered a model similar to ours, although somewhat simpler and more relevant to the solar dynamo, using multiscale expansions. After averaging over the fluctuating α effect, he still finds an effective α effect from which he obtains a dynamo which grows in the mean (as opposed to mean-square in our case). His results give the scaling, $k^{\text{peak}} \sim S$ and $\gamma \sim S^2$ which disagree with the DNS results of Yousef et al. (2008).

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APPENDIX A: AVERAGING OVER GAUSSIAN NOISE

We explain here the technique used to derive Equation (4). Let us begin by considering Gaussian vector-valued noise $\nu_j(t)$ (not necessarily white-in-time) and an arbitrary functional of that, $F(\boldsymbol{\nu})$. Then,

$$\langle F(\boldsymbol{\nu})\nu_j(t) \rangle = \int dt' \langle \nu_j(t)\nu_k(t') \rangle \left\langle \frac{\delta F}{\delta \nu_k(t')} \right\rangle, \quad (\text{A1})$$

where the average factorises by virtue of the Gaussian property of the noise. Here the operator $\delta(\cdot)/\delta \nu_k$ is the functional derivative with respect to $\boldsymbol{\nu}$. This useful identity often goes by the name *Gaussian integration by parts*; see, e.g., Zinn-Justin (1999), Section 4.2 for a proof; see also Frisch (1996) and Frisch & Wirth (1997) where this method has been used to derive closed moment equations for the Kraichnan model of passive scalar advection (Kraichnan 1968).

Let us now apply this identity to evaluate

$$\langle \alpha_{yx}(t)\bar{B}_x \rangle = \int dt' \langle \alpha_{yx}(t)\alpha_{kl}(t') \rangle \left\langle \frac{\delta \bar{B}_x}{\delta \alpha_{kl}(t')} \right\rangle, \quad (\text{A2})$$

where we have considered the magnetic field to be a functional of α_{ij} . If we view Equations (1) and (2) as stochastic ordinary differential equations (SDEs) with multiplicative noise, Equation (A2) corresponds to using the Stratanovich prescription for the SDEs, see e.g., Gardiner (1994). Substituting the covariance of α_{ij} from Equation (3) in Equation (A2), integrating the δ function over time, contracting over the Kronecker deltas, and evaluating the functional derivative using Equation (1), we obtain Equation (4).

The functional derivatives of the components of the magnetic field with respect to α_{ij} can be obtained by first formally integrating Equations (1) and (2) to obtain $\bar{B}_x(t)$ and $\bar{B}_y(t)$, respectively, and then calculating their functional derivatives with respect to α_{ij} ; see however Zinn-Justin (1999), Section 4.2, for a detailed discussion. For reference, all the non-zero functional derivatives needed are given below:

$$\begin{aligned} \frac{\delta \bar{B}_x}{\delta \alpha_{yx}} &= -\partial_z \bar{B}_x, & \frac{\delta \bar{B}_x}{\delta \alpha_{yy}} &= -\partial_z \bar{B}_y, \\ \frac{\delta \bar{B}_y}{\delta \alpha_{xx}} &= \partial_z \bar{B}_x, & \frac{\delta \bar{B}_y}{\delta \alpha_{xy}} &= \partial_z \bar{B}_y. \end{aligned} \quad (\text{A3})$$

To calculate the second moment of the magnetic field we need to evaluate terms of the general form

$$\langle \alpha_{ij} \hat{B}_k \hat{B}_l^* \rangle = D_{ij} \left\langle \frac{\delta \hat{B}_k}{\delta \alpha_{ij}} \hat{B}_l^* + \hat{B}_k \frac{\delta \hat{B}_l^*}{\delta \alpha_{ij}} \right\rangle. \quad (\text{A4})$$

The only non-zero functional derivative are given by the Fourier transform in space of both sides of Equation (A3). We show this explicitly for two examples,

$$\begin{aligned} \langle \alpha_{yx} \hat{B}_x \hat{B}_x^* \rangle &= D_{yx} \left\langle \frac{\delta \hat{B}_x}{\delta \alpha_{yx}} \hat{B}_x^* + \frac{\delta \hat{B}_x^*}{\delta \alpha_{yx}} \hat{B}_x \right\rangle \\ &= D_{yx} [-ik \langle \hat{B}_x \hat{B}_x^* \rangle + ik \langle \hat{B}_x \hat{B}_x^* \rangle] = 0, \end{aligned} \quad (\text{A5})$$

and

$$\begin{aligned} \langle \alpha_{yy} \hat{B}_x \hat{B}_y^* \rangle &= D_{yy} \left\langle \frac{\delta \hat{B}_y}{\delta \alpha_{yy}} \hat{B}_x^* + \frac{\delta \hat{B}_x^*}{\delta \alpha_{yy}} \hat{B}_y \right\rangle \\ &= ik D_{yy} \langle \hat{B}_y \hat{B}_y^* \rangle. \end{aligned} \quad (\text{A6})$$

APPENDIX B: A ZERO-DIMENSIONAL MEAN-FIELD MODEL WITH FLUCTUATING α

A simpler mean-field model in a one-mode truncation, but with fluctuating α effect, was introduced by Vishniac & Brandenburg (1997); see Equation (20)eq:VB97. For this model we define the following moments of successive orders,

$$\mathcal{C}^1 \equiv (\langle \bar{B}_x \rangle, \langle \bar{B}_y \rangle), \quad (\text{B1})$$

$$\mathcal{C}^2 \equiv (\langle \bar{B}_x^2 \rangle, \langle \bar{B}_y^2 \rangle, \langle \bar{B}_x \bar{B}_y \rangle), \quad (\text{B2})$$

$$\mathcal{C}^3 \equiv (\langle \bar{B}_x^3 \rangle, \langle \bar{B}_x^2 \bar{B}_y \rangle, \langle \bar{B}_x \bar{B}_y^2 \rangle, \langle \bar{B}_y^3 \rangle), \quad (\text{B3})$$

$$\mathcal{C}^4 \equiv (\langle \bar{B}_x^4 \rangle, \langle \bar{B}_x^3 \bar{B}_y \rangle, \langle \bar{B}_x^2 \bar{B}_y^2 \rangle, \langle \bar{B}_x \bar{B}_y^3 \rangle, \langle \bar{B}_y^4 \rangle). \quad (\text{B4})$$

Each of these moments satisfies a closed equation of the form

$$\partial_t \mathcal{C}^p = \mathbf{N}_p \mathcal{C}^p. \quad (\text{B5})$$

The matrices \mathbf{N}_p can be found by applying the technique described in Appendix A and by using the covariance of α given in Equation (21). We give below the first four matrices:

$$\mathbf{N}_1 = \begin{bmatrix} -\eta_t & 0 \\ -S & -\eta_t \end{bmatrix}, \quad (\text{B6})$$

$$\mathbf{N}_2 = \begin{bmatrix} -2\eta_t & 2D & 0 \\ 0 & -2\eta_t & -2S \\ -S & 0 & -2\eta_t \end{bmatrix}, \quad (\text{B7})$$

$$\mathbf{N}_3 = \begin{bmatrix} -3\eta_t & 0 & 6D & 0 \\ -S & -3\eta_t & 0 & D \\ 0 & -2S & -3\eta_t & 0 \\ 0 & 0 & -3S & -3\eta_t \end{bmatrix}, \quad (\text{B8})$$

$$\mathbf{N}_4 = \begin{bmatrix} -4\eta_t & 0 & 12D & 0 & 0 \\ -S & -4\eta_t & 0 & 6D & 0 \\ 0 & -2S & -4\eta_t & 0 & 2D \\ 0 & 0 & -3S & -3\eta_t & 0 \\ 0 & 0 & 0 & -4S & -4\eta_t \end{bmatrix}. \quad (\text{B9})$$

The growth rate at order p is defined to be $\mathcal{C}^p \sim \exp(p\gamma_p t)$. This gives $\gamma_1 = -\eta_t$, i.e., there is no dynamo. But this also gives dynamo modes with positive eigenvalues given by

$$\gamma_p \sim S^{2/3} D_{xx}^{1/3}, \quad p = 2, 3, \dots \quad (\text{B10})$$

The same result was obtained by Vishniac & Brandenburg (1997) for γ_2 by using a different method.

Note the striking similarity between matrix \mathbf{N}_2 in Equation (B7) and the matrix \mathbf{N}_2 in Equation (9). A trivial way of generalising Equation (B7) to one spatial dimension is to replace D and η_t in Equation (B7) by $k^2 D$ and $k^2 \eta_t$, respectively. The solution of the resultant eigenvalue problem gives the scaling, $\gamma \sim S$ and $k^{\text{peak}} \sim \sqrt{S}$. Thus, Equation (B7) for this zero dimensional model is equivalent to Equation (9) in the space-time model.