On a unified formulation of completely integrable systems

Răzvan M. Tudoran

Abstract. The purpose of this article is to show that a $\mathcal{C}^1$ differential system on $\mathbb{R}^n$ which admits a set of $n-1$ independent $\mathcal{C}^2$ conservation laws defined on an open subset $\Omega \subseteq \mathbb{R}^n$, is essentially $\mathcal{C}^1$ equivalent on an open and dense subset of Ω, with the linear differential system $u_1' = u_1,\ u_2' = u_2, \dots,\ u_n' = u_n$. The main results are illustrated in the case of two concrete dynamical systems, namely the three dimensional Lotka-Volterra system, and respectively the Euler equations from the free rigid body dynamics.

Mathematics Subject Classification (2000). 37J35; 37K10; 70H05; 70H06.

Keywords. integrable systems; Hamiltonian dynamics; linear normal forms.

1. Introduction

Recently, in [6] it is proved that an integrable $\mathcal{C}^1$ planar differential system is roughly speaking $\mathcal{C}^1$equivalent to the linear differential system $u_1' = u_1,\ u_2' = u_2$.

The purpose of this article is to generalize this result in the n dimensional case for a $\mathcal{C}^1$ differential system that admits a set of $n-1$ independent conservation laws. In the second section we show that such a system can always be realized as a Hamilton-Poisson dynamical system on a full measure open subset of $\mathbb{R}^n$ with respect to a rank 2 Poisson structure. In the third section a new time transformation will be explicitly constructed in order to bring the system to a linear differential system of the type $u_1' = u_1,\ u_2' = u_2, \dots,\ u_n' = u_n$. In the last section we illustrate the main results in the case of two concrete dynamical systems, namely the three dimensional Lotka-Volterra system, and respectively the Euler equations from the free rigid body dynamics.

For details on Poisson geometry and Hamiltonian dynamics, see, e.g. [1], [2], [11], [8], [9], [10], [12].

2. Hamiltonian divergence free vector fields naturally associated to integrable systems

In this section we give a method to construct a Hamilton-Poisson divergence free vector field, naturally associated with a given Hamilton-Poisson realization of a n dimensional differential system admitting $n-1$ independent conservation laws.

First step in this approach is to construct a Hamilton-Poisson realization of a given n dimensional differential system admitting $n-1$ independent integrals of motion.

Let us consider a $\mathcal{C}^1$ differential system on $\mathbb{R}^n$:

$$\begin{cases} \dot{x}_1 = X_1(x_1,\dots,x_n) \\ \dot{x}_2 = X_2(x_1,\dots,x_n) \\ \cdots \\ \dot{x}_n = X_n(x_1,\dots,x_n), \end{cases} \tag{2.1}$$

where $X_1, X_2, \dots, X_n \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ are arbitrary real functions. Suppose that $C_1, \dots, C_{n-2}, C_{n-1} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ are $n-1$ independent $\mathcal{C}^2$ integrals of motion of (2.1) defined on a nonempty open subset $\Omega \subseteq \mathbb{R}^n$.

Since $C_1, \dots, C_{n-2}, C_{n-1} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ are integrals of motion of the vector field $X = X_1\partial_{x_1} + \dots + X_n\partial_{x_n} \in \mathfrak{X}(\mathbb{R}^n)$, we obtain that for each $i \in \{1,\dots,n-1\}$

$$\langle \nabla C_i(x), X(x)\rangle = \sum_{j=1}^{n} \partial_{x_j} C_i \cdot \dot{x}_j = 0,$$

for every $x = (x_1,\dots,x_n) \in \Omega$, where $\langle\cdot,\cdot\rangle$ stand for the canonical inner product on $\mathbb{R}^n$, and respectively ∇ stand for the gradient with respect to $\langle\cdot,\cdot\rangle$.

Hence, by a standard multilinear algebra argument, the $\mathcal{C}^1$ vector field X is given as the $\mathcal{C}^1$ vector field $\star(\nabla C_1 \wedge \dots \wedge \nabla C_{n-1})$ multiplied by a $\mathcal{C}^1$ real function (rescaling function), where $\star$ stand for the Hodge star operator for multivector fields (see for details e.g. [5]). It may happen that the domain of definition for the rescaling function to be a proper subset of Ω. In the following we will consider the generic case when the rescaling function is defined on an open and dense subset of Ω. In order to simplify the notations, we will also denote this set by Ω.

Consequently, the vector field X can be realized on the open set $\Omega \subseteq \mathbb{R}^n$ as the Hamilton-Poisson vector field $X_H \in \mathfrak{X}(\Omega)$ with respect to the Hamiltonian function $H := C_{n-1}$ and respectively the Poisson bracket defined for $f, g \in \mathcal{C}^1(\Omega, \mathbb{R})$ by:

$$\{f,g\}_{\nu;C_1,\dots,C_{n-2}} dx_1 \wedge \dots \wedge dx_n = \nu dC_1 \wedge \dots dC_{n-2} \wedge df \wedge dg,$$

where $\nu \in \mathcal{C}^1(\Omega, \mathbb{R})$ is a given real function (rescaling). For $\nu \equiv 1$, the associated Poisson bracket it is exactly the Flaschka-Raţiu bracket. For similar Hamilton-Poisson formulations of completely integrable systems see also [3], [7].

In coordinates, the bracket $\{f,g\}_{\nu;C_1,\dots,C_{n-2}}$ is given by:

$$\{f,g\}_{\nu;C_1,\dots,C_{n-2}} = \nu \cdot \frac{\partial(C_1,\dots,C_{n-2},f,g)}{\partial(x_1,\dots,x_n)}.$$

Note that $\{C_1,\dots,C_{n-2}\}$ is a complete set of Casimirs for the Poisson bracket $\{\cdot,\cdot\}_{\nu;C_1,\dots,C_{n-2}}$.

Recall that the action of the Hamiltonian vector field $X_H \in \mathfrak{X}(\Omega)$ on an arbitrary real function $f \in \mathcal{C}^1(\Omega,\mathbb{R})$, is given by:

$$X_H(f) = \{f, H\}_{\nu;C_1,\dots,C_{n-2}} \in \mathcal{C}^1(\Omega,\mathbb{R}).$$

Hence, the differential system (2.1) can be written in Ω as a Hamilton-Poisson dynamical system of the type:

$$\begin{cases} \dot{x}_1 = \{x_1, H\}_{\nu;C_1,\dots,C_{n-2}} \\ \dot{x}_2 = \{x_2, H\}_{\nu;C_1,\dots,C_{n-2}} \\ \cdots \\ \dot{x}_n = \{x_n, H\}_{\nu;C_1,\dots,C_{n-2}}, \end{cases}$$

or equivalently

$$\begin{cases} \dot{x}_1 = \nu \cdot \dfrac{\partial(C_1,\dots,C_{n-2},x_1,H)}{\partial(x_1,\dots,x_n)} \\ \dot{x}_2 = \nu \cdot \dfrac{\partial(C_1,\dots,C_{n-2},x_2,H)}{\partial(x_1,\dots,x_n)} \\ \cdots \\ \dot{x}_n = \nu \cdot \dfrac{\partial(C_1,\dots,C_{n-2},x_n,H)}{\partial(x_1,\dots,x_n)}. \end{cases} \tag{2.2}$$

Consequently, the components of the vector field $X = X_1\partial_{x_1} + \cdots + X_n\partial_{x_n}$ which generates the differential system (2.1), are given in Ω as follows:

$$X_i = \nu \cdot \frac{\partial(C_1,\dots,C_{n-2},x_i,H)}{\partial(x_1,\dots,x_n)},$$

for $i \in \{1,\dots,n\}$.

Next result gives a method to construct a divergence free vector field out of the vector field X. The divergence operator we will use in this approach is the divergence associated with the standard Lebesgue measure on $\mathbb{R}^n$, namely $\mathcal{L}_X dx_1 \wedge \cdots \wedge dx_n = \operatorname{div} X dx_1 \wedge \cdots \wedge dx_n$, where $\mathcal{L}_X$ stand for the Lie derivative along the vector field X.

Theorem 2.1. *The vector field* $\widetilde{X} := \dfrac{1}{\nu} \cdot X$ *is a divergence free vector field on* $\Omega \setminus \mathcal{Z}(\nu)$*, where* $\mathcal{Z}(\nu) = \{(x_1,\dots,x_n) \in \Omega \mid \nu(x_1,\dots,x_n) = 0\}$.

Proof. Note that the components of the vector field $\widetilde{X} = \widetilde{X}_1\partial_{x_1} + \cdots + \widetilde{X}_n\partial_{x_n}$ are given by:

$$\widetilde{X}_i = \frac{\partial(C_1,\dots,C_{n-2},x_i,H)}{\partial(x_1,\dots,x_n)},$$

for $i \in \{1,\dots,n\}$. By definition, the vector field $\widetilde{X}$ is a Hamilton-Poisson vector field with respect to the Flaschka-Raţiu bracket, and having the same Hamiltonian H as the vector field X.

Hence, the divergence of $\widetilde{X}$ is given by:

$$\begin{aligned}\operatorname{div}(\widetilde{X}) &= \sum_{i=1}^{n} \partial_{x_i} \widetilde{X}_i = \sum_{i=1}^{n} \partial_{x_i} \frac{\partial(C_1, \ldots, C_{n-2}, x_i, H)}{\partial(x_1, \ldots, x_n)} \\ &= \sum_{i=1}^{n} (-1)^{i+n-1} \partial_{x_i} \frac{\partial(C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)},\end{aligned}$$

where the notation "$\widehat{x}_i$" means that "x_i" is omitted.

Let us now analyze the general term in the above sum. By using the derivative of a determinant we obtain the following:

$$\begin{aligned}\partial_{x_i} \frac{\partial(C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} &= \frac{\partial(\partial_{x_i} C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} + \cdots + \frac{\partial(C_1, \ldots, \partial_{x_i} C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} \\ &+ \frac{\partial(C_1, \ldots, C_{n-2}, \partial_{x_i} H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)}.\end{aligned}$$

Hence,

$$\begin{aligned}&\operatorname{div}(\widetilde{X}) = \sum_{i=1}^{n} (-1)^{i+n-1} \Big(\frac{\partial(\partial_{x_i} C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} + \cdots + \frac{\partial(C_1, \ldots, \partial_{x_i} C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} \\ &+ \frac{\partial(C_1, \ldots, C_{n-2}, \partial_{x_i} H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} \Big) \\ &= \sum_{i=1}^{n} (-1)^{i+n-1} \frac{\partial(\partial_{x_i} C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} + \cdots + \sum_{i=1}^{n} (-1)^{i+n-1} \frac{\partial(C_1, \ldots, \partial_{x_i} C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)} \\ &+ \sum_{i=1}^{n} (-1)^{i+n-1} \frac{\partial(C_1, \ldots, C_{n-2}, \partial_{x_i} H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)}.\end{aligned}$$

Next we prove that each of the above sums vanishes. In order to do that, it is enough to show that the general sum S_k vanishes, where

$$S_k := \sum_{i=1}^{n} (-1)^{i+n-1} \frac{\partial(C_1, \ldots, C_{k-1}, \partial_{x_i} C_k, C_{k+1}, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, \widehat{x}_i, \ldots, x_n)}.$$

Indeed, we obtain that:

$$
\begin{aligned}
S_k &= \sum_{i=1}^{n}(-1)^{i+n-1}\sum_{j=1}^{i-1}(-1)^{j+k}\partial^2_{x_j x_i}C_k \cdot \frac{\partial(C_1,\ldots,\widehat{C}_k,\ldots,C_{n-2},H)}{\partial(x_1,\ldots,\widehat{x}_j,\ldots,\widehat{x}_i,\ldots,x_n)}\\
&+ \sum_{i=1}^{n}(-1)^{i+n-1}\sum_{j=i+1}^{n}(-1)^{j+k-1}\partial^2_{x_j x_i}C_k \cdot \frac{\partial(C_1,\ldots,\widehat{C}_k,\ldots,C_{n-2},H)}{\partial(x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_n)}\\
&= \sum_{i=1}^{n}(-1)^{i+n-1}\sum_{j=1}^{i-1}(-1)^{j+k}\partial^2_{x_j x_i}C_k \cdot \frac{\partial(C_1,\ldots,\widehat{C}_k,\ldots,C_{n-2},H)}{\partial(x_1,\ldots,\widehat{x}_j,\ldots,\widehat{x}_i,\ldots,x_n)}\\
&+ \sum_{j=1}^{n}(-1)^{j+n-1}\sum_{i=j+1}^{n}(-1)^{i+k-1}\partial^2_{x_i x_j}C_k \cdot \frac{\partial(C_1,\ldots,\widehat{C}_k,\ldots,C_{n-2},H)}{\partial(x_1,\ldots,\widehat{x}_j,\ldots,\widehat{x}_i,\ldots,x_n)}\\
&= (-1)^{n+k-1}\sum_{1\leq j<i\leq n}[(-1)^{i+j}\partial^2_{x_j x_i}C_k + (-1)^{i+j-1}\partial^2_{x_i x_j}C_k]\cdot \frac{\partial(C_1,\ldots,\widehat{C}_k,\ldots,C_{n-2},H)}{\partial(x_1,\ldots,\widehat{x}_j,\ldots,\widehat{x}_i,\ldots,x_n)}\\
&= 0.
\end{aligned}
$$

□

Remark 2.2. *In the symplectic case, each Hamiltonian vector field is divergence free with respect to the divergence operator defined by the Liouville volume form.*

3. A unified linear formulation of integrable systems

In this section we give a unified linear formulation for the n dimensional differential systems admitting a set of $n-1$ independent conservation laws. The construction of this linear formulation makes use explicitly of the Hamilton-Poisson realization (2.2) of a differential system of type (2.1).

Let us now state the main result of this article. The notations and respectively the hypothesis are supposed to be the same as in the previous section.

Theorem 3.1. *Let (2.1) be a $\mathcal{C}^1$ differential system having a set of $n-1$ independent $\mathcal{C}^2$ conservation laws defined on an open subset $\Omega \subseteq \mathbb{R}^n$. Assume that there exists a $\mathcal{C}^1$ rescaling function ν, nonzero on an open and dense subset Ω_0 of Ω, such that the system (2.1) admits a Hamilton-Poisson realization of the type (2.2) and the Lebesgue measure of the set*

$$
\mathcal{O} := \left\{ x = (x_1,\ldots,x_n) \in \Omega_0 \mid \operatorname{div}(X)(x)\cdot \frac{\partial(1/\nu, C_1,\ldots,C_{n-2},H)}{\partial(x_1,\ldots,x_n)}(x) = 0 \right\}
$$

in Ω_0 is zero.

Then, the change of variables $((x_1,\dots,x_n),t)\mapsto((u_1,\dots,u_n),s)$ *given by*

$$\begin{cases} u_1 = 1/\nu(x_1,\dots,x_n)\\ u_2 = C_1(x_1,\dots,x_n)/\nu(x_1,\dots,x_n)\\ \dots\\ u_{n-1} = C_{n-2}(x_1,\dots,x_n)/\nu(x_1,\dots,x_n)\\ u_n = H(x_1,\dots,x_n)/\nu(x_1,\dots,x_n)\\ ds = -\operatorname{div}(X)dt, \end{cases}$$

in the open and dense subset $\Omega_{00} := \Omega_0 \setminus \mathcal{O}$ *of* Ω_0*, transforms the system* (2.2) *restricted to* Ω_{00} *into the linear differential system* $u_1' = u_1,\ u_2' = u_2,\dots,\ u_n' = u_n$*, where "prime" stand for the derivative with respect to the new time "s".*

Proof. Let us start by recalling that the change of variables $(x_1,\dots,x_n)\longmapsto(u_1,\dots,u_n)$ is of class $\mathcal{C}^1$ on Ω_{00}. More exactly, the change of variables is defined on a wider set, namely $\Omega_0\setminus\mathcal{E}$, where

$$\begin{aligned}\mathcal{E} :&= \left\{x=(x_1,\dots,x_n)\in\Omega_0 \mid \frac{\partial(u_1,\dots,u_n)}{\partial(x_1,\dots,x_n)}(x)=0\right\}\\ &= \left\{x=(x_1,\dots,x_n)\in\Omega_0 \mid (1/\nu)^{n-1}(x)\cdot\frac{\partial(1/\nu,C_1,\dots,C_{n-2},H)}{\partial(x_1,\dots,x_n)}(x)=0\right\}\\ &= \left\{x=(x_1,\dots,x_n)\in\Omega_0 \mid \frac{\partial(1/\nu,C_1,\dots,C_{n-2},H)}{\partial(x_1,\dots,x_n)}(x)=0\right\}.\end{aligned}$$

In order to determine the transformed differential system (2.1) (generated by X) through this change of variables, we need the following result. Recall by Theorem (2.1) that the vector filed $1/\nu\cdot X$ is a divergence free vector field on Ω_{00}. Since

$$\operatorname{div}(1/\nu\cdot X) = \langle\nabla(1/\nu),X\rangle + 1/\nu\cdot\operatorname{div}(X),$$

and $\operatorname{div}(1/\nu\cdot X)=0$, we obtain that:

$$\langle\nabla(1/\nu),X\rangle = -1/\nu\cdot\operatorname{div}(X). \tag{3.1}$$

After the change of variables in Ω_{00}, the system (2.1) becomes:

$$\begin{cases} \dfrac{du_1}{dt} = \dfrac{d}{dt}(1/\nu) = \langle\nabla(1/\nu),X\rangle = -\operatorname{div}(X)\cdot 1/\nu = -\operatorname{div}(X)\cdot u_1\\ \dfrac{du_2}{dt} = \dfrac{d}{dt}(1/\nu\cdot C_1) = C_1\cdot\langle\nabla(1/\nu),X\rangle + 1/\nu\cdot\langle\nabla C_1,X\rangle = -\operatorname{div}(X)\cdot u_2\\ \cdots\\ \dfrac{du_{n-1}}{dt} = \dfrac{d}{dt}(1/\nu\cdot C_{n-2}) = C_{n-2}\cdot\langle\nabla(1/\nu),X\rangle + 1/\nu\cdot\langle\nabla C_{n-2},X\rangle = -\operatorname{div}(X)\cdot u_{n-1}\\ \dfrac{du_n}{dt} = \dfrac{d}{dt}(1/\nu\cdot H) = H\cdot\langle\nabla(1/\nu),X\rangle + 1/\nu\cdot\langle\nabla H,X\rangle = -\operatorname{div}(X)\cdot u_n, \end{cases} \tag{3.2}$$

where we used the relation (3.1) and the fact that $C_1,\dots,C_{n-2},H$ are conservation laws for the vector field X, and consequently

$$\langle\nabla C_1,X\rangle = \cdots = \langle\nabla C_{n-2},X\rangle = \langle\nabla H,X\rangle = 0.$$

Now using the new time transformation $ds = -\operatorname{div}(X)dt$ on Ω_{00}, the system (3.2) becomes:

$$\begin{cases} u_1' = u_1 \\ u_2' = u_2 \\ \cdots \\ u_n' = u_n, \end{cases}$$

where $u_i' = \dfrac{du_i}{ds}$, for $i \in \{1, \ldots, n\}$. □

Remark 3.2. *If the rescaling function $\nu =: \nu_{cst.}$ is a constant function, then the Lebesgue measure of the set $\mathcal{O} = \Omega_0 = \Omega$ is nonzero in Ω_0, and hence the assumptions of the Theorem (3.1) do not hold. In this case, we search for a new $\mathcal{C}^1$ rescaling function μ defined on an open and dense subset Ω_0 of Ω, such that the vector field $\mu \cdot X$ satisfies the assumptions of the Theorem (3.1). The function μ satisfies:*

$$\operatorname{div}(\mu \cdot X) = \langle \nabla\mu, X\rangle + \mu \cdot \operatorname{div}(X) = \langle \nabla\mu, X\rangle,$$

since in the case of a constant function $\nu = \nu_{cst.}$ we obtain from Theorem (2.1) that

$$0 = \operatorname{div}(1/\nu_{cst.} \cdot X) = (1/\nu_{cst.}) \cdot \operatorname{div}(X),$$

and hence $\operatorname{div}(X) = 0$.

Consequently, we have to search for a rescaling function μ such that

$$\operatorname{div}(\mu \cdot X) = \langle \nabla\mu, X\rangle$$

it is not identically zero in Ω_0.

The second condition that μ has to satisfy is that the function

$$\frac{\partial(1/\mu, C_1, \ldots, C_{n-2}, H)}{\partial(x_1, \ldots, x_n)}$$

it is not identically zero in Ω_0.

The transformation between the differential system (2.1) and the differential system generated by the vector field $\mu \cdot X$ is done by using the new time transformation $dt = \mu(x)dt'$. More exactly, the differential system (2.1) generated by X, namely:

$$\frac{dx}{dt} = X(x),$$

is transformed via the new time transformation $dt = \mu(x)dt'$ into the system:

$$\frac{dx}{dt'} = \mu(x) \cdot X(x). \tag{3.3}$$

The differential system (3.3) can also be realized as a Hamilton-Poisson dynamical system with respect to the Poisson bracket $\{\cdot,\cdot\}_{\mu\cdot\nu_{cst.};C_1,\ldots,C_{n-2}}$ and respectively the same Hamiltonian H as for the Hamilton-Poisson realization (2.2) of the vector field X.

4. Examples

In this section we will apply the main result of this article in the case of two concrete dynamical systems, namely a 3D Lotka-Volterra system and respectively Euler's equations of the free rigid body dynamics.

Let us start with the Lotka-Volterra system. The 3D Lotka-Volterra system we consider (see e.g. [4]), is described by the following differential system:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1(x_2 + x_3) \\ \dfrac{dx_2}{dt} = x_2(-x_1 + x_3) \\ \dfrac{dx_3}{dt} = x_3(-x_1 - x_2). \end{cases} \tag{4.1}$$

If one denote:

$$X = [x_1(x_2 + x_3)]\partial_{x_1} + [x_2(-x_1 + x_3)]\partial_{x_2} + [x_3(-x_1 - x_2)]\partial_{x_3},$$

then $\operatorname{div}(X) = 2(x_3 - x_1)$.

The system (4.1) admits a Hamilton-Poisson realization of the type (2.2), where:

$$\begin{aligned} \nu(x_1, x_2, x_3) &= -\frac{x_1^2 x_3^2}{x_1 + x_2 + x_3}, \\ C(x_1, x_2, x_3) &= \frac{x_2(x_1 + x_2 + x_3)}{x_1 x_3}, \\ H(x_1, x_2, x_3) &= x_1 + x_2 + x_3. \end{aligned}$$

The sets introduced in Theorem (3.1) in the case of the Lotka-Volterra system (4.1) are given by:

$$\begin{aligned} \Omega &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_3 \neq 0\}, \\ \Omega_0 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_3 \neq 0;\ x_1 + x_2 + x_3 \neq 0\}, \\ \mathcal{O} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = x_3;\ x_1 x_3 \neq 0;\ x_1 + x_2 + x_3 \neq 0\}, \\ \Omega_{00} &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq x_3;\ x_1 x_3 \neq 0; x_1 + x_2 + x_3 \neq 0\}. \end{aligned}$$

Then by Theorem (3.1), the change of variables $((x_1, x_2, x_3), t) \mapsto ((u_1, u_2, u_3), s)$ defined by:

$$\begin{cases} u_1 = -\dfrac{x_1 + x_2 + x_3}{x_1^2 x_3^2} \\ u_2 = -\dfrac{x_2(x_1 + x_2 + x_3)^2}{x_1^3 x_3^3} \\ u_3 = -\dfrac{(x_1 + x_2 + x_3)^2}{x_1^2 x_3^2} \\ ds = 2(x_1 - x_3)dt, \end{cases}$$

for $(x_1, x_2, x_3) \in \Omega_{00}$ transforms the Lotka-Volterra system (4.1) into the linear differential system:

$$\begin{cases} \dfrac{du_1}{ds} = u_1 \\ \dfrac{du_2}{ds} = u_2 \\ \dfrac{du_3}{ds} = u_3. \end{cases}$$

Let us now analyze the Euler equations from the free rigid body dynamics (see e.g. [8], [9], [10], [11], [12]). Recall that the Euler equations from the free rigid body dynamics are given by the differential system:

$$\begin{cases} \dfrac{dx_1}{dt} = \dfrac{I_2 - I_3}{I_2 I_3} x_2 x_3 \\ \dfrac{dx_2}{dt} = \dfrac{I_3 - I_1}{I_1 I_3} x_1 x_3 \\ \dfrac{dx_3}{dt} = \dfrac{I_1 - I_2}{I_1 I_2} x_1 x_2, \end{cases} \tag{4.2}$$

where the nonzero real numbers I_1, I_2, I_3 are the components of the inertia tensor. In the following we consider the case when $I_2 \neq I_3$.

If one denote:

$$X = (\frac{I_2 - I_3}{I_2 I_3} x_2 x_3)\partial_{x_1} + (\frac{I_3 - I_1}{I_1 I_3} x_1 x_3)\partial_{x_2} + (\frac{I_1 - I_2}{I_1 I_2} x_1 x_2)\partial_{x_3},$$

then $\mathrm{div}(X) = 0$, and hence the assumptions of the Theorem (3.1) do not hold.

The system (4.2) admits a Hamilton-Poisson realization of type (2.2), where:

$$\begin{aligned} &\nu(x_1, x_2, x_3) =: \nu_{cst.}(x_1, x_2, x_3) = -1, \\ &C(x_1, x_2, x_3) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \\ &H(x_1, x_2, x_3) = \frac{1}{2}(\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3}). \end{aligned}$$

In order to correct the vector field X such that one can apply the Theorem (3.1), we use the Remark (3.2) and consequently choose a rescaling function $\mu(x_1, x_2, x_3) = x_1$, and the associated new time transformation $dt = \mu(x_1, x_2, x_3)dt'$ defined on the open and dense subset of $\mathbb{R}^3$ given by $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0\}$.

This new time transformation, transforms the system (4.2) into the differential system:

$$\begin{cases} \dfrac{dx_1}{dt'} = \dfrac{I_2 - I_3}{I_2 I_3} x_1 x_2 x_3 \\ \dfrac{dx_2}{dt'} = \dfrac{I_3 - I_1}{I_1 I_3} x_1^2 x_3 \\ \dfrac{dx_3}{dt'} = \dfrac{I_1 - I_2}{I_1 I_2} x_1^2 x_2, \end{cases} \tag{4.3}$$

Recall that the vector field which generates the differential system (4.3) is $\mu \cdot X$:

$$\mu \cdot X = (\frac{I_2 - I_3}{I_2 I_3} x_1 x_2 x_3)\partial_{x_1} + (\frac{I_3 - I_1}{I_1 I_3} x_1^2 x_3)\partial_{x_2} + (\frac{I_1 - I_2}{I_1 I_2} x_1^2 x_2)\partial_{x_3}.$$

One note that the divergence of the vector field $\mu \cdot X$ is given by:

$$\operatorname{div}(\mu \cdot X) = \frac{I_2 - I_3}{I_2 I_3} x_2 x_3.$$

For $I_2 \neq I_3$ we have that $\operatorname{div}(\mu \cdot X)$ it is not identically zero.

The system (4.3) admits a Hamilton-Poisson realization of the type (2.2), where:

$$\begin{aligned} \nu(x_1, x_2, x_3) &= \nu_{cst.}(x_1, x_2, x_3) \cdot \mu(x_1, x_2, x_3) = -x_1, \\ C(x_1, x_2, x_3) &= \frac{1}{2}(x_1^2 + x_2^2 + x_3^2), \\ H(x_1, x_2, x_3) &= \frac{1}{2}(\frac{x_1^2}{I_1} + \frac{x_2^2}{I_2} + \frac{x_3^2}{I_3}). \end{aligned}$$

The sets introduced in Theorem (3.1) in the case of the system (4.3) are given by:

$$\begin{aligned} &\Omega = \mathbb{R}^3, \\ &\Omega_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0\}, \\ &\mathcal{O} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \neq 0;\ x_2 x_3 = 0\}, \\ &\Omega_{00} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 x_2 x_3 \neq 0\}. \end{aligned}$$

Then by Theorem (3.1), the change of variables $((x_1, x_2, x_3), t') \mapsto ((u_1, u_2, u_3), s)$ defined by:

$$\begin{cases} u_1 = -\dfrac{1}{x_1} \\ u_2 = -\dfrac{x_1^2 + x_2^2 + x_3^2}{2x_1} \\ u_3 = -\dfrac{x_1}{2I_1} - \dfrac{x_2^2}{2x_1 I_2} - \dfrac{x_3^2}{2x_1 I_3} \\ ds = \dfrac{I_3 - I_2}{I_2 I_3} x_2 x_3 dt', \end{cases}$$

for $(x_1, x_2, x_3) \in \Omega_{00}$ transforms the system (4.3) into the linear differential system:

$$\begin{cases} \dfrac{du_1}{ds} = u_1 \\ \dfrac{du_2}{ds} = u_2 \\ \dfrac{du_3}{ds} = u_3. \end{cases}$$

Acknowledgment

I would like to thank Professor Metin Gürses for indicating the references [3], [7].

References

[1] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Benjamin Cummings, New York 1978.

[2] V.I. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, vol. 60, second edition, Springer 1989.

[3] A. Ay, M. Gürses and K. Zheltukhin, Hamiltonian equations in $\mathbb{R}^3$, *J. Math. Phys.*, 44(2003), 5688–5705.

[4] O. Bogoyavlenskij, Y. Itoh and T. Yukawa, Lotka-Volterra systems integrable in quadratures, *J. Math. Phys.*, 49(2008), 053501.

[5] R.W.R. Darling, *Differential forms and connections*, Cambridge University Press, 1994.

[6] J.Giné and J. Llibre, On the planar integrable differential systems, *Z. Angew. Math. Phys.*,(2011), DOI 10.1007/s00033-011-0116-5.

[7] M. Gürses, G.S. Guseinov and K. Zheltukhin, Dynamical systems and Poisson structures, *J. Math. Phys.*, 50(2009), 112703.

[8] D.D. Holm, *Geometric Mechanics I: Dynamics and Symmetry*, World Scientific: Imperial College Press, Singapore 2008.

[9] D.D. Holm, *Geometric Mechanics II: Rotating, Translating and Rolling*, World Scientific: Imperial College Press, Singapore 2008.

[10] D.D. Holm, T. Schmah and C. Stoica, *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions*, Oxford Texts in Applied and Engineering Mathematics, Oxford University Press, 2009.

[11] J.E. Marsden and T.S. Ratiu, *Introduction to mechanics and symmetry*, Texts in Applied Mathematics, vol. 17, second edition, second printing, Springer, Berlin 1999.

[12] T.S. Ratiu, R.M. Tudoran, L. Sbano, E. Sousa Dias and G. Terra, *Geometric Mechanics and Symmetry: the Peyresq Lectures; Chapter II: A Crash Course in Geometric Mechanics*, pp. 23–156, London Mathematical Society Lecture Notes Series, vol. 306, Cambridge University Press 2005.

Răzvan M. Tudoran
The West University of Timişoara,
Faculty of Mathematics and Computer Science,
Department of Mathematics,
Blvd. Vasile Pârvan, No. 4,
300223 - Timişoara,
Romania.
e-mail: `tudoran@math.uvt.ro`