The non-autonomous chiral model and the Ernst equation of General Relativity in the bidifferential calculus framework

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Abstract

We generate exact solutions of the non-autonomous chiral model, which in particular arises in General Relativity, using a general result of the bidifferential calculus approach to integrable PDDEs. In the case of the 2×2, respectively 3×3 matrix chiral model, this family of solutions admits reductions to solutions of the Ernst equation(s) describing stationary and axially symmetric (electro-) vacuum space-times. In this way we recover the Ernst potentials of multi-Kerr-NUT and multi-Demiański-Newman metrics.

1 Introduction

The bidifferential calculus framework allows to elaborate solution generating methods for a wide class of nonlinear "integrable" partial differential or difference equations (PDDEs) to a considerable extent on a universal level, i.e. resolved from specific examples. A brief account of the basic structures and some results have been presented in [1] (also see the references therein), the essentials needed in the present work are provided in Section 2. We explore in this framework the non-autonomous chiral model, with matrix dimensions $m \times m$, that results from the stationary axially symmetric Einstein ($m = 2$) and Einstein-Maxwell ($m = 3$) vacuum equations [2–11], and for $m > 3$ also appears in higher-dimensional gravity theories (see e.g. [12–17]). More precisely, we concentrate on a surprisingly simple *non-iterative* solution generating result that has been successfully applied in various other cases of integrable (soliton) equations [1, 18–20] to generate multi-soliton families. In order to make it applicable to the non-autonomous chiral model, a slight generalization is required, however (see Section 3 and Appendix A). Section 4 then elaborates it for the $m \times m$ chiral model, and Section 5 addresses reductions, in particular to the Ernst equation of General Relativity. Section 6 contains some concluding remarks.

2 Preliminaries

Basic definitions. A *graded algebra* is an associative algebra Ω over $\mathbb{C}$ with a direct sum decomposition $\Omega = \bigoplus_{r \geq 0} \Omega^r$ into a subalgebra $\mathcal{A} := \Omega^0$ and $\mathcal{A}$-bimodules Ω^r, such that $\Omega^r \, \Omega^s \subseteq \Omega^{r+s}$. A *bidifferential* *calculus* (or *bidifferential graded algebra*) is a unital graded algebra Ω equipped with two ($\mathbb{C}$-linear) graded derivations $\mathrm{d}, \bar{\mathrm{d}} : \Omega \to \Omega$ of degree one (hence $\mathrm{d}\Omega^r \subseteq \Omega^{r+1}$, $\bar{\mathrm{d}}\Omega^r \subseteq \Omega^{r+1}$), with the properties

$$\mathrm{d}_\kappa^2 = 0 \qquad \forall \kappa \in \mathbb{C} \,, \qquad \text{where} \quad \mathrm{d}_\kappa := \bar{\mathrm{d}} - \kappa \, \mathrm{d} \,, \tag{2.1}$$

and the graded Leibniz rule

$$\mathrm{d}_\kappa(\chi \, \chi') = (\mathrm{d}_\kappa \chi) \, \chi' + (-1)^r \, \chi \, \mathrm{d}_\kappa \chi' \,,$$

for all $\chi \in \Omega^r$ and $\chi' \in \Omega$.

Dressing a bidifferential calculus. Let $(\Omega, \mathrm{d}, \bar{\mathrm{d}})$ be a bidifferential calculus. Replacing d_κ in (2.1) by

$$\mathrm{D}_\kappa := \bar{\mathrm{d}} - \mathbb{A} - \kappa \, \mathrm{d} \,, \tag{2.2}$$

with a 1-form $\mathbb{A}$ (i.e. an element of Ω^1), the resulting condition $\mathrm{D}_\kappa^2 = 0$ (for all $\kappa \in \mathbb{C}$) can be expressed as

$$\mathrm{d}\mathbb{A} = 0 \qquad \text{and} \qquad \bar{\mathrm{d}}\mathbb{A} - \mathbb{A} \, \mathbb{A} = 0 \,. \tag{2.3}$$

If these equations are equivalent to a PDDE (or a system of PDDEs), we say we have a *bidifferential calculus formulation* for it. This requires that $\mathbb{A}$ depends on an independent variable (or a set of independent variables) and the derivations $\mathrm{d}, \bar{\mathrm{d}}$ involve differential or difference operators. There are several ways to reduce the two equations (2.3) to a single one. Here we only consider two of them.

1. We can solve the first of (2.3) by setting

$$\mathbb{A} = \mathrm{d}\phi \,. \tag{2.4}$$

This converts the second of (2.3) into

$$\bar{\mathrm{d}} \, \mathrm{d} \, \phi = \mathrm{d}\phi \, \mathrm{d}\phi \,. \tag{2.5}$$

This equation is obviously invariant under $\phi \mapsto \alpha\phi\alpha^{-1} + \beta$ with an invertible $\alpha \in \mathcal{A}$ satisfying $\mathrm{d}\alpha = \bar{\mathrm{d}}\alpha = 0$, and $\beta \in \mathcal{A}$ satisfying $\mathrm{d}\beta = 0$.

2. Alternatively, the second of equations (2.3) can be solved by setting

$$\mathbb{A} = (\bar{\mathrm{d}}g) \, g^{-1} \,, \tag{2.6}$$

and the first equation then reads

$$\mathrm{d}\left((\bar{\mathrm{d}}g) \, g^{-1} \right) = 0 \,. \tag{2.7}$$

This equation has the (independent left and right handed, i.e. chiral) symmetry

$$g \mapsto \alpha \, g \, \beta \,, \tag{2.8}$$

where $\alpha \in \mathcal{A}$ is d-constant[1] and $\beta \in \mathcal{A}$ is $\bar{\mathrm{d}}$-constant, and both have to be invertible. Since

$$\bar{\mathrm{d}} \left[(\mathrm{d}g^{-1}) \, g \right] = -\bar{\mathrm{d}}(g^{-1}\mathrm{d}g) = g^{-1} \left[(\mathrm{d}\bar{\mathrm{d}}g) \, g^{-1} - (\bar{\mathrm{d}}g) \, \mathrm{d}g^{-1} \right] g = g^{-1} \, \mathrm{d}[(\bar{\mathrm{d}}g) \, g^{-1}] \, g \,, \tag{2.9}$$

g solves (2.7) iff g^{-1} solves (2.7) with d and $\bar{\mathrm{d}}$ exchanged. In our central example, the non-autonomous chiral model, $g \mapsto g^{-1}$ becomes a symmetry.

[1] It may not be evident that we need not require $\bar{\mathrm{d}}\alpha = 0$ in addition, but the latter condition is indeed not necessary. The simple proof uses $\mathrm{d}\bar{\mathrm{d}} = -\bar{\mathrm{d}}\mathrm{d}$ and the Leibniz rule for d and $\bar{\mathrm{d}}$.

Linear system. The compatibility condition of the *linear* equation

$$\bar{\mathrm{d}} X = (\mathrm{d}X)\, P + \mathbb{A}\, X \tag{2.10}$$

is

$$0 = \bar{\mathrm{d}}^2 X = (\mathrm{d}X)\, [(\mathrm{d}P)P - \bar{\mathrm{d}}P] + (\bar{\mathrm{d}}\mathbb{A} - \mathbb{A}^2)\, X - (\mathrm{d}\mathbb{A})\, X\, P\ .$$

If P satisfies[2]

$$\bar{\mathrm{d}} P = (\mathrm{d}P)\, P\,, \tag{2.11}$$

this reduces to

$$(\bar{\mathrm{d}}\mathbb{A} - \mathbb{A}^2)\, X = (\mathrm{d}\mathbb{A})\, X\, P\ . \tag{2.12}$$

For the above choices of $\mathbb{A}$, this implies the respective PDDE. Hence (2.10) is the source of a corresponding Lax pair, also see Appendix B.

Miura transformation. If a pair (ϕ, g) solves the *Miura transformation* equation

$$(\bar{\mathrm{d}} g)\, g^{-1} = \mathrm{d}\phi \tag{2.13}$$

(cf. [1]), it follows (as an integrability condition) that ϕ solves (2.5) and g solves (2.7). We note that (2.13) is just the linear equation (2.10) if we identify $\mathbb{A} = \mathrm{d}\phi$, $X = g$ and set $P = 0$. If we have chosen a bidifferential calculus and a reduction condition such that (2.5) becomes equivalent to some PDDE, this does not necessarily mean that also (2.7) is equivalent to some ordinary PDDE. But for the central example of this work, the non-autonomous chiral model, such a mismatch does not occur. In fact, in Section 3 we will actually present a solution generating method for (2.13).

3 A solution generating method

Let $\bigwedge(\mathbb{C}^N)$ denote the exterior (Grassmann) algebra of the vector space $\mathbb{C}^N$ and $\mathrm{Mat}(m, n, \mathcal{B})$ the set of $m \times n$ matrices with entries in some unital algebra $\mathcal{B}$. We choose $\mathcal{A}$ as the algebra of all matrices (with entries in $\mathcal{B}$), where the product of two matrices is defined to be zero if the sizes of the two matrices do not match, and assume that $\Omega = \mathcal{A} \otimes \bigwedge(\mathbb{C}^N)$ is supplied with the structure of a bidifferential calculus. In the following, $I = I_m$ and $\boldsymbol{I} = \boldsymbol{I}_n$ denote the $m \times m$, respectively $n \times n$, identity matrix.

Proposition 3.1. *Let $\boldsymbol{P}, \boldsymbol{R}, \boldsymbol{X} \in \mathrm{Mat}(n, n, \mathcal{B})$ be invertible solutions of*

$$\bar{\mathrm{d}}\boldsymbol{P} = (\mathrm{d}\boldsymbol{P})\, \boldsymbol{P}\,, \quad \bar{\mathrm{d}}\boldsymbol{R} = \boldsymbol{R}\, \mathrm{d}\boldsymbol{R}\,, \quad \bar{\mathrm{d}}\boldsymbol{X} = (\mathrm{d}\boldsymbol{X})\, \boldsymbol{P} - (\mathrm{d}\boldsymbol{R})\, \boldsymbol{X}\,, \quad \boldsymbol{X}\,\boldsymbol{P} - \boldsymbol{R}\,\boldsymbol{X} = \boldsymbol{V}\,\boldsymbol{U}\,, \tag{3.14}$$

with d- *and* $\bar{\mathrm{d}}$*-constant* $\boldsymbol{U} \in \mathrm{Mat}(m, n, \mathcal{B})$, $\boldsymbol{V} \in \mathrm{Mat}(n, m, \mathcal{B})$*. Then*

$$\phi = \boldsymbol{U}\boldsymbol{X}^{-1}\boldsymbol{V}\,, \qquad g = I + \boldsymbol{U}(\boldsymbol{R}\,\boldsymbol{X})^{-1}\boldsymbol{V} \tag{3.15}$$

solve the Miura transformation equation (2.13), and thus (2.5), respectively (2.7).

Proof: Using the last three of (3.14) we obtain

$$\begin{aligned}
\bar{\mathrm{d}}(\boldsymbol{R}\boldsymbol{X})^{-1} &= -\boldsymbol{X}^{-1}[\bar{\mathrm{d}}\boldsymbol{X}\,\boldsymbol{X}^{-1} + \boldsymbol{R}^{-1}\bar{\mathrm{d}}\boldsymbol{R}]\,\boldsymbol{R}^{-1} \\
&= -\boldsymbol{X}^{-1}(\mathrm{d}\boldsymbol{X})\,\boldsymbol{X}^{-1}(\boldsymbol{X}\boldsymbol{P})\,(\boldsymbol{R}\boldsymbol{X})^{-1} \\
&= (\mathrm{d}\boldsymbol{X}^{-1})[\boldsymbol{I} + \boldsymbol{V}\,\boldsymbol{U}\,(\boldsymbol{R}\boldsymbol{X})^{-1}]\ .
\end{aligned}$$

Multiplication by $\boldsymbol{U}$ from the left and by $\boldsymbol{V}$ from the right, and using $\bar{\mathrm{d}}I = 0$, leads to

$$\bar{\mathrm{d}} g = \boldsymbol{U}(\mathrm{d}\boldsymbol{X}^{-1})\boldsymbol{V}\, g = (\mathrm{d}\phi)\, g\ .$$

Hence ϕ and g solve the Miura transformation equation (2.13). We did not use the first of (3.14), but it arises as an integrability condition: $0 = \bar{\mathrm{d}}^2\boldsymbol{X} = (\mathrm{d}\boldsymbol{X})\,[(\mathrm{d}\boldsymbol{P})\,\boldsymbol{P} - \bar{\mathrm{d}}\boldsymbol{P}]$. $\square$

[2] We note that, as a consequence of this equation, also P^k with any $k \in \mathbb{N}$ ($k \in \mathbb{Z}$) is a solution (if P is invertible).

Remark 3.2. The third of (3.14), which has the form of the linear equation (2.10), is almost a consequence of the fourth, which is a Sylvester equation. Indeed, as a consequence of the Sylvester equation we have

$$\begin{aligned} 0 &= \bar{\mathrm{d}}(\boldsymbol{R}\,\boldsymbol{X} - \boldsymbol{X}\,\boldsymbol{P} + \boldsymbol{V}\,\boldsymbol{U}) = (\bar{\mathrm{d}}\boldsymbol{R})\,\boldsymbol{X} + \boldsymbol{R}\,\bar{\mathrm{d}}\boldsymbol{X} - (\bar{\mathrm{d}}\boldsymbol{X})\,\boldsymbol{P} - \boldsymbol{X}\,\bar{\mathrm{d}}\boldsymbol{P} \\ &= \boldsymbol{R}\,[\bar{\mathrm{d}}\boldsymbol{X} + (\mathrm{d}\boldsymbol{R})\,\boldsymbol{X}] - [\bar{\mathrm{d}}\boldsymbol{X} + \boldsymbol{X}\,\mathrm{d}\boldsymbol{P}]\,\boldsymbol{P} \\ &= \boldsymbol{R}\,[\bar{\mathrm{d}}\boldsymbol{X} + (\mathrm{d}\boldsymbol{R})\,\boldsymbol{X} - (\mathrm{d}\boldsymbol{X})\,\boldsymbol{P}] - [\bar{\mathrm{d}}\boldsymbol{X} + (\mathrm{d}\boldsymbol{R})\,\boldsymbol{X} - (\mathrm{d}\boldsymbol{X})\,\boldsymbol{P}]\,\boldsymbol{P} + \mathrm{d}(\boldsymbol{R}\,\boldsymbol{X} - \boldsymbol{X}\,\boldsymbol{P})\,\boldsymbol{P}\,, \end{aligned}$$

where the last term vanishes. If $\boldsymbol{P}$ and $\boldsymbol{R}$ are sufficiently independent, this implies that the third of (3.14) is satisfied. In particular, this holds if $\mathcal{B}$ is the algebra of complex functions of some variables and if $\boldsymbol{P}$ and $\boldsymbol{R}$ have no eigenvalue in common. □

Appendix A explains how Proposition 3.1 arises from a more general theorem that has been applied in previous work to generate soliton solutions of several integrable PDDEs.

4 The non-autonomous chiral model

The PDE defining the non-autonomous chiral model can be obtained as a reduction of the self-dual Yang-Mills (sdYM) equation (see e.g. [21–23]). In an analogous way, a bidifferential calculus for the non-autonomous chiral model can be derived from a bidifferential calculus for the sdYM equation (also see [24]). In coordinates ρ, z, θ, it is given by

$$\mathrm{d}f = -f_z\,\zeta_1 + e^{\theta}\,(f_\rho - \rho^{-1} f_\theta)\,\zeta_2\,, \qquad \bar{\mathrm{d}}f = e^{-\theta}\,(f_\rho + \rho^{-1} f_\theta)\,\zeta_1 + f_z\,\zeta_2\,. \tag{4.1}$$

Here e.g. f_z denotes the partial derivative of a function f (of the three coordinates) with respect to z, and ζ_1, ζ_2 is a basis of $\bigwedge^1(\mathbb{C}^2)$. d and $\bar{\mathrm{d}}$ extend to matrices of functions and moreover to $\Omega = \mathcal{A} \otimes \bigwedge(\mathbb{C}^2)$ with $\mathcal{A} = \mathrm{Mat}(m, m, \mathbb{C})$, treating ζ_1, ζ_2 as constants. The coordinate θ is needed to have the properties of a bidifferential calculus, but we are finally interested in equations for objects that do not depend on it.

A (matrix-valued) function is d-constant ($\bar{\mathrm{d}}$-constant) iff it is z-independent and only depends on the variables θ, ρ through the combination $\theta + \log\rho$ (respectively $\theta - \log\rho$). It is d- and $\bar{\mathrm{d}}$-constant iff it is constant, i.e. independent of z, θ, ρ.

For an $m \times m$ matrix-valued function g, (2.7) takes the form

$$(\rho\, g_z\, g^{-1})_z + (\rho\, g_\rho\, g^{-1})_\rho - (g_\rho\, g^{-1})_\theta + (g_\theta\, g^{-1})_\rho - \rho^{-1}\,(g_\theta\, g^{-1})_\theta = 0\,.$$

Restricting g by setting

$$g = e^{c\,\theta}\,\tilde{g} \tag{4.2}$$

with any constant c and θ-independent $\tilde{g}$, for the latter we obtain the *non-autonomous chiral model* equation[3]

$$(\rho\,\tilde{g}_z\,\tilde{g}^{-1})_z + (\rho\,\tilde{g}_\rho\,\tilde{g}^{-1})_\rho = 0\,. \tag{4.3}$$

In Section 4.1, we derive a family of exact solutions by application of Proposition 3.1. In Appendix B we recover two familiar linear systems (Lax pairs) for this equation.

Miura transformation. Evaluating (2.5) with

$$\phi = e^{-\theta}\,\tilde{\phi}\,, \tag{4.4}$$

where $\tilde{\phi}$ is θ-independent, we obtain

$$\tilde{\phi}_{zz} + \tilde{\phi}_{\rho\rho} + \rho^{-1}\tilde{\phi}_\rho = [\tilde{\phi}_\rho + \rho^{-1}\tilde{\phi}, \tilde{\phi}_z]\,,$$

which is related to the non-autonomous chiral model by the Miura transformation

$$\tilde{\phi}_z = -\tilde{g}_\rho\,\tilde{g}^{-1} - c\,\rho^{-1}\,I\,, \qquad \tilde{g}_z\,\tilde{g}^{-1} = \tilde{\phi}_\rho + \rho^{-1}\,\tilde{\phi}\,. \tag{4.5}$$

[3]Changing the sign of the first term in the expression for $\mathrm{d}f$ in (4.1), we obtain a minus sign between the two terms on the left hand side of (4.3). This hyperbolic version of the chiral model shows up, in particular, in the reduction of the Einstein vacuum equations with two spacelike commuting Killing vector fields, describing gravitational plane waves [8]. Our further analysis can be adapted to this case.

Symmetries. (4.3) is invariant under each of the following transformations, and thus, more generally, any combination of them.

(1) $\tilde{g} \mapsto \alpha\, \tilde{g}\, \beta$, with any invertible constant $m \times m$ matrices α and β (cf. (2.8)).

(2) $\tilde{g} \mapsto \rho^c\, \tilde{g}$ with any constant c.

(3) $\tilde{g} \mapsto g^{-1}$ (also see (2.9)).

(4) $\tilde{g} \mapsto \tilde{g}^\dagger$, where † indicates Hermitian conjugation.

We note that $\tilde{g} \mapsto (g^\dagger)^{-1}$ is a fairly obvious symmetry. With its help, (4) follows immediately from (3).

4.1 A family of exact solutions

Let us first consider the equation $\bar{\mathrm{d}}\boldsymbol{P} = (\mathrm{d}\boldsymbol{P})\,\boldsymbol{P}$, which is the first of (3.14). Using the above bidifferential calculus, it takes the form

$$\boldsymbol{P}_z\,\boldsymbol{P} = -e^{-\theta}(\boldsymbol{P}_\rho + \rho^{-1}\boldsymbol{P}_\theta)\,, \qquad \boldsymbol{P}_z = e^{\theta}(\boldsymbol{P}_\rho - \rho^{-1}\boldsymbol{P}_\theta)\,\boldsymbol{P}\,.$$

Writing

$$\boldsymbol{P} = e^{-\theta}\,\tilde{\boldsymbol{P}}\,,$$

and assuming that $\tilde{\boldsymbol{P}}$ does not depend on θ, this translates to

$$\tilde{\boldsymbol{P}}_\rho - \rho^{-1}\tilde{\boldsymbol{P}} = -\tilde{\boldsymbol{P}}_z\,\tilde{\boldsymbol{P}}\,, \qquad \tilde{\boldsymbol{P}}_z = (\tilde{\boldsymbol{P}}_\rho + \rho^{-1}\tilde{\boldsymbol{P}})\,\tilde{\boldsymbol{P}}\,. \tag{4.6}$$

Lemma 4.1. *The following holds.*
(1) If $\tilde{\boldsymbol{P}}$ and $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ are invertible, the system (4.6) implies

$$\tilde{\boldsymbol{P}}^2 - 2\rho^{-1}\,(z\,\boldsymbol{I} + \boldsymbol{B})\,\tilde{\boldsymbol{P}} - \boldsymbol{I} = 0\,, \tag{4.7}$$

with a constant matrix $\boldsymbol{B}$.
(2) Let $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ be invertible and $\tilde{\boldsymbol{P}}_\rho$, $\tilde{\boldsymbol{P}}_z$ commute with $\tilde{\boldsymbol{P}}$. If $\tilde{\boldsymbol{P}}$ satisfies (4.7), then $\tilde{\boldsymbol{P}}$ solves (4.6).

Proof: (1) Assuming that $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ is invertible, the system (4.6) can be decoupled into

$$\tilde{\boldsymbol{P}}_\rho = \rho^{-1}\,\tilde{\boldsymbol{P}}\,(\boldsymbol{I} - \tilde{\boldsymbol{P}}^2)(\boldsymbol{I} + \tilde{\boldsymbol{P}}^2)^{-1}\,, \qquad \tilde{\boldsymbol{P}}_z = 2\rho^{-1}\,\tilde{\boldsymbol{P}}^2\,(\boldsymbol{I} + \tilde{\boldsymbol{P}}^2)^{-1}\,, \tag{4.8}$$

which can also be written as

$$(\tilde{\boldsymbol{P}}^{-1})_\rho = -\rho^{-1}\,\tilde{\boldsymbol{P}}^{-1}\,(\boldsymbol{I} - \tilde{\boldsymbol{P}}^2)(\boldsymbol{I} + \tilde{\boldsymbol{P}}^2)^{-1}\,, \qquad (\tilde{\boldsymbol{P}}^{-1})_z = -2\rho^{-1}\,(\boldsymbol{I} + \tilde{\boldsymbol{P}}^2)^{-1}\,,$$

assuming that $\tilde{\boldsymbol{P}}$ is invertible. Subtraction yields

$$(\tilde{\boldsymbol{P}} - \tilde{\boldsymbol{P}}^{-1})_\rho = -\rho^{-1}\,(\tilde{\boldsymbol{P}} - \tilde{\boldsymbol{P}}^{-1})\,, \qquad (\tilde{\boldsymbol{P}} - \tilde{\boldsymbol{P}}^{-1})_z = 2\rho^{-1}\,\boldsymbol{I}\,,$$

which can be integrated to

$$\tilde{\boldsymbol{P}} - \tilde{\boldsymbol{P}}^{-1} = 2\rho^{-1}\,(z\,\boldsymbol{I} + \boldsymbol{B})\,, \tag{4.9}$$

with a constant matrix $\boldsymbol{B}$. This implies (4.7).
(2) Let $\tilde{\boldsymbol{P}}$ satisfy (4.7) with a constant matrix $\boldsymbol{B}$. Then $\tilde{\boldsymbol{P}}$ is invertible, since the existence of a non-vanishing vector annihilated by $\tilde{\boldsymbol{P}}$ would be in conflict with (4.7). Thus (4.9) holds, which implies $[\tilde{\boldsymbol{P}}, \boldsymbol{B}] = 0$. Differentiation of (4.7) with respect to ρ, and elimination of $z\boldsymbol{I} + \boldsymbol{B}$ with the help of (4.7) or equivalently (4.9), leads to

$$0 = \tilde{\boldsymbol{P}}_\rho\,\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{P}}\tilde{\boldsymbol{P}}_\rho + 2\rho^{-2}(z\boldsymbol{I} + \boldsymbol{B})\,\tilde{\boldsymbol{P}} - 2\rho^{-1}(z\boldsymbol{I} + \boldsymbol{B})\,\tilde{\boldsymbol{P}}_\rho = (\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{P}}^{-1})\tilde{\boldsymbol{P}}_\rho + \rho^{-1}(\tilde{\boldsymbol{P}}^2 - \boldsymbol{I})\,,$$

where we used the assumption $[\tilde{\boldsymbol{P}}_\rho, \tilde{\boldsymbol{P}}] = 0$. If $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ is invertible, the resulting equation is the first of (4.8). In the same way we obtain the second of (4.8). (4.8) is equivalent to (4.6). □

Remark 4.2. The conditions $[\tilde{\boldsymbol{P}}_\rho, \tilde{\boldsymbol{P}}] = [\tilde{\boldsymbol{P}}_z, \tilde{\boldsymbol{P}}] = 0$ in part (2) of the lemma are satisfied in particular if the spectrum $\mathrm{spec}(\boldsymbol{B})$ is simple, i.e. if the eigenvalues of $\boldsymbol{B}$ are all distinct, since then the solutions of (4.7) are functions of ρ, z and the matrix $\boldsymbol{B}$ (and thus $\tilde{\boldsymbol{P}}_\rho$ and $\tilde{\boldsymbol{P}}_z$ commute with $\tilde{\boldsymbol{P}}$) [25]. But this would be unnecessarily restrictive, see section 4.2. $\square$

Remark 4.3. Under the assumption that $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ is invertible, (4.8) implies $[\tilde{\boldsymbol{P}}_\rho, \tilde{\boldsymbol{P}}] = [\tilde{\boldsymbol{P}}_z, \tilde{\boldsymbol{P}}] = 0$. For the bidifferential calculus under consideration, $\bar{\mathrm{d}}\boldsymbol{P} = (\mathrm{d}\boldsymbol{P})\,\boldsymbol{P}$ is therefore equivalent to $\bar{\mathrm{d}}\boldsymbol{P} = \boldsymbol{P}\,\mathrm{d}\boldsymbol{P}$. The latter is one of our equations for $\boldsymbol{R}$ in Proposition 3.1. Setting

$$\boldsymbol{R} = e^{-\theta} \tilde{\boldsymbol{R}} ,$$

with $\tilde{\boldsymbol{R}}$ θ-independent, invertible $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{R}}$ both have to solve (4.7). $\square$

The third of (3.14) becomes

$$\boldsymbol{X}_\rho + \rho^{-1} \boldsymbol{X}_\theta = -\boldsymbol{X}_z\, \tilde{\boldsymbol{P}} + \tilde{\boldsymbol{R}}_z\, \boldsymbol{X} , \qquad \boldsymbol{X}_z = (\boldsymbol{X}_\rho - \rho^{-1} \boldsymbol{X}_\theta)\, \tilde{\boldsymbol{P}} - (\tilde{\boldsymbol{R}}_\rho + \rho^{-1} \tilde{\boldsymbol{R}})\, \boldsymbol{X} .$$

Assuming that $\boldsymbol{U}$ and $\boldsymbol{V}$ are θ-independent, and recalling the θ-dependence of ϕ, the formula for ϕ in (3.15) requires $\boldsymbol{X} = e^\theta \tilde{\boldsymbol{X}}$ with θ-independent $\tilde{\boldsymbol{X}}$. Hence

$$\tilde{\boldsymbol{X}}_\rho + \rho^{-1} \tilde{\boldsymbol{X}} = -\tilde{\boldsymbol{X}}_z\, \tilde{\boldsymbol{P}} + \tilde{\boldsymbol{R}}_z\, \tilde{\boldsymbol{X}} , \qquad \tilde{\boldsymbol{X}}_z = (\tilde{\boldsymbol{X}}_\rho - \rho^{-1} \tilde{\boldsymbol{X}})\, \tilde{\boldsymbol{P}} - (\tilde{\boldsymbol{R}}_\rho + \rho^{-1} \tilde{\boldsymbol{R}})\, \tilde{\boldsymbol{X}} . \tag{4.10}$$

The last of (3.14) becomes the θ-independent Sylvester equation

$$\tilde{\boldsymbol{X}}\, \tilde{\boldsymbol{P}} - \tilde{\boldsymbol{R}}\, \tilde{\boldsymbol{X}} = \boldsymbol{V}\boldsymbol{U} . \tag{4.11}$$

Now Proposition 3.1 implies the following.

Proposition 4.4. *Let $n \times n$ matrices $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{R}}$ be solutions of (4.7) (with a matrix $\boldsymbol{B}$, respectively $\boldsymbol{B}'$), with the properties that they commute with their derivatives w.r. to ρ and z, and that $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ and $\boldsymbol{I} + \tilde{\boldsymbol{R}}^2$ are invertible. Furthermore, let $\mathrm{spec}(\tilde{\boldsymbol{P}}) \cap \mathrm{spec}(\tilde{\boldsymbol{R}}) = \emptyset$ and $\boldsymbol{X}$ an invertible solution of the Sylvester equation (4.11) with constant $m \times n$, respectively $n \times m$, matrices $\boldsymbol{U}$ and $\boldsymbol{V}$. Then*

$$\tilde{g} = (I + \boldsymbol{U} (\tilde{\boldsymbol{R}} \tilde{\boldsymbol{X}})^{-1} \boldsymbol{V})\, g_0 , \tag{4.12}$$

with any constant invertible $m \times m$ matrix[4] g_0, solves the non-autonomous chiral model equation (4.3).

Proof: As a consequence of the spectrum condition, a solution $\tilde{\boldsymbol{X}}$ of the Sylvester equation (4.11) exists and is unique. The further assumptions for $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{R}}$ are those of Lemma 4.1, part (2). Furthermore, (4.10) is a consequence of (4.11) if the spectrum condition holds (see also Remark 3.2). Now our assertion follows from Proposition 3.1 and the preceding calculations. $\square$

Remark 4.5. The determinant of (4.12) is obtained via Sylvester's theorem,

$$\begin{aligned} \det(\tilde{g}) &= \det(I + \boldsymbol{U} (\tilde{\boldsymbol{R}} \tilde{\boldsymbol{X}})^{-1} \boldsymbol{V}) \det(g_0) = \det(\boldsymbol{I} + \boldsymbol{V}\boldsymbol{U} (\tilde{\boldsymbol{R}} \tilde{\boldsymbol{X}})^{-1}) \det(g_0) \\ &= \det(\tilde{\boldsymbol{R}} \tilde{\boldsymbol{X}} + \boldsymbol{V}\boldsymbol{U}) \det(\tilde{\boldsymbol{R}} \tilde{\boldsymbol{X}})^{-1} \det(g_0) = \det(\tilde{\boldsymbol{X}} \tilde{\boldsymbol{P}}) \det(\tilde{\boldsymbol{R}} \tilde{\boldsymbol{X}})^{-1} \det(g_0) = \frac{\det(\tilde{\boldsymbol{P}})}{\det(\tilde{\boldsymbol{R}})} \det(g_0) , \end{aligned}$$

where we used the Sylvester equation (4.11) and assumed that it has an invertible solution. $\square$

Remark 4.6. As an obvious consequence of (4.11), $\boldsymbol{U}$ and $\boldsymbol{V}$ enter $\tilde{g}$ given by (4.12) only modulo an arbitrary scalar factor different from zero. We also note that a transformation

$$\tilde{\boldsymbol{P}} \mapsto \boldsymbol{T}_1^{-1} \tilde{\boldsymbol{P}} \boldsymbol{T}_1 , \quad \tilde{\boldsymbol{R}} \mapsto \boldsymbol{T}_2^{-1} \tilde{\boldsymbol{R}} \boldsymbol{T}_2 , \quad \boldsymbol{U} \mapsto \boldsymbol{U} \boldsymbol{T}_1 , \quad \boldsymbol{V} \mapsto \boldsymbol{T}_2^{-1} \boldsymbol{V} , \quad \tilde{\boldsymbol{X}} \mapsto \boldsymbol{T}_2^{-1} \tilde{\boldsymbol{X}} \boldsymbol{T}_1 ,$$

with constant invertible $n \times n$ matrices $\boldsymbol{T}_1, \boldsymbol{T}_2$, leaves (4.7), (4.10), (4.11) and (4.12) invariant. As a consequence, without restriction of generality, we can assume that the matrix $\boldsymbol{B}$ in (4.7), and the corresponding matrix related to $\tilde{\boldsymbol{R}}$, both have Jordan normal form. $\square$

[4]Here g_0 represents the freedom of chiral transformations.

Example 4.7. Let $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{R}}$ be diagonal, i.e.

$$\tilde{\boldsymbol{P}} = (p_i\, \delta_{ij}) \ , \qquad \tilde{\boldsymbol{R}} = (r_i\, \delta_{ij}) \ .$$

If they have no eigenvalue in common, then (4.11) has a unique solution given by the Cauchy-like matrix

$$\tilde{\boldsymbol{X}}_{ij} = \frac{(\boldsymbol{V}\boldsymbol{U})_{ij}}{p_j - r_i} \ .$$

It remains to solve (4.7) (choosing $\boldsymbol{B}$ diagonal), which yields

$$p_i = \rho^{-1} \left(z + b_i + j_i \sqrt{(z+b_i)^2 + \rho^2} \right), \quad r_i = \rho^{-1} \left(z + b'_i + j'_i \sqrt{(z+b'_i)^2 + \rho^2} \right), \tag{4.13}$$

with constants b_i, b'_i and $j_i, j'_i \in \{\pm 1\}$. Since we assume that $\{b_i\} \cap \{b'_i\} = \emptyset$, the assumptions of Proposition 4.4 are satisfied.[5] It follows that, with the above data, (4.12) solves the non-autonomous chiral model equation. □

The case where $\tilde{\boldsymbol{P}}$ or $\tilde{\boldsymbol{R}}$ is non-diagonal is exploited in the next subsection. But Example 4.7 will be sufficient to understand most of Section 5.

4.2 More about the family of solutions

Introducing matrices $\boldsymbol{A}$ and $\boldsymbol{L}$ via

$$\boldsymbol{A} = (z\,\boldsymbol{I} + \boldsymbol{B})^2 + \rho^2\, \boldsymbol{I} \, , \qquad \tilde{\boldsymbol{P}} = \rho^{-1} \left(\boldsymbol{L} + z\, \boldsymbol{I} + \boldsymbol{B} \right),$$

(4.7) translates into

$$\boldsymbol{L}^2 = \boldsymbol{A} \ . \tag{4.14}$$

According to Remark 4.6, we can take $\boldsymbol{B}$ in Jordan normal form,

$$\boldsymbol{B} = \text{block-diag}(\boldsymbol{B}_{n_1}, \ldots, \boldsymbol{B}_{n_s}) \ .$$

Let us first consider the case where $\boldsymbol{B}$ is a single $r \times r$ Jordan block,

$$\boldsymbol{B}_r = b\, \boldsymbol{I}_r + \boldsymbol{N}_r \, , \qquad \boldsymbol{N}_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} .$$

Then we have

$$\boldsymbol{A} = \mathfrak{r}^2 \left(\boldsymbol{I}_r + \boldsymbol{M}_r \right),$$

where

$$\boldsymbol{M}_r = \mathfrak{r}^{-2} \left[2(z+b)\, \boldsymbol{N}_r + \boldsymbol{N}_r^2 \right], \qquad \mathfrak{r} = \pm\sqrt{(z+b)^2 + \rho^2} \, ,$$

and thus

$$\boldsymbol{L} = \mathfrak{r} \left(\boldsymbol{I}_r + \boldsymbol{M}_r \right)^{1/2} = \mathfrak{r} \sum_{k=0}^{r-1} \binom{1/2}{k} \boldsymbol{M}_r^k \, ,$$

[5] We have to restrict ourselves to the region $\rho > 0$, of course. Furthermore, $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ can lack invertibility at most on a curve in the (ρ, z) coordinate space.

by use of the generalized binomial expansion formula, noting that $\boldsymbol{M}_r^r = 0$ as a consequence of $\boldsymbol{N}_r^r = 0$. Hence we obtain the following solution of (4.7),

$$\tilde{\boldsymbol{P}}_r = \rho^{-1} \left(z\, \boldsymbol{I}_r + \boldsymbol{B}_r + \mathfrak{r} \sum_{k=0}^{r-1} \binom{1/2}{k} \boldsymbol{M}_r^k \right), \tag{4.15}$$

which is an upper triangular Toeplitz matrix. In particular, we have

$$\begin{aligned}
\tilde{\boldsymbol{P}}_1 &= \rho^{-1} [z + b + \mathfrak{r}] \,, \\
\tilde{\boldsymbol{P}}_2 &= \rho^{-1}[z + b + \mathfrak{r}] \begin{pmatrix} 1 & \mathfrak{r}^{-1} \\ 0 & 1 \end{pmatrix} , \\
\tilde{\boldsymbol{P}}_3 &= \rho^{-1}[z + b + \mathfrak{r}] \begin{pmatrix} 1 & \mathfrak{r}^{-1} & \frac{1}{2}\rho^2\mathfrak{r}^{-3}(z+b+\mathfrak{r})^{-1} \\ 0 & 1 & \mathfrak{r}^{-1} \\ 0 & 0 & 1 \end{pmatrix} , \\
\tilde{\boldsymbol{P}}_4 &= \rho^{-1}[z + b + \mathfrak{r}] \begin{pmatrix} 1 & \mathfrak{r}^{-1} & \frac{1}{2}\rho^2\mathfrak{r}^{-3}(z+b+\mathfrak{r})^{-1} & -\frac{1}{2}(z+b)\,\rho^2\mathfrak{r}^{-5}(z+b+\mathfrak{r})^{-1} \\ 0 & 1 & \mathfrak{r}^{-1} & \frac{1}{2}\rho^2\mathfrak{r}^{-3}(z+b+\mathfrak{r})^{-1} \\ 0 & 0 & 1 & \mathfrak{r}^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix} .
\end{aligned}$$

These matrices are obviously nested and, from one to the next, only the entry in the right upper corner is new.

For the above Jordan normal form of $\boldsymbol{B}$, solutions of (4.7) are now given by[6]

$$\tilde{\boldsymbol{P}} = \text{block-diag}(\tilde{\boldsymbol{P}}_{n_1}, \ldots, \tilde{\boldsymbol{P}}_{n_s}) \,,$$

where the blocks typically involve different constants replacing b, i.e. different eigenvalues of $\boldsymbol{B}$. Since $\tilde{\boldsymbol{P}}_\rho$ and $\tilde{\boldsymbol{P}}_z$ obviously commute with $\tilde{\boldsymbol{P}}$, and since $\boldsymbol{I} + \tilde{\boldsymbol{P}}^2$ is generically invertible, Lemma 4.1, part (2), ensures that $\tilde{\boldsymbol{P}}$ solves (4.6). If $\tilde{\boldsymbol{P}}$ has the above form, and $\tilde{\boldsymbol{R}}$ a similar form, and if $\tilde{\boldsymbol{P}}$ and $\tilde{\boldsymbol{R}}$ have disjoint spectra, it remains to solve the Sylvester equation[7] (4.11) in order that (4.12) yields solutions of the non-autonomous chiral model equation. This yields a plethora of exact solutions. We postpone an example to Section 5, where additional conditions considerably reduce the freedom we have here, see Example 5.6.

5 Reductions of the non-autonomous chiral model to Ernst equations

According to Section 4, a particular involutive symmetry of the non-autonomous chiral model (4.3) is given by $\tilde{g} \mapsto \gamma\, (\tilde{g}^\dagger)^{-1}\gamma$, where γ is a constant matrix with

$$\gamma^\dagger = \gamma \,, \qquad \gamma^2 = I \,.$$

(4.3) therefore admits the generalized unitarity reduction $\tilde{g}^\dagger\, \gamma\, \tilde{g} = \gamma$, which means that $\tilde{g}$ belongs to the unitary group $U(m;\gamma)$.[8] Another reduction, associated with an involutive symmetry, is $\tilde{g}^\dagger = \tilde{g}$. Imposing both reductions simultaneously, amounts to setting

$$\tilde{g}^\dagger = \tilde{g} \,, \qquad (\gamma\tilde{g})^2 = I \,. \tag{5.1}$$

Writing

$$\tilde{g} = \gamma\, (I - 2\, \mathcal{P}) \,,$$

translates theses conditions into

$$\gamma \mathcal{P}^\dagger \gamma = \mathcal{P} \,, \qquad \mathcal{P}^2 = \mathcal{P} \,.$$

---

[6]For $\tilde{\boldsymbol{P}}$ with *simple* spectrum, any solution of (4.7) has this form. Otherwise there are additional solutions, see [25].

[7]Under the stated conditions the Sylvester equation possesses a unique solution and a vast literature exists to express it.

[8]If γ has p positive and q negative eigenvalues, this is commonly denoted $U(p, q)$.

In particular, $\mathcal{P}$ is a projector. If we require in addition that $\mathrm{rank}(\mathcal{P}) = 1$, which for a projector is equivalent to $\mathrm{tr}(\mathcal{P}) = 1$ ([26], Fact 5.8.1), the following parametrization of $\tilde{g}$ can be achieved (also see e.g. [27–30]),

$$\tilde{g} = \gamma - 2\frac{v\,v^\dagger}{v^\dagger \gamma\, v}\,, \tag{5.2}$$

where v is an m-component vector with $v^\dagger \gamma\, v \neq 0$. This parametrization is invariant under $v \mapsto c\,v$ with a non-zero constant c, so that the first component of v can be set to 1 in the generic case where it is different from zero. If γ has signature $m-1$, (5.2) is a parametrization of the symmetric space $SU(m-1,1)/S(U(m-1)\times U(1))$ [27–29]. The condition $\mathrm{tr}(\mathcal{P}) = 1$ corresponds to

$$\mathrm{tr}(\gamma\tilde{g}) = m-2\,. \tag{5.3}$$

We also note that $\det(\tilde{g}) = -\det(\gamma)$. The following result shows how the reduction conditions (5.1) and (5.3) can be implemented on the family of solutions of the non-autonomous chiral model obtained via Proposition 4.4.

Proposition 5.1. *Let $\tilde{\boldsymbol{X}}$ solve the Sylvester equation (4.11), where $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{R}}, \boldsymbol{U}, \boldsymbol{V}$, satisfy*[9]

$$(\boldsymbol{\Gamma}\,\tilde{\boldsymbol{P}})^2 = -\boldsymbol{I}\,, \quad (\boldsymbol{\Gamma}\,\tilde{\boldsymbol{R}})^2 = -\boldsymbol{I}\,, \quad g_0\gamma\,\boldsymbol{U} = \boldsymbol{U}\,\boldsymbol{\Gamma}\,, \quad \boldsymbol{\Gamma}\,\boldsymbol{V} = \boldsymbol{V}\,g_0\gamma\,, \tag{5.4}$$

with an $n\times n$ matrix $\boldsymbol{\Gamma}$ and a constant $m\times m$ matrix g_0 satisfying

$$\boldsymbol{\Gamma}^2 = \boldsymbol{I}\,, \qquad (g_0\gamma)^2 = I\,. \tag{5.5}$$

Furthermore, let $\mathrm{spec}(\tilde{\boldsymbol{P}}) \cap \mathrm{spec}(\tilde{\boldsymbol{R}}) = \emptyset$.
(1) $\tilde{g}$ given by (4.12) satisfies

$$(\gamma\tilde{g})^2 = I \qquad \textit{and} \qquad \mathrm{tr}(\gamma\tilde{g}) = \mathrm{tr}(\gamma g_0) - 2\,\mathrm{tr}(\boldsymbol{\Gamma})\,. \tag{5.6}$$

(2) If moreover the relations

$$\tilde{\boldsymbol{R}}^\dagger = \boldsymbol{\Gamma}\tilde{\boldsymbol{P}}\boldsymbol{\Gamma}\,, \qquad \boldsymbol{U}^\dagger = \boldsymbol{V}\,g_0\,, \qquad g_0^\dagger = g_0\,, \qquad \boldsymbol{\Gamma}^\dagger = \boldsymbol{\Gamma} \tag{5.7}$$

hold, then $\tilde{g}$ given by (4.12) is Hermitian.

Proof: Using (4.11), (5.4) and (5.5), we find

$$\tilde{\boldsymbol{R}}\,(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}} + \boldsymbol{\Gamma}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}}\boldsymbol{\Gamma}) - (\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}} + \boldsymbol{\Gamma}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}}\boldsymbol{\Gamma})\,\tilde{\boldsymbol{P}} = 0\,,$$

so that the spectrum condition implies

$$\boldsymbol{\Gamma}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}}\boldsymbol{\Gamma} = -\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}}\,. \tag{5.8}$$

With the help of this result we obtain

$$g_0\gamma\,(I + \boldsymbol{U}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}) = g_0\gamma + \boldsymbol{U}\boldsymbol{\Gamma}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V} = g_0\gamma - \boldsymbol{U}(\boldsymbol{\Gamma}\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}})^{-1}\boldsymbol{V} = (I - \boldsymbol{U}(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}})^{-1}\boldsymbol{V})\,g_0\gamma\,.$$

Using $(g_0\gamma)^2 = I$, the condition $(\gamma\tilde{g})^2 = I$ for (4.12) is therefore equivalent to

$$(I - \boldsymbol{U}(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}})^{-1}\boldsymbol{V})(I + \boldsymbol{U}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}) = I\,.$$

Expanding the left hand side and using the Sylvester equation (4.11) to eliminate $\boldsymbol{V}\boldsymbol{U}$, this indeed turns out to be satisfied. To complete the proof of (1), it remains to derive the trace formula. Using (4.12), (4.11) and (5.8), we obtain

$$\begin{aligned}
\mathrm{tr}(\gamma\tilde{g}) - \mathrm{tr}(\gamma g_0) &= \mathrm{tr}((\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}\,g_0\gamma\boldsymbol{U}) = \mathrm{tr}((\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}\boldsymbol{U}\boldsymbol{\Gamma}) \\
&= \mathrm{tr}((\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}(\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}} - \tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})\boldsymbol{\Gamma}) = -\mathrm{tr}(\boldsymbol{\Gamma}) + \mathrm{tr}((\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}}\boldsymbol{\Gamma}) \\
&= -\mathrm{tr}(\boldsymbol{\Gamma}) - \mathrm{tr}((\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{\Gamma}\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}}) = -2\,\mathrm{tr}(\boldsymbol{\Gamma})\,.
\end{aligned}$$

[9] These conditions are motivated by the structure of $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{R}}, \boldsymbol{U}, \boldsymbol{V}$ found in example 5.5.

In order to prove (2), we consider the Hermitian conjugate of the Sylvester equation (4.11). By use of (5.7), and with the help of $g_0\gamma\,\boldsymbol{U} = \boldsymbol{U}\,\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}\,\boldsymbol{V} = \boldsymbol{V}\,g_0\gamma$ and $(g_0\gamma)^2 = I$, it takes the form

$$\boldsymbol{V}\boldsymbol{U} = -(\boldsymbol{\Gamma}\tilde{\boldsymbol{X}}^\dagger\,\boldsymbol{\Gamma})\tilde{\boldsymbol{P}} + \tilde{\boldsymbol{R}}\,(\boldsymbol{\Gamma}\,\tilde{\boldsymbol{X}}^\dagger\,\boldsymbol{\Gamma})\ .$$

By comparison with the original Sylvester equation, the spectrum condition allows us to conclude that

$$\tilde{\boldsymbol{X}}^\dagger = -\boldsymbol{\Gamma}\tilde{\boldsymbol{X}}\,\boldsymbol{\Gamma}\ .$$

Together with (5.8) this implies

$$(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^\dagger = \tilde{\boldsymbol{X}}^\dagger\tilde{\boldsymbol{R}}^\dagger = -\boldsymbol{\Gamma}\tilde{\boldsymbol{X}}\,\boldsymbol{\Gamma}^2\tilde{\boldsymbol{P}}\boldsymbol{\Gamma} = -\boldsymbol{\Gamma}\tilde{\boldsymbol{X}}\tilde{\boldsymbol{P}}\boldsymbol{\Gamma} = \tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}}\ .$$

It follows that $\boldsymbol{U}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}\,g_0$ is Hermitian, and thus also $\tilde{g}$ given by (4.12). □

Remark 5.2. Let the matrix data $(\tilde{\boldsymbol{P}}_i, \tilde{\boldsymbol{R}}_i, \boldsymbol{U}_i, \boldsymbol{V}_i, \boldsymbol{\Gamma}_i)$ satisfy $\boldsymbol{\Gamma}_i^2 = \boldsymbol{I}_{n_i}$ and

$$(\boldsymbol{\Gamma}_i\,\tilde{\boldsymbol{P}}_i)^2 = -\boldsymbol{I}_{n_i}\ , \quad (\boldsymbol{\Gamma}_i\,\tilde{\boldsymbol{R}}_i)^2 = -\boldsymbol{I}_{n_i}\ , \quad g_0\gamma\,\boldsymbol{U}_i = \boldsymbol{U}_i\,\boldsymbol{\Gamma}_i\ , \quad \boldsymbol{\Gamma}_i\,\boldsymbol{V}_i = \boldsymbol{V}_i\,g_0\gamma\ .$$

Set $\tilde{\boldsymbol{P}} = \text{block-diag}(\tilde{\boldsymbol{P}}_1, \ldots, \tilde{\boldsymbol{P}}_N)$, $\tilde{\boldsymbol{R}} = \text{block-diag}(\tilde{\boldsymbol{R}}_1, \ldots, \tilde{\boldsymbol{R}}_N)$, $\boldsymbol{\Gamma} = \text{block-diag}(\gamma, \ldots, \gamma)$, and

$$\boldsymbol{U} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N)\ , \quad \boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{pmatrix}\ .$$

Then we have $\boldsymbol{\Gamma}^2 = \boldsymbol{I}$ and (5.4) holds. If $\operatorname{spec}(\tilde{\boldsymbol{P}}) \cap \operatorname{spec}(\tilde{\boldsymbol{R}}) = \emptyset$, the corresponding Sylvester equation has a unique solution $\tilde{\boldsymbol{X}}$. According to part (1) of Proposition 5.1, $\tilde{g}$ given by (4.12) satisfies the reduction conditions (5.6). This is a way to *superpose* solutions from the class obtained in section 3, preserving the constraints (5.4). We simply block-diagonally compose the matrix data associated with the constituents. In an obvious way, this method can be extended to part (2) of Proposition 5.1. □

Remark 5.3. Let $n = 2N$ and

$$\tilde{\boldsymbol{P}} = \begin{pmatrix} \check{\boldsymbol{P}} & 0 \\ 0 & -\check{\boldsymbol{P}}^{-1} \end{pmatrix}\ , \quad \tilde{\boldsymbol{R}} = \begin{pmatrix} \check{\boldsymbol{R}} & 0 \\ 0 & -\check{\boldsymbol{R}}^{-1} \end{pmatrix}\ , \quad \boldsymbol{\Gamma} = \begin{pmatrix} 0 & \mathrm{i}\,\boldsymbol{I}_N \\ -\mathrm{i}\,\boldsymbol{I}_N & 0 \end{pmatrix}\ ,$$

where $\check{\boldsymbol{P}}$ and $\check{\boldsymbol{R}}$ are invertible block-diagonal $N \times N$ matrices, composed of blocks of the form (4.15). Then we have $(\boldsymbol{\Gamma}\tilde{\boldsymbol{P}})^2 = -\boldsymbol{I}_n$ and $(\boldsymbol{\Gamma}\tilde{\boldsymbol{R}})^2 = -\boldsymbol{I}$. Choosing γ and g_0 such that $(g_0\gamma)^2 = I$, the conditions $g_0\gamma\boldsymbol{U} = \boldsymbol{U}\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}\boldsymbol{V} = \boldsymbol{V}g_0\gamma$ are solved by

$$\boldsymbol{U} = \begin{pmatrix} \check{\boldsymbol{U}} & \mathrm{i}\,g_0\gamma\check{\boldsymbol{U}} \end{pmatrix}\ , \quad \boldsymbol{V} = \begin{pmatrix} \check{\boldsymbol{V}} \\ -\mathrm{i}\,\check{\boldsymbol{V}}\,g_0\gamma \end{pmatrix}\ ,$$

where $\check{\boldsymbol{U}}$ and $\check{\boldsymbol{V}}$ are arbitrary constant $m \times N$, respectively $N \times m$ matrices. Writing

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} \check{\boldsymbol{X}} & \check{\boldsymbol{R}}^{-1}\check{\boldsymbol{Z}}\check{\boldsymbol{P}} \\ \check{\boldsymbol{Z}} & \check{\boldsymbol{R}}\check{\boldsymbol{X}}\check{\boldsymbol{P}} \end{pmatrix}\ ,$$

reduces the $2N \times 2N$ Sylvester equation (4.11) to the two $N \times N$ Sylvester equations

$$\check{\boldsymbol{X}}\check{\boldsymbol{P}} - \check{\boldsymbol{R}}\check{\boldsymbol{X}} = \check{\boldsymbol{V}}\check{\boldsymbol{U}}\ , \qquad \check{\boldsymbol{Z}}\check{\boldsymbol{P}} + \check{\boldsymbol{R}}^{-1}\check{\boldsymbol{Z}} = -\mathrm{i}\,\check{\boldsymbol{V}}\,g_0\gamma\check{\boldsymbol{U}}\ . \tag{5.9}$$

If $\check{\boldsymbol{X}}$ and $\check{\boldsymbol{Z}}$ are invertibel, then $\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}}$ is invertible.[10] Proposition 5.1, part (1), implies that (4.12) with the above matrix data satisfies $(\gamma\tilde{g})^2 = I$ and $\operatorname{tr}(\gamma\tilde{g}) = \operatorname{tr}(\gamma g_0)$. With a suitable choice of γ and g_0 we can achieve that (5.3) holds. To fulfil the remaining Hermiticity condition, one possibility is via part (2) of Proposition 5.1. See also Examples 5.6 and 5.12. Such solutions can be superposed in the way described in Remark 5.2.

In the special case where $\check{\boldsymbol{R}} = r\,\boldsymbol{I}_N$, the solutions of the Sylvester equations (5.9) are $\check{\boldsymbol{X}} = \check{\boldsymbol{V}}\check{\boldsymbol{U}}\,(\boldsymbol{P} - r\,\boldsymbol{I}_N)^{-1}$ and $\check{\boldsymbol{Z}} = -\mathrm{i}\,\check{\boldsymbol{V}}\,g_0\gamma\check{\boldsymbol{U}}\,(\boldsymbol{P} + r^{-1}\,\boldsymbol{I}_N)^{-1}$. These expressions are not invertible if $N > m$, so in this particular case our solution formula only works for $N \leq m$ (also see Example 5.6). □

[10] $(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}$ can be computed as a 2×2 block matrix. The problem of evaluating the original expression for $\tilde{g}$ that involves $2N \times 2N$ matrices then reduces to that of evaluating only $N \times N$ matrix expressions.

Remark 5.4. Let H be an $m \times m$-matrix that satisfies

$$H^\dagger \gamma H = \gamma . \tag{5.10}$$

If $\tilde{g}$ satisfies

$$(\gamma \tilde{g})^2 = I , \qquad \tilde{g}^\dagger = \tilde{g} ,$$

then also

$$\tilde{g}' = H \tilde{g} H^\dagger ,$$

and we have $\mathrm{tr}(\gamma \tilde{g}') = \mathrm{tr}(\gamma \tilde{g})$. If $\tilde{g}$ has the form (4.12) with Hermitian g_0 and $(\gamma g_0)^2 = I$, and if H also satisfies

$$H g_0 H^\dagger = g_0 , \tag{5.11}$$

then the effect of the transformation $\tilde{g} \mapsto \tilde{g}'$ amounts to the replacement

$$\boldsymbol{U} \mapsto \boldsymbol{U}' = H \boldsymbol{U} , \qquad \boldsymbol{V} \mapsto \boldsymbol{V}' = \boldsymbol{V} H^{-1} , \tag{5.12}$$

which leaves the Sylvester equation (4.11) invariant. □

5.1 Solutions of the Ernst equation of General Relativity

We choose $m = 2$ and

$$\gamma = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} , \tag{5.13}$$

and write

$$v = \begin{pmatrix} 1 \\ \mathrm{i}\,\mathcal{E} \end{pmatrix} ,$$

with a complex function $\mathcal{E}$ and its complex conjugate $\bar{\mathcal{E}}$. Then (5.2) takes the form

$$\tilde{g} = \frac{2}{\mathcal{E} + \bar{\mathcal{E}}} \begin{pmatrix} 1 & \frac{\mathrm{i}}{2}(\mathcal{E} - \bar{\mathcal{E}}) \\ \frac{\mathrm{i}}{2}(\mathcal{E} - \bar{\mathcal{E}}) & \bar{\mathcal{E}}\mathcal{E} \end{pmatrix} ,$$

so that

$$\mathcal{E} = \frac{1 - \mathrm{i}\,\tilde{g}_{21}}{\tilde{g}_{11}} .$$

(4.3) now becomes the *Ernst equation*

$$(\mathrm{Re}\mathcal{E})\,(\partial_\rho^2 + \rho^{-1}\partial_\rho + \partial_z^2)\,\mathcal{E} = (\mathcal{E}_\rho)^2 + (\mathcal{E}_z)^2 ,$$

where e.g. ∂_ρ denotes the partial derivative with respect to ρ. This equation determines solutions of the stationary axially symmetric Einstein vacuum equations. The following statements are easily verified.

1. Excluding $\tilde{g} = \pm\gamma$, the reduction conditions (5.1) are equivalent to

$$\tilde{g} \ \text{ real} , \qquad \det(\tilde{g}) = 1 , \qquad \tilde{g}_{12} = \tilde{g}_{21} . \tag{5.14}$$

   (5.3) is then automatically satisfied.

2. For real $\tilde{g}$, the second of the reduction conditions (5.1) implies the first. As a consequence, Proposition 5.1, part (1), already generates solutions of the Ernst equation.

We will use these observations in the following examples.

Example 5.5 (Kerr-NUT)**.** For the solution of the non-autonomous chiral model given in Example 4.7 with $n = 2$, we have

$$\det(\tilde{g}) = \frac{p_1 p_2}{r_1 r_2} \det(g_0) \,,$$

with p_i, r_i given by (4.13) (also see Remark 4.5). Choosing

$$g_0 = I_2 \,,$$

so that $\det(g_0) = 1$, the second of the reduction conditions (5.14) is solved by setting

$$p_2 = -\frac{1}{p_1} \,, \qquad r_2 = -\frac{1}{r_1} \,,$$

noting that $-1/p_i$ is given by the expression for p_i with j_i exchanged by $-j_i$. We shall write p, r instead of p_1, r_1. With $\boldsymbol{U} = (u_{ij}), \boldsymbol{V} = (v_{ij})$, the remaining constraint $\tilde{g}_{12} = \tilde{g}_{21}$ is solved by[11]

$$u_{22} = -u_{11} u_{12}/u_{21} \,, \qquad v_{22} = -v_{11} v_{21}/v_{12} \,.$$

In the following we assume that u_{11} and v_{11} are different from zero and write

$$u_{21} = u\, u_{11} \,, \qquad v_{12} = v\, v_{11} \,.$$

Then $u_{11}, u_{12}, v_{11}, v_{21}$ drop out of $\tilde{g}$. Without restriction of generality, we can therefore choose them as $u_{11} = 1$, $u_{12} = -u$, $v_{11} = 1$ and $v_{21} = -v$, hence

$$\boldsymbol{U} = \begin{pmatrix} 1 & -u \\ u & 1 \end{pmatrix} \,, \qquad \boldsymbol{V} = \begin{pmatrix} 1 & v \\ -v & 1 \end{pmatrix} \,.$$

Then $\boldsymbol{U}$ and $\boldsymbol{V}$ commute with γ. $\tilde{g}$ is *real* in particular if either of the following conditions is fulfilled.

(1) p, r, u, v are real.

(2) $\bar{r} = -\frac{1}{p}$, $v = \bar{u}$ and $j = -j' \in \{\pm 1\}$.

The Ernst potential takes the form

$$\mathcal{E} = \frac{(1+uv)\,\frac{p+r}{p-r} - \mathrm{i}\,(u-v)\,\frac{pr-1}{pr+1} + (u-\mathrm{i})(v-\mathrm{i})}{(1+uv)\,\frac{p+r}{p-r} - \mathrm{i}\,(u-v)\,\frac{pr-1}{pr+1} - (u-\mathrm{i})(v-\mathrm{i})} \,.$$

By a shift of the origin of the coordinate z, we can arrange in both cases that

$$p = \rho^{-1}\,(z + b + j\,\mathfrak{r}_+) \,, \quad r = \rho^{-1}\,(z - b + j'\,\mathfrak{r}_-) \,, \quad \mathfrak{r}_\pm := \sqrt{(z \pm b)^2 + \rho^2} \,, \quad j, j' \in \{\pm 1\} \,, \tag{5.15}$$

where $b \in \mathbb{R}$ in case (1) and $b \in \mathrm{i}\,\mathbb{R}$ in case (2). Using

$$\frac{p+r}{p-r} = \frac{1}{2b}\,(j\,\mathfrak{r}_+ + j'\,\mathfrak{r}_-) \,, \quad \frac{pr-1}{pr+1} = \frac{1}{2b}\,(j\,\mathfrak{r}_+ - j'\,\mathfrak{r}_-) \,, \tag{5.16}$$

and introducing

$$\mathfrak{a} = -b\,\frac{1+uv}{u-v} \,, \quad \mathfrak{l} = j\,b\,\frac{1-uv}{u-v} \,, \quad \mathfrak{m} = -j\,b\,\frac{u+v}{u-v} \,, \tag{5.17}$$

we obtain

$$\mathcal{E} = \frac{\mathfrak{r}_+ - jj'\,\mathfrak{r}_- - \mathrm{i}\,\frac{\mathfrak{a}}{b}\,(\mathfrak{r}_+ + jj'\,\mathfrak{r}_-) - 2\,(\mathfrak{m} + \mathrm{i}\,\mathfrak{l})}{\mathfrak{r}_+ - jj'\,\mathfrak{r}_- - \mathrm{i}\,\frac{\mathfrak{a}}{b}\,(\mathfrak{r}_+ + jj'\,\mathfrak{r}_-) + 2\,(\mathfrak{m} + \mathrm{i}\,\mathfrak{l})} \,. \tag{5.18}$$

Setting $jj' = -1$, the cases (1) and (2) now simply distinguish the *non-extreme* and the *hyperextreme Kerr-NUT* space-times (see e.g. [31]). The constants satisfy

$$\mathfrak{m}^2 + \mathfrak{l}^2 - \mathfrak{a}^2 = b^2 \,.$$

□

[11] Another solution is $u_{22} = u_{12} u_{21}/u_{11}, v_{22} = v_{12} v_{21}/v_{11}$. But this leads to $\tilde{g} = g_0$.

Example 5.6. Let $n = 2N$. With the choices made in Remark 5.3, Proposition 5.1, part (1), implies that $\tilde{g}$, given by (4.12) with $g_0 = I_2$, satisfies $(\gamma\tilde{g})^2 = I_2$ and $\mathrm{tr}(\gamma\tilde{g}) = 0$. Choosing all parameters real, it follows that $\tilde{g} = I_2 + \boldsymbol{U}(\tilde{\boldsymbol{R}}\tilde{\boldsymbol{X}})^{-1}\boldsymbol{V}$ determines a solution of the Ernst equation, provided that $\tilde{\boldsymbol{X}}$ is invertible.[12] For $N = 1$, we are back to the preceding example. For $N = 2$ let, for example,

$$\tilde{\boldsymbol{P}} = \begin{pmatrix} p & p\,\mathfrak{r}^{-1} & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & -p^{-1} & (p\mathfrak{r})^{-1} \\ 0 & 0 & & -p^{-1} \end{pmatrix}, \quad \tilde{\boldsymbol{R}} = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & -r_1^{-1} & 0 \\ 0 & 0 & 0 & -r_2^{-1} \end{pmatrix}, \quad \boldsymbol{\Gamma} = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix},$$

where $p = \rho^{-1}(z + b + \mathfrak{r})$, $\mathfrak{r} = \pm\sqrt{(z+b)^2 + \rho^2}$, and r_i is also of the form (4.13) with a constant $b'_i \neq b$. The conditions for $\boldsymbol{U}$ and $\boldsymbol{V}$ restrict these matrices to the form

$$\boldsymbol{U} = \begin{pmatrix} u_1 & u_3 & -u_2 & -u_4 \\ u_2 & u_4 & u_1 & u_3 \end{pmatrix}, \quad \boldsymbol{V} = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \\ -v_2 & v_1 \\ -v_4 & v_3 \end{pmatrix},$$

with constants u_i, v_i. If $r_1 = r_2 =: r$ (i.e. $b'_1 = b'_2$), it turns out that $\tilde{g}$ does *not* depend on r and v_i, and we obtain

$$\mathcal{E} = \frac{\mathfrak{r} + \mathrm{i}\,\mathfrak{a}\,(z+b)\,\mathfrak{r}^{-1} - (\mathfrak{m} + \mathrm{i}\,\mathfrak{l})}{\mathfrak{r} + \mathrm{i}\,\mathfrak{a}\,(z+b)\,\mathfrak{r}^{-1} + (\mathfrak{m} + \mathrm{i}\,\mathfrak{l})},$$

with the parameters

$$\mathfrak{a} = \frac{u_1^2 + u_2^2}{2(u_1u_4 - u_2u_3)}, \qquad \mathfrak{l} = \frac{u_1^2 - u_2^2}{2(u_1u_4 - u_2u_3)}, \qquad \mathfrak{m} = -\frac{u_1u_2}{u_1u_4 - u_2u_3},$$

which satisfy $\mathfrak{m}^2 - \mathfrak{a}^2 + \mathfrak{l}^2 = 0$. This is the Ernst potential of an *extreme* Kerr-NUT space-time. □

Example 5.7 (Multi-Kerr-NUT)**.** According to Remark 5.2, there is a simple way to superpose solutions by block-diagonally composing their matrix data. Let

$$\boldsymbol{U}_i = \begin{pmatrix} 1 & -u_i \\ u_i & 1 \end{pmatrix}, \quad \boldsymbol{V}_i = \begin{pmatrix} 1 & v_i \\ -v_i & 1 \end{pmatrix}, \quad \tilde{\boldsymbol{P}}_i = \begin{pmatrix} p_i & 0 \\ 0 & -1/p_i \end{pmatrix}, \quad \tilde{\boldsymbol{R}}_i = \begin{pmatrix} r_i & 0 \\ 0 & -1/r_i \end{pmatrix},$$

where $p_i \neq r_k$, $i, k = 1, \ldots, N$, are given by (4.13), and either $b_i, b'_i, u_i, v_i \in \mathbb{R}$ or $\bar{b}'_i = b_i \in \mathbb{C}$, $j'_i = -j_i$, $v_i = \bar{u}_i \in \mathbb{C}$. Set

$$\boldsymbol{U} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N), \quad \boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{pmatrix},$$

and $\tilde{\boldsymbol{P}} = \text{block-diag}(\tilde{\boldsymbol{P}}_1, \ldots, \tilde{\boldsymbol{P}}_N)$, $\tilde{\boldsymbol{R}} = \text{block-diag}(\tilde{\boldsymbol{R}}_1, \ldots, \tilde{\boldsymbol{R}}_N)$, $\boldsymbol{\Gamma} = \text{block-diag}(\gamma, \ldots, \gamma)$. With $g_0 = I_2$, all assumptions of part (1) of Proposition 5.1 hold, hence with these data (4.12) determines a family of solutions of the Ernst equation. Obviously, such a solution is a superposition of N (non-extreme, respectively hyperextreme) Kerr-NUT solutions.[13] More generally, in the same way we can superpose any number of solutions with matrix data of the form given in Example 5.6. □

5.2 Solutions of the Ernst equations in the Einstein-Maxwell case

Choosing

$$\gamma = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\mathrm{i} & 0 & 0 \\ 0 & 0 & -I_{m-2} \end{pmatrix}, \tag{5.19}$$

[12]The latter condition may indeed be violated, as shown in Remark 5.3.

[13]See e.g. [6,32,33] for other derivations, and also [34], as well as the references cited there.

and writing

$$
v = \begin{pmatrix} 1 \\ \mathrm{i}\,\mathcal{E} \\ \sqrt{2}\,\Phi \end{pmatrix} ,
$$

with a complex function $\mathcal{E}$ and a complex $(m-2)$-component vector Φ, (5.2) takes the form

$$
\tilde{g} = \frac{2}{\mathcal{E}+\bar{\mathcal{E}}+2\,\Phi^\dagger\Phi} \begin{pmatrix} 1 & \frac{\mathrm{i}}{2}(\mathcal{E}-\bar{\mathcal{E}}+2\,\Phi^\dagger\Phi) & \sqrt{2}\,\Phi^\dagger \\ \frac{\mathrm{i}}{2}(\mathcal{E}-\bar{\mathcal{E}}-2\,\Phi^\dagger\Phi) & \bar{\mathcal{E}}\mathcal{E} & \mathrm{i}\,\sqrt{2}\,\mathcal{E}\,\Phi^\dagger \\ \sqrt{2}\,\Phi & -\mathrm{i}\,\sqrt{2}\,\bar{\mathcal{E}}\,\Phi & 2\Phi\Phi^\dagger - \frac{1}{2}(\mathcal{E}+\bar{\mathcal{E}}+2\,\Phi^\dagger\Phi) I_{m-2} \end{pmatrix}
$$

(also see [30, 35]). We have

$$
\mathcal{E} = \frac{1-\mathrm{i}\,\tilde{g}_{21}}{\tilde{g}_{11}} , \qquad \Phi^\intercal = \frac{1}{\sqrt{2}\,\tilde{g}_{11}}\,(\tilde{g}_{31},\ldots,\tilde{g}_{m-2,1}) ,
$$

where $^\intercal$ denotes transposition. In the following we consider the case $m=3$, where (4.3) becomes the system of *Ernst equations*

$$
\begin{aligned}
(\mathrm{Re}\mathcal{E}+\bar{\Phi}\Phi)\,(\partial_\rho^2+\rho^{-1}\partial_\rho+\partial_z^2)\,\mathcal{E} &= (\mathcal{E}_\rho)^2+(\mathcal{E}_z)^2+2\,\bar{\Phi}\,[\Phi_\rho\,\mathcal{E}_\rho+\Phi_z\,\mathcal{E}_z] , \\
(\mathrm{Re}\mathcal{E}+\bar{\Phi}\Phi)\,(\partial_\rho^2+\rho^{-1}\partial_\rho+\partial_z^2)\,\Phi &= \mathcal{E}_\rho\,\Phi_\rho+\mathcal{E}_z\,\Phi_z+2\,\bar{\Phi}\,[(\Phi_\rho)^2+(\Phi_z)^2] ,
\end{aligned}
$$

which determine solutions of the stationary axially symmetric Einstein-Maxwell equations (without further matter fields). If $\mathcal{E}=1$ and $\Phi=0$, then $\tilde{g}$ reduces to

$$
g_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \tag{5.20}
$$

which corresponds to the Minkowski metric.

Example 5.8 (Demiański-Newman)**.** Let $n=2$ and

$$
\boldsymbol{\Gamma} = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} , \quad \tilde{\boldsymbol{P}} = \begin{pmatrix} p & 0 \\ 0 & -1/p \end{pmatrix} , \quad \tilde{\boldsymbol{R}} = \begin{pmatrix} r & 0 \\ 0 & -1/r \end{pmatrix} , \tag{5.21}
$$

with p, r as in (5.15). Solving $g_0\gamma\boldsymbol{U} = \boldsymbol{U}\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}\boldsymbol{V} = \boldsymbol{V}g_0\gamma$, and recalling that $\boldsymbol{U}$ and $\boldsymbol{V}$ enter the solution formula (4.12) only up to an overall factor, leads to

$$
\boldsymbol{U} = \begin{pmatrix} 1 & -u \\ u & 1 \\ s & \mathrm{i}\,s \end{pmatrix} , \quad \boldsymbol{V} = \begin{pmatrix} 1 & v & -t \\ -v & 1 & \mathrm{i}\,t \end{pmatrix} . \tag{5.22}
$$

According to Proposition 5.1, part (1), in order to obtain solutions of the Ernst equations it remains to determine conditions under which $\tilde{g}$ is Hermitian. By explicit evaluation one finds that this is so if one of the following sets of conditions is satisfied.

(1) $\bar{b}=-b$, $j'=-j$, $v=\bar{u}$ and $t=\bar{s}$.

(2) $b\in\mathbb{R}$ and

$$
st = -2\,\frac{v+\mathrm{i}}{\bar{u}+\mathrm{i}}\,\mathrm{Im}u , \quad |v+\mathrm{i}|^2\,\mathrm{Im}u+|u-\mathrm{i}|^2\,\mathrm{Im}v = 0 , \quad 2\,\mathrm{Im}u+|s|^2 = 0 . \tag{5.23}
$$

The Ernst potential $\mathcal{E}$ is again of the form (5.18), where now

$$
\mathfrak{a} = -b\,\frac{1+uv-st}{u-v+\mathrm{i}\,st} , \qquad \mathfrak{l} = j\,b\,\frac{1-uv}{u-v+\mathrm{i}\,st} , \qquad \mathfrak{m} = -j\,b\,\frac{u+v}{u-v+\mathrm{i}\,st} ,
$$

and using the definitions (5.15). The second Ernst potential is given by

$$\Phi = \frac{2\,(q_e + \mathrm{i}\,q_m)}{\mathfrak{r}_+ - jj'\,\mathfrak{r}_- - \mathrm{i}\,\frac{\mathfrak{a}}{b}\,(\mathfrak{r}_+ + jj'\,\mathfrak{r}_-) + 2\,(\mathfrak{m} + \mathrm{i}\,\mathfrak{l})}\,, \tag{5.24}$$

where

$$q_e = -\frac{j\,b}{\sqrt{2}}\,\frac{s\,(v-\mathrm{i}) + t\,(u+\mathrm{i})}{u - v + \mathrm{i}\,st}\,, \qquad q_m = \mathrm{i}\,\frac{j\,b}{\sqrt{2}}\,\frac{s\,(v-\mathrm{i}) - t\,(u+\mathrm{i})}{u - v + \mathrm{i}\,st}\,.$$

In both cases, the parameters $\mathfrak{a}, \mathfrak{l}, \mathfrak{m}, q_e, q_m$ are real and satisfy

$$\mathfrak{m}^2 - \mathfrak{a}^2 + \mathfrak{l}^2 - q_e^2 - q_m^2 = b^2\,. \tag{5.25}$$

Cases (1) and (2) correspond to a *hyperextreme*, respectively *non-extreme*, *Demiański-Newman* space-time (see e.g. [31]). q_e and q_m are the electric and magnetic charge, respectively. Whereas (1) can be neatly expressed via (5.7), we have been unable so far to find a corresponding formulation for the conditions (2) in terms of the matrices $\tilde{\boldsymbol{P}}, \tilde{\boldsymbol{R}}, \boldsymbol{U}, \boldsymbol{V}$, also see Remark 5.11. □

Example 5.9 (Harrison transformation)**.** We can generate solutions of (5.23) via a *Harrison transformation.* A non-extreme Kerr-NUT solution (without charge) corresponds to the data

$$\boldsymbol{U}_{\mathrm{K}} = \begin{pmatrix} 1 & -u_0 \\ u_0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{V}_{\mathrm{K}} = \begin{pmatrix} 1 & v_0 & 0 \\ -v_0 & 1 & 0 \end{pmatrix},$$

with *real* u_0, v_0. The matrix

$$H = \frac{1}{1-|c|^2}\begin{pmatrix} 1 & \mathrm{i}\,|c|^2 & \mathrm{i}\sqrt{2}\,c \\ -\mathrm{i}\,|c|^2 & 1 & \sqrt{2}\,c \\ \mathrm{i}\sqrt{2}\,\bar{c} & -\sqrt{2}\,\bar{c} & -1-|c|^2 \end{pmatrix}$$

with $c \in \mathbb{C}$ satisfies (5.10) and (5.11). Then $\boldsymbol{U}' = H\,\boldsymbol{U}_{\mathrm{K}}$ and $\boldsymbol{V}' = \boldsymbol{V}_{\mathrm{K}}\,H$ satisfy $g_0\gamma\boldsymbol{U}' = \boldsymbol{U}'\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}\boldsymbol{V}' = \boldsymbol{V}'g_0\gamma$, since $\boldsymbol{U}_{\mathrm{K}}$ and $\boldsymbol{V}_{\mathrm{K}}$ satisfy these conditions. Without effect on the solution of the chiral model, we can rescale these matrices to

$$\begin{aligned}\boldsymbol{U} &= \frac{1}{1+\mathrm{i}\,u_0\,|c|^2}\begin{pmatrix} 1+\mathrm{i}\,u_0\,|c|^2 & -(u_0 - \mathrm{i}\,|c|^2) \\ u_0 - \mathrm{i}\,|c|^2 & 1+\mathrm{i}\,u_0\,|c|^2 \\ \sqrt{2}\,(\mathrm{i}-u_0)\,\bar{c} & \mathrm{i}\sqrt{2}\,(\mathrm{i}-u_0)\,\bar{c} \end{pmatrix}, \\ \boldsymbol{V} &= \frac{1}{1-\mathrm{i}\,v_0\,|c|^2}\begin{pmatrix} 1-\mathrm{i}\,v_0\,|c|^2 & v_0 + \mathrm{i}\,|c|^2 & \sqrt{2}\,(\mathrm{i}+v_0)\,c \\ -(v_0+\mathrm{i}\,|c|^2) & 1-\mathrm{i}\,v_0\,|c|^2 & -\mathrm{i}\sqrt{2}\,(\mathrm{i}+v_0)\,c \end{pmatrix},\end{aligned}$$

which have the form (5.22) and indeed satisfy (5.23). Using (5.16), the resulting Ernst potentials $\mathcal{E}$ and Φ can be written in the form (5.18), respectively (5.24), where now

$$\mathfrak{a} = -b\,\frac{1+u_0v_0}{u_0-v_0}\,, \qquad \mathfrak{l} = j\,b\,\frac{1-u_0v_0}{u_0-v_0}\,\frac{1+|c|^2}{1-|c|^2}\,, \qquad \mathfrak{m} = -j\,b\,\frac{u_0+v_0}{u_0-v_0}\,\frac{1+|c|^2}{1-|c|^2}\,,$$

and

$$q_e = \frac{2jb\,[(u_0v_0-1)\,\mathrm{Re}c - (u_0+v_0)\,\mathrm{Im}c]}{(1-|c|^2)(u_0-v_0)}\,, \qquad q_m = -\frac{2jb\,[(u_0+v_0)\,\mathrm{Re}c + (u_0v_0-1)\,\mathrm{Im}c]}{(1-|c|^2)(u_0-v_0)}\,.$$

□

Example 5.10 (hyperextreme multi-Demiański-Newman)**.** Let

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & \mathrm{i} & & & \\ -\mathrm{i} & 0 & & & \\ & & \ddots & & \\ & & & 0 & \mathrm{i} \\ & & & -\mathrm{i} & 0 \end{pmatrix}, \qquad \begin{aligned} &\tilde{\boldsymbol{P}} = \text{block-diag}(\tilde{\boldsymbol{P}}_1, \ldots, \tilde{\boldsymbol{P}}_N) \\ &\tilde{\boldsymbol{R}} = \text{block-diag}(\tilde{\boldsymbol{R}}_1, \ldots, \tilde{\boldsymbol{R}}_N)\,, \\ &\boldsymbol{U} = (\boldsymbol{U}_1, \ldots, \boldsymbol{U}_N) \end{aligned} \qquad \boldsymbol{V} = \begin{pmatrix} \boldsymbol{V}_1 \\ \vdots \\ \boldsymbol{V}_N \end{pmatrix},$$

with

$$\tilde{\boldsymbol{P}}_i = \begin{pmatrix} p_i & 0 \\ 0 & -1/p_i \end{pmatrix} , \quad \tilde{\boldsymbol{R}}_i = \begin{pmatrix} r_i & 0 \\ 0 & -1/r_i \end{pmatrix} , \quad \boldsymbol{U}_i = \begin{pmatrix} 1 & -u_i \\ u_i & 1 \\ s_i & \mathrm{i}\, s_i \end{pmatrix} , \quad \boldsymbol{V}_i = \begin{pmatrix} 1 & \bar{u}_i & -\bar{s}_i \\ -\bar{u}_i & 1 & \mathrm{i}\, \bar{s}_i \end{pmatrix} ,$$

where $\bar{r}_i = -1/p_i$ and $j_i' = -j_i$. Then the conditions of Proposition 5.1 are obviously satisfied (with γ and g_0 given by (5.19), respectively (5.20)). It follows that (4.12) determines a solution of the (Einstein-Maxwell-) Ernst equations. This is a superposition of N hyperextreme Demiański-Newman solutions. □

Remark 5.11. Whereas the hyperextreme *multi*-Demiański-Newman solutions are obtained in a straightforward way, this is not so in the non-extreme case. So far a suitable condition on the matrix data is lacking. Similar problems are known in other approaches, see e.g. [36]. □

Example 5.12. Let $n = 2N$. With the choices made in Remark 5.3, Proposition 5.1, part (1), implies that the expression given by (4.12) with g_0 in (5.20), satisfies $(\gamma\tilde{g})^2 = I_3$ and $\mathrm{tr}(\gamma\tilde{g}) = 1$. In order to obtain a solution of the Ernst equations, it suffices to arrange that $\tilde{g}$ is Hermitian. A sufficient condition is given by part (2) of Proposition 5.1. This leads to a huge family of solutions of the Einstein-Maxwell equations. The hyperextreme Demiański-Newman solution is just the simplest example in this family. Furthermore, such solutions can be superposed in the simple way described in Remark 5.2 and Example 5.10. An exploration of the corresponding space-times would be a difficult task. □

6 Conclusions

Theorem A.3, of which Proposition 3.1 is a corollary, can actually be formulated and proved without explicit use of the two *non*linear equations involving only $\boldsymbol{P}$, respectively $\boldsymbol{R}$. In such a formulation, the theorem generates solutions of the nonlinear integrable equation (2.5), respectively (2.7), from solutions of *linear* equations. However, the equations for $\boldsymbol{P}$ and $\boldsymbol{R}$ arise as integrability conditions of the latter. In previous work [1, 18–20], we chose $\boldsymbol{P}$ and $\boldsymbol{R}$ as d- and $\bar{\mathrm{d}}$-*constant* matrices, which indeed reduces the equations that have to be solved to only *linear* ones, and we recovered (and somewhat generalized) known soliton solution families. In case of the non-autonomous chiral model and, more specifically, its reduction to the Ernst equation, it turned out to be necessary to go beyond this level, and thus to consider genuine solutions of the nonlinear equations for $\boldsymbol{P}$ and $\boldsymbol{R}$, in order to obtain relevant solutions like those associated with multi-Kerr-NUT space-times and their (electrically and magnetically) charged generalizations. This also suggests a corresponding application of the theorem to other integrable PDDEs.

There are several problems not sufficiently clarified in this work, and they are partly of a rather difficult nature. Our method typically yields huge classes of exact solutions and the task remains to understand their behavior and to reveal their properties, at best in relation to the structure of the (constant) matrices that parametrize the family of solutions. In the present work we simply identified sub-families of solutions of the Ernst equation(s) by comparing the resulting Ernst potential(s) with that of well-known exact solutions already found via different methods. There seems to be more, however, in particular solutions associated with matrix data involving Jordan blocks. Perhaps the solutions resulting from such data can be obtained alternatively via certain limits of solutions from the family associated with diagonal matrix data. But at present it is far from clear what the generated class of solutions really embraces, not to talk about the possibility to make sense of the limit[14] $n \to \infty$ (where $n \times n$ is the size of the matrices that parametrize the solutions).

We addressed the non-autonomous chiral model and the Ernst equations in a new way, starting from a universal non-iterative solution generating result within the bidifferential calculus approach. The resulting solutions are parametrized by matrices, subject to conditions required by reductions, for which there may not always be a nice way to implement them (which is apparently the case for non-extreme multi-Demiański-Newman solutions). The only task which then remains is to solve a Sylvester equation, which generically can always be done. This reducibility of the solution generating problem to a Sylvester equation is a common feature of many integrable PDDEs. But this is the first time we came across a Sylvester equation involving non-constant matrix data. A particularly nice feature of this approach to (soliton-like) solutions of

[14] See e.g. [37, 38] for results on the operator Sylvester equation.

integrable PDDEs is the fact that solutions can be superposed by simply composing their matrix data into bigger block-diagonal matrices.

In Appendix B we recovered two familiar Lax pairs for the non-autonomous chiral model from the general linear equation (2.10) in the bidifferential calculus framework. Our way toward exact solutions in Section 4 is more closely related to Maison's Lax pair than to that of Belinski and Zakharov. We eliminated the θ-dependence, whereas in the Lax pair of Belinski and Zakharov the θ-dependence is kept and it involves derivatives with respect to this "spectral parameter".

Our results extend beyond the Einstein-Maxwell case and are also applicable to higher-dimensional gravity theories (see e.g. [12–17]).

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Appendix A: Via a Darboux transformation and a projection to a non-iterative solution generating result

Lemma A.1. *Let P be invertible. The transformation*

$$(\phi, g) \quad \mapsto \quad (\phi', g') = (\phi + X\,P\,X^{-1}, X\,P\,X^{-1}\,g)\,, \tag{A.1}$$

where X is an invertible solution of (2.10) with $\mathbb{A} = \mathrm{d}\phi = (\bar{\mathrm{d}}g)\,g^{-1}$, and $\bar{\mathrm{d}}P = (\mathrm{d}P)\,P$, maps a solution of the Miura transformation equation (2.13) into another solution.

Proof: Using (2.10) and $\bar{\mathrm{d}}P = (\mathrm{d}P)\,P$, a direct computation leads to

$$(\bar{\mathrm{d}}g')\,{g'}^{-1} - \mathrm{d}\phi' = \mathbb{A} - \mathrm{d}\phi - X\,P\,X^{-1}[\mathbb{A} - (\bar{\mathrm{d}}g)\,g^{-1}]\,XP^{-1}X^{-1}\,,$$

which vanishes if $\mathbb{A} = \mathrm{d}\phi = (\bar{\mathrm{d}}g)\,g^{-1}$. □

(A.1) is an essential part of a *Darboux transformation*, cf. [1]. In the following we will use this result to derive a theorem which covers Proposition 3.1 as a special case, see Remark A.5.

Lemma A.2. *Let $(\boldsymbol{\phi}, \boldsymbol{g})$ be a solution of the Miura transformation equation (2.13) in* $\mathrm{Mat}(n,n,\mathcal{B})$. *Let $\boldsymbol{U} \in \mathrm{Mat}(m,n,\mathcal{B})$ and $\boldsymbol{V} \in \mathrm{Mat}(n,m,\mathcal{B})$ be* d- *and* $\bar{\mathrm{d}}$*-constant. If*

$$\boldsymbol{\phi} = \boldsymbol{V}\boldsymbol{U}\,\hat{\boldsymbol{\phi}}\,, \tag{A.2}$$

with some $\hat{\boldsymbol{\phi}} \in \mathrm{Mat}(n,n,\mathcal{B})$, then

$$\phi = \boldsymbol{U}\hat{\boldsymbol{\phi}}\boldsymbol{V}\,, \qquad g = (\boldsymbol{U}\boldsymbol{g}^{-1}\boldsymbol{V})^{-1} \tag{A.3}$$

solve the Miura transformation equation (2.13) in $\mathrm{Mat}(m,m,\mathcal{B})$.

Proof: Since $(\boldsymbol{\phi}, \boldsymbol{g})$ is assumed to solve (2.13), we have

$$\bar{\mathrm{d}}\boldsymbol{g}^{-1} = -\boldsymbol{g}^{-1}\,\mathrm{d}\boldsymbol{\phi} = -\boldsymbol{g}^{-1}\,\boldsymbol{V}\boldsymbol{U}\,\mathrm{d}\hat{\boldsymbol{\phi}}\,.$$

Multiplying by $\boldsymbol{U}$ from the left and by $\boldsymbol{V}$ from the right, we obtain

$$\bar{\mathrm{d}}g^{-1} = -g^{-1}\,\mathrm{d}\phi\,,$$

which is equivalent to (2.13). □

Theorem A.3. *Let $(-\boldsymbol{R}, \boldsymbol{S})$ be a solution of the Miura transformation equation (2.13) in* $\mathrm{Mat}(n,n,\mathcal{B})$, *i.e.*

$$\bar{\mathrm{d}}\boldsymbol{S} = -(\mathrm{d}\boldsymbol{R})\,\boldsymbol{S}\,, \tag{A.4}$$

and $\boldsymbol{S}$ invertible. Let $\boldsymbol{X}$ be an invertible solution of the linear equation (2.10), now in $\mathrm{Mat}(n,n,\mathcal{B})$ *and with invertible $\boldsymbol{P}$, hence*

$$\bar{\mathrm{d}}\boldsymbol{X} = (\mathrm{d}\boldsymbol{X})\,\boldsymbol{P} - (\mathrm{d}\boldsymbol{R})\,\boldsymbol{X}\,, \qquad \bar{\mathrm{d}}\boldsymbol{P} = (\mathrm{d}\boldsymbol{P})\,\boldsymbol{P}\,. \tag{A.5}$$

In addition we require that

$$\boldsymbol{X}\,\boldsymbol{P} - \boldsymbol{R}\,\boldsymbol{X} = \boldsymbol{V}\,\boldsymbol{U}\,\boldsymbol{Y}\,, \qquad \bar{\mathrm{d}}\boldsymbol{R} = \boldsymbol{R}\,\mathrm{d}\boldsymbol{R}\,, \qquad \bar{\mathrm{d}}\boldsymbol{Y} = (\mathrm{d}\boldsymbol{Y})\,\boldsymbol{P}\,, \tag{A.6}$$

where $\boldsymbol{U} \in \mathrm{Mat}(m,n,\mathcal{B})$ *and* $\boldsymbol{V} \in \mathrm{Mat}(n,m,\mathcal{B})$ *are* d- *and* $\bar{\mathrm{d}}$-*constant, and* $\boldsymbol{Y} \in \mathrm{Mat}(n,n,\mathcal{B})$. *Then also*

$$\phi = \boldsymbol{U}\boldsymbol{Y}\boldsymbol{X}^{-1}\boldsymbol{V} \qquad \text{and} \qquad g = (\boldsymbol{U}\boldsymbol{S}^{-1}\boldsymbol{X}\boldsymbol{P}^{-1}\boldsymbol{X}^{-1}\boldsymbol{V})^{-1}$$

solve the Miura transformation equation (2.13), and thus (2.5), respectively (2.7).

Proof: Since we assume that $(-\boldsymbol{R}, \boldsymbol{S})$ solves the Miura transformation equation (2.13) in $\mathrm{Mat}(n,n,\mathcal{B})$, according to Lemma A.1 this also holds for the pair

$$\boldsymbol{\phi} = -\boldsymbol{R} + \boldsymbol{X}\boldsymbol{P}\boldsymbol{X}^{-1}\,, \qquad \boldsymbol{g} = \boldsymbol{X}\boldsymbol{P}\boldsymbol{X}^{-1}\,\boldsymbol{S}\,.$$

Using the first of (A.6), we find that (A.2) holds with $\hat{\boldsymbol{\phi}} = \boldsymbol{Y}\,\boldsymbol{X}^{-1}$. Now (A.3) yields the asserted formulas for ϕ and g. According to Lemma A.2, ϕ and g solve the Miura transformation equation (2.13).

Together with (A.5), the first of (A.6) implies

$$\boldsymbol{V}\boldsymbol{U}\,(\bar{\mathrm{d}}\boldsymbol{Y} - (\mathrm{d}\boldsymbol{Y})\boldsymbol{P}) = (\boldsymbol{R}\,\mathrm{d}\boldsymbol{R} - \bar{\mathrm{d}}\boldsymbol{R})\,\boldsymbol{X}\,,$$

which is satisfied if the last two conditions of (A.6) hold. □

Remark A.4. This theorem generalizes a previous result in [1], which has been applied in [1, 18–20] with d- and $\bar{\mathrm{d}}$-constant $\boldsymbol{P}, \boldsymbol{R}$, in which case only *linear* equations have to be solved in order to generate solutions of (2.5), respectively (2.7).

The above derivation shows that the theorem may be regarded as a combination of a Darboux transformation (Lemma A.1), on the level of matrices of arbitrary size, and a projection mechanism (Lemma A.2). The projection idea can be traced back to work of Marchenko [39]. More generally, the above result can be formulated in terms of suitable operators, replacing the matrices that involve a size n. □

Remark A.5. In the above theorem we set[15] $\boldsymbol{S} = \boldsymbol{R}^{-1}$ and $\boldsymbol{Y} = \boldsymbol{P}$, and we rename $\boldsymbol{X}\boldsymbol{P}^{-1}$ to $\boldsymbol{X}$. Then ϕ is given by the expression in (3.15). The expression for g in the theorem takes the form

$$g = (\boldsymbol{U}\boldsymbol{R}\,\boldsymbol{X}(\boldsymbol{X}\boldsymbol{P})^{-1}\boldsymbol{V})^{-1} = (I - \boldsymbol{U}(\boldsymbol{X}\boldsymbol{P})^{-1}\boldsymbol{V})^{-1}\,(\boldsymbol{U}\boldsymbol{V})^{-1}\,,$$

assuming temporarily invertibility of $\boldsymbol{U}\boldsymbol{V}$. Together with ϕ, this remains a solution of (2.13) if we drop the last factor, so that

$$g = (I - \boldsymbol{U}(\boldsymbol{X}\boldsymbol{P})^{-1}\boldsymbol{V})^{-1}\,.$$

This expression also makes sense if $\boldsymbol{U}\boldsymbol{V}$ is not invertible. We can still translate it into a simpler form. From the first of (A.6), which now has the form of the last of (3.14), we obtain

$$(\boldsymbol{R}\,\boldsymbol{X})^{-1} - (\boldsymbol{X}\,\boldsymbol{P})^{-1} = (\boldsymbol{R}\,\boldsymbol{X})^{-1}\boldsymbol{V}\,\boldsymbol{U}\,(\boldsymbol{X}\,\boldsymbol{P})^{-1}\,.$$

Multiplication by $\boldsymbol{U}$ from the left and by $\boldsymbol{V}$ from the right, and use in our last formula for g, leads to the expression for g in (3.15). □

Appendix B: Linear systems for the non-autonomous chiral model

B.1 Maison's Lax pair

Using the bidifferential calculus determined by (4.1), (2.10) with $\mathbb{A} = (\bar{\mathrm{d}}g)\,g^{-1}$ takes the form

$$X_\rho + \rho^{-1}X_\theta = (g_\rho + \rho^{-1}g_\theta)\,g^{-1}X - X_z\,P\,e^\theta\,, \qquad X_z = g_z\,g^{-1}X + (X_\rho - \rho^{-1}X_\theta)\,P\,e^\theta\,,$$

[15] We note that (A.4) and the second of (A.6) imply $\bar{\mathrm{d}}(\boldsymbol{R}\boldsymbol{S}) = 0$, i.e. $\boldsymbol{R}\boldsymbol{S}$ is $\bar{\mathrm{d}}$-constant.

and (2.11) reads

$$e^{-\theta}\,(P_\rho + \rho^{-1} P_\theta) = -P_z\, P\,, \qquad P_z = e^{\theta}\,(P_\rho - \rho^{-1} P_\theta)\, P\;.$$

Disregarding a constant solution (cf. section B.2), we can eliminate the θ-dependence in the latter equations via

$$P = e^{-\theta}\,\tilde{P}\,,$$

with $\tilde{P}$ independent of θ, and obtain

$$\tilde{P}_\rho = \rho^{-1}\tilde{P}\,(I - \tilde{P}^2)\,(I + \tilde{P}^2)^{-1}\,, \qquad \tilde{P}_z = 2\rho^{-1}\tilde{P}^2\,(I + \tilde{P}^2)^{-1}\;.$$

Furthermore, setting

$$X = e^{c_1\theta}\,\tilde{X}\,, \qquad g = e^{c_2\theta}\,\tilde{g}\,,$$

with $\tilde{X}, \tilde{g}$ independent of θ, the above linear system becomes

$$\begin{aligned}
\tilde{X}_\rho(I + \tilde{P}^2) &= (\tilde{g}_\rho + c_2\,\rho^{-1}\tilde{g})\,\tilde{g}^{-1}\tilde{X} - \tilde{g}_z\,\tilde{g}^{-1}\,\tilde{X}\,\tilde{P} - c_1\,\rho^{-1}\tilde{X}\,(I - \tilde{P}^2)\,,\\
\tilde{X}_z\,(I + \tilde{P}^2) &= \tilde{g}_z\,\tilde{g}^{-1}\tilde{X} + (\tilde{g}_\rho + c_2\,\rho^{-1}\tilde{g})\,\tilde{g}^{-1}\tilde{X}\,\tilde{P} - 2c_1\,\rho^{-1}\tilde{X}\,\tilde{P}\;.
\end{aligned}$$

Choosing

$$\tilde{P} = p\,I$$

with a function $p(\rho, z)$, the equations for $\tilde{P}$ can easily be integrated, which results in[16]

$$p = \rho^{-1}\,\left(z + b \pm \sqrt{(z+b)^2 + \rho^2}\right)\,,$$

where b is an arbitrary constant. In terms of

$$\hat{X} = \tilde{g}^{-1}\tilde{X}$$

the above linear system, simplified by setting $c_1 = c_2 = 0$, then takes the form

$$\hat{X}_\rho = -\frac{p}{1+p^2}\,(\tilde{g}^{-1}\tilde{g}_z + p\,\tilde{g}^{-1}\tilde{g}_\rho)\,\hat{X}\,, \qquad \hat{X}_z = \frac{p}{1+p^2}\,(\tilde{g}^{-1}\tilde{g}_\rho - p\,\tilde{g}^{-1}\tilde{g}_z)\,\hat{X}\;.$$

This system is equivalent to a linear system for the non-autonomous chiral model, first found by Maison in 1979 [4] (also see [24]).

B.2 The Belinski-Zakharov Lax pair

Using instead of θ the variable

$$\lambda = -\rho\, e^{\theta}\,,$$

(4.1) translates into

$$\mathrm{d}f = -f_z\,\zeta_1 - \rho^{-1}\lambda\, f_\rho\,\zeta_2\,, \qquad \bar{\mathrm{d}}f = -(\rho\,\lambda^{-1} f_\rho + 2\,f_\lambda)\,\zeta_1 + f_z\,\zeta_2\;. \tag{B.7}$$

We consider the linear system (2.10) with $P = I$, which trivially solves (2.11), i.e.

$$\bar{\mathrm{d}}X = \mathbb{A}\,X + \mathrm{d}X\;.$$

Writing

$$\mathbb{A} = -\frac{\rho}{\lambda}\,A\,\zeta_1 + B\,\zeta_2\,, \tag{B.8}$$

[16] We note that p is $\tilde{\boldsymbol{P}}_1$ in Section 4.2.

the integrability condition (2.12) takes the form

$$B_\rho - A_z = [A, B]\,, \qquad (\rho A)_\rho + (\rho B)_z = 0\,,$$

assuming that A, B are λ-independent. Solving the first (zero curvature) condition by

$$A = g_\rho\, g^{-1}\,, \qquad B = g_z\, g^{-1}\,,$$

the second equation becomes the non-autonomous chiral model equation

$$(\rho\, g_\rho\, g^{-1})_\rho + (\rho\, g_z\, g^{-1})_z = 0\;.$$

The above linear equation leads to

$$X_\rho = \frac{\rho\,\mathcal{U} + \lambda\,\mathcal{V}}{\rho^2 + \lambda^2}\, X - \frac{2\rho\lambda}{\rho^2 + \lambda^2}\, X_\lambda\,, \quad X_z = \frac{\rho\,\mathcal{V} - \lambda\,\mathcal{U}}{\rho^2 + \lambda^2}\, X + \frac{2\lambda^2}{\rho^2 + \lambda^2}\, X_\lambda\,,$$

where

$$\mathcal{U} = \rho\, A\,, \qquad \mathcal{V} = \rho\, B\;.$$

This is the Belinski-Zakharov Lax pair [6] (also see [8], chapter 8). We note that the "spectral parameter" λ has its origin in a coordinate of the self-dual Yang-Mills equation. We also note that $\mathbb{A} = (\bar{\mathrm{d}}g)\, g^{-1}$ (using $g_\lambda = 0$).

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