Integrability of the Pentagram Map

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June 2011

Abstract

The pentagram map was introduced by R. Schwartz in 1992 for convex planar polygons. Recently, V. Ovsienko, R. Schwartz, and S. Tabachnikov proved Liouville integrability of the pentagram map for generic monodromies by providing a Poisson structure and the sufficient number of integrals in involution on the space of twisted polygons.

In this paper we prove algebraic-geometric integrability for any monodromy, i.e., for both twisted and closed polygons. For that purpose we show that the pentagram map can be written as a discrete zero-curvature equation with a spectral parameter, study the corresponding spectral curve, and the dynamics on its Jacobian. We also prove that on the symplectic leaves Poisson brackets discovered for twisted polygons coincide with the symplectic structure obtained from Krichever-Phong's universal formula.

Introduction

The pentagram map was introduced by R. Schwartz in [1] as a map defined on convex polygons understood up to projective equivalence on a real projective plane. Here is a picture of this map for a pentagon and a hexagon:

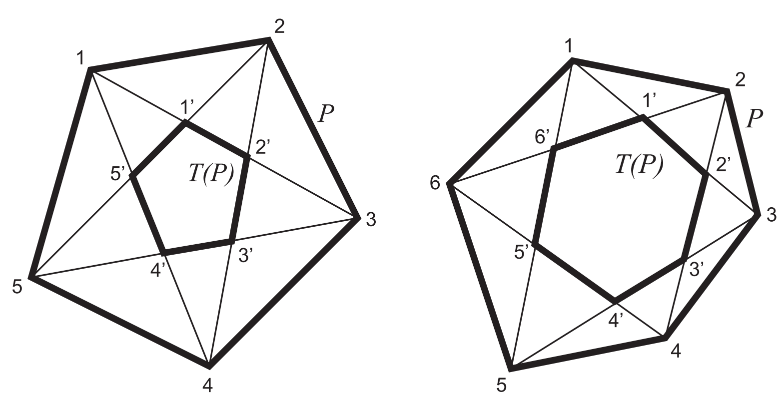

Figure 1: The pentagram map defined on a pentagon and a hexagon

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This map sends an i-th vertex to the intersection of 2 diagonals: $(i-1, i+1)$ and $(i, i+2)$. The definition implies that this map is invariant under projective transformations.

Surprisingly, this simple map stands at the intersection of many branches of mathematics: dynamical systems, integrable systems, projective geometry, and cluster algebras. In this paper we focus on integrability of the pentagram map.

Its integrability was thoroughly studied in the paper [3], where the authors considered the pentagram map on a more general space $\mathcal{P}_n$ of the so-called twisted polygons. A twisted polygon is a piecewise linear curve, which is not necessarily closed, but has a monodromy relating its vertices after n steps (we state its precise definition in the next section). They proved the Arnold-Liouville integrability for the pentagram map on this space:

Theorem 0.1 ([3]). *There exists a Poisson structure on the space $\mathcal{P}_n$ invariant under the pentagram map. The number of vertices of the polygons is n, and $n \geq 4$. When n is even, the Poisson brackets have 4 independent Casimirs, and $n-2$ invariant functions in involution. When n is odd, there are only 2 Casimirs, and $2q$ (where $q = [n/2]$) invariant functions in involution.*

The total dimension of $\mathcal{P}_n$ for all monodromies together is $2n$, and this theorem implies the Arnold-Liouville complete integrability on $\mathcal{P}_n$. I.e., a Zariski open subset of $\mathcal{P}_n$ is foliated into tori, and the time evolution is a quasiperiodic motion on these tori. The authors of [3] posed an open question about integrability for regular closed polygons. Closed polygons form a submanifold $\mathcal{C}_n$ of codimension 8 in $\mathcal{P}_n$, but it is difficult to find out what happens with Poisson brackets and integrability on this submanifold. One of the main results of the present study is a solution of this problem (see Theorem C below) in the complexified case.

Note that R.Schwartz conjectured that the pentagram map is a quasi-periodic motion in [1], introduced the integrals of motion and proved their algebraic independence in [2].

The central component of the algebraic-geometric integrability is a Lax representation with a spectral parameter, which is introduced for the pentagram map in Theorem 2.2. This Lax representation implies our main results, which can be formulated in the following 4 theorems.

Theorem A. *Consider a spectral curve $\Gamma_0 \subset \mathbb{CP}^2$ defined by the equation:*

$$R(z,k) = k^3 - k^2 \left(\sum_{j=0}^{q} J_j z^{j-q} \right) + k \left(\sum_{j=0}^{q} I_j z^{q-j} \right) z^{-n} - z^{-n} = 0,$$

where $I_j, J_j, 0 \leq j \leq q$ are complex parameters. Let the normalization of Γ_0 be Γ. The genus Γ is $g = n-2$ for even n, and $g = n-1$ for odd n. The Jacobian of Γ is $J(\Gamma)$.

Then there exists the spectral map $S : \mathcal{P}_n \to (\Gamma, J(\Gamma))$, which has a non-degenerate Jacobian matrix at a generic point, i.e., locally, S is one-to-one.

Notice that we consider polygons on a complex projective plane instead of a real projective plane, which does not change any formulas for the pentagram map.

Next theorem, along with the previous one, establishes the algebraic-geometric integrability:

Theorem B. *Let* $[D_{0,0}] \in J(\Gamma)$ *be the point that corresponds to a twisted polygon at time* $t = 0$ *under the map* S*, and* $[D_{0,t}]$ *be the point describing the twisted polygon at time* t*. Then* $[D_{0,t}]$ *is related to* $[D_{0,0}]$ *by the formulas:*

- *when* n *is odd,*
$$[D_{0,t}] = [D_{0,0} - tO_1 + tW_2] \in J(\Gamma),$$
- *when* n *is even,*
$$[D_{0,t}] = \left[D_{0,0} - tO_1 + \left[\frac{1+t}{2}\right] W_2 + \left[\frac{t}{2}\right] W_3\right].$$

For odd n *the time evolution in* $J(\Gamma)$ *is along a straight line, whereas for even* n *the evolution resembles a "staircase."*

The point $O_1 \in \Gamma$ *corresponds to* ($z = 0$, k *is finite*)*, and the points* $W_2, W_3 \in \Gamma$ *correspond to* ($z = \infty, k = 0$)*.*

Theorem C. *Closed polygons are singled out by the condition that* $(z, k) = (1, 1)$ *is a triple point of* Γ*. The latter is equivalent to 5 linear relations on* I_j, J_j*:*

$$\sum_{j=0}^{q} I_j = \sum_{j=0}^{q} J_j = 3, \quad \sum_{j=0}^{q} jI_j = \sum_{j=0}^{q} jJ_j = 3q - n, \quad \sum_{j=0}^{q} j^2 I_j = \sum_{j=0}^{q} j^2 J_j.$$

The genus of Γ *drops to* $g = n - 5$ *when* n *is even, and to* $g = n - 4$ *when* n *is odd. The dimension of the Jacobian* $J(\Gamma)$ *drops by* 3 *for closed polygons. Theorem A holds with this genus adjustment, and Theorem B holds verbatim for closed polygons.*

The relations on I_j, J_j found in Theorem 4 in [3] are equivalent to those in Theorem C.

Corollary. The dimension of the phase space $\mathcal{C}_n$ in the periodic case is $2n - 8$. In the complexified case, $\mathcal{C}_n$ is fibred over the base of dimension $2q - 3$. The coordinates on the base are $I_j, J_j, 0 \leq j \leq n - 1$, subject to the constraints from Theorem C. The fibres are Jacobians (complex tori) of dimension $2q - 3$ for odd n, and of dimension $2q - 5$ for even n. Note that the restriction of the symplectic form (which corresponds to the Poisson brackets on the symplectic leaves) to the space $\mathcal{C}_n$ is always degenerate, therefore the Arnold-Liouville theorem is not directly applicable for closed polygons. Nevertheless, the algebraic-geometric methods guarantee that the pentagram map exhibits quasi-periodic motion on a Jacobian.

Finally, we prove that:

Theorem D. *Krichever-Phong's universal formula (defined in [5, 6]) provides a pre-symplectic 2-form on the space* $\mathcal{P}_n$*. This 2-form becomes a symplectic form of rank* $2g$ *after the restriction to the leaves:* $\delta I_q = \delta J_q = 0$ *for odd* n*, and* $\delta I_0 = \delta I_q = \delta J_0 = \delta J_q = 0$ *for even* n*. These leaves coincide with the symplectic leaves of the Poisson structure found in [3]. The symplectic form is invariant under the pentagram map and coincides with the inverse of the Poisson structure restricted to the symplectic leaves.*

We would also like to point out that there is some similarity between the pentagram map and the integrable model [7] which corresponds to the $\mathcal{N} = 2$ SUSY $SU(N)$ Yang-Mills theory with a hypermultiplet in the antisymmetric representation.

1 Definition of the pentagram map

In this section, we give a definition of a twisted polygon, following [3], introduce coordinates on the space of such polygons, and give formulas of the map in terms of these coordinates.

Definition 1.1. A *twisted n-gon* is a map $\phi : \mathbb{Z} \to \mathbb{CP}^2$, such that $\phi(k+n) = M \circ \phi(k)$ for any k, and $M \in PSL(3, \mathbb{C})$ is a projective transformation of the plane $\mathbb{CP}^2$. M is called the *monodromy* of ϕ. Two twisted n-gons are equivalent if there is a transformation $g \in PSL(3, \mathbb{C})$, such that $g \circ \phi_1 = \phi_2$. The space of n-gons considered up to $PSL(3, \mathbb{C})$ transformations is called $\mathcal{P}_n$.

Notice that the monodromy is transformed as $M \to gMg^{-1}$ under transformations $g \in PSL(3, \mathbb{C})$. The dimension of $\mathcal{P}_n$ is $2n$, because a twisted n-gon depends on $2n$ variables representing coordinates of $\phi(k), 0 \leq k \leq n-1$, on a monodromy matrix M (8 additional parameters), and the equivalence relation reduces the dimension by 8.

The coordinates on $\mathcal{P}_n$ are introduced in the following way. If we assume that n is not divisible by 3, then there exists the unique lift of the points $\phi(k) \in \mathbb{P}^2$ to the vectors $V_k \in \mathbb{C}^3$ provided that $\det(V_j, V_{j+1}, V_{j+2}) = 1$ for all j. We associate a difference equation to the sequence of vectors V_k:

$$V_{j+3} = a_j V_{j+2} + b_j V_{j+1} + V_j \text{ for all } j.$$

The sequences (a_j) and (b_j) are n-periodic, i.e., $a_{j+n} = a_j, b_{j+n} = b_j$ for all j. The monodromy is a matrix $M \in SL(3, \mathbb{C})$, such that $V_{j+n} = MV_j$ for all j. The variables $a_i, b_i, 0 \leq i \leq n-1$ are coordinates on the space $\mathcal{P}_n$.

Notice, that sometimes this map is not defined. For example, it happens when 3 consecutive points lie on one line. When it is defined, and $n = 3m+1$ or $n = 3m+2$, the map is given by the formulas:

$$T^*(a_i) = a_{i+2} \prod_{l=1}^{m} \frac{1 + a_{i+3l+2} b_{i+3l+1}}{1 + a_{i-3l+2} b_{i-3l+1}}, \quad T^*(b_i) = b_{i-1} \prod_{l=1}^{m} \frac{1 + a_{i-3l} b_{i-3l-1}}{1 + a_{i+3l} b_{i+3l-1}}. \tag{1.1}$$

The proof of these formulas is a direct calculation, which has been performed [1] in [3].

2 A Lax representation and the geometry of the spectral curve

The key ingredient of the algebraic-geometric integrability is a Lax representation with a spectral parameter. First, we show that the map (1.1) has such a representation. It implies the conservation of all invariant functions from Theorem 0.1. The Lax representation organizes these invariant functions in the form of the so-called spectral curve. We investigate some properties of the spectral curve, which are important for our purposes.

[1] There is a typo in the formula (4.14) for $T^*(b_i)$ in [3].

In the continuous case, a zero-curvature equation is a compatibility condition for an over-determined system of linear differential equations, hence the name (for example, see [9] for details). In the discrete case, a system of differential equations becomes a system of linear difference equations on functions $\Psi_{i,t}, i,t \geq 0$ of an auxiliary variable z (called a *spectral parameter*):

$$\begin{cases} L_{i,t}(z)\Psi_{i,t}(z) = \Psi_{i+1,t}(z) \\ P_{i,t}(z)\Psi_{i,t}(z) = \Psi_{i,t+1}(z). \end{cases} \tag{2.1}$$

The indices i and t are integers and represent discrete space and time variables. The initial polygon corresponds to $t = 0$. It is convenient to represent several functions $\Psi_{i,t}, i,t \geq 0$ and their relationship on a diagram:

$$\begin{array}{ccccccccc}
\Psi_{i,t+1} & \xrightarrow{L_{i,t+1}} & \Psi_{i+1,t+1} & \longrightarrow \dots \longrightarrow & \Psi_{i+n-1,t+1} & \xrightarrow{L_{i+n-1,t+1}} & \Psi_{i+n,t+1} \\
{\scriptstyle P_{i,t}}\uparrow & & {\scriptstyle P_{i+1,t}}\uparrow & & {\scriptstyle P_{i+n-1,t}}\uparrow & & {\scriptstyle P_{i+n,t}}\uparrow \\
\Psi_{i,t} & \xrightarrow{L_{i,t}} & \Psi_{i+1,t} & \longrightarrow \dots \longrightarrow & \Psi_{i+n-1,t} & \xrightarrow{L_{i+n-1,t}} & \Psi_{i+n,t}
\end{array}$$

Equations (2.1) form an over-determined system, whose compatibility condition imposes a relation on the functions $L_{i,t}$ and $P_{i,t}$. This relation is called a discrete zero-curvature equation.

Definition 2.1. A *discrete zero-curvature equation* is the compatibility condition for system (2.1), which reads explicitly as:

$$L_{i,t+1}(z) = P_{i+1,t}(z)L_{i,t}(z)P_{i,t}^{-1}(z), \tag{2.2}$$

where $L_{i,t}$ is called a *Lax function.*

Theorem 2.2. *A Lax function for the pentagram map is*

$$L_{i,t}(z) = \begin{pmatrix} -b_i & 1 & 0 \\ -a_i/z & 0 & 1/z \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_i \\ 0 & z & a_i \end{pmatrix}^{-1}$$

The variables $a_i, b_i, 0 \leq i \leq n-1$, depend on time t. Their dependence on t is not indicated in the notation, but it will always be clear from the context which moment of time they correspond to.

Proof. The proof is to check that formulas (1.1) are equivalent to equation (2.2) for an appropriate choice of the function $P_{i,t}$. The matrix function $P_{i,t}$ is different for $n = 3m+1$ and $n = 3m+2$. When $n = 3m+1$, the function $P_{i,t}(z)$ is

$$P_{i,t} = \begin{pmatrix} -a_i\lambda_{i-1} & 0 & \lambda_{i-1} \\ \lambda_{i-3} & -a_{i+1}\lambda_i & b_{i-1}\lambda_{i-3} \\ 0 & z\lambda_{i-2} & 0 \end{pmatrix}, \text{ where } \lambda_i = \prod_{l=1}^{m}(1 + a_{i+3l+1}b_{i+3l}).$$

If $n = 3m+2$, then $P_{i,t}(z)$ is

$$P_{i,t} = \begin{pmatrix} -a_i\lambda_i\lambda_{i-1}(1+a_{i+1}b_i) & 0 & \lambda_i\lambda_{i-1}(1+a_{i+1}b_i) \\ \lambda_i\lambda_{i-2}(1+a_{i+1}b_i) & -a_{i+1}\lambda_i\lambda_{i+1}(1+a_{i+2}b_{i+1}) & b_{i-1}\lambda_i\lambda_{i-2}(1+a_{i+1}b_i) \\ 0 & z\lambda_{i+1}\lambda_{i-1}(1+a_{i+2}b_{i+1}) & 0 \end{pmatrix}.$$

Notice that $P_{i+n,t} \equiv P_{i,t}$ and $L_{i+n,t} \equiv L_{i,t}$ for all i. The rest of the proof is a straightforward calculation using the formulas:

- for $n = 3m+1$: $\quad T^*(a_i) = a_{i+2}\dfrac{\lambda_{i+1}}{\lambda_{i-1}}, \quad T^*(b_i) = b_{i-1}\dfrac{\lambda_{i-3}}{\lambda_{i-1}}, \quad \dfrac{1+a_{i+1}b_i}{1+a_ib_{i-1}}\lambda_i = \lambda_{i-3},$
- for $n = 3m+2$: $\quad T^*(a_i) = a_{i+2}\dfrac{\lambda_{i+1}}{\lambda_i}, \quad T^*(b_i) = b_{i-1}\dfrac{\lambda_{i-2}}{\lambda_{i-1}}, \quad \dfrac{1+a_{i+3}b_{i+2}}{1+a_{i+1}b_i}\lambda_{i+2} = \lambda_{i-1}.$

□

A discrete analogue of the monodromy matrix is a monodromy operator:

Definition 2.3. *Monodromy operators* $T_{0,t}, T_{1,t}, ..., T_{n-1,t}$ are defined as the following ordered products of the Lax functions:

$$T_{0,t} = L_{n-1,t}L_{n-2,t}...L_{0,t},$$
$$T_{1,t} = L_{0,t}L_{n-1,t}L_{n-2,t}...L_{1,t},$$
$$T_{2,t} = L_{1,t}L_{0,t}L_{n-1,t}L_{n-2,t}...L_{2,t},$$
$$...$$
$$T_{n-1,t} = L_{n-2,t}L_{n-3,t}...L_{0,t}L_{n-1,t}.$$

Similarly to the continuous case, one can define Floquet-Bloch solutions:

Definition 2.4. A *Floquet-Bloch solution* $\psi_{i,t}$ of a difference equation $\psi_{i+1,t} = L_{i,t}\psi_{i,t}$ is an eigenvector of the monodromy operator: $T_{i,t}\psi_{i,t} = k\psi_{i,t}$.

Definition 2.5. A *spectral curve* of the monodromy operator $T_{i,t}(z)$ is

$$R(k,z) = \det\left(T_{i,t}(z) - kI\right) = 0.$$

The Floquet-Bloch solutions are parameterized by the points (k,z) of the spectral curve.

Theorem 2.6. *The spectral curve for the pentagram map is*

$$R(k,z) = k^3 - k^2 tr T_{i,t} + k\, tr(T_{i,t}^{-1})z^{-n} - z^{-n} = 0, \tag{2.3}$$

where the functions $tr(T_{i,t}^{-1})$ and $tr T_{i,t}$ are equal to

$$tr(T_{i,t}^{-1}) = \sum_{j=0}^{q} I_j z^{q-j}, \quad tr T_{i,t} = \sum_{j=0}^{q} J_j z^{j-q}. \tag{2.4}$$

Here q is an integer part of $n/2$, i.e., $q = [n/2]$. The coefficients I_j, J_j are polynomials in $a_i, b_i, 0 \le i \le n-1$, and they coincide with the invariants introduced in [3].

The spectral curve is independent on i and t.

Proof. If k_1, k_2, k_3 are eigenvalues of the matrix $T_{i,t}$, then we have:

$$\operatorname{tr} T_{i,t} = k_1 + k_2 + k_3, \qquad \operatorname{tr}(T_{i,t}^{-1}) = k_1^{-1} + k_2^{-1} + k_3^{-1}, \qquad \det T_{i,t} = k_1 k_2 k_3.$$

Since $\det L_{i,t} = 1/z$, equation (2.3) follows from Vieta's formula. Equation (2.2) implies that the monodromy operators satisfy the discrete-time Lax equation:

$$T_{i,t+1}(z) = P_{i,t}(z) T_{i,t}(z) P_{i,t}^{-1}(z),$$

i.e., monodromies $T_{i,t}$ are conjugated to each other for different t. Consequently, the function $\det(T_{i,t}(z) - kI)$ is independent on t. The monodromy operators $T_{i,t}(z)$ with a fixed t and different i's are also conjugated to each other, therefore $R(k,z)$ is independent on i.

The definition of I_j, J_j in [3] is:

$$\operatorname{tr}(N_0 N_1 ... N_{n-1}) = \sum_{j=0}^{q} I_j s^{w(j)}, \qquad \operatorname{tr}(N_{n-1}^{-1} ... N_1^{-1} N_0^{-1}) = \sum_{j=0}^{q} J_j s^{-w(j)}, \tag{2.5}$$

$$\text{where } N_j = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_j/s \\ 0 & 1 & a_j s \end{pmatrix}, \qquad w(j) = n + 3j - 3q.$$

We observe that $L_j^{-1} = (gN_jg^{-1})/s$, where $g = \operatorname{diag}(s, s^2, 1)$, if we identify $z = s^{-3}$. It implies that formulas (2.4) and (2.5) are identical. □

We will need the explicit expressions for some of the invariant functions (Proposition 5.3 in [3]):

$$\text{for any } n, \quad I_q = \prod_{j=0}^{n-1} a_j, \quad J_q = (-1)^n \prod_{j=0}^{n-1} b_j,$$

$$\text{for even } n, \quad I_0 = \prod_{j=0}^{q-1} b_{2j} + \prod_{j=0}^{q-1} b_{2j+1}, \quad J_0 = (-1)^q \prod_{j=0}^{q-1} a_{2j} + (-1)^q \prod_{j=0}^{q-1} a_{2j+1}.$$

Theorem 2.7. *A homogenous polynomial $R(k,z,w) = 0$ corresponding to (2.3) defines an algebraic curve Γ_0 in $\mathbb{CP}^2$. For generic values of the parameters I_i, J_i, this curve is singular only at 2 points: $(1:0:0)$, $(0:1:0) \in \mathbb{CP}^2$. Its normalization Γ is a Riemann surface of genus $g = 2(n-q-1)$.*

Proof. A homogenous polynomial that corresponds to equation (2.3) is

$$R(k,z,w) = k^3 z^n - \sum_{j=0}^{q} J_j k^2 z^{n+j-q} w^{1-j+q} + \sum_{j=0}^{q} I_j k z^{q-j} w^{n+2+j-q} - w^{n+3}.$$

The equation $R(k,z,w) = 0$ defines an algebraic curve in $\mathbb{CP}^2$, which we denote by Γ_0. Singular points are the points where $\partial_k R = \partial_z R = \partial_w R = R = 0$. One can check that the only singular points with $w = 0$ are the points $(1:0:0)$, $(0:1:0) \in \mathbb{CP}^2$. Let us show that there are no singular points in the affine chart $(k : z : 1)$. By Euler's theorem on homogenous functions, we have $k\partial_k R + z\partial_z R + w\partial_w R = (n+3)R$. Therefore, we have a system of 3 equations for the singular points:

$$\begin{cases} \partial_k R = 3k^2 z^n - \sum_{j=0}^{q} 2J_j k z^{n+j-q} + \sum_{j=0}^{q} I_j z^{q-j} = 0 \\ \partial_z R = nk^3 z^{n-1} - \sum_{j=0}^{q}(n+j-q)J_j k^2 z^{n+j-q-1} + \sum_{j=0}^{q-1}(q-j)I_j k z^{q-j-1} = 0 \\ R = k^3 z^n - \sum_{j=0}^{q} J_j k^2 z^{n+j-q} + \sum_{j=0}^{q} I_j k z^{q-j} - 1 = 0. \end{cases}$$

These polynomials may have a solution in common only if I_j, J_j satisfy some non-trivial polynomial equation as follows, for example, from Sylvester's resultant formula. This polynomial can not vanish identically, because the system of equations has no solutions for $I_0 = J_0 = ... = I_q = J_q = 0$. Therefore, for generic values of the parameters I_j, J_j there are no singular points in the chart $(k : z : 1)$. For the same reason, one may assume that all branch points of Γ_0 on z-plane are simple, since the branch points of index 3 are given by 3 equations: $R = \partial_k R = \partial_k^2 R = 0$.

According to the normalization theorem, there always exists the unique Riemann surface Γ with a map $\sigma : \Gamma \to \Gamma_0$ biholomorphic away from the singular points. We will always work with the normalized curve Γ. The genus g of Γ is called the geometric genus of the algebraic curve Γ_0. To find it, we have to analyze the type of singularities of Γ_0, i.e., find the formal series solutions at the singular points.

Lemma 2.8. *The singularities of the curve Γ_0 are as follows:*

- *if n is even, the equation $R(k, z, 1) = 0$ has 3 distinct formal series solutions at $z = 0$:*

$$O_1 : \quad k_1 = \frac{1}{I_q} - \frac{I_{q-1}}{I_q^2} z + O(z^2),$$

$$O_2 : \quad k_2 = (-1)^q \left(\prod_{j=0}^{q-1} a_{2j} \right) \frac{1}{z^q} + O\left(\frac{1}{z^{q-1}} \right),$$

$$O_3 : \quad k_3 = (-1)^q \left(\prod_{j=0}^{q-1} a_{2j+1} \right) \frac{1}{z^q} + O\left(\frac{1}{z^{q-1}} \right),$$

  and also 3 solutions at $z = \infty$:

$$W_1 : \quad k_1 = J_q + \frac{J_{q-1}}{z} + O\left(\frac{1}{z^2} \right),$$

$$W_2 : \quad k_2 = \left(\prod_{j=0}^{q-1} b_{2j} \right)^{-1} \frac{1}{z^q} + O\left(\frac{1}{z^{q+1}} \right),$$

$$W_3 : \quad k_3 = \left(\prod_{j=0}^{q-1} b_{2j+1} \right)^{-1} \frac{1}{z^q} + O\left(\frac{1}{z^{q+1}} \right).$$

- *if n is odd, the equation $R(k,z,1)=0$ has 3 distinct Puiseux series solutions at $z=0$:*

$$O_1: \quad k_1 = \frac{1}{I_q} - \frac{I_{q-1}}{I_q^2} z + O(z^2),$$

$$O_2: \quad k_2 = \frac{\sqrt{-I_q}}{z^{n/2}} + \frac{J_0}{2z^{(n-1)/2}} + O\left(\frac{1}{z^{(n-2)/2}}\right),$$

$$k_3 = -\frac{\sqrt{-I_q}}{z^{n/2}} + \frac{J_0}{2z^{(n-1)/2}} + O\left(\frac{1}{z^{(n-2)/2}}\right),$$

and 3 solutions at $z=\infty$:

$$W_1: \quad k_1 = J_q + \frac{J_{q-1}}{z} + O\left(\frac{1}{z^2}\right),$$

$$W_2: \quad k_2 = \frac{1}{\sqrt{-J_q}}\frac{1}{z^{n/2}} + \frac{I_0}{2J_q}\frac{1}{z^{(n+1)/2}} + O\left(\frac{1}{z^{(n+2)/2}}\right),$$

$$k_3 = -\frac{1}{\sqrt{-J_q}}\frac{1}{z^{n/2}} + \frac{I_0}{2J_q}\frac{1}{z^{(n+1)/2}} + O\left(\frac{1}{z^{(n+2)/2}}\right).$$

If $\sigma : \Gamma \to \Gamma_0$ is a normalization of Γ_0, the singularities of Γ_0 correspond to several points on Γ:

- *for odd n,* $\quad \sigma^{-1}(1:0:0) = O_2, \quad \sigma^{-1}(0:1:0) = \{W_1, W_2\},$
- *for even n,* $\quad \sigma^{-1}(1:0:0) = \{O_2, O_3\}, \quad \sigma^{-1}(0:1:0) = \{W_1, W_2, W_3\}.$

The point $O_1 \in \Gamma$ is non-singular.

Proof. The proof is a computation using equation (2.3). □

Now we can complete the proof of Theorem 2.7. First, we find the number of branch points of Γ, and then we use the Riemann-Hurwitz formula to find the genus of Γ.

The number of branch points of Γ on z-plane equals the number of zeroes of the function:

$$\partial_k R(k,z) = 3k^2 - 2k\left(\frac{J_0}{z^q} + \frac{J_1}{z^{q-1}} + ... + \frac{J_{q-1}}{z} + J_q\right) + \left(\frac{I_0}{z^{n-q}} + \frac{I_1}{z^{n-q+1}} + ... + \frac{I_{q-1}}{z^{n-1}} + \frac{I_q}{z^n}\right)$$

with an exception of the singular points. The function $\partial_k R(z,k)$ is meromorphic on Γ, therefore the number of its zeroes equals the number of its poles. For any n, $\partial_k R$ has poles of total order $3n$ at $z=0$, and $\partial_k R$ has zeroes of total order n at $z=\infty$. For even n the Riemann-Hurwitz formula implies that $2-2g = 6-(3n-n)$, thus the genus of Γ is $g=n-2$. For odd n we have $2-2g = 6-(3n-n+2)$, and $g=n-1$. The difference between odd and even values of n occurs because O_2, W_2 are branch points for odd n. □

3 Direct and inverse spectral transforms

In this section we prove Theorems A and B. Recall that Theorem A reads as follows:

Theorem A. *At a generic point, the spectral map $S : \mathcal{P}_n = (a_i, b_i, 0 \leq i \leq n-1) \to (\Gamma, [D])$ has a non-degenerate Jacobian matrix, i.e., locally, it is one-to-one. The spectral curve Γ is defined in Theorem 2.7, and $[D]$ is a point in the Jacobian $J(\Gamma)$.*

Remark 3.1. Γ is determined by $2q+2$ parameters: $I_j, J_j, 0 \leq j \leq q$, and a point $[D]$ in the Jacobian of Γ is determined by g parameters, therefore the dimensions of the spaces on the left and the right hand sides of the map S match. Since the map S is locally biholomorphic, it implies the functional (and algebraic) independence of the invariants $I_j, J_j, 0 \leq j \leq q$. Their independence was proved in [3] by a different method.

The proof of Theorem A consists of two parts: the direct spectral transform (the construction of the map S itself) and the inverse spectral transform (the construction of the map S^{-1}). Combined together, they imply the functional independence of the parameters $I_j, J_j, 0 \leq j \leq q$ and coordinates in $J(\Gamma)$.

3.1 Direct spectral transform.

Given a set of parameters $(a_i, b_i, 0 \leq i \leq n-1)$, we construct the spectral curve and the Floquet-Bloch solution $\psi_{0,0}$. The vector function $\psi_{0,0}$ is defined up to a multiplication by a scalar function. To get rid of this ambiguity, we *normalize* $\psi_{0,0}$ by dividing it by the sum of its components. As a result, the vector function $\psi_{0,0}$ always satisfies the identity: $\sum_{i=1}^{3} \psi_{0,0,i} \equiv 1$. The Abel map assigns a point in the Jacobian $J(\Gamma)$ of the curve Γ to each divisor on Γ. We denote the pole divisor of $\psi_{0,0}$ by $D_{0,0}$, and the corresponding point in $J(\Gamma)$ by $[D_{0,0}]$. A pair Γ and $[D_{0,0}] \in J(\Gamma)$ is called the *spectral data*, and it is used to define the map S.

Notice that once we define the function $\psi_{0,0}$, all other functions $\psi_{i,t}$ with $i, t \geq 0$ are uniquely determined using equations (2.1). However, in Theorem B below we need to normalize each vector $\psi_{i,t}$, and we denote the normalized vectors by $\tilde{\psi}_{i,t}$. The vectors $\psi_{0,0}$ and $\tilde{\psi}_{0,0}$ are identical in this notation. The following proposition establishes the number of poles of the *normalized* Floquet-Bloch solution with any values of i, t.

Proposition 3.2. *A Floquet-Bloch solution $\tilde{\psi}_{i,t}$ is a meromorphic vector function on Γ. It is uniquely defined if we require $\sum_{j=1}^{3} \tilde{\psi}_{i,t,j} \equiv 1$. Its pole divisor $D_{i,t}$ has degree $g+2$.*

Proof. Firstly, we show that $\tilde{\psi}_{i,t}$ is a meromorphic function. By definition, it is a solution to the linear equation: $(T_{i,t} - k)u = 0$. By Cramer's rule, the components of the vector u are rational functions in the entries of the matrix $T_{i,t} - k$ and, consequently, they are rational functions in k and z. The normalized solution (u divided by the sum of its components $u_1 + u_2 + u_3$) is also a rational function in k and z, i.e., a meromorphic function on Γ.

Secondly, we find the behavior of $\tilde{\psi}_{i,t}$ at the branch points. Let the expansion of $k(z)$ at the branch point $(k_0, z_0) \in \Gamma$ be $k(z) = k_0 \pm k_1\sqrt{z - z_0} + O(z - z_0)$. If $k_1 = 0$, then the equation $R(k,z)=0$ implies that $\partial_z R(k_0,z_0)=0$, i.e., the point $(k_0,z_0)\in\Gamma$ is singular. It is not possible by Theorem 2.7, so $k_1\neq 0$. One can check that the corresponding expansion of $\tilde\psi_{i,t}$ at the branch point is $\tilde\psi_{i,t}=v\pm w\sqrt{z-z_0}+O(z-z_0)$, where the vectors v and w are determined as follows:

$$T_{i,t}(z_0)v=k_0v,\qquad (T_{i,t}(z_0)-k_0)w=k_1v,\qquad \sum_{i=1}^3 v_i=1,\qquad \sum_{i=1}^3 w_i=0.$$

The latter equations determine v,w uniquely, and they imply that k_0 corresponds to a Jordan block of the matrix $T_{i,t}(z_0)$.

Thirdly, we find the number of the poles of $\tilde\psi_{i,t}$. If $u_1+u_2+u_3=0$, then the function $\tilde\psi_{i,t}$ may develop a pole. For generic values of the parameters a_i,b_i, we may assume that these poles are distinct from the branch points of Γ. Let $k_i, 1\le i\le 3$ be the solutions of equation (2.3) for a fixed value of z. Then $Q_i=(k_i,z), 1\le i\le 3$, correspond to 3 points on Γ, and we can form a matrix $\tilde\Psi_{i,t}(z)=\{\tilde\psi_{i,t}(Q_1),\tilde\psi_{i,t}(Q_2),\tilde\psi_{i,t}(Q_3)\}$. Obviously, this matrix depends on the ordering of the roots k_1,k_2,k_3. However, an auxiliary function $F(z)=\det{}^2\tilde\Psi_{i,t}(z)$ is independent on that ordering. Consequently, $F(z)$ is a well-defined meromorphic function on Γ. Generically, it is not singular at the points $z=0$ and $z=\infty$, which follows from Proposition 3.3 below. One can check using the above series expansion of $\tilde\psi_{i,t}$ that $F(z)$ has zeroes precisely at the branch points of Γ, and that these zeroes are simple. In Theorem 2.7 we found that the number of the branch points of Γ is $\nu=2g+4$. The pole divisor of $F(z)$ equals $2\pi(D_{i,t})$. Consequently, we have $deg\, D_{i,t}=\nu/2=g+2$. □

3.2 Inverse spectral transform.

The construction of the map S^{-1} consists of 3 parts (which we describe in detail below):

- Proposition 3.3 establishes analytic properties of the Floquet-Bloch solution ψ_i. These properties allow us to reconstruct the components $\psi_{i,j}, 1\le j\le 3$, up to a multiplication by constants. Since the construction of S^{-1} doesn't depend on time t, we drop the index t in Propositions 3.3, 3.4, 3.7.

- Given a spectral curve Γ with marked points $O_i, W_i, 1\le i\le l$, where $l=2$ or 3 depending on whether n is odd or even, and a point $[D]\in J(\Gamma)$, Proposition 3.4 allows us to reconstruct Lax matrices $L'_j, 0\le j\le n-1$:

$$L'_j(z)=\begin{pmatrix}0&0&c'_j\\ d'_j&0&b'_j\\ 0&e'_jz&a'_j\end{pmatrix}^{-1},0\le j\le n-1,\qquad \prod_{i=0}^{n-1}c'_id'_ie'_i=1.$$

- If n is not divisible by 3, Proposition 3.7 allows us to perform the unique reduction from L'_j to L_j, which completes the construction of S^{-1}. It will be evident from the construction that $S\circ S^{-1}=Id$, which concludes the proof of Theorem A.

Proposition 3.3. *The divisors of the functions $\psi_{i,j}, 0\le i\le n-1, 1\le j\le 3$ have the following properties:*

- *when n is odd,*

$$(\psi_{i,1}) \geq -D + (1-i)O_2 + (1+i)W_2, \qquad (\psi_{i,2}) \geq -D - iO_2 + W_1 + (1+i)W_2,$$
$$(\psi_{i,3}) \geq -D + (2-i)O_2 + iW_2;$$

- *when n is even,*

$$(\psi_{2k,1}) \geq -D - kO_2 + (1-k)O_3 + kW_2 + (1+k)W_3,$$
$$(\psi_{2k,2}) \geq -D - kO_2 - kO_3 + W_1 + kW_2 + (1+k)W_3,$$
$$(\psi_{2k,3}) \geq -D + (1-k)O_2 + (1-k)O_3 + kW_2 + kW_3,$$
$$(\psi_{2k+1,1}) \geq -D - kO_2 - kO_3 + (1+k)W_2 + (1+k)W_3,$$
$$(\psi_{2k+1,2}) \geq -D - (1+k)O_2 - kO_3 + W_1 + (1+k)W_2 + (1+k)W_3,$$
$$(\psi_{2k+1,3}) \geq -D - kO_2 + (1-k)O_3 + kW_2 + (1+k)W_3.$$

Proof. First we establish the necessary properties of ψ_0. They are different for even and odd n. When n is even, the expansion of $T_{0,t}(z)$ at $z=0$ is:

$$T_{0,t}(z) = \begin{pmatrix} (-1)^q \prod_{i=0}^{q-1} a_{2i} & 0 & (-1)^{q-1} \prod_{i=1}^{q-1} a_{2i} \\ C_1 & (-1)^q \prod_{i=0}^{q-1} a_{2i+1} & C_2 \\ 0 & 0 & 0 \end{pmatrix} \frac{1}{z^q} + O\left(\frac{1}{z^{q-1}}\right),$$

where C_1, C_2 are some non-trivial polynomials in a_i, b_i.

Using Lemma 2.8, the definition of the Floquet-Bloch solution, and the identity $\psi_{0,1} + \psi_{0,2} + \psi_{0,3} \equiv 1$, one can check that ψ_0 is holomorphic at the points O_1, O_2, O_3 and that

$$\psi_0(O_3) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi_{0,3}(O_2) = 0, \qquad a_0 = \lim_{Q \to O_1} \frac{\psi_{0,3}(Q)}{\psi_{0,1}(Q)}.$$

Similarly, the expansion of $T_{0,t}^{-1}(z)$ at $z = \infty$ is:

$$T_{0,t}^{-1}(z) = \begin{pmatrix} 0 & \prod_{i=1}^{q-1} b_{2i} & 0 \\ 0 & \prod_{i=0}^{q-1} b_{2i} & 0 \\ \prod_{i=1}^{q-1} b_{2i-1} & C_3 & \prod_{i=1}^{q} b_{2i-1} \end{pmatrix} z^q + O(z^{q-1}),$$

which, along with the identity $T_0^{-1}\psi_0 = k^{-1}\psi_0$, implies that $\psi(Q)$ is holomorphic at the points W_1, W_2, W_3 and that

$$\psi_0(W_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \psi_{0,2}(W_1) = 0, \qquad b_0 = \lim_{Q \to W_2} \frac{\psi_{0,2}(Q)}{\psi_{0,1}(Q)}, \qquad b_{n-1} = -\lim_{Q \to W_1} \frac{\psi_{0,1}(Q)}{\psi_{0,3}(Q)}.$$

We provide a similar analysis for odd n:

$$T_{0,t}(z) = \begin{pmatrix} O(z^{-q}) & \dfrac{\prod_{i=1}^{q}(-a_{2i-1})}{z^q} + O(z^{1-q}) & O(z^{-q}) \\ \dfrac{\prod_{i=0}^{q}(-a_{2i})}{z^{q+1}} + O(z^{-q}) & O(z^{-q}) & \dfrac{\prod_{i=1}^{q}(-a_{2i})}{z^{q+1}} + O(z^{-q}) \\ O(z^{-q}) & O(z^{1-q}) & O(z^{-q}) \end{pmatrix},$$

$$T_{0,t}^{-1}(z) = \begin{pmatrix} O(z^q) & O(z^q) & z^q \prod_{i=1}^{q} b_{2i} + O(z^{q-1}) \\ z^q \prod_{i=0}^{q-1} b_{2i} + O(z^{q-1}) & O(z^q) & z^q \prod_{i=0}^{q} b_{2i} + O(z^{q-1}) \\ O(z^q) & z^{q+1} \prod_{i=1}^{q} b_{2i-1} + O(z^q) & O(z^q) \end{pmatrix},$$

which implies that:

$$\psi_0(O_2) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi_0(W_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \psi_{0,2}(W_1) = 0, \qquad \psi'_{0,3}(O_2) = 0, \tag{3.1}$$

$$a_0 = \lim_{Q \to O_1} \frac{\psi_{0,3}(Q)}{\psi_{0,1}(Q)}, \qquad b_0 = \lim_{Q \to W_2} \frac{\psi_{0,2}(Q)}{\psi_{0,1}(Q)}, \qquad b_{n-1} = -\lim_{Q \to W_1} \frac{\psi_{0,1}(Q)}{\psi_{0,3}(Q)}. \tag{3.2}$$

Notice that a cyclic permutation of indices $(n-1, n-2, ..., 1, 0)$ changes $T_i \to T_{i+1}$ and $\tilde{\psi}_i \to \tilde{\psi}_{i+1}$. For even n, it also permutes $\tilde{\psi}_i(O_2) \leftrightarrow \tilde{\psi}_i(O_3)$ and $\tilde{\psi}_i(W_2) \leftrightarrow \tilde{\psi}_i(W_3)$. This observation allows us to write formulas similar to (3.2) for $a_i, b_i, i > 0$:

$$a_i = \lim_{Q \to O_1} \frac{\psi_{i,3}(Q)}{\psi_{i,1}(Q)}, \qquad b_{2k} = \lim_{Q \to W_2} \frac{\psi_{2k,2}(Q)}{\psi_{2k,1}(Q)}, \qquad b_{2k+1} = \lim_{Q \to W_3} \frac{\psi_{2k+1,2}(Q)}{\psi_{2k+1,1}(Q)}. \tag{3.3}$$

We can use the vectors $\psi_i, i > 0$ instead of $\tilde{\psi}_i, i > 0$ in formulas (3.3), because they do not depend on the normalization. Formulas for b_{2k} and b_{2k+1} coincide for odd n.

Now using formulas (3.3) and the equation $\psi_{i+1} = L_i \psi_i$, one can check that the components of ψ_i have the required properties. $\square$

Proposition 3.4. *Given the spectral curve Γ with marked points $O_i, W_i, 1 \leq i \leq l$, where $l = 2$ for odd n, and $l = 3$ for even n, and a point $[D]$ in the Jacobian $J(\Gamma)$, one can recover a sequence of n matrices:*

$$L'_j(z) = \begin{pmatrix} 0 & 0 & c'_j \\ d'_j & 0 & b'_j \\ 0 & e'_j z & a'_j \end{pmatrix}^{-1}, 0 \leq j \leq n-1, \qquad \prod_{i=0}^{n-1} c'_i d'_i e'_i = 1.$$

This sequence is unique up to gauge transformations: $L'_j \to g_{j+1} L'_j g_j^{-1}$, where $g_j, 0 \leq j \leq n-1$, are non-degenerate diagonal matrices.

Proof. The procedure to reconstruct the matrices $L'_j, 0 \leq j \leq n-1$, consists of 3 steps:

1. We pick an arbitrary divisor D of degree $g+2$ in the equivalence class $[D] \in J(\Gamma)$.

2. We observe that the degree of all divisors in Proposition 3.3 is $-g$. According to the Riemann-Roch theorem, it means that each function $\psi_{i,j}$ is determined up to a multiplication by a constant. We pick arbitrary non-zero constants, and thus obtain a sequence of vectors ψ_i.

3. For each i, we find the matrix L'_i from the equation $\psi_{i+1} = L'_i \psi_i$. One can check using Proposition 3.3 that the matrices L'_i are uniquely determined for all i at this step.

The remaining part is to prove that $\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1$ and that the matrices L'_j are defined uniquely up to gauge transformations.

Since $\psi'_n = k\psi'_0$, the determinant of the product $T'_0 = L'_{n-1}L'_{n-2}...L'_0$ equals $k_1k_2k_3 = z^{-n}$. On the other hand, we have $\det T'_0 = \left(z^n \prod_{i=0}^{n-1} c'_i d'_i e'_i\right)^{-1}$. Consequently, $\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1$.

Assume that we have a divisor D' of degree $g+2$ equivalent to D. Two divisors are equivalent if and only if there is a meromorphic function f on Γ with zeroes at D and with poles at D'. Therefore, a choice of the divisor D' instead of D at step 1 is equivalent to multiplying all functions $\psi_i, 0 \le i \le n$, by the function f. Clearly, such multiplication does not change the matrices L'_i, which we obtain at step 3.

A different choice of constants at step 2 is equivalent to a transformation $\psi_i \to g_i\psi_i$, where g_i is a non-degenerate diagonal matrix. As a result, the matrix L'_i, which we obtain at step 3, is transformed to $g_{i+1}L'_i g_i^{-1}$, i.e., we obtain a gauge-equivalent sequence of matrices L'_i. □

Remark 3.5. The reason to introduce the marked points O_i, W_i in the statement of Proposition 3.4 is the following. When n is even, there is no natural way to distinguish the points O_2, O_3 and W_2, W_3. Since the divisors in Proposition 3.3 are not symmetric with respect to swaps $O_2 \leftrightarrow O_3$, $W_2 \leftrightarrow W_3$, different markings of the curve Γ may result in 4 non-equivalent sequences of matrices $L'_j, 0 \le j \le n-1$.

Remark 3.6. Note that one can define the spectral transform S' for the matrices $L'_j, 0 \le j \le n-1$, in the same way as the transform S in Section 3.1. The space $\mathcal{P}'_n$ of the matrices L'_j is of dimension $5n-1$ and it is parameterized by the variables

$$(a'_i, b'_i, c'_i, d'_i, e'_i, 0 \le i \le n-1)$$

subject to the constraint $\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1$. Then at generic points there is a bijection: $\mathcal{P}'_n/G \leftrightarrow (\Gamma, [D], O_i, W_i, 1 \le i \le l)$, where G denotes the action by gauge transformations defined in Proposition 3.4. The last statement is a particular case of the general construction proposed in [8].

Proposition 3.7. *If n is not divisible by 3, any sequence of n matrices:*

$$L'_j(z) = \begin{pmatrix} 0 & 0 & c'_j \\ d'_j & 0 & b'_j \\ 0 & e'_j z & a'_j \end{pmatrix}^{-1}, 0 \le j \le n-1, \qquad \prod_{i=0}^{n-1} c'_i d'_i e'_i = 1$$

may be transformed to a unique sequence of matrices $L_j(z)$ (defined in Theorem 2.2) with help of gauge transformations: $L_j = g_{j+1}L'_j g_j^{-1}$, where $g_j = diag(\alpha_j, \beta_j, \gamma_j)$ $(0 \le j \le n-1,\ g_n = g_0)$ are diagonal matrices.

Proof. The equation $L_j = g_{j+1} L'_j g_j^{-1}$ reads as:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_j \\ 0 & z & a_j \end{pmatrix} = g_{j+1} \begin{pmatrix} 0 & 0 & c'_j \\ d'_j & 0 & b'_j \\ 0 & e'_j z & a'_j \end{pmatrix} g_j^{-1},$$

and it implies a system of equations for $\alpha_j, \beta_j, \gamma_j, 0 \le j \le n-1$:

$$c'_j \frac{\alpha_j}{\gamma_{j+1}} = d'_j \frac{\beta_j}{\alpha_{j+1}} = e'_j \frac{\gamma_j}{\beta_{j+1}} = 1.$$

The latter system of equations has a one-parameter family of solutions provided that

$$\prod_{i=0}^{n-1} c'_i d'_i e'_i = 1.$$

The parameter appears because a multiplication of all matrices g_j by an arbitrary constant: $g_j \to \mu g_j$ leaves the above equations invariant. The variables a_j, b_j are independent on μ due to their defining equations:

$$a_j = a'_j \frac{\gamma_j}{\gamma_{j+1}}, \qquad b_j = b'_j \frac{\beta_j}{\gamma_{j+1}}.$$

□

Remark 3.8. Another set of coordinates was proposed in [3]. It is related to a_i, b_i via the formulas:

$$x_i = \frac{a_{i-2}}{b_{i-2} b_{i-1}}, \qquad y_i = -\frac{b_{i-1}}{a_{i-2} a_{i-1}}. \tag{3.4}$$

Notice, however, that the coordinates x_i, y_i may be defined independently of a_i, b_i. As opposed to a_i, b_i, they are well-defined for all n, and there is no "non-divisibility by 3" requirement. A Lax representation also exists for the variables x_i, y_i. It is related to the function $L_j(z)$ that we are using in the following way:

$$\tilde{L}_j = -\frac{b_{j+1}}{a_j} \left(g_{j+1}^{-1} L_j g_j \right) = \begin{pmatrix} 1/x_{j+2} & -1/x_{j+2} & 0 \\ 1/z & 0 & 1/z \\ -y_{j+2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1/y_{j+2} \\ -x_{j+2} & 0 & -1/y_{j+2} \\ 0 & z & 1/y_{j+2} \end{pmatrix}^{-1},$$

where $g_j = \mathrm{diag}(1, b_j, -a_j)$ is a gauge matrix. The matrix $P_{i,t}$, which determines the time evolution, becomes independent on n:

$$\tilde{P}_{i,t}(z) = \begin{pmatrix} 1 - x_{i+2} y_{i+2} & 0 & 1 - x_{i+2} y_{i+2} \\ x_{i+1} y_{i+1} (1 - x_{i+2} y_{i+2}) & 1 - x_{i+1} y_{i+1} & 1 - x_{i+2} y_{i+2} \\ 0 & -z y_{i+2} (1 - x_{i+3} y_{i+3}) & 0 \end{pmatrix}.$$

All theorems of this paper hold with minor changes in the coordinates x_i, y_i without the "non-divisibility by 3" requirement. Note that Proposition 3.7 is an obstacle to integrability in the coordinates a_i, b_i, when n is a multiple of 3.

3.3 Time evolution.

The remaining part of this section is to describe the time evolution of the pentagram map and to prove:

Theorem B. *The equivalence class of the pole divisor $D_{i,t}$ of $\tilde{\psi}_{i,t}$ changes as:*

- *when n is odd,*

$$[D_{i,t}] = [D_{0,0} - tO_1 + iO_2 + (t-i)W_2] \in J(\Gamma),$$

- *when n is even,*

$$[D_{i,t}] = \left[D_{0,0} - tO_1 + \left[\frac{1+i}{2}\right] O_2 + \left[\frac{i}{2}\right] O_3 + \left[\frac{1+t-i}{2}\right] W_2 + \left[\frac{t-i}{2}\right] W_3\right],$$

where $\deg D_{i,t} = g+2$, and $D_{0,0} \equiv D$ determines the point in $J(\Gamma)$ at $t=0$. For odd n the time evolution in $J(\Gamma)$ takes place along a straight line, whereas for even n the evolution goes along a "staircase" (i.e., its square goes along a straight line).

The time evolution of the pentagram map is described by the equation: $\psi_{i,t+1} = P_{i,t}\psi_{i,t}$, where t is an integer parameter. The value $t=0$ corresponds to an initial n-gon. Proposition 3.9 describes the time evolution at the level of divisors:

Proposition 3.9. *The divisors of the functions $\psi_{i,t,j}, 0 \le i \le n-1, 1 \le j \le 3$ have the following properties:*

- *when n is odd,*

$$(\psi_{i,t,1}) \ge -D + tO_1 + (1-i)O_2 + (1+i-t)W_2,$$

$$(\psi_{i,t,2}) \ge -D + tO_1 - iO_2 + W_1 + (1+i-t)W_2,$$

$$(\psi_{i,t,3}) \ge -D + tO_1 + (2-i)O_2 + (i-t)W_2,$$

- *when n is even,*

$$(\psi_{i,t,1}) \ge -D + tO_1 + \left[\frac{1-i}{2}\right] O_2 + \left[\frac{2-i}{2}\right] O_3 + \left[\frac{1+i-t}{2}\right] W_2 + \left[\frac{2+i-t}{2}\right] W_3,$$

$$(\psi_{i,t,2}) \ge -D + tO_1 + \left[\frac{-i}{2}\right] O_2 + \left[\frac{1-i}{2}\right] O_3 + W_1 + \left[\frac{1+i-t}{2}\right] W_2 + \left[\frac{2+i-t}{2}\right] W_3,$$

$$(\psi_{i,t,3}) \ge -D + tO_1 + \left[\frac{2-i}{2}\right] O_2 + \left[\frac{3-i}{2}\right] O_3 + \left[\frac{i-t}{2}\right] W_2 + \left[\frac{1+i-t}{2}\right] W_3,$$

where $[x]$ is the greatest integer less than or equal to x.

Proof. The proof is a direct verification using Proposition 3.3, formulas for $P_{i,t}$, and 2 additional formulas:

$$\psi_j(O_1) \propto \left(1, \frac{1}{a_{j+1}} + b_j, a_j\right)^T, \qquad b_{j-1} = -\lim_{Q \to W_1} \frac{\psi_{j,1}(Q)}{\psi_{j,3}(Q)},$$

where "$\propto$" means "proportional to." □

Propositions 3.4, 3.7, and 3.9 allow us to reconstruct the time evolution of an n-gon completely.

Now we are in a position to prove Theorem B itself:

Proof. The vector functions $\psi_{i,t}$ with $i, t \neq 0$ are not normalized. The normalized vectors are equal to $\tilde{\psi}_{i,t} = \psi_{i,t}/f_{i,t}$, where $f_{i,t} = \sum_{j=1}^{3} \psi_{i,t,j}$. According to Proposition 3.9, the divisor of each function $f_{i,t}$ is:

- for odd n,
$$(f_{i,t}) = D_{i,t} - D_{0,0} + tO_1 - iO_2 + (i-t)W_2,$$

- for even n,
$$(f_{i,t}) = D_{i,t} - D_{0,0} + tO_1 + \left[\frac{-i}{2}\right] O_2 + \left[\frac{1-i}{2}\right] O_3 + \left[\frac{i-t}{2}\right] W_2 + \left[\frac{1+i-t}{2}\right] W_3.$$

Since the divisor of any meromorphic function is equivalent to [0], the result of the theorem follows. □

Remark 3.10. For even n, it is not sufficient to know $[D_{i,t}]$ to recover the polygon uniquely for each t. Proposition 3.4 additionally requires the marking of the spectral curve Γ for each t. Only one marking is possible for odd n, but 4 are possible for even n. Notice that Proposition 3.9 contains more information about the time evolution. If one specifies the marking at $t = 0$, Proposition 3.9 determines the time evolution completely.

Remark 3.11. If we use a different normalization $\psi_{0,0,0} \equiv 1$ (i.e., if we divide the vector function $\psi_{0,0}$ by the first component instead of the sum of all components), the divisor D becomes:

- $D = D_g + O_2 + W_2$ for odd n,

- $D = D_g + O_3 + W_3$ for even n,

where D_g is a generic divisor of degree g on Γ.

4 Periodic case - closed polygons

In this section we prove:

Theorem C. *Closed polygons are singled out by the condition that* $(z, k) = (1, 1)$ *is a triple point of* Γ*. The latter is equivalent to 5 linear relations on* I_j, J_j*:*

$$\sum_{j=0}^{q} I_j = \sum_{j=0}^{q} J_j = 3, \quad \sum_{j=0}^{q} jI_j = \sum_{j=0}^{q} jJ_j = 3q - n, \quad \sum_{j=0}^{q} j^2 I_j = \sum_{j=0}^{q} j^2 J_j. \tag{4.1}$$

The genus of Γ *drops to* $g = n - 5$ *when* n *is even, and to* $g = n - 4$ *when* n *is odd. The dimension of the Jacobian* $J(\Gamma)$ *drops by* 3 *for closed polygons. Theorem A holds with this genus adjustment, and Theorem B holds verbatim for closed polygons.*

Proof. The monodromy matrix from the definition of the twisted n-gon equals $T_{0,t}(1)$. Clearly, an n-gon is closed if and only if $T_{0,t}(1) = I$. The latter condition implies that $(z, k) = (1, 1)$ is a self-intersection point for Γ_0. The algebraic conditions implying that $(1, 1)$ is a triple point are:

- $R(1, 1) = 0,$
- $\partial_k R(1, 1) = \partial_z R(1, 1) = 0,$
- $\partial_k^2 R(1, 1) = \partial_z^2 R(1, 1) = \partial_{kz}^2 R(1, 1) = 0.$

They are equivalent to 5 linear relations among I_j, J_j:

$$\sum_{j=0}^{q} I_j = \sum_{j=0}^{q} J_j = 3, \quad \sum_{j=0}^{q} jI_j = \sum_{j=0}^{q} jJ_j = 3q - n, \quad \sum_{j=0}^{q} j^2 I_j = \sum_{j=0}^{q} j^2 J_j.$$

Equivalent relations were found in Theorem 4 in [3].

The proofs of Theorems A and B apply, mutatis mutandis, to the periodic case with one change: a count of the number of branch points ν of Γ and the corresponding calculation for the genus g of Γ.

As before, the function $\partial_k R$ has poles of total order $3n$ above $z = 0$, and zeroes of total order n about $z = \infty$. Now since $R(z, k)$ has a triple point $(1, 1)$, $\partial_k R$ has a double zero at $(1, 1)$. But $z = 1$ is not a branch point of the normalization Γ. Consequently, $\partial_k R$ has double zeroes on 3 sheets of Γ above $z = 1$. The Riemann-Hurwitz formula for even n becomes: $2-2g = 6-\nu$, $\nu = 3n-n-6 = 2n-6$, and for odd n: $2-2g = 6-\nu$, $\nu = 3n-n-6+2 = 2n-4$. Therefore, we have $g = n - 5$ for even n, and $g = n - 4$ for odd n.

Remark 3.1 implies that there are no other relations among $I_i, J_i, 0 \le i \le q$, except for (4.1) in the periodic case. The dimension of the Jacobian $J(\Gamma)$ is 3 less than for twisted polygons. Therefore, closed polygons form a subspace of codimension 8 in $\mathcal{P}_n$. □

Corollary 4.1. The dimension of the phase space $\mathcal{C}_n$ in the periodic case is $2n-8$. In the complexified case, $\mathcal{C}_n$ is fibred over the base of dimension $2q-3$. The coordinates on the base are $I_j, J_j, 0 \le j \le n-1$, subject to the constraints from Theorem C. The fibres are Jacobians (complex tori) of dimension $2q-3$ for odd n, and of dimension $2q-5$ for even n. Note that the restriction of the symplectic form (which corresponds to the Poisson brackets on the symplectic leaves) to the space $\mathcal{C}_n$ is always degenerate, therefore the Arnold-Liouville theorem is not directly applicable for closed polygons. Nevertheless, the algebraic-geometric methods guarantee that the pentagram map exhibits quasi-periodic motion on a Jacobian.

5 The symplectic form

Definition 5.1 ([5, 6])**.** *Krichever-Phong's universal formula* defines a pre-symplectic form on the space of Lax operators, i.e., on the space $\mathcal{P}_n$. It is given by the expression:

$$\omega = -\frac{1}{2} \sum_{z=0,\infty} \operatorname{res} \operatorname{Tr} \left(\Psi_0^{-1} T_0^{-1} \delta T_0 \wedge \delta \Psi_0 \right) \frac{dz}{z}.$$

The matrix $\Psi_{0,t}$ is defined in Proposition 3.2. In this section we drop the index t, because all variables correspond to the same moment of time.

The *leaves* of the 2-form ω are defined as submanifolds of $\mathcal{P}_n$, where the expression $\delta \ln k dz/z$ is holomorphic. The latter expression is considered as a one-form on the spectral curve Γ.

Remark 5.2. A heuristic principle justified by many examples is that when ω is restricted to these leaves, it becomes a symplectic form of rank $2g$, where g is the genus of Γ. Moreover, one can prove ([8]) that ω does not depend on the normalization of the eigenvectors used to construct the matrix $\Psi_{0,t}$, and on gauge transformations $L_j \to g_{j+1} L_j g_j^{-1},\ g_j \in GL(3, \mathbb{C})$, when restricted to the leaves.

Remark 5.3. There exist different variations of the universal formula, which provide 2 or even more compatible Hamiltonian structures for some integrable systems. However, it seems likely that other modifications of the universal formula lead to degenerate 2-forms for the pentagram map.

Theorem D. *The 2-form ω defined above equals:*

$$\omega = \sum_{(i,j) \in \Lambda} (\delta \ln a_i \wedge \delta \ln a_j - \delta \ln b_i \wedge \delta \ln b_j),$$

where the set Λ consists of pairs $(i, j), 0 \le i \le n-1, i < j \le n-1$, such that either both i and j are even, or i is odd and j is arbitrary.

The 2-form ω is a symplectic form of rank $2g$, where g is the genus of the spectral curve Γ, when restricted to the leaves: $\delta I_q = \delta J_q = 0$ for odd n, and $\delta I_0 = \delta I_q = \delta J_0 = \delta J_q = 0$ for even n. These leaves coincide with the symplectic leaves of the Poisson bracket found in Proposition 4.12 in [3]. The inverse of ω coincides with the Poisson bracket on the symplectic leaves.

Proof. First we find the equations which define the leaves of the 2-form ω.

Lemma 5.4. *The one-form $\delta \ln k dz/z$ is holomorphic on the spectral curve Γ when restricted to the leaves: $\delta I_q = \delta J_q = 0$ for odd n, and $\delta I_0 = \delta I_q = \delta J_0 = \delta J_q = 0$ for even n.*

Proof. Using Lemma 2.8, one can calculate the principal parts of the one-form $\delta \ln k dz/z$ at the points O_2, W_2 for odd n, and at the points O_2, O_3, W_2, W_3 for even n. The equations defining the leaves follow after equating these principal parts to zero. □

Now we proceed to the computation of ω. Note that

$$\mathrm{Tr}\left(\Psi_0^{-1}T_0^{-1}\delta T_0 \wedge \delta\Psi_0\right) = \sum_{k=0}^{n-1} \mathrm{Tr}\left(\Psi_0^{-1}L_0^{-1}...L_k^{-1}\delta L_k L_{k-1}...L_0 \wedge \delta\Psi_0\right) =$$

$$= \sum_{k=0}^{n-1} \mathrm{Tr}\left(\Psi_k^{-1}L_k^{-1}\delta L_k \wedge \delta\Psi_k\right) - \sum_{k=0}^{n-2} \mathrm{Tr}\left(L_0^{-1}...L_k^{-1}\delta L_k \wedge \delta(L_{k-1}...L_0)\right),$$

where $\Psi_k = L_{k-1}...L_0\Psi_0$ (this transformation is similar to the one used in [8]). Notice that the last sum does not have any poles except at the points $z = 0$ and $z = \infty$ and vanishes after the summation over both residues. Therefore,

$$\omega = -\frac{1}{2}\sum_{j=0}^{n-1} \mathop{\mathrm{res}}_{0,\infty} \mathrm{Tr}\left(\Psi_j^{-1}L_j^{-1}\delta L_j \wedge \delta\Psi_j\right)\frac{dz}{z}.$$

To compute ω, we use a normalization of ψ_0 in which $\psi_{0,1} \equiv 1$. It corresponds to the case when the first line of Ψ_0 is $(1,1,1)$. Note that a different normalization is used in Proposition 3.3. Therefore, when we use Proposition 3.3 in this proof, we have to keep in mind Remark 3.11. In particular, this remark implies that the vector ψ_i may acquire additional poles at the points O_2, W_2 or O_3, W_3.

The matrices $\Psi_j, j > 0$, are not normalized. A normalized matrix $\tilde{\Psi}_j, j > 0$, is related to Ψ_j by a diagonal matrix F_j: $\tilde{\Psi}_j = \Psi_j F_j$. The matrices $F_j, j > 0$, may have poles or zeroes at $z = 0, \infty$. We have the formula:

$$\mathrm{Tr}\left(\Psi_j^{-1}L_j^{-1}\delta L_j \wedge \delta\Psi_j\right) = \mathrm{Tr}\left(\tilde{\Psi}_j^{-1}L_j^{-1}\delta L_j \wedge \delta\tilde{\Psi}_j\right) - \mathrm{Tr}\left(\tilde{\Psi}_j^{-1}L_j^{-1}\delta L_j\tilde{\Psi}_j \wedge \delta \ln F_j\right)$$

Notice that the product $L_j^{-1}\delta L_j$ is

$$L_j^{-1}\delta L_j = \begin{pmatrix} 0 & 0 & 0 \\ -\delta b_j & 0 & 0 \\ -\delta a_j & 0 & 0 \end{pmatrix},$$

and the first line of $\delta\tilde{\Psi}_j$ is always zero due to the normalization. Consequently, we obtain the formula:

$$\omega = \frac{1}{2}\sum_{j=0}^{n-1} \mathop{\mathrm{res}}_{0,\infty} \mathrm{Tr}\left(\tilde{\Psi}_j^{-1}L_j^{-1}\delta L_j\tilde{\Psi}_j \wedge \delta \ln F_j\right)\frac{dz}{z}.$$

We can rewrite the last formula as:

$$\omega = \frac{1}{2}\sum_{j=0}^{n-1}\sum_i \operatorname*{res}_{O_i,W_i} \operatorname{Tr}\left(\psi_j^* L_j^{-1}\delta L_j \tilde{\psi}_j \wedge \delta \ln f_j\right)\frac{dz}{z},$$

where ψ_j^* is an eigen-covector: $\psi_j^* T_j = k\psi_j^*$. Covectors are normalized by $\psi_j^*\tilde{\psi}_j = 1$, and $\tilde{\psi}_{j,1} \equiv 1$. One can check that $\psi_j^* L_j^{-1}\delta L_j\tilde{\psi}_j = -\psi_{j,3}^*\delta a_j - \psi_{j,2}^*\delta b_j$. The formula for ω becomes:

$$\omega = -\frac{1}{2}\sum_{j=0}^{n-1}\sum_i \operatorname*{res}_{O_i,W_i}(\psi_{j,3}^*\delta a_j + \psi_{j,2}^*\delta b_j)\wedge\delta\ln f_j\frac{dz}{z} = \sum_i \omega_{O_i} + \sum_i \omega_{W_i}. \tag{5.1}$$

We use formula (5.1) to compute ω. We compute the terms ω_{O_i} and ω_{W_i} with different i separately, and then sum them up.

Lemma 5.5. *The contribution from the point O_1 is independent on the parity of n and is given by:*

$$\omega_{O_1} = -\frac{1}{2}\sum_{j=2}^{n-1}\delta\ln a_j \wedge \delta\ln\left(\prod_{k=1}^{j} a_k\right).$$

Proof. First, we prove 2 formulas:

$$\tilde{\psi}_j(O_1) = \left(1, \frac{1}{a_{j+1}} + b_j, a_j\right)^T, \qquad \psi_j^*(O_1) = (0,0,1/a_j), \tag{5.2}$$

then we find $f_j(O_1)$, and compute ω_{O_1} using formula (5.1).

The vectors ψ_0, ψ_0^* and the matrix T_0 are related to $\tilde{\psi}_j$, ψ_j^*, T_j by a permutation of the variables $a_0,...,a_{n-1}$ and $b_0,...,b_{n-1}$. Therefore, formulas (5.2) are equivalent to 2 formulas (which we prove below):

$$\psi_0(O_1) = \left(1, \frac{1}{a_1} + b_0, a_0\right)^T, \quad \psi_0^*(O_1) = (0,0,1/a_0).$$

Proposition 3.3 and formulas (3.2) imply that $\psi_0(O_1) = (1, x, a_0)^T$ for some constant x. Using the value of T_0^{-1} at $z = 0$:

$$T_0^{-1}(0) = \begin{pmatrix} 0 & 0 & I_q/a_0 \\ 0 & 0 & (1+a_1b_0)I_q/(a_0a_1) \\ 0 & 0 & I_q \end{pmatrix},$$

and the formula $T_0^{-1}(0)\psi_0(O_1) = I_q\psi_0(O_1)$, we find that $x = (1/a_1) + b_0$.

One can check that the equation $\psi_0^* T_0 = k\psi_0^*$ implies that $\psi_0^*(O_1) = (0,0,y)$ for some constant y. Since $\psi_0^*\psi_0 = 1$, we find that $y = 1/a_0$.

To find $f_j(O_1)$, we have to compare $\tilde{\psi}_j$ and $L_{j-1}...L_0\psi_0$ at the point O_1. One can check that $L_0\psi_0(O_1) = (1/a_1, *, 1)^T$. Therefore, $f_1(O_1) = a_1$. When $i > 0$, we have $L_i\tilde{\psi}_i = (1/a_{i+1}, *, 1)^T$. Consequently, we find that $f_i(O_1)/f_{i-1}(O_1) = a_i$. Multiplying the latter equations with $2 \le i \le j$ by each other, we obtain that $f_j(O_1) = \prod_{k=1}^{j} a_k$.

Substituting $f_j(O_1)$ and $\psi_j^*(O_1)$ into formula (5.1), we obtain ω_{O_1}. □

Similarly, the contribution from the point W_1 is given by:

Lemma 5.6. *For both even and odd n,*

$$\omega_{W_1} = \frac{1}{2}\sum_{j=1}^{n-1} \delta \ln b_j \wedge \delta \ln \left(\prod_{k=0}^{j-1} b_k \right).$$

Proof. In the same way as in Lemma 5.5, we find that

$$\tilde{\psi}_j(W_1) = (1, 0, -1/b_{j-1})^T, \qquad \psi_j^*(W_1) = (1, -1/b_j, 0), \qquad f_j(W_1) = (-1)^j \prod_{k=0}^{j-1} b_k^{-1},$$

which implies the formula for ω_{W_1}. □

The computation at the points O_2, O_3, W_2, W_3 is trickier, because it differs for even and odd n.

Lemma 5.7. *If n is odd, then*

$$\omega_{O_2} = -\frac{1}{2}\sum_{j=1}^{n-1} \delta \ln a_j \wedge \delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=0}^{q} a_{k+2i} \right).$$

Proof. First, we need to prove 2 formulas:

$$\tilde{\psi}_j = \left(1, \frac{(-1)^{q+1} \prod_{i=0}^{q} a_{j+2i}}{\sqrt{-I_q}} \frac{1}{\sqrt{z}} + O(1), \frac{(-1)^q \prod_{i=0}^{q-1} a_{j+2i}}{\sqrt{-I_q}} \sqrt{z} + O(z) \right)^T \text{ at } O_2, \tag{5.3}$$

$$\psi_j^*(O_2) = \left(\frac{1}{2}, 0, -\frac{1}{2a_j} \right). \tag{5.4}$$

Note that a cyclic permutation of the variables $a_j \to a_{j+1}, b_j \to b_{j+1}$ (for all j) permutes the eigenvectors and covectors as follows: $\tilde{\psi}_j \to \tilde{\psi}_{j+1}$, $\psi_j^* \to \psi_{j+1}^*$. Therefore, we only need to find $\tilde{\psi}_0$ at O_2 and $\psi_0^*(O_2)$ to prove formulas (5.3) and (5.4).

Proposition 3.3 implies that $\psi_0 = (1, \alpha/\sqrt{z} + O(1), \beta\sqrt{z} + O(z))^T$ around the point O_2. Since $T_0\psi_0 = \left(\sqrt{-I_q} z^{-n/2} + O(z^{-q})\right)\psi$, we find that

$$\alpha = \frac{(-1)^{q+1} \prod_{i=0}^{q} a_{2i}}{\sqrt{-I_q}}.$$

One can check that $\left(T_0^{-1}\right)_{32} = I_q z/a_{n-1} + O(z^2)$, and since $T_0^{-1}\psi_0 = \psi_0 O(z^{n/2})$ in the neighborhood of O_2, we deduce that $\beta = -\alpha/a_{n-1}$. Formula (5.3) with $j = 0$ is proven.

The equation $\psi_0^*\psi_0 = 1$ implies that $\psi_0^* = (\alpha' + O(\sqrt{z}), \beta'\sqrt{z} + O(z), \gamma' + O(\sqrt{z}))$ at the point O_2. Using the identity $\psi_0^* T_0 = \left(\sqrt{-I_q} z^{-n/2} + O(z^{-q})\right)\psi_0^*$, we find that

$$\beta' \prod_{i=0}^{q}(-a_{2i}) = \alpha'\sqrt{-I_q}, \qquad \beta' \prod_{i=1}^{q}(-a_{2i}) = \gamma'\sqrt{-I_q}, \qquad \alpha' + \beta' \frac{(-1)^{q+1} \prod_{i=0}^{q} a_{2i}}{\sqrt{-I_q}} = 1.$$

Solving these equations for α', β', γ', we obtain that $\psi_0^*(O_2) = (1/2, 0, -1/(2a_0))$.

Now we find the value of $\delta \ln f_j(O_2)$. Since $(L_0\psi_0)_1 = \psi_{0,2} - b_0\psi_{0,1}$, we obtain that $\delta \ln f_1(O_2) = -\delta \ln \alpha$. The argument similar to the one used in the proof of Lemma 5.5, along with the condition $\delta I_q = 0$, implies that

$$\delta \ln f_j(O_2) = -\delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=0}^{q} a_{k+2i} \right).$$

Finally, using formula (5.1), we obtain that

$$\omega_{O_2} = \frac{1}{2} \sum_{j=1}^{n-1} 2 \cdot \frac{1}{2} \delta \ln a_j \wedge \delta \ln f_j(O_2) = -\frac{1}{2} \sum_{j=1}^{n-1} \delta \ln a_j \wedge \delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=0}^{q} a_{k+2i} \right).$$

The coefficient "2" in the last formula appears because O_2 is a branch point. The local parameter around the point O_2 is $\sqrt{z}$, and one has to use the formula $2(d\sqrt{z})/\sqrt{z}$ instead of dz/z to compute the residue at O_2. □

Lemma 5.8. *If n is odd, then*

$$\omega_{W_2} = -\frac{1}{2} \sum_{j=1}^{n-1} \delta \ln b_j \wedge \delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=1}^{q} b_{k+2i+1} \right).$$

Proof. The computation of ω_{W_2} is very similar to that of ω_{O_2} in Lemma 5.7. By computing ψ_0, $\psi_0^*(W_2)$, $\delta \ln f_1(W_2)$, we find the expressions for $\delta \ln f_j(W_2)$ and $\psi_j^*(W_2)$ with arbitrary j:

$$\delta \ln f_j(W_2) = -\delta \ln \left(\prod_{k=0}^{j-1} \prod_{i=1}^{q} b_{k+2i+1} \right), \qquad \psi_j^*(W_2) = \left(0, \frac{1}{2b_j}, 0 \right). \tag{5.5}$$

From Proposition 3.3 and formula (3.2) it follows that $\psi_0 = (1, b_0 + \beta/\sqrt{z} + O(1/z), \alpha\sqrt{z} + O(1))^T$ near the point W_2. From the identity $(T_0^{-1}\psi_0)_1 = k^{-1}\psi_{0,1}$ we find that $\alpha \prod_{i=1}^q b_{2i} = \sqrt{-J_q}$. The identity $(T_0\psi_0)_1 = k\psi_{0,1}$, along with the formulas:

$$T_0(W_2) = \begin{pmatrix} J_q & -J_q/b_0 & 0 \\ 0 & 0 & 0 \\ -J_q/b_{n-1} & J_q/(b_0 b_{n-1}) & 0 \end{pmatrix}, \qquad T_0(z)_{13} = J_q/(b_0 b_1 z) + O(z^{-2}) \text{ near } W_2,$$

implies that $\beta b_1 = \alpha$. Solving the above equations for β, we find that $\beta = \left(\prod_{j=1}^q b_{2j+1} \right) / \sqrt{-J_q}$.

Since $(L_0\psi_0)_1 = \beta/\sqrt{z} + O(1/z)$, we obtain that $\delta \ln f_1(W_2) = -\delta \ln \beta$. On the symplectic leaf we have $\delta J_q = 0$, therefore $\delta \ln f_1(W_2) = -\delta \ln \left(\prod_{i=1}^q b_{2i+1} \right)$.

Now we find the covector ψ_0^* at the point W_2. The identity $\psi_0^*\psi_0 \equiv 1$ implies that $\psi_0^* = (A + O(1/\sqrt{z}), B + O(1/\sqrt{z}), C/\sqrt{z} + O(1/z))$, and that $A + b_0 B + \alpha C = 1$. The identity $(\psi_0^* T_0^{-1})_2 = k^{-1}\psi_{0,2}^*$ implies that $C \prod_{i=1}^q b_{2i-1} = B\sqrt{-J_q}$. One can check that since the product $\psi_0^* T_0$ has zero of order n at W_2, it must be that $A = 0$. Solving the above equations for B, we find that $B = 1/(2b_0)$, and that $\psi_0^*(W_2) = (0, 1/(2b_0), 0)$.

The arguments identical to those used in Lemmas 5.5, 5.7 prove formulas (5.5). Substituting formulas (5.5) into formula (5.1), we obtain:

$$\omega_{W_2} = \frac{1}{2}\sum_{j=1}^{n-1} 2\cdot\frac{1}{2}\delta\ln b_j \wedge \delta\ln f_j(W_2) = -\frac{1}{2}\sum_{j=1}^{n-1}\delta\ln b_j \wedge \delta\ln\left(\prod_{k=0}^{j-1}\prod_{i=1}^{q} b_{k+2i+1}\right).$$

The coefficient "-2" appears in the last formula because the local parameter at W_2 is $z^{-1/2}$, and the formula $-2d(z^{-1/2})/z^{-1/2}$ should be used instead of dz/z to compute the residue. □

Now we find the contribution to ω from the points O_2, O_3, W_2, W_3 for even n.

Lemma 5.9. *If n is even, then*

$$\omega_{O_2} = -\frac{1}{2}\sum_{j=1}^{q-1}\delta\ln a_{2j}\wedge\delta\ln\prod_{k=0}^{j-1} a_{2k}, \qquad \omega_{O_3} = -\frac{1}{2}\sum_{j=1}^{q-1}\delta\ln a_{2j+1}\wedge\delta\ln\prod_{k=0}^{j-1} a_{2k+1}.$$

Proof. We prove that the following identities hold:

$$\psi_{2j}^*(O_2) = (1, 0, -1/a_{2j}), \quad \delta\ln f_{2j}(O_2) = -\delta\ln\prod_{k=0}^{j-1} a_{2k}, \quad \psi_{2j+1}^*(O_2) = (0,0,0),$$

$$\psi_{2j+1}^*(O_3) = (1, 0, -1/a_{2j+1}), \quad \delta\ln f_1(O_3) = -\delta\ln\eta,$$

$$\delta\ln f_{2j+1}(O_3) = -\delta\ln\left(\eta\prod_{k=0}^{j-1} a_{2k+1}\right), \quad \psi_{2j}^*(O_3) = (0,0,0).$$

These identities and formula (5.1) imply the lemma. The parameter η vanishes from the final formulas on the symplectic leaf $\delta\ln(a_1 a_3 \ldots a_{2q-1}) = 0$.

Note that a cyclic permutation of the variables $a_j \to a_{j+1}, b_j \to b_{j+1}$ (for all j) permutes the eigenvectors and covectors as follows: $\tilde{\psi}_{2j}(O_2) \to \tilde{\psi}_{2j+1}(O_3)$, $\tilde{\psi}_{2j}(O_3) \to \tilde{\psi}_{2j+1}(O_2)$, $\psi_{2j}^*(O_2) \to \psi_{2j+1}^*(O_3)$, $\psi_{2j}^*(O_3) \to \psi_{2j+1}^*(O_2)$. The use of these permutations and the usual argument to find the functions $f_j, j > 0$, imply that the identities above are equivalent to the following 4 formulas (which we prove below):

$$\delta\ln f_2(O_2) = -\delta\ln a_0, \quad \psi_0^*(O_2) = (1, 0, -1/a_0), \quad \delta\ln f_1(O_3) = -\delta\ln\eta, \quad \psi_0^*(O_3) = (0,0,0).$$

Proposition 3.3 implies that $\psi_0 = (1, O(1), O(z))^T$ at the point O_2. One can check that the principal part of $(L_1 L_0 \psi_0)_1$ at O_2 is $-a_0/z$, which implies that $\delta\ln f_2(O_2) = -\delta\ln a_0$.

Let the covector $\psi_0^*(O_2)$ be (α, β, γ). The equation $(\psi_0^* T_0)_1 = k\psi_{0,1}^*$ implies that $\beta = 0$. Since $\psi_0^*(O_2)\psi_0(O_2) = 1$, we find that $\alpha = 1$. One can check that since the product $\psi_0^* T_0^{-1}$ has zero of order q at O_2, it must be that $\gamma = -1/a_0$. Therefore, we obtain that $\psi_0^*(O_2) = (1, 0, -1/a_0)$.

Proposition 3.3 implies that $\psi_0 = (1, \eta/z + O(1), O(1))^T$ at the point O_3. The principal part of $(L_0\psi_0)_1$ at O_3 is η/z, therefore $\delta \ln f_1(O_3) = -\delta \ln \eta$. Since the product $\psi_0^*\psi_0$ is holomorphic at O_3, it must be that $\psi_{0,2}^*(O_3) = 0$ and $\psi_0^*(O_3) = (\alpha, 0, \beta)$ for some α, β. One can check that the equation $\psi_0^* T_0 = k\psi_0^*$ implies that $\alpha = \beta = 0$, thus $\psi_0^*(O_3) = (0, 0, 0)$. □

Lemma 5.10. *If n is even, then*

$$\omega_{W_2} = \frac{1}{2}\sum_{j=1}^{q-1} \delta \ln b_{2j} \wedge \delta \ln \prod_{k=0}^{j} b_{2k}, \qquad \omega_{W_3} = \frac{1}{2}\sum_{j=1}^{q-1} \delta \ln b_{2j+1} \wedge \delta \ln \prod_{k=0}^{j} b_{2k+1}.$$

Proof. The proof of this lemma is very similar to the proof of Lemma 5.9. We prove that:

$$\psi_{2j}^*(W_2) = (0, 1/b_{2j}, 0), \quad \psi_{2j+1}^*(W_2) = (0,0,0), \quad \delta \ln f_{2j}(W_2) = \delta \ln \prod_{k=1}^{j} b_{2k},$$

$$\psi_{2j+1}^*(W_3) = (0, 1/b_{2j+1}, 0), \quad \psi_{2j}^*(W_3) = (0,0,0),$$

$$\delta \ln f_1(W_3) = \delta \ln \xi, \quad \delta \ln f_{2j+1}(W_3) = \delta \ln \left(\xi \prod_{k=1}^{j} b_{2k+1} \right).$$

These identities, along with formula (5.1) and the equations for the symplectic leaves: $\delta \ln (b_0 b_2 ... b_{2q-2}) = \delta \ln (b_1 b_3 ... b_{2q-1}) = 0$, imply the statement of the lemma. Due to a cyclic permutation $a_j \to a_{j+1}, b_j \to b_{j+1}$, $\tilde{\psi}_{2j}(W_2) \to \tilde{\psi}_{2j+1}(W_3)$, $\tilde{\psi}_{2j}(W_3) \to \tilde{\psi}_{2j+1}(W_2)$, $\psi_{2j}^*(W_2) \to \psi_{2j+1}^*(W_3)$, $\psi_{2j}^*(W_3) \to \psi_{2j+1}^*(W_2)$ (for all j), we only need to prove 4 formulas:

$$\delta \ln f_2(W_2) = \delta \ln b_2, \quad \psi_0^*(W_2) = (0, 1/b_0, 0), \quad \delta \ln f_1(W_3) = \delta \ln \xi, \quad \psi_0^*(W_3) = (0,0,0).$$

Proposition 3.3 implies that $\psi_0 = (1, b_0 + O(1/z), O(1))^T$ at the point W_2. By definition of f_2, we have $\tilde{\psi}_2 = f_2 L_1 L_0 \psi_0$. We deduce that $f_2 \psi_{0,1} = (L_0^{-1} L_1^{-1} \tilde{\psi}_2)_1 = b_2 z + O(z^2)$ at W_2. Since $\psi_{0,1} = 1$, we find that $\delta \ln f_2(W_2) = \delta \ln b_2$.

Let the covector $\psi_0^*(W_2)$ be (α, β, γ). One can check that the highest order terms of the equation $\psi_0^* T_0^{-1} = k^{-1}\psi_0^*$ imply that $\alpha = \gamma = 0$. Since $\psi_0^*\psi_0 = 1$, we find that $\beta = 1/b_0$, and $\psi_0^*(W_2) = (0, 1/b_0, 0)$.

Proposition 3.3 implies that $\psi_0 = (1, O(1), O(z))^T$ at the point W_3. Therefore, $(L_0\psi_0)_1$ is a constant, which we denote by $1/\xi$. Hence, $\delta \ln f_1(W_3) = \delta \ln \xi$. Since $\psi_0^*\psi_0$ is holomorphic at W_3, it must be that $\psi_0^*(W_3) = (\alpha, \beta, 0)$ for some α, β. One can check that $\psi_0^* T_0 = k\psi_0^*$ implies $\alpha = \beta = 0$. Therefore, $\psi_0^*(W_3) = (0,0,0)$. □

Proof of Theorem D (continued).

Finally, by using the contributions at different points that we have found in Lemmas 5.5-5.10, one can show that their sum equals the expression in the statement of the theorem for both even and odd n.

The remaining part of the proof is to show that the inverse of ω coincides with the Poisson structure found in [3]. It is easier to do using the variables x_i, y_i defined by (3.4).

As is shown in Proposition 4.12 in [3], the Poisson bracket in the variables x_i, y_i is:

$$\{x_i, x_j\} = (\delta_{i,j-1} - \delta_{i,j+1})x_i x_j, \qquad \{y_i, y_j\} = (\delta_{i,j+1} - \delta_{i,j-1})y_i y_j,$$

and all other brackets vanish. The symplectic leaves for these brackets have positive codimension, therefore the corresponding 2-form is not unique. One of the possible 2-forms is

$$\omega_0 = \sum_{j=0}^{q-1} \delta \ln x_{2j+1} \wedge \delta \ln \left(\prod_{k=0}^{j} x_{2k} \right) - \sum_{j=0}^{q-1} \delta \ln y_{2j+1} \wedge \delta \ln \left(\prod_{k=0}^{j} y_{2k} \right).$$

Substituting formulas (3.4) into ω_0 and using the equations for the symplectic leaves, one can show that ω_0 equals ω. Consequently, ω has the same rank as ω_0 when restricted to the symplectic leaves, i.e., its rank is $2g$. □

Acknowledgements

I am grateful to I.Krichever and B.Khesin for important comments and their help in improving this paper. This work was partially supported by the NSERC research grant.

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