Pole solutions in formation of the Saffman-Teylor "finger" with one half of the channel width without surface tension

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Laplacian growth without surface tension has nice analytical solutions which replace its complex integro-differential motion equations by simple differential equations of poles motion in a complex plane. The main problem of such solution is existing of finite time singularities. To prevent such singularities nonzero surface tension usually is used. But such nonzero surface tension destroys analytical solutions. However more elegant way exists to solve the problem. First of all, we can introduce some small poles noise to system. Secondary, for regularization of problem we throw out all new poles that can give finite time singularity. It can be strictly proved that asymptotic solution for such system is a single finger. Moreover the qualitative consideration demonstrate that finger with $\frac{1}{2}$ of the channel width is statistically stable. So all properties of such solution are completely the same as for the solution with a nonzero surface tension under a numerical noise. Surprisedly, flame front propagation problem has the same pole solutions and qualitative behavior.

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I. INTRODUCTION

The problem of pattern formation is one of the most rapidly developing branches of nonlinear science today [1]. Of special interest is the study of the front dynamics between two phases (interface) that arises in a variety of nonequilibrium physical systems. If, as it usually happens, the motion of the interface is slow in comparison with the processes that take place in the bulk of both phases (such as heat transfer, diffusion, etc.), the scalar field governing the evolution of the interface is a harmonic function. It is natural then, to call the whole process *Laplacian growth*. Depending on the system, this harmonic scalar field is a temperature (in the freezing of a liquid or Stefan problem), a concentration (in solidification from a supersaturated solution), an electrostatic potential (in electrodeposition), a pressure (in flows through porous media), a probability (in diffusion-limited aggregation), etc.

The mathematical problem of Laplacian growth without surface tension exhibits a family of exact analytical solutions in terms of logarithmic poles in the complex plane. We show that this family of solutions has a remarkable property: generic initial conditions in channel geometry which begin with arbitrarily many features exhibit an inverse cascade into a single finger.

The main problem of such solution is existing of finite time singularities. To prevent such singularities nonzero surface tension usually is used. But such nonzero surface tension destroys analytical solutions.

However more elegant way exists to solve the problem. First of all, we can introduce some small noise to system. (It can be considered as a poles flux from infinity.) Secondary, for regularization of problem we throw out all new poles that can give finite time singularity. It can be strictly proved that asymptotic solution for such system is a single finger. Moreover the qualitative consideration demonstrate that finger with $\frac{1}{2}$ of the channel width is statistically stable. So all properties of such solution are completely the same as for the solution with a nonzero surface tension under a numerical noise.

The rest of the paper is organized as follows. We begin by presenting arguments about Saffman-Teylor "finger" formation with one half of the channel size(Section II).Next Section III describes asymptotic single Saffman-Teylor "finger" formation without surface tension. And finally (Section IV) we give summary and conclusions.

II. SAFFMAN-TEYLOR "FINGER" FORMATION WITH ONE HALF OF THE CHANNEL SIZE

This derivation is similar to [2]. The case of Laplacian growth in the channel without surface tension was in details considered by Mark Mineev-Weinstein and Dawson [3]. In this case the problem has the beautiful analytical solution. Moreover they assumed that all major effects in the case with vanishingly small surface tension may be received also without surface tension. It would allow applying to vanishingly small surface tension case the powerful analytical methods developed for the no surface tension case. However without additional assumptions this hypothesis may not be accepted.

The first objection is related to finite time singularities for some initial conditions. Actually for overcoming this difficulty the regular item with surface tension was introduced. This surface tension item is resulting in loss of the analytical decision. However regularization may be carried out much more simply - simply by rejecting the initial conditions which result to these singularities. The second objection is given in work Siegel and Tanveer [4]. There it is shown, that in numerical simulations in a case with any (even vanishingly small) surface tension any initial thickness "finger" extends up to $\frac{1}{2}$ thickness width of the channel. The analytical solution in a case without a surface tension results in constant thickness of the "finger" equal to its initial size that may be arbitrary. Siegel and Tanveer however did not take into account the simple fact, that numerical noise introduces small perturbation or to the initial condition, or even during "finger" growth, which is equivalent to the remote poles, and with respect to this perturbation the analytical solution with constant "finger" thickness is unstable.

By Mark Mineev-Weinstein [5] it was shown, that similar pole perturbations can give, at the some initial conditions, extending up to the Siegel and Tanveer solutions. This positive aspect of the paper [5] was mentioned by Sarkissian and Levine in them Comment [6]. Summing up, it is possible to tell, that for identity of the results with and without surface tension it is necessary to introduce a permanent source of the new remote poles: it may be either external noise or infinite number of poles in an initial condition. What from these methods is preferred it is a open question yet.

In the case of flame front propagation it was shown [9], that external noise is necessary for an explanation of flame front velocity increase with the sizes of system: the infinite number of poles in an initial condition can not give this result. It is interesting to know, what is situation in the channel Laplacian growth. One of main results of Laplacian growth in the channel with a small surface tension is Saffman-Teylor "finger" formation with the thickness equal to $\frac{1}{2}$ thickness of the channel. And to use the analytical result received for zero surface tension, it is necessary to prove, that formation of the "finger" with thickness equal to $\frac{1}{2}$ thickness of the channel takes place without surface tension also.

In our teamwork with Mark Mineev-Weinstein [7] it was shown, that for finite number of poles at almost all allowed (in the sense of not approaching to finite time singularities) initial conditions, except for small number of some degenerated initial conditions, they have asymptotic as some "finger" with any possible thickness. It should be mentioned, that the solutions and asymptotic found in [7] for finite number of poles are though also idealization, but quite have real sense for any finite intervals of time between appearance of the new poles introduced into system by external noise or connected to an entrance to the system of remote poles of an initial condition, including infinite number of such poles. The theorem proved in [7] and may be again applied for this final set of new and old poles is again received asymptotic, being again "finger", but already with possible new, distinct from former, thickness. Thus, introduction of a source of new poles results only in possible drift of thickness of the final "finger", but not changing of type of this solution.

It should be mentioned, that instead of periodical boundary conditions, much more realistic "noflux" boundary conditions may be introduced [8] (This paper repeats the result about a single finger asymptotic formerly already proved in the papers [7] and [9] for periodic boundary conditions.), forbidding a stream through a wall which insert additional, probably useful, restrictions on a positions, number and parameters of new and old poles (explaining, for example, why the sum of all complex parameters α_i for poles give the real value α for the pole solution (5) in [5]), not influencing, however, as shown in [9], on correctness and applicability proved in [7] results and methods of their including.

Given in [5] by Mark Mineev-Weinstein "proof", that steady asymptotic for Laplacian growth in a channel with zero surface tension is single "finger" with thickness equal to $\frac{1}{2}$ thickness of the channel, is unequivocally erroneous: completely the same method which was used in [5] to prove and demonstrate instability of "finger" with thickness distinct from $\frac{1}{2}$ with respect to introducing the new remote poles, instability of "finger" with thickness equal to $\frac{1}{2}$ may be proved and demonstrated! This objection was repeatedly stated to Mark Mark Mineev-Weinstein before the publication of his paper [5], however has not found any answer there. Moreover, in our teamwork [7] was is shown, that for finite number of poles any thickness "finger" is possible as asymptotic.

It does not mean, nevertheless, that privileged role of "finger" with thickness $\frac{1}{2}$ cannot be proved in the case of surface tension absence, but means only that such the proof are not given in [5]. Let us try to give these correct arguments here. The general pole solution (5) in work [5] is characterized by the real parameter α being the sum of the complex parameters α_i for poles. Thickness of the asymptotic finger is simple function of α: (Thickness = $1 - \frac{\alpha}{2}$). The value ($\alpha = 1$) corresponds to thickness $\frac{1}{2}$. As far as possible thickness of the "finger" is between 0 and 1, possible α value is in an interval between 0 and 2: ($0 < \alpha < 2$). The value $\alpha = 1$ corresponding to the finger width $\frac{1}{2}$ is exactly in the middle of this interval. What occurs to quite possible initial pole conditions with α outside of limits from 0 up to 2? They are "not allowed" because of already known to us finite time singularities [7]. Also a part of solutions inside of interval $0 < \alpha < 2$ results to the similar finite time singularities.

Exact necessary conditions, whether defining the initial pole condition as "allowed", i.e. singular, is still a open problem. How number of these "allowed" initial pole conditions (to be exact speaking, their percent from the full number of the possible initial pole conditions corresponding to the given real value α) is distributed inside of this interval?

From the reasons of a continuity and symmetry with respect to $\alpha = 1$ (Fig. 1) it is possible to conclude, that this distribution has a minimum in point $\alpha = 1$ (thickness $\frac{1}{2}$!), the value which is the most remote from both borders of interval $0 < \alpha < 2$, being increased to borders $\alpha = 2$ or 0, and reaching 100 percent from all pole solutions outside of these borders. I.e. thickness $\frac{1}{2}$ is the most probable because for this thickness value the minimal percent of potentially capable to give such thickness value initial conditions is "not allowed", i.e. results to singularities. Source of new poles results to the drift of finger thickness, but this thickness drift is closed to the

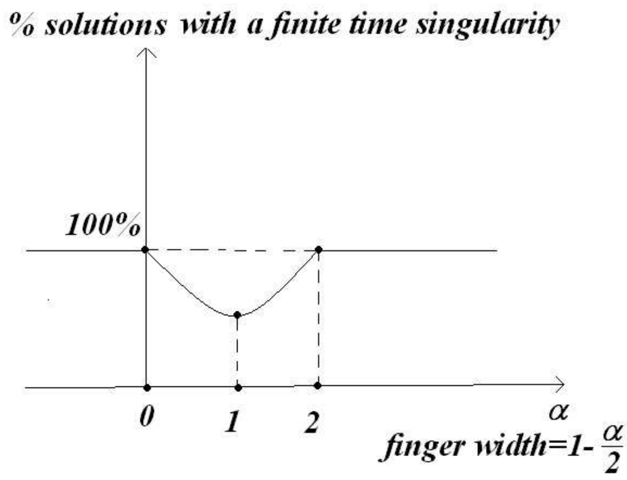

FIG. 1: A width of a finger is equal to $1 - \frac{\alpha}{2}$ (The channel width is assumed 1). The graph for current α gives percent of all possible solutions resulting in a finite time singularity. Maximum value is equal to 100 percent and correspondents to $\alpha \leq 0$ or $\alpha \geq 2$. Minimum is in the middle point $\alpha = 1$ between $\alpha = 0$ (the finger width 1) and $\alpha = 2$ (the finger width 0). So in the minimum the finger width is $\frac{1}{2}$

most probable and average size equal to The similar result is obtained in the case of Saffman-Teylor "finger" with vanishingly small surface tension and with some external noise. As it was desirable to be proved. It should be mentioned that these formulated arguments are only qualitative and the strict proof are also necessary.

III. ASYMPTOTIC SINGLE SAFFMAN-TEYLOR "FINGER" FORMATION WITHOUT SURFACE TENSION

In the absence of surface tension, whose effect is to stabilize the short-wavelength perturbations of the interface, the problem of 2D Laplacian growth is described as follows

$$(\partial_x^2 + \partial_y^2)u = 0 \ . \tag{1}$$

$$u \mid_{\Gamma(t)} = 0 \ , \partial_n u \mid_{\Sigma} = 1 \ . \tag{2}$$

$$v_n = \partial_n u \mid_{\Gamma(t)} \ . \tag{3}$$

Here $u(x, y; t)$ is the scalar field mentioned, $\Gamma(t)$ is the moving interface, Σ is a fixed external boundary, ∂_n is a component of the gradient normal to the boundary (i.e. the normal derivative), and v_n is a normal component of the velocity of the front.

Now we introduce physical "no-flux" boundary conditions. It means no flux across the lateral boundaries of the channel. This requires that the moving interface orthogonally intersects the walls of the channel. However, unlike the case of periodic boundary conditions, the end points at the two boundaries do not necessarily have the same horizontal coordinate. Nevertheless this can be also considered as a periodic problem where the period equals *twice* the width of the channel. But only half of this periodic strip should be considered as the physical channel, whereas the second half is its unphysical mirror image.

Then we introduce a time-dependent conformal map f from the lower half of a "mathematical" plane, $\xi \equiv \zeta + i\eta$, to the domain of the physical plane, $z \equiv x + iy$, where the Laplace equation 1 is defined as $\xi \stackrel{f}{\longrightarrow} z$. We also require that $f(t, \xi) \approx \xi$ for $\xi \longrightarrow \zeta - i\infty$. Thus the function $z = f(t, \zeta)$ describes the moving interface. From Eqs. (1), (2), (3) for function $f(t, \xi)$ we obtain the *Laplacian Growth Equation*

$$Im(\frac{\partial f(\xi, t)}{\partial \xi} \overline{\frac{\partial f(\xi, t)}{\partial t}}) = 1 \mid_{\xi = \zeta - i0} , f_\zeta \mid_{\zeta - i\infty} = 1 \ . \tag{4}$$

Let us look for a solution of Eq. (4) in the nextfollowing form

$$f(\xi, t) = \lambda\xi - i\tau(t) - i\sum_{l=1}^{N} \alpha_l \log(e^{i\xi} - e^{i\xi_l(t)}), \tag{5}$$

$$\alpha = \sum_{l=1}^{N} \alpha_l = 1 - \lambda, \tag{6}$$

where $\tau(t)$ is some real function of time, α_l is a complex constant, $\xi_l = \zeta_l + i\eta_l$ denotes the position of the pole with the number l and N is the number of poles.

For our "no-flux" boundary condition we must add the condition that for every pole $\xi_l = \zeta_l + i\eta_l$ with α_l exists a pole $\xi_l = -\zeta_l + i\eta_l$ with $\overline{\alpha_l}$.

So we can conclude from this condition for pairs of poles and eq. (6) that λ is a real constant.

We will prove below that necessary condition for no existence of a finite time singularity for a pole solution is

$$-1 < \lambda < 1 \ , \tag{7}$$

Also for the function $F(i\xi, t) = if(\xi, t)$ for "no-flux" boundary condition

$$\overline{F(i\xi, t)} = F(\overline{i\xi}, t) \tag{8}$$

We want to prove that the final state will be only one finger if no finite time singularity appears during poles evolutions.

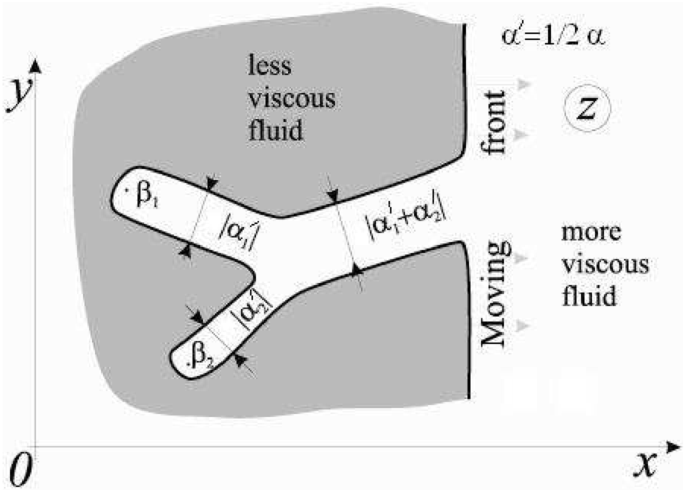

FIG. 2: Geometrical interpretation of the complex constants of motion $\alpha_k' = \frac{1}{2}\alpha_k$ and β_k; $k = 1, ..., N$.

A. Asymptotic behavior of the poles in the mathematical plane

This derivation is similar to [7] but we assume "no-flux" boundary conditions here (in analogy with [8]). The main purpose of this chapter is to investigate the asymptotic behavior of the poles in the mathematical plane. We want to demonstrate that for time $t \mapsto \infty$, all poles go to the two boundary points for no-flux boundary conditions. The equation for the interface is

$$f(\xi, t) = \lambda\xi - i\tau(t) - i\sum_{l=1}^{N}\alpha_l \log(e^{i\xi} - e^{i\xi_l(t)}),$$

$$\sum_{l=1}^{N}\alpha_l = 1 - \lambda, -1 < \lambda < 1 \ . \tag{9}$$

By substitution of Eq. (9) in the *Laplacian Growth Equation*

$$Im(\frac{\partial f(\xi,t)}{\partial \xi}\overline{\frac{\partial f(\xi,t)}{\partial t}}) = 1 \mid_{\xi=\zeta-i0} \ , \tag{10}$$

we can find the equations of pole motion (FIG. 2):

$$\beta_k = \tau(t) + (1 - \sum_{k=1}^{N}\overline{\alpha_k})\log\frac{1}{a_l} + \sum_{k=1}^{N}\overline{\alpha_k}\log(\frac{1}{a_l} - \overline{a_k}) = const \tag{11}$$

and

$$\tau = t - \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\overline{\alpha_k}\alpha_l \log(1 - \overline{a_k}a_l) + C_0 \ , \tag{12}$$

where $a_l = e^{i\xi_l}$, C_0 is a constant.

From eqs. (11) we can find

$$(1-\lambda)\tau - \sum_{l=1}^{N}\alpha_l \log a_l +$$
$$\sum_{k=1}^{N}\sum_{l=1}^{N}\overline{\alpha_k}\alpha_l \log(1 - \overline{a_k}a_l) = const \ . \tag{13}$$

From eqs. (12) and (13) we can obtain

$$Im(\sum_{l=1}^{N}\alpha_l \log a_l) = const \tag{14}$$

and

$$t = (\frac{1+\lambda}{2})\tau + \frac{1}{2}Re(\sum_{l=1}^{N}\alpha_l \log a_l) + C_1/2 \ , \tag{15}$$

where C_1 and α_l is a constant, $\xi_l(t)$ is the position of the poles, $a_l = e^{i\xi_l(t)}$.

In Appendix A we will prove from eq.(12) that $\tau \mapsto \infty$, if $t \mapsto \infty$ and if any finite time singularity does not exist.

The equations of pole motion are following from eqs. (11) are following

$$\tau + i\overline{\xi_k} + \sum_l \alpha_l \log(1 - e^{i(\xi_l - \overline{\xi_k})}) = const, \tag{16}$$

or in a different form:

$$\zeta_k + \sum_l(\alpha_l'' \log \mid 1 - e^{i(\xi_l - \overline{\xi_k})} \mid +$$
$$\alpha_l' \arg(1 - e^{i(\xi_l - \overline{\xi_k})})) = const, \tag{17}$$

$$\tau + \eta_k + \sum_l(\alpha_l' \log \mid 1 - e^{i(\xi_l - \overline{\xi_k})} \mid -$$
$$\alpha_l'' \arg(1 - e^{i(\xi_l - \overline{\xi_k})})) = const, \tag{18}$$

where

$$\xi_l = \zeta_l + i\eta_l, \eta_l > 0 \ . \tag{19}$$

$$\alpha_l = \alpha_l' + i\alpha_l'' \ . \tag{20}$$

Let us transform

$$\arg(1 - e^{i(\xi_l - \overline{\xi_k}}) =$$
$$\arg([1 - e^{i(\zeta_l - \zeta_k)}e^{-(\eta_l + \eta_k)}]) =$$
$$\arg[1 - a_{lk}e^{i\varphi_{lk}}] \tag{21}$$

$$\varphi_{lk} = \zeta_l - \zeta_k, a_{lk} = e^{-(\eta_l + \eta_k)} \tag{22}$$

$\arg[1 - a_{lk}e^{i\varphi_{lk}}]$ is a single valued function of φ_{lk}, i.e.

$$-\frac{\pi}{2} \leq \arg[1 - a_{lk} e^{i\varphi_{lk}}] \leq \frac{\pi}{2} \ . \tag{23}$$

We multiply eq. (18) by α''_k and eq. (17) by α'_k and take difference we obtain the following equation:

$$\alpha'_k \zeta_k - \alpha''_k \tau + \sum_{l\neq k} ((\alpha''_l \alpha'_k - \alpha''_k \alpha'_l) \log \mid 1 - e^{i(\xi_l - \overline{\xi_k})} \mid + (\alpha'_l \alpha'_k + \alpha''_l \alpha''_k) \arg(1 - e^{i(\xi_l - \overline{\xi_k})})) = const. \tag{24}$$

We want to investigate asymptotic behavior of poles $\tau \mapsto \infty$.

We have the divergent terms $\alpha''_k \tau$ in this equation. From the eq. (24) only term $\log \mid 1 - e^{i(\xi_k - \overline{\xi_k})} \mid$ can eliminate this divergence. The necessary condition for it is $\eta_k \mapsto 0$ for $\tau \mapsto \infty, 1 \leq k \leq N$.

We may assume that for $t \mapsto \infty$, N' groups of poles exist ($N' \leq N$) ($\varphi_{lk} \mapsto 0$ for all members of a group). The N' is currently arbitrary and even can be equal to N. N_l is the number of poles in each group, $1 < l < N'$.

For each group by summation of eqs. (24) over all group poles we obtain

$$\alpha^{gr\prime}_k \zeta^{gr}_k - \alpha^{gr\prime\prime}_k \tau + \sum_{l\neq k} ((\alpha^{gr\prime\prime}_l \alpha^{gr\prime}_k - \alpha^{gr\prime\prime}_k \alpha^{gr\prime}_l) \log \mid 1 - e^{i(\xi^{gr}_l - \overline{\xi^{gr}_k})} \mid + (\alpha^{gr\prime}_l \alpha^{gr\prime}_k + \alpha^{gr\prime\prime}_l \alpha^{gr\prime\prime}_k) \arg(1 - e^{i(\xi^{gr}_l - \overline{\xi^{gr}_k})})) = const, \tag{25}$$

where

$$\alpha^{gr\prime\prime}_l = \sum_k^{N_l} \alpha''_k \ , \tag{26}$$

$$\alpha^{gr\prime}_l = \sum_k^{N_l} \alpha'_k \ . \tag{27}$$

We have no merging between defined groups for large τ so we investigate the motion of poles with this assumption

$$\mid \zeta^{gr}_l - \zeta^{gr}_k \mid \gg \eta^{gr}_l + \eta^{gr}_k, 1 \leq l, k \leq N \ . \tag{28}$$

For $l \neq k$, $\eta^{gr}_k \mapsto 0$, $\varphi^{gr}_{lk} = \zeta^{gr}_l - \zeta^{gr}_k$ we obtain

$$\log \mid 1 - e^{i(\xi^{gr}_l - \overline{\xi^{gr}_k})} \mid \approx \log \mid 1 - e^{i(\zeta^{gr}_l - \zeta^{gr}_k)} \mid = \log 2 + \frac{1}{2} \log \sin^2 \frac{\varphi^{gr}_{lk}}{2} \tag{29}$$

and

$$\arg(1 - e^{i(\xi^{gr}_l - \overline{\xi^{gr}_k})}) \approx \arg(1 - e^{i(\zeta^{gr}_l - \zeta^{gr}_k)}) = \frac{\varphi^{gr}_{lk}}{2} + \pi n - \frac{\pi}{2} \ . \tag{30}$$

We choose n in Eq.(30) so that Eq.(23) is correct. Substituting these results to the eqs. (25) we obtain

$$C_k = \alpha^{gr\prime} \zeta^{gr}_k - \alpha^{gr\prime\prime}_k \tau + \sum_{l\neq k} [(\alpha^{gr\prime\prime}_l \alpha^{gr\prime}_k - \alpha^{gr\prime\prime}_k \alpha^{gr\prime}_l) \log \mid \sin \frac{\varphi^{gr}_{lk}}{2} \mid + (\alpha^{gr\prime}_l \alpha^{gr\prime}_k + \alpha^{gr\prime\prime}_l \alpha^{gr\prime\prime}_k) \frac{\varphi^{gr}_{lk}}{2}]. \tag{31}$$

B. Theorem about coalescence of the poles

From eqs. (31) we can conclude

(i) By summation of eqs.(31) (or exactly from eq. (14)) we obtain

$$\sum_k \alpha^{gr\prime}_k \zeta^{gr}_k = const \ . \tag{32}$$

(ii) For $\mid \varphi^{gr}_{lk} \mid \mapsto 0, 2\pi$, we obtain $\log \mid \sin \frac{\varphi^{gr}_{lk}}{2} \mid \mapsto \infty$, meaning that the poles can not pass off each other;

(iii) From (ii) we conclude that $0 < \mid \varphi^{gr}_{lk} \mid < 2\pi$

(iv) From (i) and (iii), $\zeta^{gr}_k \mapsto \infty$ is impossible;

(v) In eq.(31) we must compensate the second divergent term. From (iv) and (iii) we can do it only if $\alpha^{gr\prime\prime}_l = \sum_k^{N_l} \alpha''_k = 0$ for all l.

So from eq. (31) we obtain

$$\sum_k^{N_l} \alpha''_k = 0 \ , \tag{33}$$

$$\dot{\varphi^{gr}_{lk}} = 0 \ , \tag{34}$$

$$\varphi^{gr}_{lk} \neq 0 \ , \tag{35}$$

$$\dot{\zeta^{gr}_k} = 0 \ . \tag{36}$$

For the asymptotic motion of poles in the group N_m we obtain from eqs. (33), (34), (35), (36) taking leadind terms in eqs. (15), (16)

$$\tau = \frac{2}{\lambda + 1} t \ , \tag{37}$$

$$0 = \dot{\tau} + \sum_l^{N_m} \alpha_l \frac{\dot{\eta}_k + \dot{\eta}_l + i(\dot{\zeta}_k - \dot{\zeta}_l)}{\eta_k + \eta_l + i(\zeta_k - \zeta_l)} \ . \tag{38}$$

The solution to these equations is

$$\eta_k = \eta^0_k e^{-\frac{1}{\alpha^{gr\prime}_m} \frac{2}{1+\lambda} t} \ , \tag{39}$$

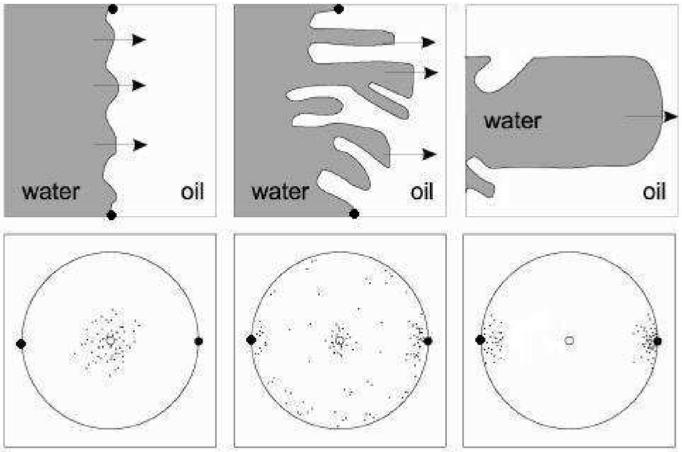

FIG. 3: Three consecutive stages of viscous fingering in the Hele-Shaw cell: initial (left), intermediate (center), and asymptotic (right). The physical plane z is shown in the upper pictures, while the lower pictures depict a distribution of moving poles $a_k(t)$ in the unit circle $|\omega| < 1$ on the mathematical plane ω. The open circle indicates the repeller, $\omega = 0$, while the solid circle indicates the attractor, $\omega = 1$, of poles whose dynamics is given by (11-12).

$$\varphi_{lk} = {\varphi_{lk}}^{0} e^{-\frac{1}{\alpha_m^{gr\prime}}\frac{2}{1+\lambda}t} \;, \tag{40}$$

$$\dot{\zeta}_k = 0 \;. \tag{41}$$

So we may conclude that for eliminating the divergent term we need

$$\alpha_l^{gr\prime\prime} = \sum_k^{N_l} \alpha_k^{\prime\prime} = 0, \tag{42}$$

$$\alpha_l^{gr\prime}(1+\lambda) > 0 \tag{43}$$

for all l.

C. The final result

With the no-flux boundary condition we have a pair of the poles whose condition of eq. (42) is correct so all these pairs must merge. Because of the symmetry of the problem these poles can merge only on the boundaries of the channel $\zeta = 0, \pm\pi$. Therefore we obtain two groups of the poles on boundaries $N' = 2$, $m = 1, 2$, $N_1 + N_2 = N$, $\alpha_1^{gr\prime} + \alpha_2^{gr\prime} = 1 - \lambda$.

Consequently we obtain the solution (on two boundaries FIG. 3):

$$\eta_k^{(1)} = \eta_k^{(1),0} e^{-\frac{1}{\alpha_1^{gr\prime}}\frac{2}{1+\lambda}t} \;, \tag{44}$$

$$\varphi_{lk}^{(1)} = {\varphi_{lk}}^{(1),0} e^{-\frac{1}{\alpha_1^{gr\prime}}\frac{2}{1+\lambda}t} \;, \tag{45}$$

$$\zeta_k^{(1)} = 0 \;; \tag{46}$$

$$\eta_k^{(2)} = \eta_k^{(2),0} e^{-\frac{1}{\alpha_2^{gr\prime}}\frac{2}{1+\lambda}t} \;, \tag{47}$$

$$\varphi_{lk}^{(2)} = {\varphi_{lk}}^{(2),0} e^{-\frac{1}{\alpha_2^{gr\prime}}\frac{2}{1+\lambda}t} \;, \tag{48}$$

$$\zeta_k^{(2)} = \pm\pi \;; \tag{49}$$

$$\alpha_1^{gr\prime}(1+\lambda) > 0, \tag{50}$$

$$\alpha_2^{gr\prime}(1+\lambda) > 0. \tag{51}$$

By summation of eqs. (50) and (50) and using eq. 6 we obtain

$$(1-\lambda)(1+\lambda) = 1 - \lambda^2 > 0. \tag{52}$$

This immediately gives us the formerly formulated condition (7) for λ.

$\frac{\lambda+1}{2} = 1 - \frac{\alpha}{2}$ has a explicit physical sense. It is the portion of the channel occupied by the moving liquid. We see that for no finite time singularity and for $t \mapsto \infty$ we obtain one finger with wide $\frac{\lambda+1}{2}$.

IV. CONCLUSIONS

Analytical pole solution for Laplasian growth gives sometimes finite time singularities. But nice solution of this problem exists. First of all, we introduce some small noise to system. This noise can be considered as a poles flux from infinity. Secondary, for regularization of problem we throw out all new poles that can give a finite time singularity. It can be strictly proved that asymptotic solution for such system is a single finger. Moreover the qualitative consideration demonstrate that finger with $\frac{1}{2}$ of the channel width is statistically stable. So all properties of such solution are completely the same as for the solution with a nonzero surface tension under a numerical noise.

Surprisedly, flame front propagation problem (in spite of absolutely different physics and mathematic equations for motion) has also the analytical pole solutions and demonstrates the same qualitative behavior of these solutions [9–13].

V. APPENDIX A

We need to prove that $\tau \mapsto \infty$, if $t \mapsto \infty$ and if any finite time singularity does not exist. Formula for τ is following:

$$\tau = t + [-\frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\overline{\alpha_k}\alpha_l \log(1-\overline{a_k}a_l)] + C_0 \ , \qquad (53)$$

where $\mid a_l \mid < 1$ for all l.

Let us prove that the second term in this formula is greater than zero:

$$\begin{aligned} &-\frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\overline{\alpha_k}\alpha_l \log(1-\overline{a_k}a_l) = \\ &-\frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\overline{\alpha_k}\alpha_l \sum_{n=1}^{\infty}(-\frac{(\overline{a_k}a_l)^n}{n}) = \\ &\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}(\sum_{k=1}^{N}\overline{\alpha_k}(\overline{a_k})^n)(\sum_{l=1}^{N}\alpha_l(a_l)^n) = \\ &\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\overline{(\sum_{l=1}^{N}\alpha_l(a_l)^n)}(\sum_{l=1}^{N}\alpha_l(a_l)^n) > 0 \end{aligned} \qquad (54)$$

So the second term in eq. (53) always more then zero and, consequently, $\tau \mapsto \infty$, if $t \mapsto \infty$ for no finite time singularity.

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