Integrable deformations of Lotka-Volterra systems

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Abstract

The Hamiltonian structure of a class of three-dimensional (3D) Lotka-Volterra (LV) equations is revisited by showing that the quadratic Poisson structure underlying its integrability structure is just a real three-dimensional Poisson-Lie group. As a consequence, the Poisson coalgebra map $\Delta^{(2)}$ that is given by the group multiplication provides the keystone for the explicit construction of a family of 3N-dimensional integrable systems that, under certain constraints, contain N sets of deformed versions of the 3D LV equations. Moreover, by considering the most generic Poisson-Lie structure on this group, a two-parametric integrable perturbation of the 3D LV system through polynomial and rational perturbation terms is explicitly found.

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1 Introduction

The Lotka-Volterra (LV) system of differential equations

$$\dot{x}_i = x_i \left[\sum_{j=1}^{M} a_{ij}\, x_j + d_i \right] \qquad i = 1, \ldots, M \tag{1.1}$$

where a_{ij} and d_i are constants, were initially introduced for $M = 2$ in order to model prey-predator and chemical reaction dynamics [1, 2], but nowadays it is well known that the generic system (1.1) plays an outstanding role in many different nonlinear models with a wide range of applications (see, for instance [3, 4] and references therein).

In particular, the three-dimensional ($M = 3$) LV system has attracted much attention, in many cases focused in the search for the specific values of the parameters a_{ij} and d_i for which a constant of the motion does exist (see [5, 6, 7, 8, 9, 10, 11]). Obviously, such integrability is guaranteed in case that the LV system could be obtained as the Hamilton equations $\dot{x} = \{x, \mathcal{H}\}$ provided by a Hamiltonian $\mathcal{H}$ and a suitable three-dimensional (generalized) Poisson structure [12, 13, 14, 15, 16, 17, 18]. In particular, in [12] it was shown that an integrable LV system previously described in [7] and given by

$$\begin{aligned} \dot{x} &= x(Cy + z + \lambda) \\ \dot{y} &= y(x + Az + \mu) \\ \dot{z} &= z(Bx + y + \nu) \end{aligned} \tag{1.2}$$

where

$$A\,B\,C = -1 \qquad \text{and} \qquad \nu = \mu B - \lambda AB \tag{1.3}$$

was bihamiltonian. One of the Hamiltonian functions leading to (1.2) is given by

$$\mathcal{H} = A\,B\,x + y - A\,z + \nu\,\ln y - \mu \ln z \tag{1.4}$$

together with the quadratic Poisson structure

$$\{x, y\} = C\,xy \qquad \{x, z\} = BC\,xz \qquad \{y, z\} = -yz \tag{1.5}$$

which has the following Casimir function (that is indeed a constant of the motion for the LV system):

$$\mathcal{C} = AB \ln x - B \ln y + \ln z. \tag{1.6}$$

Moreover, the second Hamiltonian structure for the same system is obtained by taking $\mathcal{C}$ as the Hamiltonian and by considering a different cubic Poisson bracket, for which $\mathcal{H}$ is just the Casimir function (see [12] for details).

The aim of this paper is to generalize the abovementioned quadratic Hamiltonian structure of the LV equations by showing that it can be interpreted as a Poisson-Lie bracket on a certain Lie group. Therefore, these 3D LV equations can be understood as an instance of integrable Poisson-Lie dynamics, and this property will enables us to get integrable deformations of the LV equations in two different ways. On one hand, by using the group multiplication (which defines by construction a Poisson map between two copies of the group manifold) in order to get a embedding of certain integrable perturbations of LV equations into a higher dimensional integrable system. On the other hand, by looking for deformations of the underlying quadratic Poisson-Lie bracket that are preserved under the group multiplication.

The structure of the letter is as follows. In the next Section we present the Poisson-Lie group features of the quadratic Hamiltonian structure of LV equations. In Section 3 we use the group multiplication map $\Delta^{(2)}$ in order to obtain integrable perturbations of the 3D LV equations through the embedding method. Finally, in Section 4 the most generic deformation of the quadratic Poisson-Lie structure is given, and a new two-parametric integrable deformation of the LV equations containing polynomial and rational perturbation terms is explicitly obtained.

2 Lotka-Volterra equations as Poisson-Lie dynamics

Let us consider the following quadratic Poisson algebra $\mathcal{P}$ [10, 11] that generalizes (1.5)

$$\{X,Y\} = \alpha XY \qquad \{X,Z\} = \beta XZ \qquad \{Y,Z\} = \gamma YZ. \tag{2.1}$$

The Casimir function for $\mathcal{P}$ reads

$$\mathcal{C} = X^{-\gamma} Y^{\beta} Z^{-\alpha}. \tag{2.2}$$

If we consider the Hamiltonian function -which also generalizes (1.4)-

$$\mathcal{H} = a_1\, X + a_2\, Y + a_3\, Z + b_1\, \log X + b_2\, \log Y + b_3\, \log Z \tag{2.3}$$

we get the following LV equations as the integrable dynamical system given by $\dot{F} = \{F, \mathcal{H}\}$:

$$\begin{aligned} \dot{X} &= X[\alpha a_2\, Y + \beta a_3\, Z + (\alpha b_2 + \beta b_3)] \\ \dot{Y} &= Y[-\alpha a_1\, X + \gamma a_3\, Z + (\gamma b_3 - \alpha b_1)] \\ \dot{Z} &= Z[-\beta a_1\, X - \gamma a_2\, Y - (\beta b_1 + \gamma b_2)]. \end{aligned} \tag{2.4}$$

These are Lotka-Volterra equations that depend on 9 free parameters, and the system is always integrable since the Hamiltonian and the Casimir function are constants of the motion. Note that if the three a_i parameters are all of them different from zero, then they can be reabsorbed in the Hamiltonian and equations through a linear change of variables, and the following skew-symmetric form of the LV system is obtained (see [10, 11])

$$\begin{aligned} \dot{X} &= X[\alpha\, Y + \beta\, Z + (\alpha b_2 + \beta b_3)] \\ \dot{Y} &= Y[-\alpha\, X + \gamma\, Z + (\gamma b_3 - \alpha b_1)] \\ \dot{Z} &= Z[-\beta\, X - \gamma\, Y - (\beta b_1 + \gamma b_2)]. \end{aligned} \tag{2.5}$$

Moreover, this system can be also shown to be equivalent (through a further linear change of variables) to the more usual 6-parameter LV equations given by (1.2) (the so-called ABC-system). In fact, if $\alpha, \beta, \gamma \neq 0$ and we define

$$a_1 = -\frac{1}{\alpha}, \quad a_2 = -\frac{1}{\gamma}, \quad a_3 = \frac{1}{\beta} \tag{2.6}$$

then from (2.4) we get the ABC system with

$$\begin{aligned} &A = \frac{\gamma}{\beta}, && B = \frac{\beta}{\alpha}, && C = -\frac{\alpha}{\gamma} \\ &\lambda = \alpha b_2 + \beta b_3, && \mu = -\alpha b_1 + \gamma b_3, && \nu = -\beta b_1 - \gamma b_2 \end{aligned} \tag{2.7}$$

which fulfils the conditions

$$A\,B\,C = -1, \qquad \nu = \mu B - \lambda A B \tag{2.8}$$

that characterize the bi-Hamiltonian case described in [12].

At this point it is important to stress that, as we shall see in Section 3, after deforming the Poisson bracket (2.1) the abovementioned linear changes of variables that relate the different presentations of LV systems will no longer work. Therefore in the rest of the paper we will consider as the LV system the 9-parameter fully generic case (2.4).

2.1 The LV Poisson algebra as a Poisson-Lie group

Let us consider the Lie group G generated by the 3D multiparametric Lie algebra $g_{\alpha,\beta,\gamma}$ given by

$$[e_0, e_1] = \frac{\gamma}{\beta}\, e_1 \qquad [e_0, e_2] = -\frac{\gamma}{\alpha}\, e_2 \qquad [e_1, e_2] = 0. \tag{2.9}$$

A three-dimensional representation of this Lie algebra is given by:

$$\rho(e_1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \rho(e_0) = \begin{pmatrix} \frac{\gamma}{\beta} & 0 & 0 \\ 0 & -\frac{\gamma}{\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.10}$$

In this representation, the three-dimensional form of a generic Lie group element M reads:

$$M = \exp\left(E_1 \rho(e_1)\right) \exp\left(E_2 \rho(e_2)\right) \exp\left(E_0 \rho(e_0)\right) = \begin{pmatrix} \exp(\frac{\gamma}{\beta}\, E_0) & 0 & E_1 \\ 0 & \exp(-\frac{\gamma}{\alpha}\, E_0) & E_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.11}$$

By changing to the new group parameters $X = \exp(E_0), Y = E_1, Z = E_2$, we obtain

$$M = \begin{pmatrix} X^{\frac{\gamma}{\beta}} & 0 & Y \\ 0 & X^{-\frac{\gamma}{\alpha}} & Z \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.12}$$

Now, if we compute the product of two group elements $M_2 \cdot M_1$ we get the group law as

$$\begin{pmatrix} X_2^{\frac{\gamma}{\beta}} & 0 & Y_2 \\ 0 & X_2^{-\frac{\gamma}{\alpha}} & Z_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} X_1^{\frac{\gamma}{\beta}} & 0 & Y_1 \\ 0 & X_1^{-\frac{\gamma}{\alpha}} & Z_1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} (X_2\, X_1)^{\frac{\gamma}{\beta}} & 0 & X_2^{\frac{\gamma}{\beta}}\, Y_1 + Y_2 \\ 0 & (X_2\, X_1)^{-\frac{\gamma}{\alpha}} & X_2^{-\frac{\gamma}{\alpha}}\, Z_1 + Z_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.13}$$

Now by identifying $X_1, Y_1, Z_1, X_2, Y_2, Z_2$ with the tensor product variables

$$\begin{aligned} X_1 &= X \otimes 1, & X_2 &= 1 \otimes X \\ Y_1 &= Y \otimes 1, & Y_2 &= 1 \otimes Y \\ Z_1 &= X \otimes 1, & Z_2 &= 1 \otimes X \end{aligned} \tag{2.14}$$

we can interpret the entries in (2.13) as elements of $M \otimes M$. This means that if we write the entries of this product matrix as

$$M_2 \cdot M_1 := \begin{pmatrix} [\Delta^{(2)}(X)]^{\frac{\gamma}{\beta}} & 0 & \Delta^{(2)}(Y) \\ 0 & [\Delta^{(2)}(X)]^{-\frac{\gamma}{\alpha}} & \Delta^{(2)}(Z) \\ 0 & 0 & 1 \end{pmatrix} \tag{2.15}$$

we get the definition of the so-called coproduct (or comultiplication map) $\Delta^{(2)} : M \to M \otimes M$, which in this case reads:

$$\begin{aligned} \Delta^{(2)}(X) &= X_2 X_1 \\ \Delta^{(2)}(Y) &= X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2 \\ \Delta^{(2)}(Z) &= X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2. \end{aligned} \tag{2.16}$$

Therefore, the coproduct $\Delta^{(2)}$ is nothing but an algebraic way of rewriting the group law as a mapping defined on the group element entries (see [19] for details).

Now, the key observation is the following invariance property: If we assume that the algebra of smooth functions on the group coordinates $\{X,Y,Z\}$ is endowed with the LV Poisson structure $\mathcal{P}$ given by (2.1), then the coproduct $\Delta^{(2)}$ is a Poisson algebra homomorphism with respect to $\mathcal{P}$, *i.e.*, the following relations hold

$$\begin{aligned}
\{\Delta^{(2)}(X),\Delta^{(2)}(Y)\} &= \alpha\,\Delta^{(2)}(X)\,\Delta^{(2)}(Y)\\
\{\Delta^{(2)}(X),\Delta^{(2)}(Z)\} &= \beta\,\Delta^{(2)}(X)\,\Delta^{(2)}(Z)\\
\{\Delta^{(2)}(Y),\Delta^{(2)}(Z)\} &= \gamma\,\Delta^{(2)}(Y)\,\Delta^{(2)}(Z),
\end{aligned} \tag{2.17}$$

where in (2.17) $\{,\}$ denotes the natural Poisson structure on $\mathcal{P}\otimes\mathcal{P}$ given by

$$\{a\otimes b, c\otimes d\} = \{a,c\}\otimes bd + ac\otimes\{b,d\}. \tag{2.18}$$

The previous statement is proven by a straightforward computation and means that $(C^\infty(G),\mathcal{P})$ is a Poisson-Lie group (equivalently, that $(\mathcal{P},\Delta^{(2)})$ is a Poisson coalgebra [19, 20]). Therefore, from this perspective we can conclude that the LV equations (2.4) are just a specific example of Poisson-Lie dynamics defined on $(C^\infty(G),\mathcal{P})$ and generated by the Hamiltonian

$$\mathcal{H} = a_1\,X + a_2\,Y + a_3\,Z + b_1\,\log X + b_2\,\log Y + b_3\,\log Z. \tag{2.19}$$

3 Integrable deformations induced from the coproduct map

In the sequel we show that the Poisson-Lie nature of the bracket $\mathcal{P}$ provides a straightforward deformation approach to LV equations which is based on their embedding within higher dimensional integrable systems. These systems will be obtained by taking into account that Poisson algebras endowed with a coproduct map $\Delta^{(2)}$ give rise to a systematic way of constructing integrable systems with an arbitrary number of degrees of freedom (for a detailed exposition of such coalgebra approach to integrability, including its relation with other approaches to integrability and many explicit examples, see [20, 21, 22, 23]).

In order to make this construction explicit, let us work out the $N=2$ case by considering as the new Hamiltonian $\mathcal{H}^{(2)}$ the coproduct $\Delta^{(2)}$ of the $N=1$ Hamiltonian (2.3), namely

$$\begin{aligned}
&\mathcal{H}^{(2)} := \Delta^{(2)}(\mathcal{H})\\
&= a_1\,\Delta^{(2)}(X) + a_2\,\Delta^{(2)}(Y) + a_3\,\Delta^{(2)}(Z) + b_1\,\Delta^{(2)}(\log X) + b_2\,\Delta^{(2)}(\log Y) + b_3\,\Delta^{(2)}(\log Z)\\
&= a_1\,(X_2X_1) + a_2\left(X_2^{\frac{\gamma}{\beta}}Y_1 + Y_2\right) + a_3\left(X_2^{-\frac{\gamma}{\alpha}}Z_1 + Z_2\right)\\
&\qquad + b_1\log\,(X_2X_1) + b_2\log\left(X_2^{\frac{\gamma}{\beta}}Y_1 + Y_2\right) + b_3\log\left(X_2^{-\frac{\gamma}{\alpha}}Z_1 + Z_2\right).
\end{aligned} \tag{3.1}$$

Now, by considering the natural definition (2.18) of the Poisson bracket on two copies of the group manifold, we get the following Hamilton equations for this 6D dynamical system:

$$
\begin{aligned}
\dot{X}_1 &= X_1\left(\alpha\, a_2 Y_1 X_2^{\frac{\gamma}{\beta}} + \beta\, a_3 Z_1 X_2^{-\frac{\gamma}{\alpha}}\right) + X_1\left[\alpha Y_1\left(\frac{b_2 X_2^{\frac{\gamma}{\beta}}}{X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2}\right) + \beta Z_1\left(\frac{b_3 X_2^{-\frac{\gamma}{\alpha}}}{X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2}\right)\right] \\
\dot{Y}_1 &= Y_1\left(-\alpha a_1 X_1 X_2 + \gamma a_3 Z_1 X_2^{-\frac{\gamma}{\alpha}}\right) + \gamma Y_1\left[\frac{b_3 Z_1 X_2^{-\frac{\gamma}{\alpha}}}{X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2}\right] - \alpha\, b_1 Y_1 \\
\dot{Z}_1 &= Z_1\left(-\beta\, a_1 X_1 X_2 - \gamma Y_1 X_2^{\frac{\gamma}{\beta}}\right) - \gamma\, Z_1\left[\frac{b_2 Y_1 X_2^{\frac{\gamma}{\beta}}}{X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2}\right] - \beta\, b_1 Z_1 \\
\dot{X}_2 &= X_2(\alpha a_2 Y_2 + \beta a_3 Z_2) + X_2\left[\alpha Y_2\left(\frac{b_2}{X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2}\right) + \beta Z_2\left(\frac{b_3}{X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2}\right)\right] \\
\dot{Y}_2 &= Y_2(-\alpha a_1 X_2 X_1 + \gamma a_3 Z_2) + \gamma Y_2\left[\frac{b_3\, Z_2}{X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2}\right] \\
&\quad -\alpha Y_2\left[b_1 + \frac{\gamma}{\beta} X_2^{\frac{\gamma}{\beta}} Y_1\left(a_2 + \frac{b_2}{X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2}\right) - \frac{\gamma}{\alpha} X_2^{-\frac{\gamma}{\alpha}} Z_1\left(a_3 + \frac{b_3}{X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2}\right)\right] \\
\dot{Z}_2 &= Z_2(-\beta a_1 X_2 X_1 - \gamma a_2 Y_2) - \gamma Z_2\left[\frac{b_2\, Y_2}{X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2}\right] \\
&\quad -\beta Z_2\left[b_1 + \frac{\gamma}{\beta} X_2^{\frac{\gamma}{\beta}} Y_1\left(a_2 + \frac{b_2}{X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2}\right) - \frac{\gamma}{\alpha} X_2^{-\frac{\gamma}{\alpha}} Z_1\left(a_3 + \frac{b_3}{X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2}\right)\right]
\end{aligned}
\tag{3.2}
$$

This system is completely integrable since by making use of the Poisson coalgebra approach it is immediate to show [20] that, besides the Hamiltonian $\mathcal{H}^{(2)}$, there exist three more integrals of the motion in involution. They are the two Casimir functions for each subset of group variables together with the coproduct of the Casimir function, namely:

$$\mathcal{C}_1 = X_1^{-\gamma} Y_1^{\beta} Z_1^{-\alpha} \qquad \mathcal{C}_2 = X_2^{-\gamma} Y_2^{\beta} Z_2^{-\alpha} \tag{3.3}$$

$$\mathcal{C}^{(2)} := \Delta^{(2)}(\mathcal{C}) = (X_1 X_2)^{-\gamma}\left(X_2^{\frac{\gamma}{\beta}} Y_1 + Y_2\right)^{\beta}\left(X_2^{-\frac{\gamma}{\alpha}} Z_1 + Z_2\right)^{-\alpha}. \tag{3.4}$$

It is straightforward to realize that the previous six equations are two 'twisted' sets of integrable deformations of the 3D LV system. In fact, the equations for $\{X_2, Y_2, Z_2\}$ are just an integrable deformation of the LV equations provided that the constraint $\dot{X}_1 = 0$ is imposed, and the perturbation terms contain both the $\{X_2, Y_2, Z_2\}$ variables as well as the $\{Y_1, Z_1\}$ ones. Complementarily, the equations for $\{X_1, Y_1, Z_1\}$ are another integrable deformation the of LV equations under the constraint $\dot{X}_2 = 0$. Obviously, this dynamical twisting comes indeed from the explicit form of the coproduct map $\Delta^{(2)}$ or, in geometric terms, from the fact that the group law that defines the coproduct is non-abelian.

3.1 3N-dimensional generalization

Higher dimensional generalizations of this construction are straightforwardly obtained from the definition of the N-th coproduct map $\Delta^{(N)}$. By following the same procedure as in the $N = 2$ case -see (2.13) and (2.15)-, the N-th coproduct map is obtained by computing the product of N group matrices $M_N \cdot M_{N-1} \cdot \dots \cdot M_2 \cdot M_1$ and reads

$$
\begin{aligned}
\Delta^{(N)}(X) &= \prod_{i=1}^{N} X_i \\
\Delta^{(N)}(Y) &= \sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\gamma/\beta} \\
\Delta^{(N)}(Z) &= \sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\gamma/\alpha}
\end{aligned}
\tag{3.5}
$$

which is again, by construction, a Poisson homomorphism for the N-th tensor product of the LV bracket $\mathcal{P}$, as it can be explicitly checked.

Now, if we define the Hamiltonian as the N-th coproduct of $\mathcal{H}$

$$
\begin{aligned}
&\mathcal{H}^{(N)} := \Delta^{(N)}(\mathcal{H}) \\
&= a_1\,\Delta^{(N)}(X) + a_2\,\Delta^{(N)}(Y) + a_3\,\Delta^{(N)}(Z) + b_1\,\Delta^{(N)}(\log X) + b_2\,\Delta^{(N)}(\log Y) + b_3\,\Delta^{(N)}(\log Z) \\
&= a_1\left(\prod_{i=1}^{N} X_i\right) + a_2\left(\sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\gamma/\beta}\right) + a_3\left(\sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\gamma/\alpha}\right) \\
&+ b_1 \log\left(\prod_{i=1}^{N} X_i\right) + b_2 \log\left(\sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\gamma/\beta}\right) + b_3 \log\left(\sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\gamma/\alpha}\right)
\end{aligned}
\tag{3.6}
$$

the 3N Hamilton equations read ($k = 1, \dots, N$)

$$
\begin{aligned}
\dot{X}_k &= X_k\left(\alpha\, a_2 Y_k \prod_{j=k+1}^{N} X_j^{\frac{\gamma}{\beta}} + \beta\, a_3 Z_k \prod_{j=k+1}^{N} X_j^{-\frac{\gamma}{\alpha}}\right) + X_k\left[\alpha Y_k\left(\frac{b_2 \prod_{j=k+1}^{N} X_j^{\frac{\gamma}{\beta}}}{\sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\frac{\gamma}{\beta}}}\right) + \beta\, Z_K\left(\frac{b_3 \prod_{j=k+1}^{N} X_j^{-\frac{\gamma}{\alpha}}}{\sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\frac{\gamma}{\alpha}}}\right)\right] \\
\dot{Y}_k &= Y_k\left(-\alpha\, a_1 X_k \prod_{\substack{j=1 \\ j\neq k}}^{N} X_j + \gamma\, a_3 Z_k \prod_{j=k+1}^{N} X_j^{-\frac{\gamma}{\alpha}}\right) + \gamma Y_k\left[\frac{b_3\, Z_k \prod_{j=k+1}^{N} X_j^{-\frac{\gamma}{\alpha}}}{\sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\frac{\gamma}{\alpha}}}\right] \\
&\quad -\alpha\, Y_k\left[b_1 + \frac{\gamma}{\beta}\sum_{i=1}^{k-1} Y_i \prod_{j=i+1}^{N} X_j^{\frac{\gamma}{\beta}}\left(a_2 + \frac{b_2}{\sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\frac{\gamma}{\beta}}}\right) - \frac{\gamma}{\alpha}\sum_{i=1}^{k-1} Z_i \prod_{j=i+1}^{N} X_j^{-\frac{\gamma}{\alpha}}\left(a_3 + \frac{b_3}{\sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\frac{\gamma}{\alpha}}}\right)\right] \\
\dot{Z}_k &= Z_k\left(-\beta\, a_1 X_k \prod_{\substack{j=1 \\ j\neq k}}^{N} X_j - \gamma\, a_2 Y_k \prod_{j=k+1}^{N} X_j^{\frac{\gamma}{\beta}}\right) - \gamma\, Z_k\left[\frac{b_2\, Y_k \prod_{j=k+1}^{N} X_j^{\frac{\gamma}{\beta}}}{\sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\frac{\gamma}{\beta}}}\right] \\
&\quad -\beta\, Z_k\left[b_1 + \frac{\gamma}{\beta}\sum_{i=1}^{k-1} Y_i \prod_{j=i+1}^{N} X_j^{\frac{\gamma}{\beta}}\left(a_2 + \frac{b_2}{\sum_{i=1}^{N} Y_i \prod_{j=i+1}^{N} X_j^{\frac{\gamma}{\beta}}}\right) - \frac{\gamma}{\alpha}\sum_{i=1}^{k-1} Z_i \prod_{j=i+1}^{N} X_j^{-\frac{\gamma}{\alpha}}\left(a_3 + \frac{b_3}{\sum_{i=1}^{N} Z_i \prod_{j=i+1}^{N} X_j^{-\frac{\gamma}{\alpha}}}\right)\right]
\end{aligned}
\tag{3.7}
$$

Again, by making use of the general results concerning Poisson coalgebra integrability [20] it is straightforward to prove that set of integrals of the motion in involution for this system are the

Hamiltonian $\mathcal{H}^{(N)}$ together with the $N + (N-1)$ functions given by the Casimirs on each group element and the r-th coproducts ($r = 2, \ldots, N$) of the abstract Casimir $\mathcal{C}$. Explicitly,

$$\mathcal{C}_i = X_i^{-\gamma} Y_i^{\beta} Z_i^{-\alpha} \qquad i = 1, \ldots, N \tag{3.8}$$

$$\begin{aligned}\mathcal{C}^{(r)} &:= \Delta^{(r)}(\mathcal{C}) = [\Delta^{(r)}(X)]^{-\gamma}\, [\Delta^{(r)}(Y)]^{\beta}\, [\Delta^{(r)}(z)]^{-\alpha} \\ &= \left[\prod_{i=1}^{r} X_i\right]^{-\gamma} \left[\sum_{i=1}^{r} Y_i \prod_{j=i+1}^{r} X_j^{\gamma/\beta}\right]^{\beta} \left[\sum_{i=1}^{r} Z_i \prod_{j=i+1}^{r} X_j^{-\gamma/\alpha}\right]^{-\alpha} \quad r = 2, \ldots, N \end{aligned}\tag{3.9}$$

where we have used the superscript (r) to denote the comultiplication map between M and the tensor product of r copies of M.

Note that for each fixed subset of three variables $\{X_k, Y_k, Z_k\}$ we recover an integrable deformation of the LV system provided that the $(N-1)$ constraints $\dot{X}_j = 0$ for all $j \neq k$ are imposed. These constraints reflect again the 'twisting' between the N different sets of perturbed LV equations that arise as a consequence of the Poisson invariance of the system under the N-th coproduct map.

4 Integrable perturbations from a deformed Poisson-Lie group

This group-theoretical interpretation of the LV equations (2.4) raises also the natural question concerning the existence of deformations of the Poisson structure $\mathcal{P}$ (2.1) that could be also invariant under the coproduct map $\Delta^{(2)}$. Equivalently, this is just the problem of finding other PL structures on the group G, whose existence would provide new integrable deformations of the LV equations.

This question can be answered in the affirmative, and the following result can be obtained through direct computation:

Proposition. The most generic quadratic Poisson structure in $\{X, X^{\frac{\gamma}{\beta}}, X^{-\frac{\gamma}{\alpha}}, Y, Z, 1\}$ and for which the comultiplication $\Delta^{(2)}$ is a Poisson map is given by the brackets

$$\begin{aligned} \{X, Y\} &= \alpha XY + \delta X(1 - X^{\frac{\gamma}{\beta}}) \\ \{X, Z\} &= \beta XZ + \epsilon X(1 - X^{-\frac{\gamma}{\alpha}}) \\ \{Y, Z\} &= \gamma YZ + \frac{\gamma\epsilon}{\beta} Y + \frac{\gamma\delta}{\alpha} Z + \frac{\gamma\delta\epsilon}{\beta\alpha}\left(1 - X^{\frac{\gamma}{\beta} - \frac{\gamma}{\alpha}}\right). \end{aligned}\tag{4.1}$$

We shall call this Poisson bivector as $\mathcal{P}_{\delta,\epsilon}$. Moreover, the Casimir function for $\mathcal{P}_{\delta,\epsilon}$ is found to be

$$\mathcal{C}_{\delta,\epsilon} = \left[\delta(1 - X^{\frac{\gamma}{\beta}}) + \alpha Y\right]^{-\frac{\beta}{\alpha}} \left[\epsilon(X^{\frac{\gamma}{\alpha}} - 1) + \beta Z X^{\frac{\gamma}{\alpha}}\right]. \tag{4.2}$$

Consequently, we can say that $(\mathcal{P}_{\delta,\epsilon}, \Delta^{(2)})$ is a multiparametric PL structure on G whose limit $\delta, \epsilon \to 0$ leads to the LV Poisson structure $\mathcal{P}$.

With this result in mind, an integrable (δ, ϵ)-deformation of the LV equations is straightforwardly obtained by considering again the same Hamiltonian (2.3) and the new Poisson bracket $\mathcal{P}_{\delta,\epsilon}$. In this way we obtain the perturbed LV equations

$$
\begin{aligned}
\dot{X} &= X\left[\alpha a_2 Y+\beta a_3 Z+(\alpha b_2+\beta b_3)\right] \\
&\qquad +\delta X\left(1-X^{\frac{\gamma}{\beta}}\right)\left(a_2+\frac{b_2}{Y}\right)+\epsilon X\left(1-X^{-\frac{\gamma}{\alpha}}\right)\left(a_3+\frac{b_3}{Z}\right) \\
\dot{Y} &= Y\left[-\alpha a_1 X+\gamma a_3 Z+(-\alpha b_1+\gamma b_3)\right]+\delta\left[(X^{\frac{\gamma}{\beta}}-1)(a_1X+b_1)+\frac{\gamma}{\alpha}(a_3 Z+b_3)\right] \\
&\quad +\frac{\epsilon\gamma}{\beta}\left[Y\left(a_3+\frac{b_3}{Z}\right)+\frac{\delta}{\alpha}\left(a_3+\frac{b_3}{Z}\right)\left(1-X^{\frac{\gamma}{\beta}-\frac{\gamma}{\alpha}}\right)\right] \\
\dot{Z} &= Z\left[-\beta a_1 X-\gamma a_2 Y+(-\gamma b_2-\beta b_1)\right]-\frac{\delta\gamma}{\alpha}Z\left[a_2+\frac{b_2}{Y}\right] \\
&\quad +\epsilon\left[(X^{-\frac{\gamma}{\alpha}}-1)(a_1X+b_1)-\frac{\gamma}{\beta}(a_2 Y+b_2)\right]+\frac{\delta\epsilon\gamma}{\alpha\beta}\left[(X^{\frac{\gamma}{\beta}-\frac{\gamma}{\alpha}}-1)\left(a_2+\frac{b_2}{Y}\right)\right]
\end{aligned}
\tag{4.3}
$$

whose integrability is preserved despite the presence of both polynomial and rational perturbation terms, since both $\mathcal{H}$ and the deformed Casimir $\mathcal{C}_{\delta,\epsilon}$ are integrals of the motion in involution for the system.

Obviously, the (δ,ϵ)-integrable deformation of the N sets of twisted LV equations given in Section 3 can be obtained by using $\mathcal{P}_{\delta,\epsilon}$ (instead of $\mathcal{P}$) and by making use of the same N-th coproduct map (3.5) and Hamiltonian (3.6) since $\mathcal{P}_{\delta,\epsilon}$ is invariant under the same group multiplication. For the sake of conciseness we do not present the explicit form of the set of 3N equations here, but we would line to stress that in this deformed case the following (2N-1) integrals of the motion for the system can be explicitly obtained from the coalgebra construction and read

$$
\mathcal{C}_i=\left[\delta\left(1-X_i^{\frac{\gamma}{\beta}}\right)+\alpha Y_i\right]^{-\frac{\beta}{\alpha}}\left[\epsilon\left(X_i^{\frac{\gamma}{\alpha}}-1\right)+\beta Z_i X_i^{\frac{\gamma}{\alpha}}\right]\quad i=1,2,\ldots,N \tag{4.4}
$$

$$
\Delta^{(r)}(C)=\left[\delta\left(1-\Delta^{(r)}(X)^{\frac{\gamma}{\beta}}\right)+\alpha\Delta^{(r)}(Y)\right]^{-\frac{\beta}{\alpha}}\left[\epsilon\left(\Delta^{(r)}(X)^{\frac{\gamma}{\alpha}}-1\right)+\beta\Delta^{(r)}(Z)\Delta^{(r)}(X)^{\frac{\gamma}{\alpha}}\right]\ \ r=2,\ldots,N. \tag{4.5}
$$

Indeed, these are the appropriate coalgebra deformations of the integrals (3.8) and (3.9).

It would be certainly intreresting to find specific LV models in which the (δ,ϵ)-perturbation terms appearing (4.3) could be dynamically meaningful. Also, the generalization of the Poisson-Lie group approach here presented to the case of 4D (and higher dimensional) LV systems is under investigation.

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