NATURAL SELECTION WITH HABITS AND LEARNING IN HETEROGENEOUS ECONOMIES

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Abstract. We study natural selection in complete financial markets, populated by heterogeneous agents. We allow for a rich structure of heterogeneity: Individuals may differ in their beliefs concerning the economy, information and learning mechanism, risk aversion, impatience (time preference rate) and degree of habits. We develop new techniques for studying long run behavior of such economies, based on the Strassen's functional law of iterated logarithm. In particular, we explicitly determine an agent's survival index and show how the latter depends on the agent's characteristics. We use these results to study the long run behavior of the equilibrium interest rate and the market price of risk.

1. Introduction

An important area of financial economics is devoted to the study of equilibrium asset pricing. By imposing the law of supply and demand, security prices, consumption rules and other economic concepts are determined in terms of the underlying variables of the model. These primitives are the characteristics of a set of agents, who are assumed to be investing in a financial market and aiming to maximize their utility functions. There is a major distinction between the so-called representative-consumer models and models with heterogeneous agents. While representative-consumer models enjoy a transparent solution due to the assumption that there is only one type of agents, models with heterogeneous consumers are substantially more complicated to analyze. The reason for this is the complexity of the equilibrium risk sharing generated by the agents' heterogeneity.

There is a vast literature dedicated to the study of heterogeneous equilibrium models (see, e.g., Chan and Kogan [6], Xiouros and Zapatero [28] and the references therein). Numerous papers have studied the long run behavior of asset prices and risk sharing rules in such economies. See, e.g., Blume and Easley [4], Cvitanić,

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Jouini, Malamud, and Napp [5], Nishide and Rogers [22], Sandroni [24], and Yan [29, 30]. In particular, many papers are devoted to the natural selection hypothesis, going back to the ideas of Friedman [13]. The natural selection hypothesis can be stated informally as "If you are so smart, why aren't you rich?". Formally, natural selection in financial markets examines agents' long run survival[1] in equilibrium models. Intuitively, this hypothesis is based on the idea that agents with inaccurate forecasts will eventually be eliminated from the economy.

This naturally raises interesting and important theoretical questions: Can investors with irrational beliefs survive in the long run and have a fundamental impact on the economy, or would they be driven out of the economy? Would investors with a high level of risk aversion vanish in the long run or would they dominate the market, in a growing economy? In this paper, we provide complete answers to these questions for a large class of models.

We study natural selection and long run behavior of asset prices in complete financial markets, populated by heterogeneous agents. We allow for a rich structure of heterogeneity: Individuals may differ in their beliefs concerning the economy, information and learning mechanism, risk aversion, impatience (time preference rate) and degree of habits. Each individual in our model is represented by a generalized version of the *catching-up-with-the-Jonses* power utility function of Chan and Kogan [6]. This model of preferences is sometimes referred to in the literature as *exogenous habit-formation*, since it incorporates the impact of a certain given stochastic process on individual's consumption policy. Agents are assumed to possess only a partial information regarding the events associated with the evolution of the market. More specifically, the stochastic dynamics of the mean growth-rate of the economy[2] is *unobservable* and the agents' information set consists of the aggregate endowment and a publicly observable signal. The agents are rational in the sense that they use a standard Kalman filter to update their expectations about the economy's growth rate. However, agents may be irrational in the way they interpret the public signal: Some of them may be over- (or, under-)confident about the informativeness of the public signal. We use the standard way of modeling over-(or, under-)confidence, originated by Dumas, Kurshev, and Uppal [10] and Scheinkman and Xiong [25]: We assume that agents' beliefs concerning the instantaneous correlation of the public signal with the economy's growth rate may differ from the actual value of this correlation.[3] The heterogeneous filtering rules yield highly non-trivial dynamics for the individual consumption and the equilibrium state price density, determined by the market clearing condition. In particular,

[1] An agent is said to survive in the long run if the ratio of his consumption to the aggregate consumption stays positive with positive probability as time goes to infinity.

[2] We assume that the mean growth rate follows an Ornstein-Uhlenbeck process.

[3] This is a a realistic assumption as correlations are extremely difficult to estimate empirically.

subjective probability densities describing agents' beliefs give rise to multiple new state variables, governing the dynamics of the economy.

As in Yan [29], we show that an agent's long run survival is determined by a single number, the survival index, and we explicitly calculate this index in terms of an agent's characteristics. We show that the agent with the lowest survival index dominates the economy in the long run and all other agents vanish. We then show that the interest rate and the market price of risk behave asymptotically as those of an economy populated solely by the single surviving agent.

To derive our results, we develop new techniques based on ergodicity of certain stochastic processes and Strassen's functional law of iterated logarithm. To the best of our knowledge, these methods have never been used in the general equilibrium literature before.

We now discuss related articles. The most closely related to ours are the papers by Yan [29] and Cvitanić, Jouini, Malamud, and Napp [5].[4] Namely, these authors consider a special case of our model corresponding to the case when there is no learning and agents having standard CRRA preferences without any habit formation. In terms of modeling heterogeneous beliefs and learning, our model closely follows the one of Dumas, Kurshev, and Uppal [10] and Scheinkman and Xiong [25], who considered a special case of our model: A two-agent economy with standard CRRA utility functions, and the public signal being a pure noise, uncorrelated with the economy's growth rate. Chan and Kogan [6] consider a special case of our model with homogeneous catching-up-with-the-Joneses habit levels and a continuum of agents with heterogeneous risk aversions. Xiouros and Zapatero [28] derive a closed form expression for the equilibrium state price density in the Chan and Kogan [6] model. Cvitanić and Malamud [7] study how long run risk sharing depends on the presence of multiple agents with different levels of risk aversion. Kogan, Ross, Wang and Westerfield [18] and Cvitanić and Malamud [8] study interaction of survival and price impact in economies where agents derive utility only from terminal consumption. Fedyk, Heyerdahl-Larsen and Walden [12] extend the model of Yan [29] by allowing for many assets. Kogan, Ross, Wang and Westerfield [19] study the link between survival and price impact in the presence of intermediate consumption and allow for general utilities with unbounded relative risk aversion and a general dividend process. Another quite large direction of the complete market risk sharing literature concentrates on the equilibrium effects of heterogeneous beliefs. With CRRA agents differing only in their beliefs, the equilibrium state price density can be derived in a closed form and thus many equilibrium properties can be analyzed in detail. See, e.g., Basak [1, 2], Jouini and Napp [16, 17], Jouini, Martin and Napp [15] and Xiong and Yan [27].

[4] Bhamra and Uppal [3], Dumas [9], and Wang [26] considered the same model, but only with two agent types and heterogeneity coming only from risk aversion.

The paper is organized as follows. In Section 2, we introduce the model and provide some preliminary results that will be employed in subsequent sections. Section 3 is devoted to a brief description of the equilibrium state price density in homogeneous and heterogeneous settings, and to the derivation of formulas for the interest rate and market price of risk. In Section 4, we present the main results of the paper and discuss some special cases and corollaries. Section 5 deals with some auxiliary results (limit theorems for certain stochastic processes) that are crucial for the proof of the main result. In Section 6, we provide a proof for the main result. Finally, in Section 7 we establish long run results for the interest rate and the market price of risk. Some of the results appearing in Sections 5 and 7 are of an independent mathematical interest.

2. Preliminaries

We consider a continuous-time Arrow-Debreu economy with an infinite horizon, in which heterogeneous agents maximize their utility functions from consumption. The uncertainty in our model is captured by a (complete) probability space $(\Omega, \mathcal{F}_\infty, P)$ and a continuous filtration $\mathcal{F} := (\mathcal{F}_t)_{t\in[0,\infty)}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We fix three standard and independent Wiener processes $(W_t^{(i)})_{t\in[0,\infty)}$, $i = 1, 2, 3$, adapted to the filtration $\mathcal{F}$. There are N different types of agents in the economy, labeled by $i = 1, ..., N$. Each agent i is equipped with a non-negative endowment process $\left(e_t^i\right)_{t\in[0,\infty)}$, which is adapted with respect to the filtration $\mathcal{G}$ (see (2.7)). We denote by $D_t := \sum_{i=1}^N e_t^i$ the *aggregate endowment* process and assume that $(D_t)_{t\in[0,\infty)}$ satisfies

$$\frac{dD_t}{D_t} = \mu_t^D dt + \sigma^D dW_t^{(1)} \ , \ D_0 = 1 \tag{2.1}$$

or equivalently,

$$D_t = \exp\left(\int_0^t \mu_s^D ds - \frac{1}{2}(\sigma^D)^2 t + \sigma^D W_t^{(1)}\right), \tag{2.2}$$

where the constant $\sigma^D > 0$ represents the volatility. The *mean-growth rate* $(\mu_t^D)_{t\in[0,\infty)}$ is an Ornstein-Uhlenbeck process that solves uniquely the SDE:

$$d\mu_t^D = -\xi(\mu_t^D - \overline{\mu})dt + \sigma^\mu dW_t^{(2)}, \tag{2.3}$$

that is

$$\mu_t^D = \overline{\mu} + (\mu_0 - \overline{\mu})\, e^{-\xi t} + \sigma^\mu e^{-\xi t} \int_0^t e^{\xi s} dW_s^{(2)}, \tag{2.4}$$

where $\overline{\mu}, \mu_0$ and σ^μ are some real numbers and $\xi > 0$. The numbers $\overline{\mu}, \mu_0$ will be referred to as the *average* and *initial* mean-growth rate respectively.

2.1. **The Financial Market.** We consider a financial market[5] that consists of at least two long-lived securities: A risky asset $(S_t)_{t\in[0,+\infty)}$ and a bank account $(S_t^0)_{t\in[0,+\infty)}$. In addition there are other (not explicitly modeled) assets guaranteeing that the market is dynamically complete.[6] The bond is in zero net supply and the stock is a claim to the total endowment of the economy $(D_t)_{t\in[0,\infty)}$ and has a net supply of one share. The risk-less bond is denoted by $S_t^0 = e^{\int_0^t r_s ds}$, where $(r_t)_{t\in[0,\infty)}$ is the risk-free rate process. We assume that there exists a *unique positive state price density* denoted by $(M_t)_{t\in[0,\infty)}$, that is, a positive adapted process that satisfies

$$M_t = E\left[e^{\int_t^u r_s ds} M_u \big| \mathcal{G}_t\right],$$

for all $u > t$, and

$$S_t = E\left[\int_t^\infty \frac{M_u}{M_t} D_u du \big| \mathcal{G}_t\right],$$

for all $t > 0$, where the filtration $\mathcal{G} := (\mathcal{G}_t)_{t\in[0,\infty)}$ is defined in (2.7). Note that our assumption excludes arbitrage opportunities in the model and implies that the market is *complete*. The state price density as well as all other parameters are to be derived endogenously in equilibrium.

2.2. **Preferences and Equilibrium.** Agent i is maximizing his intertemporal von Neumann-Morgenstern expected utility

$$\sup_{(c_{it})_{t\in[0,\infty)}} E^{P^i}\left[\int_0^\infty e^{-\rho_i t} U_i(c_{it}) dt\right],$$

from consumption, under the constraints that the consumption stream $(c_{it})_{t\in[0,\infty)}$ is a positive adapted process with respect to $\mathcal{G}$ (which is defined in (2.7)) and lies in the budget set:

$$E\left[\int_0^\infty c_t^i M_t dt\right] \le E\left[\int_0^\infty \epsilon_t^i M_t dt\right].$$

Here, $E^{P_i}[\cdot]$ stands for the expectation with respect to the subjective probability measure P_i of agent i. The exact form of P_i is specified in (2.15). We assume that

[5]More precisely, the model (to be set below) can be implemented by a complete securities market with a unique state price density derived in equilibrium. See Duffie and Huang (1986) for a detailed exposition of this issue.

[6]Formally, as there will be only two Brownian motions driving the economy, one additional asset can complete the market. However, since the price of this asset would be determined endogenously, one would have to verify endogenous completeness. This can be done using the techniques of Hugonnier, Malamud and Trubowitz [14]. Otherwise, we can just assume that there are sufficiently many (derivative) assets, completing the market.

all agents are represented by 'catching up with the Jonses[7]' preferences:

$$U_i\left(c_{it}\right) = \frac{1}{1-\gamma_i}\left(\frac{c_{it}}{H_{it}}\right)^{1-\gamma_i}.$$

Here, $(H_{it})_{t\in[0,\infty)}$ is an exogenously given stochastic process that represents the subjective index of 'standard of living'. We consider here a more general specification for H_{it} than the one in Chan and Kogan [6]. Namely, we set $H_{it} := X_t^{\beta_i} = e^{\beta_i \cdot x_t}$, for some $\beta_i > 0$, where

$$x_t = e^{-\lambda t}\cdot\left(x_0 + \lambda\cdot\int_0^t e^{\lambda s}\cdot\log(D_s)ds\right), \tag{2.5}$$

or equivalently, $(x_t)_{t\in[0,\infty)}$ is the unique solution of the SDE:

$$dx_t = \lambda(\log(D_t) - x_t)dt.$$

In this setting, the process $(X_t)_{t\in[0,\infty)}$ is the index representing the 'standard of living' in the economy. For each agent i, the number β_i measures the impact of the index X_t on the agent; in particular, when $\beta_i = 0$, the agent is not influenced by the index at all. For large β_i, the influence is somewhat heavy. In complete markets, the optimal consumption stream can be easily derived as in the following statement.

Proposition 2.1. *The optimal consumption stream of agent i in a complete market represented by a state price density $(\xi_t)_{t\in[0,\infty)}$, is given by*

$$c_{it} = e^{\frac{\rho_i}{\gamma_i}t}\cdot\xi_t^{-\frac{1}{\gamma_i}}Z_{it}^{\frac{1}{\gamma_i}}\cdot H_{it}^{\frac{\gamma_i-1}{\gamma_i}}\cdot c_{i0},$$

and

$$E\left[\int_0^\infty c_{it}\xi_t\right] = E\left[\int_0^\infty \epsilon_t^i\xi_t\right],$$

where the process $(Z_{it})_{t\in[0,\infty)}$ is given in (2.15).

Proof of Proposition 2.1. The assertion follows by standard duality arguments involving the first-order conditions. □

Finally, we introduce the notion of Arrow-Debreu equilibrium.

Definition 2.1. *An equilibrium is a pair $\left((c_{it})_{t\in[0,\infty)}, (\xi_t)_{t\in[0,\infty)}\right)$ such that:*
a. Each stream $(c_{it})_{t\in[0,\infty)}$ is the optimal consumption stream of agent i and

[7]This paradigm of a utility function was first introduced in Abel (1990), and is commonly referred to in the literature as a utility with exogenous habits. This specification describes a decision maker who experiences an impact of the "index of standard of living' index. Our specification is based on the now standard model of Chan and Kogan [6] for the standard of living in our economy.

$(M_t)_{t\in[0,\infty)}$ is the state price density that represents the market.
b. The market clearing condition is satisfied:

$$\sum_{i=1}^{N} c_{it} = D_t, \tag{2.6}$$

for all $t > 0$.

2.3. **Diverse Beliefs and Learning.** The are two processes in the economy that are *observable* by all agents. The first one is the aggregate endowment process $(D_t)_{t\in[0,\infty)}$, and the second one is a certain public signal:

$$s_t = \phi W_t^{(2)} + \sqrt{1-\phi^2} W_t^{(3)},$$

for some $\phi \in [0,1)$. That is, the public signal exhibits a non-negative correlation $\phi \in [0,1)$ with the shock governing the mean growth-rate process. The corresponding filtration is denoted by

$$\mathcal{G}_t := \sigma\left(\{s_u; u \le t\} \bigcup \{D_u; u \le t\}\right). \tag{2.7}$$

In contrast to this, the mean-growth rate process is *unobservable*. That is, neither of the agents possesses an access to the data revealing the dynamics of the process $(\mu_t^D)_{t\in[0,\infty)}$. Furthermore, agents may have diverse beliefs concerning the average and initial mean growth-rate. More precisely, each agent i reckons that the initial mean-growth rate is some $\mu_{0i} \in \mathbb{R}$ and that the average mean-growth rate is some $\overline{\mu}_i \in \mathbb{R}$. That is to say, before filtering, agent i assigns in his mind the following model for μ_t^D:

$$\overline{\mu}_i + (\mu_{i0} - \overline{\mu}_i)\, e^{-\xi t} + \sigma^{\mu} e^{-\xi t} \int_0^t e^{\xi s} dW_s^{(2)}. \tag{2.8}$$

Furthermore, individuals may have irrational perception of the information supplied by the signal. I.e., each agent i believes that the public signal $(s_t)_{t\in[0,\infty)}$, has a correlation $\phi_i \in [-1,1)$ with $(W_t^{(2)})_{t\in[0,\infty)}$, when if fact, the correlation is $\phi \in [0,1)$. Therefore, under the belief of agent i, the following model is attributed to the signal s_t:

$$\phi_i W_t^{(2)} + \sqrt{1-\phi_i^2} W_t^{(3)}. \tag{2.9}$$

We denote by Q^i the measure corresponding to agent's $i-$th beliefs regarding the models in (2.8) and (2.9), where $W_t^{(1)}, W_t^{(2)}$ and $W_t^{(3)}$ are independent Wiener processes under Q^i. Consequently, agents are in the process of learning and filtering out the dynamics of the mean-growth rate, which is deduced by using the theory of optimal filtering.

Definition 2.2. *The process*

$$\mu_{it}^D := E^{Q^i}\left[\overline{\mu}_i + (\mu_{i0} - \overline{\mu}_i)\, e^{-\xi t} + \sigma^{\mu} e^{-\xi t} \int_0^t e^{\xi s} dW_s^{(2)} \big| \mathcal{G}_t\right],$$

is called the subjective mean growth-rate of agent i.

Proposition 2.2. *We have*

$$\mu_{it}^D=\frac{\mu_{i0}}{y_{it}}+\frac{\xi\overline{\mu}_i}{y_{it}}\int_0^t y_{iu}du+\frac{1}{\left(\sigma^D\right)^2 y_{it}}\int_0^t\frac{\nu_{iu}y_{iu}}{D_u}dD_u+\frac{\sigma^\mu\phi_i}{y_{it}}\int_0^t y_{iu}ds_u, \tag{2.10}$$

where

$$y_{it}=\exp\left(\xi t+\frac{1}{(\sigma^D)^2}\int_0^t\nu_{is}ds\right), \tag{2.11}$$

and the variance process

$$\nu_{it}:=E^{Q^i}\left[\left(\mu_t^D-E^{Q^i}\left[\mu_t^D|\mathcal{G}_t\right]\right)^2|\mathcal{G}_t\right],$$

is deterministic and given by

$$\nu_{it}=\alpha_{i2}\cdot(\sigma^D)^2\cdot\frac{e^{(\alpha_{i2}-\alpha_{i1})t}-1}{e^{(\alpha_{i2}-\alpha_{i1})t}-\alpha_{i2}/\alpha_{i1}}, \tag{2.12}$$

where

$$\alpha_{i2}=\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2\cdot(1-\phi_i^2)}-\xi,$$

and

$$\alpha_{i1}=-\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2\cdot(1-\phi_i^2)}-\xi.$$

Proof of Proposition 2.2. Observe that Theorem 12.7 in Lipster and Shiryaev [21] implies that $\left(\mu_{it}^D\right)_{t\in[0,\infty)}$ satisfies the following SDE:

$$d\mu_{it}^D=-\xi\left(\mu_{it}^D-\overline{\mu}_i\right)dt+\frac{\nu_{it}}{\left(\sigma^D\right)^2}\left(\frac{dD_t}{D_t}-\mu_{it}^D dt\right)+\sigma^\mu\phi_i ds_t, \tag{2.13}$$

where the variance process ν_{it} is detected through the following Riccatti ODE:

$$\nu_{it}'=-2\xi\nu_{it}+(\sigma_\mu)^2\cdot\left(1-\phi_i^2\right)-\frac{1}{(\sigma^D)^2}\cdot\nu_{it}^2,$$

with $\nu_{i0}=0$. One can solve the above equation and verify that ν_{it} is given by (2.12). Now, we shall solve the SDE (2.13). By definition, we have $y_{it}'=(\xi+\frac{\nu_{it}}{(\sigma^D)^2})y_{it}$, and $y_{i0}=1$. Notice that the preceding observation combined with Ito's formula implies that

$$d\left(y_{it}\mu_{it}^D\right)=\xi\overline{\mu}_i y_{it}dt+\frac{\nu_{it}}{\left(\sigma^D\right)^2}y_{it}\frac{dD_t}{D_t}+\sigma^\mu\phi_i y_{it}ds_t,$$

completing the proof. □

Remark 2.1. *Dumas et. al.* [10] *consider the static version of (2.10). That is, the functions ν_{it} and y_{it} are substituted by the corresponding asymptotic limits. This can be justified by Lemma 5.3 of the current paper.*

We denote by $i=0$ a fictional agent who is rational in the sense that he knows the correct average and initial mean growth-rate, and the correlation parameter ϕ. Let us denote by $\mu^D_{0t} := E^P\left[\mu^D_t \middle| \mathcal{G}_t\right]$ the estimated mean growth-rate of this agent. As in Proposition 2.2, we have

$$\mu^D_{0t} = \frac{\mu_0}{y_{0t}} + \frac{\xi\overline{\mu}}{y_{0t}}\int_0^t y_{0u}du + \frac{1}{\left(\sigma^D\right)^2 y_{0t}}\int_0^t \frac{\nu_{0u}y_{0u}}{D_u}dD_u + \frac{\sigma^\mu \phi}{y_{0t}}\int_0^t y_{0u}ds_u,$$

where y_{0t} and ν_{0t} are defined similarly as in (2.11) and (2.12). It follows by Theorem 8.1 in Lipster and Shyraev [20] that $W^{(0)}_t = W^{(1)}_t - \int_0^t \frac{\mu^D_{0s}-\mu^D_s}{\sigma^D}ds$ is a $P-$Brownian motion with respect to the filtration $\mathcal{G}$. We set

$$\delta_{it} := \frac{\mu^D_{it} - \mu^D_{0t}}{\sigma^D}, \tag{2.14}$$

to be the $i-$th agent's *error* in the mean-growth estimation. The dynamics of $(D_t)_{t\in[0,\infty)}$ from $i-$th agent's perspective, admits the form

$$\frac{dD_t}{D_t} = \mu^D_{it}dt + \sigma^D dW^{(0)}_{it},$$

where,

$$dW^{(0)}_{it} = dW^{(0)}_t - \delta_{it}dt$$

is a Brownian motion (by Girsanov's theorem) under the equivalent probability measure[8] P_i and the filtration $\mathcal{G}$, where

$$Z_{it} := E\left[\frac{dP^i}{dP}\middle|\mathcal{G}_t\right] = \exp\left(\int_0^t \delta_{is}dW^{(0)}_s - \frac{1}{2}\int_0^t \delta^2_{is}ds\right). \tag{2.15}$$

3. The Equilibrium State Price Density

In the current section we briefly depict the structure of the equilibrium state price density in both settings of homogeneous and heterogeneous economies.

3.1. **Homogeneous Economy.** Consider an economy where all agents are of the same type i, and denote by $(M_{it})_{t\in[0,\infty)}$ the corresponding equilibrium state price density. The homogeneity of the economy combined with the completeness of the market allows to derive the corresponding state price density explicitly.

Lemma 3.1. *The equilibrium state price density in a market populated by one agent of type i is given by*

$$\begin{aligned} M_{it} = e^{-\rho_i t}D_t^{-\gamma_i}Z_{it}H_{it}^{\gamma_i-1} = \\ \exp\left(-\int_0^t\left(\rho_i + \gamma_i\left(\mu^D_s - \frac{1}{2}(\sigma^D)^2\right) + \frac{1}{2}\delta^2_{is}\right)ds\right) \\ \exp\left(\left(\gamma_i - 1\right)\beta_i x_t + \int_0^t \delta_{is}dW^{(0)}_s - \gamma_i\sigma^D W^{(1)}_t\right). \end{aligned} \tag{3.1}$$

[8] One can check that the process $(Z_{it})_{t\in[0,\infty)}$ is a true martingale by verifying Novikov's condition on a small interval and then applying a similar argument as the one in Example 3, page 233, Lipster and Shiryaev [20].

Proof of Lemma 3.1. The assertion follows by employing the market clearing condition and Lemma 2.1. □

We derive now the risk free-rate and the market price of risk in a homogeneous economy.

Lemma 3.2. *The risk free rate and the market price of risk in an economy populated by one agent of type i, are given respectively by*

$$r_{it} := \rho_i + \gamma_i \mu_{it}^D - \frac{1}{2}\left(\sigma^D\right)^2 \gamma_i\left(\gamma_i+1\right) - \beta_i\left(\gamma_i-1\right)\left(x_t - \log\left(D_t\right)\right),$$

and

$$\theta_{it} := \gamma_i \sigma^D - \delta_{it}.$$

Proof of Lemma 3.2. Consider the process:

$$Y_{it} := -\int_0^t (\rho_i + \gamma_i(\mu_{0s}^D - \frac{1}{2}(\sigma^D)^2) + \frac{1}{2}\delta_{is}^2)ds + (\gamma_i-1)\,\beta_i x_t + \int_0^t (\delta_{is} - \gamma_i\sigma^D)dW_s^{(0)}.$$

The dynamics of M_{it} are given by

$$\frac{dM_{it}}{M_{it}} = dY_{it} + \frac{1}{2}d\langle Y_i, Y_i\rangle_t.$$

where

$$dY_{it} = -\left(\rho_i + \gamma_i\left(\mu_t^D - \frac{1}{2}(\sigma^D)^2\right) + \frac{1}{2}\delta_{it}^2\right)dt +$$
$$\beta_i(\gamma_i-1)\cdot(\log(D_t) - x_t)dt + \left(\delta_{it} - \gamma_i\sigma^D\right)dW_t^{(0)},$$

and

$$d\langle Y_i, Y_i\rangle_t = \left(\delta_{it} - \gamma_i\sigma^D\right)^2 dt.$$

The rest of the proof follows by the fact that the risk free rate and market price of risk coincide with minus the drift and minus the volatility of the SPD respectively. □

3.2. **Heterogeneous Economy.** Consider an economy populated by N different types of agents. By Lemma 2.1, the optimal consumption stream of agent i is given by

$$c_{it} = e^{-\frac{\rho_i}{\gamma_i}t} \cdot M_t^{-\frac{1}{\gamma_i}} Z_{it}^{\frac{1}{\gamma_i}} \cdot H_{it}^{\frac{\gamma_i-1}{\gamma_i}} \cdot c_{i0} = c_{i0}\left(\frac{M_{it}}{M_t}\right)^{1/\gamma_i} \cdot D_t, \tag{3.2}$$

where $(M_t)_{t\in[0,\infty)}$ stands for the corresponding heterogeneous equilibrium state price density, and M_{it} is given by (3.1). Therefore, the market clearing condition (2.6) admits the form

$$\sum_{i=1}^N c_{i0}\cdot\left(\frac{M_{it}}{M_t}\right)^{1/\gamma_i} = 1. \tag{3.3}$$

Example 3.1. *Consider a homogeneous risk aversion economy, i.e.,* $\gamma_1 = ... = \gamma_N = \gamma$. *Then, the equilibrium state price density is given explicitly by*

$$M_t = \left(\sum_{i=1}^{N} \frac{c_{i0} e^{-\rho_i t/\gamma} Z_{it}^{1/\gamma} H_{it}^{\frac{\gamma-1}{\gamma}}}{D_t} \right)^{\gamma}.$$

Furthermore, if the habits are homogeneous, that is, $\beta_1 = ... = \beta_N = \beta$, *we have*

$$M_t = e^{(\gamma-1)\beta x_t} \left(\sum_{i=1}^{N} \frac{c_{i0} e^{\rho_i t/\gamma} Z_{it}^{1/\gamma}}{D_t} \right)^{\gamma}.$$

If the beliefs among the agents are not varying, i.e., $Z_{1t} = ... = Z_{Nt} = Z_t$, *then, we have*

$$M_t = Z_t \left(\sum_{i=1}^{N} \frac{c_{i0} e^{\rho_i t/\gamma} H_{it}^{\frac{\gamma-1}{\gamma}}}{D_t} \right)^{\gamma}.$$

Finally, we provide formulas for the risk free rate and the market price of risk.

Proposition 3.3. *We have*

$$\theta_t = \sum_{i=1}^{N} \omega_{it} \theta_{it},$$

and

$$r_t = \sum_{i=1}^{N} \omega_{it} r_{it} + \frac{1}{2} \sum_{i=1}^{N} (1 - 1/\gamma_i) \omega_{it} \left(\theta_{it} - \theta_t \right)^2,$$

where

$$\omega_{it} := \frac{1/\gamma_i c_{it}}{\sum_{j=1}^{N} 1/\gamma_j c_{jt}} \tag{3.4}$$

denotes the r.

Proof of Proposition 3.3. The proof is identical to the proof of Proposition 4.1 in Cvitanić et. al. [5]. □

4. The Main Result: The long run Surviving Consumer

The current section is devoted to the study of the long run behavior of the optimal consumption shares in a heterogeneous economy. We establish the existence of a surviving consumer in the market, i.e., an agent whose optimal consumption asymptotically behaves as the aggregate consumption. This dominating individual is determined through the *survival index*. That is, a quantity depending on individuals' characteristics and specifies the surviving agent versus the extincting agents in the economy.

Definition 4.3. *The survival index of agent i is given by*

$$\kappa_i := \rho_i + \left(\overline{\mu} - \frac{1}{2}(\sigma^D)^2\right)(\gamma_i + (1-\gamma_i)\beta_i) + \frac{1}{2}\left(\frac{\overline{\mu}_i - \overline{\mu}}{\sigma^D}\right)^2 + \frac{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2(1-\phi_i^2)}}. \tag{4.1}$$

The following is assumed throughout the entire paper.

Assumption 4.1. *There exists an agent I_K whose survival index is the lowest one, namely $\kappa_{I_K} < \kappa_i$, for all $i \neq I_K$.*

We are now ready to state our main result.

Theorem 4.1. *In equilibrium, the only surviving agent in the long run is the one with the lowest survival index, i.e.,*

$$\lim_{t\to\infty} \frac{c_{it}}{D_t} = 0,$$

for all $i \neq I_K$, and

$$\lim_{t\to\infty} \frac{c_{I_K t}}{D_t} = 1.$$

The survival index is a complicated function of the individuals' underlying parameters. In order to isolate the effects of various agents' characteristics on the long run survival, we will discuss special cases in which agents differ with respect to only one or few particular parameters.

4.1. **The Effect of Risk-Aversion and Habits.**

Let the initial priors $(\overline{\mu}_i)_{i=1,...,N}$ and the over-confidence parameters $(\phi_i)_{i=1,...,N}$ be fixed and identical for all agents. As it will be seen in the proof of Theorem 4.1, the survival index is invariant under additive translation, and thus it is determined in the current setting by

$$\rho_i + \left(\mu - \frac{(\sigma^D)^2}{2}\right) \cdot (\gamma_i + (1-\gamma_i)\beta_i).$$

If $\beta_1 = ... = \beta_N = 0$, the survival index is the same as in Cvitanić et. al. [5]. In particular, in a growing economy (i.e. $\mu - \frac{(\sigma^D)^2}{2} > 0$) the least risk-averse agent will survive in the long run, as in the models of Yan [29], and Cvitanić et. al. [5]. The presence of habits may change the behavior. Here, if the habit is sufficiently strong ($\beta_i > 1$), the effect completely reverses: It is the most risk-averse agent who survives in the long run. Effectively, 'catching up with Jonses' preferences change an agent's risk aversion from γ_i to

$$\gamma_i + (1-\gamma_i)\beta_i.$$

Therefore, for strong habits, agents with a high risk-aversion effectively behave as agents with a low risk aversion. When risk aversion is homogeneous, the effect of habits strength on survival depends on whether risk aversion is above or below 1. If risk aversion is above 1, we get the surprising and at the first sight counter-intuitive result that agents with stronger habits survive in the long run. The reason for this is that the presence of habits forces the agent to trade more aggressively and make bets on very good realizations of the dividend in order to sustain the aggregate habit level generated by the 'catching up with the Jonses' preferences. This makes an agent with strong habits effectively less risk averse. This is beneficial for survival in a growing economy.

4.2. **The Effect of Diverse Beliefs.** Consider an economy where agents may differ only with respect to their average mean growth-rate estimations $(\overline{\mu}_i)_{i=1,\dots,N}$ and their correlation parameters $(\phi_i)_{i=1,\dots,N}$. In this case, the survival index admits the form

$$\kappa_i = \frac{1}{2}\left(\frac{\overline{\mu}_i - \overline{\mu}}{\sigma^D}\right)^2 + \frac{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2(1-\phi\phi_i)}{2\sqrt{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2(1-\phi_i^2)}}. \tag{4.2}$$

Note that in this case the survival index is a decreasing function of the correlation parameter ϕ_i in the interval $[-1,\phi]$, and an increasing function in the interval $(\phi,1]$. Therefore, in an economy where the only distinction between agents is their correlation parameters, the surviving agent is derived as follows. If either all agents are over-confident ($\phi < \phi$, for all $i = 1,...,N$) or under-confident ($\phi > \phi$, for all $i = 1,...,N$) in the signal ϕ, then, the survival index is given by

$$|\phi_i - \phi|,$$

and thus the individual with the most accurate guess of the correct correlation will dominate the market. If some agents are over-confident and some are under-confident in the signal, the situation becomes more complex. For simplicity, let us analyze the case of an economy which consists of two agents: The first agent underestimates the correlation and believes that it is $\phi_1 \in [-1,\phi]$, whereas the second agent overestimates the correlation by $\phi_2 \in [\phi,1]$. Let us set $a := \left(\frac{\xi\sigma^D}{\sigma^\mu}\right)^2$. If $\phi_1 \in \left[-1, 2\frac{a\phi(1+a)}{a\phi^2+(a+1-\phi)^2} - 1\right]$, the second agent will survive. Now, assume that $\phi_1 \in \left[2\frac{a\phi(1+a)}{a\phi^2+(a+1-\phi)^2} - 1, \phi\right]$. Then, if $\phi_2 \in \left[\phi, \frac{2(a+1)\phi-\left(a+1+\phi^2\right)\phi_1}{a+1+\phi^2-2\phi\phi_1}\right]$, then the second agent will survive; otherwise, that is, if $\phi_2 \in \left[\frac{2(a+1)\phi-\left(a+1+\phi^2\right)\phi_1}{a+1+\phi^2-2\phi\phi_1}, 1\right]$, the first agent will survive. To demonstrate the above scheme numerically, let us consider the case where $a = 1$ and $\phi = 1/2$ (see Figure 1). If $\phi_1 \in [-1,-0.2]$, then the second agent survives. If $\phi_1 \in [-0.2, 0.5]$, then: If $\phi_2 \in \left[0.5, \frac{8-9\phi_1}{9-4\phi_1}\right]$, the second agent is the one to survive. Otherwise, that is, if $\phi_2 \in \left[\frac{8-9\phi_1}{9-4\phi_1}, 1\right]$ then the first agent will survive. The preceding fact yields an economically surprising observation: Too overconfident agents will not survive when they compete with agents that believe in a weak negative correlation. Assume for instance that the second agent believes that the correlation is some $\phi_2 \in [8/9, 1]$. Then, if $\phi_1 \in [\frac{8-9\phi_2}{9-4\phi_2}, 0]$, the first agent will survive, despite of the negative correlation.

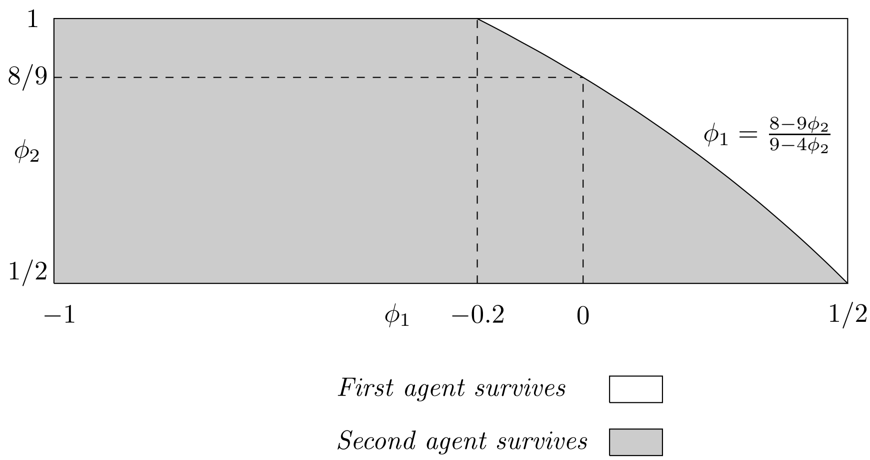

FIGURE 1. The long-run surviving consumer

If the only source of heterogeneity in the economy is the belief regarding the average mean-growth rate, then the survival index depends only on the error between the subjective mean-growth-rate and the correct one, namely,

$$\kappa_i = \left|\overline{\mu} - \overline{\mu_i}\right|.$$

Therefore, the consumer with the best forecast of the average mean-growth rate is the one to dominate the market.

4.3. **The Relative Level of Absolute Risk Tolerance.** As in Cvitanić et. al. [5], we define the *relative level of absolute risk tolerance* of agent i by

$$w_{it} := \frac{1/\gamma_i \cdot c_{it}}{\sum_{j=1}^{N} 1/\gamma_j \cdot c_{jt}}.$$

The following is an immediate consequence of Theorem 4.1.

Corollary 4.1. *We have*

$$\lim_{t\to\infty} w_{it} = 0,$$

for all $i \neq I_k$, *and*

$$\lim_{t\to\infty} w_{I_K t} = 1.$$

Proof of Corollary 4.1. Note that (3.2) implies that

$$w_{it} = \frac{c_{it}}{D_t} \cdot \frac{1/\gamma_i}{\sum_{j=1}^{N} 1/\gamma_j \cdot c_{0j}(M_{jt}/M_t)^{1/\gamma_j}}.$$

The identity (3.3) yields

$$\frac{1}{\sum_{j=1}^{N}\frac{1}{\gamma_j}c_{0j}(M_{jt}/M_t)^{1/\gamma_j}} \le \max_{k=1,\dots,N}\gamma_k.$$

The preceding observations combined with Theorem 4.1 and the equality $\sum_{i=1}^{N}\omega_{it} = 1$ complete the proof of Theorem 4.1. $\square$

5. Auxiliary Results

In the present section we introduce some results that will be crucial for proving Theorem 4.1. First, we introduce the following estimates indicating that y_{it}, $1/y_{it}$, their derivatives, and ν_{it} are close to certain functions, of a simpler form. The errors in these estimates are shown to be decaying exponentially fast to 0, as $t \to \infty$.

Lemma 5.3. *We have*

$$\left|\nu_{it} - \alpha_{i2}(\sigma^D)^2\right| \le Ce^{-2(\alpha_{i2}+\xi)t}, \tag{5.1}$$

$$\left|y_{it} - \exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{(\alpha_{i2}+\xi)t}\right| \le Ce^{-(\alpha_{i2}+\xi)t}, \tag{5.2}$$

$$\left|y'_{it} - (\alpha_{i2}+\xi)\exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{(\alpha_{i2}+\xi)t}\right| \le Ce^{-(\alpha_{i2}+\xi)t}, \tag{5.3}$$

$$\left|\frac{1}{y_{it}} - \exp\left(\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{-(\alpha_{i2}+\xi)t}\right| \le Ce^{-3(\alpha_{i2}+\xi)t}, \tag{5.4}$$

$$\left|\left(\frac{1}{y_{it}}\right)' + (\alpha_{i2}+\xi)\exp\left(\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{-(\alpha_{i2}+\xi)t}\right| \le Ce^{-3(\alpha_{i2}+\xi)t}, \tag{5.5}$$

for all $t > 0$ and some constant $C > 0$.

Proof of Lemma 5.3. Inequality (5.1) is due to the fact that $\left|\nu_{it} - \alpha_{i2}(\sigma^D)^2\right| = \left|\frac{(\alpha_{i1}-\alpha_{i2})\alpha_{i2}(\sigma^D)^2}{\alpha_{i1}e^{2(\alpha_{i2}+\xi)t}-\alpha_{i2}}\right|$. Next, by definition (see Proposition 2.2), it follows that y_{it} admits the form

$$y_{it} = \exp\left(\left(\alpha_{i2}+\xi\right)t - \frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\left(1 - e^{-2(\alpha_{i2}+\xi)t}\right)\right).$$

One checks that the inequality $e^x - 1 \le (e-1)x$, for all $0 \le x \le 1$ concludes the validity of (5.2). Recall that y_{it} satisfies the ODE $y'_{it} = \left(\xi + \frac{\nu_{it}}{(\sigma^D)^2}\right)y_{it}$, and thus we can estimate

$$\left|y'_{it} - (\alpha_{i2}+\xi)\exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{(\alpha_{i2}+\xi)t}\right| \le$$

$$\exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{(\alpha_{i2}+\xi)t}\left|\frac{\nu_{it}}{(\sigma^D)^2}-\alpha_{i2}\right|+$$
$$\left(\xi+\frac{\nu_{it}}{(\sigma^D)^2}\right)\left|y_{it}-\exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{(\alpha_{i2}+\xi)t}\right|,$$

which implies (5.3) by applying inequalities (5.1) with (5.2). Inequalities (5.4) and (5.5) are proved in a similar manner. □

For each $d \geq 1$, we denote by $\left(C_0\left([0,1];\mathbb{R}^d\right),||\cdot||_\infty\right)$ the space of all $\mathbb{R}^d$-valued continuous functions on the interval $[0,1]$ vanishing at 0 endowed with the sup topology.

Definition 5.4. *We denote by $K^{(d)}$ the space of all functions $f=(f_1,...,f_d)\in C_0\left([0,1];\mathbb{R}^d\right)$, such that each component f_i is absolutely continuous, and*

$$\sum_{i=1}^{d}\int_0^T (f_i'(x))^2dx \leq 1.$$

We note that $K^{(d)}$ is a compact subset of $C_0\left([0,1];\mathbb{R}^d\right)$ (see Proposition 2.7, page 343, in Revuz and Yor [23]). The next result deals with the asymptotics of certain multiple stochastic integrals.

Lemma 5.4. *Let $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$ be two arbitrary standard Brownian motions and denote $Z_t=\int_0^t e^{-s}\cdot W_{\frac{1}{2}\cdot(e^{2s}-1)}dB_s$. Then, we have*
(i)

$$\langle Z\rangle_\infty := \lim_{t\to\infty}\langle Z\rangle_t=\infty.$$

(ii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{au}dW_udB_s}{t}=0,$$

for any $a>0$.
(iii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{au}\int_0^u e^{bx}dW_xdudB_s}{t}=0,$$

for all $a,b>0$.

Proof of Lemma 5.4. (i) First, note that a change of variable implies that $\langle Z\rangle_t=\int_0^t e^{-2s}\cdot\left(W_{\frac{1}{2}\cdot(e^{2s}-1)}\right)^2ds=\int_0^{\frac{1}{2}(e^{2t}-1)}\frac{W_u^2}{(1+u)^2}du$. Consider the functional $F: C_0([0,1];\mathbb{R})\to\mathbb{R}_+$, which is given by

$$F(f):=\int_0^1\frac{f^2(x)}{(1+x)^2}dx.$$

Note that F is a continuous functional. Indeed, for a fixed $f\in C_0([0,1];\mathbb{R})$ and all $\varepsilon>0$, let $\delta=\varepsilon(2||f||_\infty+\varepsilon)$ and observe that if $||f-g||_\infty<\delta$, for some

$g \in C_0([0,1];\mathbb{R})$, then $|F(f) - F(g)| < \varepsilon$. It follows by Strassen's functional law of iterated logartigm (see Revuz and Yor [23], page 346, Theorem 2.12) that P-a.s,

$$\limsup_{N\to\infty} F\left(\frac{1}{\sqrt{2N\log\log(N)}} W_{Nt}\right) = \sup_{h\in K^{(1)}} F(h).$$

Notice that $\sup_{h\in K^{(1)}} F(h) \geq F(\tilde{h}) > 0$, where $\tilde{h}(x) = x$. Therefore, we have

$$\limsup_{N\to\infty} F\left(\frac{1}{\sqrt{2N\log\log(N)}} W_{Nt}\right)$$

$$= \limsup_{N\to\infty} \frac{\int_0^1 \frac{W_{Nt}}{(1+t)^2} dt}{2N\log\log N} = \limsup_{N\to\infty} \frac{\int_0^N \frac{W_u^2}{(N+u)^2} du}{2\log\log(N)} > 0.$$

Furthermore,

$$\limsup_{N\to\infty} \frac{\int_0^N \frac{W_u^2}{(1+u)^2} du}{2\log\log(N)} \geq \limsup_{N\to\infty} \frac{\int_0^N \frac{W_u^2}{(N+u)^2} du}{2\log\log(N)} > 0.$$

In particular, it follows that $\limsup_{N\to\infty} \int_0^N \frac{W_u^2}{(1+u)^2} du = \infty$, but, since the function $\int_0^N \frac{W_u^2}{(1+u)^2} du$ is monotone increasing in N, it follows that $\lim_{N\to\infty} \int_0^N \frac{W_u^2}{(1+u)^2} du = \infty$. This accomplishes the proof of part (i).
(ii) Denote $Y_t = \int_0^t e^{-s} \int_0^s e^u dW_u dB_s$ and $X_s = \int_0^s e^u dW_u$. Note that $\langle X\rangle_t = \frac{1}{2}\cdot\left(e^{2t}-1\right)$, therefore, since X_t is a martingale vanishing at 0 and $\langle X\rangle_\infty = \infty$, it follows by the Dambis, Dubins-Schwartz theorem (shortly DDS, see Revuz and Yor [23], page 181, Theorem 1.6) that $X_t = \widetilde{W}_{\frac{1}{2}\cdot(e^{2t}-1)}$, for a certain Brownian motion $\widetilde{W}_t$. Therefore, we can rewrite

$$Y_t = \int_0^t e^{-s}\cdot \widetilde{W}_{\frac{1}{2}\cdot(e^{2s}-1)} dB_s,$$

and thus by part (i), we have, $\lim_{t\to\infty}\langle Y\rangle_t = \langle Y\rangle_\infty = \infty$. Therefore, the DDS theorem implies that $Y_t = \widetilde{B}_{\langle Y\rangle_t}$, for some Brownian motion $\widetilde{B}_t$. Now, denote $\phi(x) = \sqrt{2x\log\log x}$ and let us rewrite $\frac{Y_t}{t} = \frac{\widetilde{B}_{\langle Y\rangle_t}}{\phi(\langle Y\rangle_t)}\cdot\frac{\phi(\langle Y\rangle_t)}{t}$. By the law of iterated logarithm, we have $\limsup_{t\to\infty}\frac{|\widetilde{B}_{\langle Y\rangle_t}|}{\phi(\langle Y\rangle_t)} \leq 1$, and hence it is enough to concentrate on the asymptotics of the second term:

$$\frac{\phi\left(\langle Y\rangle_t\right)}{t} = \sqrt{2\cdot\frac{\int_0^t e^{-2s}\cdot\left(\widetilde{W}_{\frac{1}{2}\cdot(e^{2t}-1)}\right)^2 ds\cdot\log\log\left(\int_0^t e^{-2s}\cdot\left(\widetilde{W}_{\frac{1}{2}\cdot(e^{2t}-1)}\right)^2 ds\right)}{t^2}}.$$

Note that $\phi(\frac{1}{2}(e^{2s}-1)) \leq e^s\sqrt{\log 2s}$ and thus, the law of iterated logarithm implies that

$$\limsup_{t\to\infty}\frac{\phi\left(\langle Y\rangle_t\right)}{t} \leq \limsup_{t\to\infty}\sqrt{\frac{\log(2t)\log\log(t\log 2t)}{t}} = 0.$$

This accomplishes the proof of part (ii).
(iii) By Fubini's Theorem, we have

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{au}\int_0^u e^{bx}dW_x du dB_s}{t}=$$

$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{bx}dW_x dB_s}{t}-\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{(a+b)x}dW_x dB_s}{t}=0,$$

where the last equality follows by part (ii). This completes the proof of Lemma 5.4. □

We proceed with the following statement.

Lemma 5.5. *Let* $(W_t)_{t\in[0,\infty)}$ *be a standard Brownian motion. Then, we have*
(i)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{ax}dW_x ds}{t}=0,$$

for all $a>0$.
(ii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{au}\int_0^u e^{bx}dW_x du ds}{t}=0,$$

for all $a,b>0$.

Proof of Lemma 5.5. (i) By using integration by parts and Fubini's Theorem, we get

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{au}dW_u ds}{t}=\lim_{t\to\infty}\frac{\int_0^t \left(W_s-ae^{-as}\int_0^s e^{au}W_u du\right)ds}{t}=$$

$$\lim_{t\to\infty}\frac{\int_0^t W_s ds-a\int_0^t\left(e^{au}W_u\int_u^t e^{-as}ds\right)du}{t}=\lim_{t\to\infty}\frac{\int_0^t e^{au}W_u du}{te^{at}}=0,$$

where the last equality follows by the law of large numbers.
(ii) As in (i), one checks that the limit is equal to

$$\frac{1}{a}\lim_{t\to\infty}\left(\int_0^t e^{-bs}\int_0^s e^{bx}dW_x ds-\int_0^t e^{-(a+b)s}\int_0^s e^{(a+b)x}dW_x ds\right),$$

which vanishes according to (i). □

In the next limit theorems, the main tool is ergodicity of certain stochastic processes. Similar ideas as below (even though we have provided a straightforward argument) could be applicable to deduce the previous lemma.

Lemma 5.6. *Let $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$ be two independent Brownian motions. Then, the following holds*
(i)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-as}\int_0^s e^{ax}dW_x e^{-bs}\int_0^s e^{bx}dB_x ds}{t}=0,$$

for all $a,b>0$.
(ii)

$$\lim_{t\to\infty}\frac{\int_0^t \left(e^{-as}\int_0^s e^{ax}dW_x\right)^2 ds}{t}=\frac{1}{2a},$$

for all $a>0$.
(iii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{ax}dW_x\int_0^s e^{bx}dW_x ds}{t}=\frac{1}{a+b},$$

for all $a,b>0$.

Proof of Lemma 5.6. (i) First observe that $\int_0^s e^{ax}dW_x$ is a martingale with $\langle\int_0^{\cdot} e^{ax}dW_x\rangle_t=\frac{e^{2at}-1}{2a}$, and thus by the DDS Theorem, we have $\int_0^t e^{ax}dW_x=\widetilde{W}_{\frac{e^{2at}-1}{2a}}$ for some Brownian motion $(\widetilde{W}_t)_{t\in[0,\infty)}$. A similar argument implies that $\int_0^t e^{bx}dB_x=\widetilde{B}_{\frac{e^{2bt}-1}{2b}}$, for a Brownian motion $(\widetilde{B}_t)_{t\in[0,\infty)}$. The construction in the DDS Theorem implies that $(\widetilde{B}_t)_{t\in[0,\infty)}$ and $(\widetilde{B}_t)_{t\in[0,\infty)}$ are independent. Recall that $e^{-at}\widetilde{W}_{e^{2at}}$ and $e^{-bt}\widetilde{B}_{e^{2bt}}$ are two independent stationary Orenstein-Uhlenbeck processes and thus the process $e^{-(a+b)t}\widetilde{W}_{e^{2at}}\widetilde{B}_{e^{2bt}}$ is stationary. Therefore, an ergodic theorem for stationary processes implies that

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\widetilde{W}_{e^{2as}}\widetilde{B}_{e^{2bs}}ds}{t}=e^{-(a+b)}E\left[\widetilde{W}_{e^{2a}}\widetilde{B}_{e^{2b}}\right]=0. \tag{5.6}$$

Next, the process $(W'_t)_{t\in[0,\infty)}$ given by $W'_t=\sqrt{2a}\widetilde{W}_{\frac{t}{2a}}$ for $t<1$, and $W'_t=\sqrt{2a}\widetilde{W}_{\frac{t-1}{2a}}+\sqrt{2a}\widetilde{W}_{\frac{1}{2a}}$ for $t>1$ is a Brownian motion. Thus, we have $\widetilde{W}_{\frac{e^{2as}-1}{2a}}=\frac{1}{\sqrt{2a}}W'_{e^{2as}}-\widetilde{W}_{\frac{1}{2a}}$, for all $s>1$. We define the process $(B'_t)_{t\in[0,\infty)}$ in a similar manner. We emphasize that $(W'_t)_{t\in[0,\infty)}$ and $(\widetilde{W}_t)_{t\in[0,\infty)}$ are independent of $(B'_t)_{t\in[0,\infty)}$ and $(\widetilde{B}_t)_{t\in[0,\infty)}$. Thus we can rewrite (5.6) as

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\left(\frac{1}{\sqrt{2a}}W'_{e^{2as}}-\widetilde{W}_{\frac{1}{2a}}\right)\left(\frac{1}{\sqrt{2b}}B'_{e^{2bs}}-\widetilde{B}_{\frac{1}{2a}}\right)ds}{t}.$$

Next, the law of iterated logarithm implies that for every $\varepsilon>0$ there exsits an $\mathcal{F}_\infty$-measurable random variable $N(\varepsilon):\Omega\to\mathbb{R}_+$ such that for all $s>N(\varepsilon)$, $\left|\frac{W_{e^{2as}}}{e^{as}\sqrt{\log(2as)}}\right|<1+\varepsilon$, and hence

$$\lim_{t\to\infty}\frac{\int_0^t\left|e^{-as-bs}W'_{e^{2as}}\right|ds}{t}\le(1+\varepsilon)\lim_{t\to\infty}\frac{\int_0^t\frac{\log(as)}{e^{bs}}ds}{t}=0.$$

This fact combined with (5.6) accomplishes the proof of part (i).
(ii) As in (i), $\int_0^s e^{ax} dW_x = \widetilde{W}_{\frac{e^{2as}-1}{2a}}$ and $\widetilde{W}_{\frac{e^{2as}-1}{2a}} = \frac{1}{\sqrt{2a}} W'_{e^{2as}} - \widetilde{W}_{\frac{1}{2a}}$. Next, ergodicity yields

$$\lim_{t\to\infty} \frac{\int_0^t \left(e^{-as}\widetilde{W}_{e^{2as}}\right)^2 ds}{t} = \frac{1}{e^{2a}} E\left[\widetilde{W}^2_{e^{2a}}\right] = 1.$$

Finally, the above limit combined with similar arguments to those appearing in (i) conclude the proof.
(iii) The idea of the proof is to rewrite the required limit in terms of limits of the same form as those in (ii). First, observe that $e^{-at}\int_0^s e^{au} dW_u = W_s - ae^{-at}\int_0^s e^{au} W_u du$. Thus we can rewrite,

$$\int_0^t \left(e^{-as}\int_0^s e^{au} dW_u\right)^2 ds \tag{5.7}$$

$$= \int_0^t W_s^2 ds - 2a\int_0^t e^{-as} W_s \int_0^s e^{au} W_u du ds + a^2 \int_0^t e^{-2as}\left(\int_0^s e^{au} W_u du\right)^2 du.$$

Observe that Fubini's Theorem implies that

$$\int_0^t e^{-2as}\left(\int_0^s e^{au} W_u du\right)^2 du = \int_0^t \int_0^t e^{ax+ay} W_x W_y \int_{\max\{x,y\}}^t e^{-2as} ds dx dy \tag{5.8}$$

$$= \frac{1}{a}\int_0^t e^{-as} W_s \int_0^s e^{au} W_u du ds - \frac{1}{2ae^{2at}}\left(\int_0^t W_x e^{ax} dx\right)^2.$$

This fact combined with (5.7) and (5.8) implies that

$$\lim_{t\to\infty} \frac{\int_0^t \left(e^{-at}\int_0^s e^{au} dW_u\right)^2 ds}{t}$$

$$= \lim_{t\to\infty} \frac{\int_0^t W_s^2 ds - a\int_0^t e^{-as} W_s \int_0^s e^{au} W_u du ds - \frac{a}{2e^{2at}}\left(\int_0^t e^{as} W_s ds\right)^2}{t}.$$

By using similar arguments and exploiting the preceding observations, one can check that

$$\lim_{t\to\infty} \frac{\int_0^t e^{-(a+b)s}\int_0^s e^{ax} dW_x \int_0^s e^{bx} dW_x ds}{t} = \tag{5.9}$$

$$\frac{a}{a+b}\lim_{t\to\infty} \frac{\int_0^t W_s^2 ds - a\int_0^t e^{-as} W_s \int_0^s e^{au} W_u du ds + \frac{a}{2e^{2at}}\left(\int_0^t e^{as} W_s ds\right)^2}{t} +$$

$$\frac{b}{a+b}\lim_{t\to\infty} \frac{\int_0^t W_s^2 ds - b\int_0^t e^{-as} W_s \int_0^s e^{au} W_u du ds + \frac{a}{2e^{2at}}\left(\int_0^t e^{as} W_s ds\right)^2}{t}.$$

The latter fact combined with part (ii) completes the proof. □

The next statement is heavily based on the above lemma.

Lemma 5.7. *Let $(W_t)_{t\in[0,\infty)}$ and $(B_t)_{t\in[0,\infty)}$ be two independent Brownian motions. Then, we have*
(i)

$$\lim_{t\to\infty}\frac{\int_0^t\left(e^{-(a+b)s}\int_0^s e^{ax}\int_0^x e^{bu}dW_u dx\right)^2 ds}{t}=\frac{1}{2b(a+b)(a+2b)},$$

for all $a,b>0$.
(ii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{au}dW_u\int_0^s e^{bu}\int_0^u e^{ax}dW_x duds}{t}=\frac{1}{2a(2a+b)},$$

for all $a,b>0$.
(iii)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\int_0^s e^{(a-\xi)u}\int_0^u e^{\xi u}dW_x du\int_0^s e^{(b-\xi)u}\int_0^u e^{\xi u}dW_x duds}{t}=$$

$$\frac{1}{(a-\xi)(b-\xi)}\left(\frac{1}{a+b}+\frac{1}{2\xi}-\frac{1}{a+\xi}-\frac{1}{b+\xi}\right),$$

for all $a,b,\xi>0$.
(iv)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-2(a+b)s}\int_0^s e^{ay}\int_0^y e^{bu}dW_u dy\int_0^s e^{(a+b)x}dW_x ds}{t}=\frac{1}{2(a+b)(a+2b)},$$

for all $a,b>0$.
(v)

$$\lim_{t\to\infty}\frac{\int_0^t e^{-2(a+b)s}\int_0^s e^{ay}\int_0^y e^{bu}dW_u dy\int_0^s e^{(a+b)x}dB_x ds}{t}=0,$$

for all $a,b>0$.

Proof of Lemma 5.7. (i) Notice that $\int_0^s e^{ax}\int_0^x e^{bu}dW_u dx=\frac{1}{a}\int_0^s e^{bu}(e^{as}-e^{au})dW_u$. Therefore, the required limit is equal to

$$\frac{1}{a^2}\lim_{t\to\infty}\frac{\int_0^t\left(e^{-bs}\int_0^s e^{bu}dW_u\right)^2 ds}{t}-\frac{2}{a^2}\lim_{t\to\infty}\frac{\int_0^t e^{-(a+2b)s}\int_0^s e^{(a+b)u}dW_u\int_0^s e^{bu}dW_u ds}{t}$$

$$+\frac{1}{a^2}\lim_{t\to\infty}\frac{\int_0^t\left(e^{-(a+b)s}\int_0^s e^{(a+b)u}dW_u\right)^2 ds}{t}.$$

Parts (ii) and (iii) in Lemma 5.6 complete the proof of (i).
(ii) As before, one checks that the limit is equal to

$$\frac{1}{b}\lim_{t\to\infty}\frac{\int_0^t e^{-2as}\left(\int_0^s e^{ax}dW_x\right)^2 ds}{t}$$

$$-\frac{1}{b}\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{au}dW_u\int_0^s e^{(a+b)x}dW_x ds}{t},$$

and the rest is a consequence of parts (ii) and (iii) of Lemma 5.6.
(iii) The limit is equal to

$$\frac{1}{(a-\xi)(b-\xi)}\left(\lim_{t\to\infty}\frac{\int_0^t \left(e^{-au}\int_0^u e^{ax}dW_x\right)^2 du+\int_0^t e^{-(a+b)u}\int_0^u e^{ax}dW_x\int_0^u e^{bx}dW_x du}{t}\right.$$

$$\left.-\lim_{t\to\infty}\frac{\int_0^t e^{-(a+\xi)u}\int_0^u e^{\xi x}dW_x\int_0^u e^{ax}dW_x du+\int_0^t e^{-(b+\xi)u}\int_0^u e^{\xi x}dW_x\int_0^u e^{bx}dW_x du}{t}\right).$$

The rest then follows by applying items (ii) and (iii) of Lemma 5.6.
(iv) One checks that the required limit is equal to

$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{bu}dW_u\int_0^s e^{(a+b)x}dW_x ds}{t}+$$
$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(a+b)s}\left(\int_0^s e^{(a+b)u}dW_u\right)^2 ds}{t},$$

and the rest follows by parts (ii) and (iii) of Lemma 5.6.
(v) As in (i), one checks that the limit is equal to

$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-(2a+b)s}\int_0^s e^{bu}dW_u\int_0^s e^{(a+b)x}dB_x ds}{t}-$$
$$\frac{1}{a}\lim_{t\to\infty}\frac{\int_0^t e^{-2(a+b)s}\int_0^s e^{(a+b)u}dW_u\int_0^s e^{(a+b)x}dB_x ds}{t},$$

which vanishes due to part (i) of Lemma 5.6. □

6. Proof of The Main Result

We provide now a proof for Theorem 4.1. Fix an arbitrary $i \neq I_K$. Recall that $\sum_{j=1}^N c_{jt} = D_t$, and thus it suffices to show that $\lim_{t\to\infty}\frac{c_{it}}{D_t}=0$. Note that (3.3) implies that $M_t \geq c_{I_K 0}^{\gamma_{I_K}}\cdot M_{I_K t}$. Therefore, identity (3.2) yields

$$\frac{c_{it}}{D_t} = c_{0i}\cdot\left(\frac{M_{it}}{M_t}\right)^{1/\gamma_i} \leq \frac{c_{i0}}{c_{I_K 0}^{\gamma_{I_K}/\gamma_i}}\cdot\left(\frac{M_{it}}{M_{I_K t}}\right)^{1/\gamma_i}.$$

In virtue of identity (3.1), we have

$$\frac{M_{it}}{M_{I_K t}} = \exp\left(a_i(t)-a_{I_K}(t)\right),$$

where

$$a_j(t) := (\gamma_j-1)\beta_j x_t + \left(\frac{(\sigma^D)^2}{2}\gamma_j-\rho_j\right)t+$$
$$\int_0^t\left(-\gamma_j\mu_s^D-\frac{\delta_{js}^2}{2}\right)ds+\int_0^t \delta_{js}dW_s^{(0)}-\gamma_j\sigma^D W_t^{(1)},$$

for all $j=1,...,N$. Therefore, in order to complete the proof of the statement, it suffices to show that

$$\lim_{t\to\infty}\frac{a_i(t)-a_{I_K}(t)}{t} = \kappa_{I_K}-\kappa_i < 0.$$

To this end, we proceed with the computation of the following limits.

Part I. We claim that

$$\lim_{t\to\infty}\frac{x_t}{t}=\overline{\mu}-\frac{1}{2}(\sigma^D)^2. \tag{6.1}$$

Recall that by (2.5) and (2.2), we have

$$\lim_{t\to\infty}\frac{x_t}{t}=\lim_{t\to\infty}\frac{x_0+\lambda\cdot\int_0^t e^{\lambda s}\left(\int_0^s\mu_u^D du-\frac{1}{2}(\sigma^D)^2 s+\sigma^D W_s^{(1)}\right)ds}{te^{\lambda t}}.$$

Now, note that the law of large numbers implies that $\lim_{t\to\infty}\frac{\int_0^t e^{\lambda s}W_s^{(1)}ds}{te^{\lambda t}}=0$. Next, it is evident that $\lim_{t\to\infty}\frac{x_0}{te^{\lambda t}}=0$ and $\lim_{t\to\infty}\frac{\int_0^t se^{\lambda s}ds}{te^{\lambda t}}=1/\lambda$. Let us show now that

$$\lim_{t\to\infty}\frac{\int_0^t\mu_u^D du}{t}=\overline{\mu}. \tag{6.2}$$

By (2.4), we get

$$\lim_{t\to\infty}\frac{\int_0^t\mu_u^D du}{t}=\lim_{t\to\infty}\frac{\int_0^t\left(\overline{\mu}+(\mu_0-\overline{\mu})\,e^{-\xi s}+\sigma^\mu\int_0^s e^{\xi(u-s)}dW_u^{(2)}\right)ds}{t}.$$

Clearly, we have $\lim_{t\to\infty}\frac{\int_0^t\left(\overline{\mu}+(\mu_0-\overline{\mu})e^{-\xi s}\right)ds}{t}=\overline{\mu}$. Furthermore, part (i) of Lemma 5.5 yields $\lim_{t\to\infty}\frac{\int_0^t\int_0^s e^{\xi(u-s)}dW_u^{(2)}ds}{t}=0$. This asserts the validity of (6.2). Next, by L'hôpital's rule, we get

$$\lim_{t\to\infty}\frac{\int_0^t e^{\lambda s}\int_0^s\mu_u^D duds}{te^{\lambda t}}=\lim_{t\to\infty}\frac{\int_0^t\mu_s^D ds}{\lambda t+1}=\frac{\overline{\mu}}{\lambda},$$

proving (6.1).

Part II. We claim that

$$\lim_{t\to\infty}\frac{\int_0^t\left(\delta_{I_K s}-\delta_{is}\right)dW_s^{(1)}}{t}=0.$$

By definition (see (2.14)), it suffices to verify that

$$\lim_{t\to\infty}\frac{\int_0^t\mu_{js}^D dW_s^{(1)}}{t}=0,$$

holds for all $j=1,...,N$. It is not hard to check by employing Lemma 5.3 combined with the law of large numbers, that the preceding limit does not change when the functions y_{iu}, $\frac{1}{y_{iu}}$ and ν_{iu} are substituted by $e^{(\alpha_{i2}+\xi)t}$, $e^{-(\alpha_{i2}+\xi)t}$ and $\alpha_{i2}(\sigma^D)^2$ respectively. In view of latter observation, by definition (see (2.10)), we need to show that

$$\lim_{t\to\infty}\frac{\int_0^t\left((\overline{\mu}_i-\overline{\mu})\left(1-e^{-\xi s}\right)+(\mu_{0i}-\mu_0)\,e^{-\xi s}+\frac{\xi\overline{\mu}_i}{\xi+\alpha_{i2}}\left(1-e^{-(\xi+\alpha_{i2})s}\right)\right)dW_s^{(1)}}{t}$$

$$+\alpha_{i2}\lim_{t\to\infty}\frac{\int_0^t e^{-(\xi+\alpha_{i2})s}\int_0^s e^{(\xi+\alpha_{i2})u}\left(\overline{\mu}+(\mu_0-\overline{\mu})\,e^{-\xi u}\right)dudW_s^{(1)}}{t}$$

$$+\alpha_{i2}\sigma^{\mu}\lim_{t\to\infty}\frac{\int_0^t e^{-(\xi+\alpha_{i2})s}\int_0^s e^{\alpha_{i2}u}\int_0^u e^{\xi x}dW_x^{(2)}dudW_s^{(1)}}{t}$$
$$+\sigma^{\mu}\phi_i\lim_{t\to\infty}\frac{\int_0^t e^{-(\xi+\alpha_{i2})s}\int_0^s e^{(\xi+\alpha_{i2})u}ds_u dW_s^{(1)}}{t}=0.$$

One checks that the first two summands vanish by the law of large numbers. The third and fourth limits vanish by part (iii) and (ii) of Lemma 5.4, respectively. This completes the proof of the second part.

Part III. We have,

$$\frac{1}{2}\lim_{t\to\infty}\frac{\int_0^t\left(\delta_{is}^2-\delta_{I_Ks}^2\right)ds}{t}=\frac{1}{2}\left(\frac{\overline{\mu}_i-\overline{\mu}}{\sigma^D}\right)^2+\frac{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2(1-\phi\phi_i)}{2\sqrt{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2(1-\phi_i^2)}}$$
$$-\frac{1}{2}\left(\frac{\overline{\mu}_{I_K}-\overline{\mu}}{\sigma^D}\right)^2-\frac{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2(1-\phi\phi_{I_K})}{2\sqrt{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2\left(1-\phi_{I_K}^2\right)}}.$$

This can be derived by applying Lemmata 5.4, 5.5, 5.6 and 5.7. The proof is now accomplished by combining the above three parts, some routine algebraic transformations and the law of large numbers. □

7. Interest Rate and Market Price of Risk: Further long run Results

The current section deals with asymptotic results for the interest rate and the market price of risk in heterogeneous economies. More precisely, it is shown that asymptotically, the latter parameters behave as those associated with a homogeneous economy populated by the dominating consumer. Under some mild conditions, we prove that the distance between these parameters in a heterogeneous economy and those associated with *any* of the non-dominating consumer homogeneous economies, becomes unbounded as time goes to infinity.

7.1. **Market Price of Risk.**

The next statement provides a full characterization of the market price of risk asymptotics in heterogeneous economies.

Theorem 7.2. *(i) We have*

$$\lim_{t\to\infty}|\theta_t-\theta_{I_Kt}|=0.$$

(ii) If $\phi_i=\phi_{I_K}$, for some $i\neq I_K$, then

$$\lim_{t\to\infty}(\theta_t-\theta_{it})=\sigma^D\left(\gamma_{I_K}-\gamma_i\right)-\frac{1}{\sigma^D}\left(\overline{\mu}_{I_K}-\overline{\mu}_i\right).$$

If ϕ_i (for some $i\neq I_K$) is such that

$$\frac{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2(1-\phi\phi_i)}{2\sqrt{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2(1-\phi_i^2)}}\neq\frac{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2(1-\phi\phi_{I_K})}{2\sqrt{\xi^2+\left(\sigma^{\mu}/\sigma^D\right)^2\left(1-\phi_{I_K}^2\right)}},$$

then

$$\limsup_{t\to\infty} |\theta_t - \theta_{it}| = +\infty.$$

Proof of Theorem 7.2. (i) First, we shall prove that $\lim_{t\to\infty} \omega_{it}\theta_{it} = 0$, for all $i \neq I_K$. As in Section 5, the DDS Theorem implies the existence of a Brownian motion $\left(\widetilde{B}(t)\right)_{t\in[0,\infty)}$ such that

$$e^{-at}\int_0^t e^{as} dB_s = e^{-at}\widetilde{B}\left(\frac{e^{2at}-1}{2a}\right),$$

where $a > 0$ is some constant and $(B_t)_{t\in[0,\infty)}$ is a Brownian motion. By exploiting the preceding fact, one checks that $\lim_{t\to\infty} \frac{\mu^D_{it}}{\sqrt{\log t}} < \infty$, for all $i = 0, ..., N$, which implies that

$$\limsup_{t\to\infty} \frac{\theta_{jt}}{\sqrt{\log t}} < \infty, \tag{7.1}$$

for all $j = 1, ..., N$. On the other hand, it was shown in Theorem 4.1 that $\omega_{it} \leq \frac{c_{it}}{D_t} \max_{i=1,...,N} \gamma_i$, for all $i \neq I_K$. We have in particular proved in Section 6 that $\frac{c_{it}}{D_t} \leq e^{-a_i t}$, for some $a_i > 0$, for all $i \neq I_K$. This implies that

$$\omega_{it} \leq e^{-a_i t} \max_{i=1,\ldots,N} \gamma_i, \tag{7.2}$$

holds for all $i \neq I_K$, and thus by (7.1) we have $\omega_{it}\theta_{it} \leq e^{-a'_i t}$, for all $i \neq I_K$, and some constant $a'_i > 0$. Therefore, by Proposition 3.3, we have

$$|\theta_t - \omega_{I_K t}\theta_{I_K t}| = \sum_{i=1, i\neq I_K}^{N} \omega_{it}\theta_{it} \leq \sum_{i=1, i\neq I_K}^{N} e^{-a'_i t}. \tag{7.3}$$

Finally, observe that $|\theta_t - \theta_{I_K t}| \leq |\theta_t - \omega_{I_K t}\theta_{I_K t}| + \sum_{i=1, i\neq I_K}^{N} \omega_{it}\theta_{I_K t}$, since $\sum_{i=1}^{N} \omega_{it} = 1$. The proof of part (i) is now concluded by (7.1), (7.2) and (7.3).
(ii) If $\phi_i = \phi_{I_K}$, by part (i) we can substitute θ_T by $\theta_{I_K T}$. The assertion follows then by noting that

$$\lim_{t\to\infty} |\theta_{I_K t} - \theta_{it}| = \lim_{t\to\infty} \left| \sigma^D (\gamma_i - \gamma_{I_K}) + \frac{1}{\sigma^D} (\mu_{I_K t} - \mu_{it}) \right|.$$

Assume now that

$$\frac{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2 (1 - \phi\phi_i)}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2 (1 - \phi_i^2)}} \neq \frac{\xi^2 + \left(\sigma^\mu/\sigma^D\right)^2 (1 - \phi\phi_{I_K})}{2\sqrt{\xi^2 + (\sigma^\mu/\sigma^D)^2 \left(1 - \phi_{I_K}^2\right)}}.$$

By part (i), the claim is equivalent to proving that

$$\limsup_{t\to\infty} |\mu^D_{I_K t} - \mu^D_{it}| = +\infty. \tag{7.4}$$

First, one checks by employing Lemma 5.3 that the limit (7.4) does not change when substituting ν_{it}, y_{it} and $\frac{1}{y_{it}}$ by $\alpha_{i2}(\sigma^D)^2$, $\exp\left(-\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{(\alpha_{i2}+\xi)t}$ and $\exp\left(\frac{\alpha_{i2}}{\alpha_{i1}}e^{-\frac{\alpha_{i2}}{\alpha_{i1}}}\right)e^{-(\alpha_{i2}+\xi)t}$ respectively. Next, note that Fubini's Theorem yields

$$\frac{\alpha_{i2}}{e^{(\xi+\alpha_{i2})T}}\int_0^T e^{\alpha_{i2}u}\int_0^u e^{\xi x}dW_x^{(2)}du =$$
$$\frac{1}{e^{\xi T}}\int_0^T e^{\xi u}dW_u^{(2)} - \frac{1}{e^{(\alpha_{i2}+\xi)T}}\int_0^T e^{(\alpha_{i2}+\xi)u}dW_u^{(2)}.$$

By exploiting the latter observations and the DDS Theorem, one checks that

$$\limsup_{t\to\infty}\left|\mu_{it}^D-\mu_{I_kt}^D\right| = \limsup_{t\to\infty}\left|f_i(t)-f_{I_k}(t)\right|,$$

where

$$f_i(t)=\frac{1}{\sqrt{(\alpha_{i2}+\xi)\,t}}\left(\sigma^D\alpha_{i2}B^{i1}(t)-\sigma^\mu\left(\phi\phi_i-1\right)B^{i2}(t)+\sigma^\mu\phi_i\sqrt{1-\phi^2}B^{i3}(t)\right).$$

Here, $B^{i1}(t), B^{i2}(t)$ and $B^{i3}(t)$ denote three independent Brownian motions. By applying further the DDS Theorem, we can rewrite

$$f_i(t)=\frac{1}{\sqrt{(\alpha_{i2}+\xi)\,t}}B^{(i)}\left(l_it\right), \tag{7.5}$$

where $B^{(i)}(t)$ is a Brownian motion, and

$$l_i=\left(\sigma^D\alpha_{i2}\right)^2+\left(\sigma^\mu\right)^2\left(1-\phi\phi_i\right)^2+\left(\sigma^\mu\phi_i\right)^2\left(1-\phi^2\right).$$

Finally, one checks that $\limsup_{t\to\infty}|f_i(t)-f_{I_k}(t)|=\infty$ by using the law of iterated logarithm and (7.5) combined with the fact that $\frac{l_i}{\alpha_{i2}+\xi}=-2\xi\sigma^D+2(\sigma^D)^2\frac{\xi^2+\left(\sigma^\mu/\sigma^D\right)^2(1-\phi\phi_i)}{2\sqrt{\xi^2+(\sigma^\mu/\sigma^D)^2\left(1-\phi_i^2\right)}}$. This completes the proof of Theorem 7.2. □

7.2. **Interest Rate.** Analogously to Theorem 7.2, we analyze in the next statement the asymptotics of the interest rate in heterogeneous economies.

Theorem 7.3. *(i) We have*

$$\lim_{t\to\infty}\left|r_t-r_{I_Kt}\right|=0.$$

(ii) If $\gamma_i=\gamma_{I_K}$, $\beta_i=\beta_{I_K}$, and $\phi_i=\phi_{I_K}$, for some $i\neq I_K$, then

$$\lim_{t\to\infty}\left(r_t-r_{it}\right)=\rho_{I_K}-\rho_i+\gamma_{I_K}\left(\overline{\mu}_{I_K}-\overline{\mu}_i\right).$$

If at least one of the conditions: $\gamma_i=\gamma_{I_K}$, $\beta_i=\beta_{I_K}$ and

$$\frac{\xi^2+\left(\sigma^\mu/\sigma^D\right)^2\left(1-\phi\phi_i\right)}{2\sqrt{\xi^2+\left(\sigma^\mu/\sigma^D\right)^2\left(1-\phi_i^2\right)}}=\frac{\xi^2+\left(\sigma^\mu/\sigma^D\right)^2\left(1-\phi\phi_{I_K}\right)}{2\sqrt{\xi^2+\left(\sigma^\mu/\sigma^D\right)^2\left(1-\phi_{I_K}^2\right)}}$$

does not hold, for some $i\neq I_K$, then

$$\limsup_{t\to\infty}\left|r_t-r_{it}\right|=+\infty.$$

Proof of Theorem 7.3. (i) By definition, we have

$$r_t - \omega_{it} r_{it} = \sum_{j=1, j\neq I_K}^{N} \omega_{jt} r_{jt} + \frac{1}{2}\sum_{j=1}^{N}(1-1/\gamma_j)\omega_{jt}\left(\theta_{jt}-\theta_t\right)^2,$$

for all $i = 1, ..., N$. We start with treating the second term. Observe that Theorem 4.1, part (i) of Theorem 7.2, (7.4) and (7.2), imply that

$$\sum_{j=1}^{N}(1-1/\gamma_j)\omega_{jt}\left(\theta_{jt}-\theta_t\right)^2 \leq e^{-a't},$$

for some constant $a' > 0$. Next, note that (7.2) yields

$$\sum_{j=1, j\neq I_K}^{N} |\omega_{jt} r_{jt}| \leq \sum_{j=1, j\neq I_K}^{N} e^{-a_j t}\left|r_{jt}\right|.$$

As in the proof of Theorem 4.1, one can check that $\limsup_{t\to\infty}\frac{r_{jt}}{t} < +\infty$, for all $j = 1, ..., N$, and thus we conclude that

$$|r_t - \omega_{I_K t} r_{I_K t}| \leq e^{-a't},$$

for some constant $a' > 0$. Finally, the proof of item (i) is accomplished by employing the inequality $|r_t - r_{I_K t}| \leq |r_t - \omega_{I_K t} r_{I_K t}| + r_{I_K t}|1-\omega_{I_K t}|$ combined with the fact that $1 = \sum_{j=1}^{N}\omega_{jt}$, (7.2) and the fact that $\limsup_{t\to\infty}\frac{r_{jt}}{t} < +\infty$, for all $j = 1, ..., N$.
(ii) If $\phi_i = \phi_{I_K}$, $\gamma_i = \gamma_{I_K}$ and $\beta_i = \beta_{I_K}$, for some $i \neq I_K$, the claim follows by combining part (i) with the fact that

$$|r_{it} - r_{I_K t}| = \left|\rho_{I_K} - \rho_i + \gamma_{I_K}\left(\mu^D_{I_K t} - \mu^D_{it}\right)\right|.$$

Now, if at least one of the indicated conditions, fails for some $i \neq I_K$, the proof is in the same spirit as the one of item (ii) of Theorem 7.2. The only distinction is as follows. If $\lambda = \xi$, one can check that the problem can be reduced to proving that

$$\limsup_{t\to\infty} e^{-\lambda t}\left(\sigma^D\int_0^t e^{\lambda u} dW_u^{(1)} + \int_0^t\int_0^s e^{\lambda u} dW_u^{(2)} ds\right) = +\infty. \tag{7.6}$$

If $\lambda = 0$, we need to prove that

$$\limsup_{t\to\infty}\left(\sigma^D W_t^{(1)} + \int_0^t W_s^{(2)} ds\right) = +\infty.$$

Let $G : C_0([0,1];\mathbb{R}) \to \mathbb{R}$ be a functional given by $G(f) = \int_0^1 f(x)dx$. Note that G is continuous, since $|G(f) - G(g)| \leq ||f-g||_\infty$ holds for all $f, g \in C_0([0,1];\mathbb{R})$. By Strassen's functional law of iterated logarithm, we have

$$\limsup_{N\to\infty} G\left(\frac{1}{\sqrt{2N\log\log N}} W^{(2)}_{Nx}\right) = \limsup_{N\to\infty}\frac{\int_0^N W_u^{(2)} du}{N^{3/2}\sqrt{2\log\log N}} = \max_{f\in K^{(1)}} G(f),$$

where the subspace $K^{(1)}$ is given in Definition 5.4. Note that $\max_{f\in K^{(1)}} G(f) \geq G(f_0) > 0$, where $f_0(x) = x$. The preceding observation combined with the fact

$\lim_{t\to\infty}\frac{W_t^{(1)}}{t^{3/2}\sqrt{\log\log t}}=0$ asserts that (7.6) holds for $\lambda=0$. Now, assume that $\lambda\neq 0$. By the DDS Theorem, (7.6) is equivalent to

$$\limsup_{t\to\infty} e^{-\lambda t}\left(\sigma^D B^{(1)}\left(\frac{e^{2\lambda t}-1}{2\lambda}\right)+\int_0^t B^{(2)}\left(\frac{e^{2\lambda s}-1}{2\lambda}\right)ds\right)=+\infty,$$

where $B^{(1)}$ and $B^{(2)}$ denote two standard independent Brownian motions. By a change of variables, the claim is equivalent to

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\left(\sigma^D B^{(1)}(t)+\int_0^t \frac{B^{(2)}(u)}{1+2\lambda u}du\right)=+\infty. \tag{7.7}$$

The LIL yields $\lim_{t\to\infty}\frac{\int_1^t \frac{B^{(1)}(u)}{u(1+2\lambda u)}du}{\sqrt{t}}=0$, and thus (7.7) can be rewritten as

$$\limsup_{t\to\infty}\frac{1}{\sqrt{t}}\left(\sigma^D B^{(1)}(t)+\frac{1}{2\lambda}\int_1^t \frac{B^{(2)}(u)}{u}du\right)=+\infty. \tag{7.8}$$

Fix some $0<\varepsilon<1$. Consider the functional $H: C_0\left([0,1];\mathbb{R}^2\right)\to\mathbb{R}$, which is given by

$$H(f,g):=\sigma^D f(1)+\frac{1}{2\lambda}\int_\varepsilon^1\frac{g(u)}{u}du.$$

Note that H is continuous, since $\left|H(f,g)-H(\widehat{f},\widehat{g})\right|\leq\sigma^D\|f-\widehat{f}\|_\infty-\frac{\log\varepsilon}{2\lambda}\|g-\widehat{g}\|_\infty$ is satisfied for all $f,g,\widehat{f},\widehat{g}\in C_0\left([0,1];\mathbb{R}^2\right)$. Next, Strassen's functional law of iterated logarithm yields

$$\limsup_{N\to\infty} H\left(\frac{1}{\sqrt{2N\log\log N}}\cdot\left(B^{(1)}(Nt),B^{(2)}(Nt)\right)\right)=\max_{(f,g)\in K^{(2)}}H(f,g),$$

where $K^{(2)}$ is introduced in Definition 5.4. Observe that $\max_{(f,g)\in K^{(2)}}H(f,g)\geq H(h(x),h(x))>0$, where $h(x)=x$. Therefore, we obtain that

$$\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_{\varepsilon N}^N\frac{B^{(2)}(u)}{u}du\right)>0. \tag{7.9}$$

We claim now that

$$\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N\frac{B^{(2)}(u)}{u}du\right)>0.$$

Assume in a contrary that this is not the case. Then, Kolmogorov's 0-1 law implies that

$$P\left(\limsup_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N\frac{B^{(2)}(u)}{u}du\right)>0\right)=0.$$

Therefore, by exploiting the symmetry of the Brownian motion, we obtain that

$$\lim_{N\to\infty}\frac{1}{\sqrt{2N\log\log N}}\left(\sigma^{(D)}B^{(1)}(N)+\frac{1}{2\lambda}\int_1^N\frac{B^{(2)}(u)}{u}du\right)=0,$$

holds $P-$a.s. But, since σ^D and λ were arbitrary, we obtain that

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left(\sigma^{(D)} B^{(1)}(N) + \frac{1}{2\lambda}\int_{\varepsilon N}^{N} \frac{B^{(2)}(u)}{u} du \right) =$$

$$\limsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \left(\left(\sigma^{(D)} - \sqrt{\varepsilon}\right) B^{(1)}(N) + \frac{1}{2\lambda}\int_{1}^{N} \frac{B^{(2)}(u)}{u} du \right.$$
$$\left. + \widetilde{B}^{(1)}(\varepsilon N) - \frac{1}{2\lambda}\int_{1}^{\varepsilon N} \frac{B^{(2)}(u)}{u} du \right) = 0,$$

where $\widetilde{B}^{(1)}(t) = \sqrt{\varepsilon} B^{(1)}\left(\frac{t}{\varepsilon}\right)$ is a Brownian Motion (independent of $B^{(2)}$), and $\varepsilon > 0$ is sufficiently small. This is a contradiction to (7.9) proving (7.7). □

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