A No-Go Theorem for the Consistent Quantization of the Massive Gravitino on Robertson-Walker Spacetimes and Arbitrary Spin 3/2 Fields on General Curved Spacetimes

Thomas-Paul Hack[1,a] and **Mathias Makedonski**[1,b]

[1] II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany.

[a] thomas-paul.hack@desy.de, [b] mathias.makedonski@desy.de

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Abstract. We first introduce a set of conditions which assure that a free spin $\frac{3}{2}$ field with $m \geq 0$ can be consistently ('unitarily') quantized on all four-dimensional curved spacetimes, i.e. also on spacetimes which are not assumed to be solutions of the Einstein equations. We discuss a large – and, as we argue, exhaustive – class of spin $\frac{3}{2}$ field equations obtained from the Rarita-Schwinger equation by the addition of non-minimal couplings and prove that no equation in this class fulfils all sufficient conditions.

Afterwards, we investigate the situation in supergravity, where the curved background is usually assumed to satisfy the Einstein equations and, hence, detailed knowledge on the spacetime curvature is available. We provide a necessary condition for the unitary quantization of a spin $\frac{3}{2}$ Majorana field and prove that this condition is not met by supergravity models in four-dimensional Robertson-Walker spacetimes if local supersymmetry is broken. Our proof is model-independent as we merely assume that the gravitino has the standard kinetic term.

1 Introduction

Since the first consistent formulation of classical supergravity [DeZu76, FNF76], supergravity theories have been studied with increasing interest, both from the conceptual and from the phenomenological point of view, see e.g. [VNi81, Nill83]. On the conceptual side, most works have been concerned with the quantization of supergravity on backgrounds constituted by locally supersymmetric solutions of the field equations, i.e. Minkowski spacetime and (anti-) de Sitter spacetime. Indeed, it is known that a unitary quantization of supergravity is possible on such spacetimes, see e.g. [VNi81, Mal97], but, to our knowledge, this statement has never been proven for general spacetimes.

However, the universe we live in is not a spacetime of constant curvature like Minkowski or (anti-)de Sitter spacetime, but rather a more general and less symmetric one. Hence, if supergravity theories are supposed to describe (part of) nature, one has to clarify whether these theories can exist as quantum theories on spacetimes other than those with constant curvature. Even setting aside this fundamental point of view and taking a more pragmatic approach, one has to acknowledge that the gravitino, part of any supergravity theory, is a good candidate for dark matter [PaPr81] and, hence, at least the status of quantum supergravity on cosmological Robertson-Walker spacetimes has to be clarified.

Admittedly, the prevailing attitude in particle physics is that gravity is a weak force, such that spacetime curvature can be neglected or treated perturbatively in particle physics computations and analyses – the expectation is that the results of analyses in standard quantum field theory on Minkowski spacetime can be understood as 'Minkowski limits' of more general quantum field theory in curved spacetime treatments, and that one is very close to that Minkowski limit in most cases. Hence, one usually does not worry much about deriving the existence and/or consistency of this limit from first principles.

This is true enough for the quantum field theories describing the particles we have observed yet, i.e. particles of spin ≤ 1, but may not be true for particles of spin > 1. Indeed, there are examples of physical theories with small parameters which are not perturbative in these parameters. Consider e.g. the quantum field theory of a single massive vector field with mass m. This theory describes three degrees of freedom for $m > 0$, however small m may be, but only two degrees of freedom if $m = 0$. This qualitative change in the $m > 0$ theory can not be captured by a perturbative expansion around the $m = 0$ theory. [1]

In this work, we prove that a similar phenomenon occurs in supergravity – as soon as the smallest bit of (non-constant) spacetime curvature is present, which usually means that local supersymmetry is broken, supergravity can not exist as a consistent, i.e. unitary, quantum theory. In more detail, we proove that the free gravitino can not exist as a unitary quantum field in this case, which means that supergravity can not be quantized *perturbatively* – but it may still be *non-perturbatively* quantizable.

Usually, unitarity of a quantum field theory is not a big issue in the context of *non-interacting* theories, so what can go wrong? As we shall explain in more detail in the main body of this work, in a potential quantum theory of a free spin $\frac{3}{2}$ field, the anticommutator of the quantum field $\boldsymbol{\psi}^\alpha$ and its adjoint $\overline{\boldsymbol{\psi}}_\alpha$ corresponds to an expression of the form $A^\dagger A + AA^\dagger$, such that the canonical anticommutation relations can be written as

$$A^\dagger A + AA^\dagger = c\underline{1}$$

with c a c-number. If we take the expectation value of this equation in an arbitrary state $|\Omega\rangle$ with $\langle\Omega|\Omega\rangle = 1$, we find

$$c = \langle\Omega|(A^\dagger A + AA^\dagger)|\Omega\rangle = \langle A\Omega|A\Omega\rangle + \langle A^\dagger\Omega|A^\dagger\Omega\rangle \geq 0\,.$$

Hence, $c \geq 0$ is required by consistency. As we shall explain, c is given in terms of the *anticommutator function* of the spin $\frac{3}{2}$ field, which in turn is just the difference of the advanced and retarded Green's functions of the free spin $\frac{3}{2}$ field equation. Thus, a consistent quantization is possible if this anticommutator function is such that $c \geq 0$, whereas, if the anticommutator function does not bear this property, a consistent interpretation of $\boldsymbol{\psi}^\alpha$ and $\overline{\boldsymbol{\psi}}_\alpha$ being operators on some Hilbert space is impossible, at least if one wants to impose the usual anticommutation relations required in canonical

[1] A rather mathematical analogon is the function $f(x) = 0^x := \lim_{y\to 0} e^{x\log y}$, which is identically zero for $x > 0$, whereas $f(0) = 1$.

quantization. Indeed, one of the two main results of this work is that $c \geq 0$ does not hold for the gravitino field in flat four-dimensional Robertson-Walker spacetimes of non-constant curvature. Our result depends entirely on the kinetic term in the field equation for the free gravitino and is thus independent of the details of the supergravity model, i.e. the other fields and the interactions, and, if local supersymmetry is broken, on the detailed breaking mechanism.

Before we reach this result, we take a broader point of view and consider the question whether *arbitrary* free spin $\frac{3}{2}$ fields, i.e. arbitrary linear equations for a spin $\frac{3}{2}$ field, can lead to a consistent quantum field theory on *all* curved spacetimes, that is, also on spacetimes which are not required to be satisfy the Einstein equations. This approach is usually taken in quantum field theory on curved spacetimes (see, e.g. [BiDa82, Wa95], and [Hac10] for a recent review) and is, although more general at first glance, different from the supergravity setting. Namely, in the latter case, one usually assumes that the background spacetime satisfies the Einstein equations for consistency [DeZu76, Tow77], whereas, in quantum field theory on curved spacetimes, one generally first checks that everything works without a detailed knowledge of the background spacetime and then considers the situation of special backgrounds in a second step.

Following this route, we introduce four conditions on possible spin $\frac{3}{2}$ field equations in four-dimensional curved spacetimes and prove that they are sufficient to ensure that a consistent quantization of the associated field theory is possible. In brief, these conditions are

- *(Irreducibility)* The field equation describes an *elementary* spin $\frac{3}{2}$ particle, i.e. propagates the correct number of degrees of freedom.
- *(Covariance)* The number of degrees of freedom is independent of the background spacetime curvature.
- *(Causality)* The solutions of the field equation propagate causally.
- *(Selfadjointness)* The adjoint field equation can be obtained by partial integration.

Our proof of the sufficiency of these conditions holds only for topologically trivial curved spacetimes, but we argue that an extension of the proof to spacetimes of non-trivial topology is possible.

Given these conditions, we consider a large class of linear spin $\frac{3}{2}$ field equations obtained from the minimally coupled Rarita-Schwinger equation by the addition of non-minimal couplings, and prove that no field equation in this class satisfies all four sufficient conditions. While proving this it becomes apparent that an enlargement of this class of field operators can hardly improve the result, such that, on practical grounds, one could claim that no modified Rarita-Schwinger operator can satisfy all these conditions.

As we consider modified Rarita-Schwinger operators, our no-go theorem only covers field equations in the *Rarita-Schwinger representation* $(1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)$ of $SL(2, \mathbb{C})$, where the spinor is written with one Dirac index and one Lorentz index, but does not encompass field equations in the other possible spin $\frac{3}{2}$ representation, the *Buchdahl representation* $(\frac{3}{2}, 0) \oplus (1, \frac{1}{2})$, where the equations are written in terms of two-spinors [Buc58, Buc82]. These two representations are equivalent in Minkowski spacetime and for free fields, but fail to be so in curved spacetimes, see e.g. [IlSch99]. Indeed, Buchdahl has written down a set of equations in the Buchdahl representation [Buc82, Wün85] which have the advantage that they solve the consistency problem for higher spin field equations [Buc58] simultaneously for all spins. These equations have been analysed in great detail, see [IlSch99] for a review, and the possibility to obtain a consistent quantum theory for these equations has been explored [Müh07, Mak11]. However, the results to date have not been promising and, although not proving a no-go theorem for the Buchdahl equations and modifications thereof, we shall argue why a consistent spin $\frac{3}{2}$ quantum field theory on the basis of the Buchdahl equations seems unlikely to exist.

After addressing the general issue of consistent spin $\frac{3}{2}$ quantum field theories in curved spacetimes, we focus on supergravity theories. As the four sufficient consistency conditions are only sensible if one wants to quantize the spin $\frac{3}{2}$ field *without* any knowledge of the background spacetime curvature, our general no-go theorem does not cover supergravity theories. Moreover, the conditions used in this theorem are only proven to be sufficient, but they may not be necessary. To improve on these drawbacks, we introduce a new condition, which is a weaker version of *(Selfadjointness)* – we require that the canonical *classical* current associated to the field equation is covariantly conserved. Afterwards, we prove that this condition is a necessary condition for a free Majorana spin $\frac{3}{2}$ field theory to be consistently quantizable, no matter how much information of the spacetime curvature enters the quantization. Using this result, we reach the second main conclusion of this work, and find that the gravitino can not be unitarily quantized on flat four-dimensional Robertson-Walker backgrounds which are not Minkowski or (anti-)de Sitter spacetime.

Our paper is organised as follows. In section 2 we recall the quantization of the free Rarita-Schwinger field in Minkowski spacetime, and discuss why it is consistent, i.e. unitary. Afterwards, we discuss the issue of consistency of spin $\frac{3}{2}$ quantum theories in curved spacetimes in section 3, and introduce four conditions to assure this consistency. In section 4, we discuss various spin $\frac{3}{2}$ field equations present in the literature and argue why the fail to satisfy the consistency conditions. We then consider a class of modified Rarita-Schwinger equations for spin $\frac{3}{2}$ fields in section 5 and prove a no-go theorem for their consistent quantization. Finally, in section 6 we address the issue of supergravity theories and prove a no-go theorem for their consistent quantization on Robertson-Walker spacetimes. The appendices A to F contain background material and the proof of a few fiducial technical results.

The reader only interested in the results on supergravity theories may readily skip the sections 4 and 5, whereas the sections 2 and 3 may be helpful to understand the general obstructions in constructing a consistent spin $\frac{3}{2}$ quantum field theory.

In the following, we shall denote Lorentz/spacetime indices by small Greek letters, whereas Dirac spinor indices will be suppressed throughout. All expressions are valid in an arbitrary basis of the considered vector bundles unless otherwise noted. We work with spacetime signature $(+,-,-,-)$ and our conventions and notations regarding Dirac spinors and curvature tensors are collected in the appendices A and B.

2 The free Rarita-Schwinger field on flat spacetime and conditions for a consistent quantization

2.1 The classical free Rarita-Schwinger field on flat spacetime

We briefly review the classical theory of the free Rarita-Schwinger field ψ^α in flat spacetime, and already present it in a form suitable for our analysis in this work. The Rarita-Schwinger field is a function on Minkowski spacetime which carries one spacetime index and one Dirac index, i.e. mathematically speaking, $\psi^\alpha \in \Gamma(R\mathbb{M})$ is a smooth (i.e. infinitely often differentiable) section of the Rarita-Schwinger bundle $R\mathbb{M} := D\mathbb{M} \otimes T\mathbb{M}$ over $\mathbb{M}$, where $T\mathbb{M}$ and $D\mathbb{M}$ denote the tangent and Dirac bundles over $\mathbb{M}$ respectively, and $\Gamma(B)$ ($\Gamma_0(B)$) denotes the smooth sections (smooth sections with compact support, i.e. 'test sections') of a bundle B. To endow this field with dynamics, one imposes the Rarita-Schwinger equation [RaSch41]

$$\mathcal{R}_0\psi^\alpha := (-i\partial\!\!\!/ + m)\psi^\alpha = 0 \tag{1}$$

and the constraint

$$\psi\!\!\!/ := \gamma_\alpha\psi^\alpha = 0\,. \tag{2}$$

The latter constraint corresponds to removing the spin $\frac{1}{2}$ piece $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ from the reducible representation

$$\left(\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right)\otimes\left(\frac{1}{2},\frac{1}{2}\right)=\left(1,\frac{1}{2}\right)\oplus\left(\frac{1}{2},1\right)\oplus\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)$$

of $SL(2,\mathbb{C})$ corresponding to $D\mathbb{M}\otimes T\mathbb{M}$. Upon contracting the Rarita-Schwinger equation with γ^α and applying the constraint,

$$\partial_\alpha\psi^\alpha = 0 \tag{3}$$

follows, and, applying the operator $i\partial\!\!\!/ + m\underline{1}$ to the Rarita-Schwinger equation yields the Klein-Gordon equation

$$(\Box + m^2)\psi^\alpha = 0.$$

If $m = 0$, there is a gauge freedom present. Indeed,

$$\psi^\alpha \qquad \text{and} \qquad \psi^\alpha + \partial^\alpha\chi\,,\ \chi\in\Gamma(D\mathbb{M})\ \&\ \partial\!\!\!/\chi = 0$$

are gauge-equivalent [RaSch41]; we shall see in the discussion of the quantization why "gauge-solutions" $\psi^\alpha = \partial^\alpha\chi$ represents "unphysical" degrees of freedom. In analogy to the Dirac field case, one can show that the Dirac-conjugated Rarita-Schwinger field $\overline{\psi}^\alpha$ (cf. appendix A for the exact definition) solves the conjugated Rarita-Schwinger equation $(i\overleftarrow{\partial\!\!\!/}+m)\overline{\psi}^\alpha=0$ iff $\mathcal{R}_0\psi^\alpha=0$.

As $(i\partial\!\!\!/+m\underline{1})\mathcal{R}_0$ is the unit matrix times the Klein-Gordon operator, i.e. a hyperbolic differential operator, one can proove that unique advanced and retarded Green's operators/functions for the Rarita-Schwinger operator exist [BGP07, Müh10]. Denoting them by ${G^{\pm}}^\alpha_{\ \beta}$, their operator versions fulfil

$$\mathcal{R}_0 {G^{\pm}}^\alpha_{\ \beta} f^\beta = {G^{\pm}}^\alpha_{\ \beta}\mathcal{R}_0 f^\beta = f^\alpha \qquad \forall f^\alpha\in\Gamma_0(R\mathbb{M})$$

and have the usual causal support properties, i.e. ${G^{+}}^\alpha_{\ \beta} f^\beta$ (${G^{-}}^\alpha_{\ \beta} f^\beta$) is only non-vanishing in the forward (backward) lightcone of the support of f^β. By defining

$${G^{\pm}}_\alpha^{\ \beta}(\bar f_\beta, g^\alpha) := \langle f^\beta, {G^{\pm}}^\alpha_{\ \beta} g^\beta\rangle := \int_{\mathbb{M}} dx\, \bar f_\beta(x)\left[{G^{\pm}}_\alpha^{\ \beta} g^\alpha\right](x)$$

and interpreting ${G^{\pm}}_\alpha^{\ \beta}(\bar f_\beta, g^\alpha)$ as

$${G^{\pm}}_\alpha^{\ \beta}(\bar f_\beta, g^\alpha) = \int_{\mathbb{M}\times\mathbb{M}} dxdy\, \bar f_\beta(x){G^{\pm}}_\alpha^{\ \beta}(x,y)g^\alpha(y)\,,$$

one obtains the Green's functions ${G^{\pm}}_\alpha^{\ \beta}(x,y)$ associated to the Green's operators ${G^{\pm}}_\alpha^{\ \beta}$. These fulfil $\mathcal{R}_{0,x}{G^{\pm}}_\alpha^{\ \beta}(x,y)=\delta_\alpha^\beta\delta(x,y)$, which is equivalent to $\mathcal{R}_0{G^{\pm}}^\alpha_{\ \beta}f^\beta=f^\alpha$. For our discussion, viewing the Green's solutions of the Rarita-Schwinger operator as operators rather than functions (bi-distributions) is more convenient since it allows for a concise notation. The difference

$$G^\alpha_\beta := {G^{-}}^\alpha_{\ \beta} - {G^{+}}^\alpha_{\ \beta}$$

defines the causal propagator G^α_β, a surjective map [BGP07, thm. 3.4.7][2] from $\Gamma_0(R\mathbb{M})$ to the solutions of the Rarita-Schwinger equation with compactly supported initial conditions, whose space we denote by $S(\mathcal{R}_0,\mathbb{M})$. In other words, every solution of $\mathcal{R}_0\psi^\alpha=0$ with compactly supported initial conditions on a Cauchy surface/equal-time surface, i.e. every "Rarita-Schwinger wave packet", is of the form $\psi^\alpha=G^\alpha_\beta f^\beta$ with some $f^\beta\in\Gamma_0(R\mathbb{M})$. Such f^β is non-unique, as the kernel of G^α_β is non-trivial. Indeed, $G^\alpha_\beta g^\beta=0$ for all g^β of the form $g^\beta=\mathcal{R}_0\tilde g^\beta$ for some $\tilde g^\beta\in\Gamma_0(R\mathbb{M})$. In the quantized Rarita-Schwinger field theory, the causal propagator is employed to defined covariant canonical anticommutation relations (CAR) as we shall discuss in the next subsection, which is why $G_\alpha^\beta(x,y)$ is often called "anticommutator function". Note that $G_\alpha^\beta(x,y)=0$ if x and y are spacelike separated, which is why $G_\alpha^\beta(x,y)=0$ is indeed a physically sensible choice of anticommutator function.

[2]Theorem 3.4.7 in [BGP07] does not prove the surjectivity of the causal propagator for Rarita-Schwinger operators, but the proof holds analogously for all strictly hyperbolic operators.

The above synopsis of the solution theory of $\mathcal{R}_0\psi^\alpha = 0$ has been independent of the constraint $\not{\psi} = 0$. In fact, it is important to clearly fix the convention that $\not{\psi} = 0$ is considered as an additional constraint on solutions, i.e. elements of $S(\mathcal{R}_0, \mathbb{M})$. To distinguish constrained and unconstrained solutions, we define

$$\not{S}(\mathcal{R}_0, \mathbb{M}) := \{\psi^\alpha \in S(\mathcal{R}_0, \mathbb{M}) \,|\, \not{\psi} = 0\}$$

$$\not{\Gamma}_0(R\mathbb{M}) := \{f^\alpha \in \Gamma_0(R\mathbb{M}) \,|\, G^\beta_\alpha f^\alpha \in \not{S}(\mathcal{R}_0, \mathbb{M})\} \,.$$

To make sure that the classical Rarita-Schwinger field theory is non-trivial, one has to check that $\not{S}(\mathcal{R}_0, \mathbb{M})$ does not contain only the zero solution. To this avail, one can derive that, for all $\psi^\alpha \in S(\mathcal{R}_0, \mathbb{M})$ their contraction $\not{\psi}$ fulfils

$$\left(\Box + m^2\right) \not{\psi} = 0 \,.$$

Since this is again a hyperbolic differential equation, demanding that $\not{\psi}$ and its derivatives are vanishing on a Cauchy surface is enough to assure that $\not{\psi} = 0$ on the full spacetime without further restrictions on ψ^α itself. With other words, $\not{\psi} = 0$ can be regarded as a constraint on the initial conditions of elements of $S(\mathcal{R}_0, \mathbb{M})$ and one can check that the constrained space of initial conditions is non-trivial; hence, $\not{S}(\mathcal{R}_0, \mathbb{M})$ is non-trivial either.

Our way to introduce and define the Rarita-Schwinger equation mimics the original definition by Rarita and Schwinger [RaSch41], whereas in modern treatments of the subject, a slightly different approach is taken. Namely, instead of specifying the differential equation (1) plus the constraint $\gamma_\alpha \psi^\alpha = 0$, one specifies only a differential equation which is such that its solutions automatically fulfil this constraint. Since these two approaches yield different results upon minimal coupling to spacetime curvature, we briefly review the modern definition. To this avail, let $\psi^\alpha \in \Gamma(RM)$ fulfil

$$\mathcal{R}_1\psi^\mu := \left(i\varepsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\partial_\rho + m\gamma^{[\mu}\gamma^{\sigma]}\right)\psi_\sigma = 0 \,, \tag{4}$$

where [ ] denotes idempotent antisymmetrisation and $\varepsilon^{\mu\nu\rho\sigma}$ is the unique antisymmetric tensor with $\varepsilon^{0123} = 1$. Let now $m > 0$. Using $\varepsilon^{\mu\nu\rho\sigma}\gamma^5 = \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$ and

$$\begin{aligned} \gamma^{[\mu_1}\cdots\gamma^{\mu_n]} &= \gamma^{[\mu_1}\cdots\gamma^{\mu_{n-1}]}\gamma^{\mu_n} - (n-1)\gamma^{[\mu_1}\cdots\gamma^{\mu_{n-2}}g^{\mu_{n-1}]\mu_n} \\ &= \gamma^{\mu_1}\gamma^{[\mu_2}\cdots\gamma^{\mu_n]} - (n-1)\gamma^{[\mu_2}\cdots\gamma^{\mu_{n-1}}g^{\mu_n]\mu_1} \,, \end{aligned} \tag{5}$$

one finds that contracting $\mathcal{R}_1\psi^\alpha = 0$ with ∂_α yields $\not{\partial}\not{\psi} = \partial_\nu\psi^\nu$ for all solutions of (4). Inserting this into (4) and contracting the result with γ_α leads to $\not{\psi} = 0$ on shell and, hence, to $\partial_\nu\psi^\nu = 0$ on shell. Finally, inserting these identities into (4), one finds that, on solutions, $\mathcal{R}_1 \equiv \mathcal{R}_0$, hence, $\not{S}(\mathcal{R}_0, \mathbb{M}) = S(\mathcal{R}_1, \mathbb{M})$. In the massless case, one can use part of the gauge equivalence of ψ^α and $\psi^\alpha + \partial^\alpha\chi$ with arbitrary $\chi \in \Gamma(D\mathbb{M})$ to require $\not{\psi} = 0$ (see e.g. [VNi81]), which, once more, leads to the on-shell identities $\partial_\mu\psi^\mu = 0$, $\mathcal{R}_1 \equiv \mathcal{R}_0$.

2.2 The consistency of the free Rarita-Schwinger quantum field in flat spacetime

To canonically quantize a classical field theory, one imposes canonical (anti)commutation relations. In the case at hand, the covariant CAR of the Rarita-Schwinger field are specified by the causal propagator G^{β}_{α}, i.e.

$$\{\boldsymbol{\psi}^{\beta}(x), \overline{\boldsymbol{\psi}}_{\alpha}(y)\} := \boldsymbol{\psi}^{\beta}(x)\overline{\boldsymbol{\psi}}_{\alpha}(y) + \overline{\boldsymbol{\psi}}_{\alpha}(y)\boldsymbol{\psi}^{\beta}(x) = iG^{\beta}_{\alpha}(x,y)\underline{1}\,, \tag{6}$$

where we shall denote the (anticommuting) quantum field $\boldsymbol{\psi}^{\alpha}$ with a bold-faced letter here and in the following, in order to distinguish it from the (c-number) classical solutions ψ^{α} of the field equation. By means of the results discussed in appendix D, these covariant CAR can be seen to be equivalent to the often imposed equal-time CAR. To understand this, we first recall that a quantum field $\boldsymbol{\psi}^{\alpha}(x)$ at a spacetime point x is too singular to be a well-defined operator on some Hilbert space. In other words, if $|\Omega\rangle$ is a normalised state, i.e. a Hilbert space vector of unit norm, $\boldsymbol{\psi}^{\alpha}(x)|\Omega\rangle$ is not normalisable any more, which is related to the fact that the anticommutator function $iG^{\beta}_{\alpha}(x,y)$ is singular if $x = y$ (in fact, if $x - y$ is lightlike). To cure this, one can "smear" the covariant field $\boldsymbol{\psi}^{\alpha}(x)$, i.e. integrate it with a test section $\bar{f}_{\alpha}$. In contrast, in the equal-time formalism, the quantum field $\boldsymbol{\Psi}^{\alpha}(t,\vec{x})$ is a well-defined operator once integrated with a solution of the Rarita-Schwinger equation rather than a test section, hence, we employ the capitalised notation to distinguish the equal-time quantum field $\boldsymbol{\Psi}^{\alpha}$ from the covariant quantum field $\boldsymbol{\psi}^{\alpha}$. In more detail, the smeared fields

$$\boldsymbol{\psi}^{\alpha}(\bar{f}_{\alpha}) := \int dx\, \bar{f}_{\alpha}(x)\boldsymbol{\psi}^{\alpha}(x) \quad \text{and} \quad \boldsymbol{\Psi}^{\alpha}\left(\overline{G^{\beta}_{\alpha}f_{\beta}}\right) := \int d\vec{x}\, \overline{\left[G^{\beta}_{\alpha}f_{\beta}\right]}(t,\vec{x})\boldsymbol{\Psi}^{\alpha}(t,\vec{x})$$

are considered to represent the operators, such that the discussion in appendix D can be subsumed as

$$i\int dxdy\, \bar{f}_{\beta}(x)G^{\beta}_{\alpha}(x,y)g^{\alpha}(y) = iG^{\beta}_{\alpha}(\bar{f}_{\beta}, g^{\alpha}) = \{\boldsymbol{\psi}^{\beta}(\bar{f}_{\beta}), \overline{\boldsymbol{\psi}}_{\alpha}(g^{\alpha})\} = \tag{7}$$

$$= \{\boldsymbol{\Psi}^{\beta}(\overline{G^{\mu}_{\beta}f_{\mu}}), \overline{\boldsymbol{\Psi}}_{\alpha}(G^{\alpha}_{\nu}g^{\nu})\} = -\int d\vec{x}\, \overline{\left[G^{\mu}_{\beta}f_{\mu}\right]}(t,\vec{x})\gamma^{0}\left[G^{\alpha}_{\nu}g^{\nu}\right](t,\vec{x})\,.$$

Hence, the covariant CAR (6) are equivalent to the equal-time CAR

$$\{\boldsymbol{\Psi}^{\alpha}(t,\vec{x}_1), \boldsymbol{\Pi}_{\beta}(t,\vec{x}_2)\} = i\delta^{\alpha}_{\beta}\delta(\vec{x}_1,\vec{x}_2)\underline{1}\,,$$

where $\boldsymbol{\Pi}_{\alpha} := i\overline{\boldsymbol{\Psi}}_{\alpha}\gamma^{0}$ is the momentum canonically conjugated to $\boldsymbol{\Psi}^{\alpha}$.

We shall now discuss a non-trivial condition for the above canonical quantum theory of the free Rarita-Schwinger field in flat spacetime to be consistent, i.e. 'unitary'. To this avail, let us note that the proposed anticommutator of a smeared Rarita-Schwinger field and its adjoint

$$\{\overline{\boldsymbol{\psi}}_{\alpha}(f^{\alpha}), \boldsymbol{\psi}^{\beta}(\bar{f}_{\beta})\}$$

is bound to be a positive operator, i.e. to have positive or vanishing expectation value in any quantum state $|\Omega\rangle$. Indeed, Hermitean conjugation (i.e. conjugation with respect to the Hilbert space scalar product) acts on the quantized Rarita-Schwinger fields as

$$\big(\boldsymbol{\psi}^\alpha(\bar{f}_\alpha)\big)^\dagger := \overline{\boldsymbol{\psi}}_\alpha(f^\alpha) \qquad \big(\overline{\boldsymbol{\psi}}_\alpha(f^\alpha)\big)^\dagger := \boldsymbol{\psi}^\alpha(\bar{f}_\alpha) \qquad \forall f^\alpha \in \Gamma_0(R\mathbb{M})\,,$$

which corresponds to the classical identity

$$\langle f^\alpha, \psi^\beta\rangle^* = \left(\int\limits_{\mathbb{M}} dx\, \bar{f}_\alpha(x)\psi^\alpha(x)\right)^* = \langle \psi^\beta, f^\alpha\rangle\,,$$

with * denoting complex conjugation; hence, the considered anticommutator can be written as

$$\{\overline{\boldsymbol{\psi}}_\alpha(f^\alpha), \boldsymbol{\psi}^\beta(\bar{f}_\beta)\} = A^\dagger A + AA^\dagger \qquad A := \boldsymbol{\psi}^\beta(\bar{f}_\beta)\,,$$

and this expression fulfils

$$\langle\Omega|(A^\dagger A + AA^\dagger)|\Omega\rangle = \langle A\Omega|A\Omega\rangle + \langle A^\dagger\Omega|A^\dagger\Omega\rangle \geq 0\,.$$

Since $\{\overline{\psi}_\alpha(f^\alpha), \psi^\beta(\bar{f}_\beta)\}$ is equal to the smeared causal propagator times the unit operator $\underline{1}$, and $c\underline{1}$ has positive expectation values iff $c > 0$, the quantization of the free Rarita-Schwinger field on Minkowski spacetime is only consistent if i times the causal propagator is a positive semidefinite distribution on "constrained test sections", i.e.

$$iG^\beta_\alpha(\bar{f}_\beta, f^\alpha) \geq 0 \qquad \forall f^\alpha \in \not{\Gamma}_0(R\mathbb{M})\,.$$

It is important to require this condition for all test sections f^α, since $\boldsymbol{\psi}^\beta(\bar{f}_\beta)$ for different f^α represent different operators in general. In more detail, if $|\Omega\rangle$ is the vacuum state, $\boldsymbol{\psi}^\beta(\bar{f}_\beta)|\Omega\rangle$ corresponds to a single particle state associated to the classical wave packet $\psi^\alpha = G^\alpha_\beta f^\beta$. Hence, the consistency condition discussed above is loosely equivalent to demanding that all wave packet quantum states have positive norm (they may have zero norm if the corresponding classical wave packet ψ^α is vanishing).

We shall now discuss why the causal propagator bears the positivity property required by consistency in Minkowski spacetime and at the same time obtain a more hands-on understanding of why the smeared anticommutator of a Fermionic quantum field should have positive expectation values. By (7), we have to check if

$$-\int d\vec{x}\, \overline{\left[G^\mu_\beta f_\mu\right]}(t,\vec{x})\gamma^0 \left[G^\alpha_\nu f^\nu\right](t,\vec{x}) \geq 0 \qquad \forall f^\alpha \in \not{\Gamma}_0(R\mathbb{M})\,,$$

i.e. if

$$-\int d\vec{x}\, \overline{\psi}_\alpha(t,\vec{x})\gamma^0\psi^\alpha(t,\vec{x}) \geq 0$$

for all wave-packet solutions ψ_α of $\mathcal{R}_0\psi^\alpha = 0$ which fulfil $\not{\psi} = 0$. Since the Rarita-Schwinger equation is diagonal in the Lorentz-index, every $\psi^\alpha \in S(\mathcal{R}_0, R\mathbb{M})$ can be expanded as

$$\psi^\alpha(t,\vec{x}) = \int d\vec{k}\, \hat{\psi}^\alpha_{i,\vec{k}}\, v^i_{\vec{k}}(t,\vec{x})\,,$$

where $\hat{\psi}^\alpha_{i,\vec{k}}$ are coefficients rapidly decreasing in $\vec{k}$ for large $\vec{k}$ and $v^i_{\vec{k}}(t,\vec{x})$ are orthonormal and complete modes of the Dirac field given in appendix C. These properties of $v^i_{\vec{k}}(t,\vec{x})$ imply that

$$-\int d\vec{x}\, \overline{\psi}_\alpha(t,\vec{x})\gamma^0\psi^\alpha(t,\vec{x}) = -\int d\vec{k} \sum_i \left(\hat{\phi}_{\alpha,i,\vec{k}}\right)^\dagger \hat{\psi}^\alpha_{i,\vec{k}}\,.$$

We now recall that $\not{\psi} = 0$ implies $\partial_\alpha\psi^\alpha = 0$. Hence, the mode expansion coefficients of ψ^α fulfil

$$k^+_\alpha\hat{\psi}^\alpha_{1,\vec{k}} = k^+_\alpha\hat{\psi}^\alpha_{2,\vec{k}} = k^-_\alpha\hat{\psi}^\alpha_{3,\vec{k}} = k^-_\alpha\hat{\psi}^\alpha_{4,\vec{k}} = 0\,,$$

where

$$(k^\pm)^\alpha = \begin{pmatrix} \pm\omega \\ \vec{k} \end{pmatrix} \qquad \omega = \sqrt{\vec{k}^2 + m^2}\,.$$

Let now $m > 0$, then $(k^\pm)^\alpha$ is timelike, hence, $\hat{\psi}^\alpha_{i,\vec{k}}$ must be spacelike for all i and $\vec{k}$ on account of the linear independence of the Dirac modes. Consequently,

$$-\int d\vec{k} \sum_i \left(\hat{\psi}_{\alpha,i,\vec{k}}\right)^\dagger \hat{\psi}^\alpha_{i,\vec{k}} > 0$$

and we find that iG^α_β is strictly positive on the constrained test sections. If $m = 0$, $(k^\pm)^\alpha$ is lightlike and the coefficients $\hat{\psi}^\alpha_{i,\vec{k}}$ are thus either spacelike or proportional to $(k^\pm)^\alpha$. The constrained solutions ψ^α with the latter property are precisely the "gauge solutions" $\psi^\alpha = \partial^\alpha\chi$ briefly mentioned in the previous subsection. If one divides them out from the allowed solution space (which heuristically corresponds to removing "zero norm states"), iG^α_β is strictly positive on the resulting constrained solution space of "gauge-equivalent" solutions.

3 Conditions for a consistent canonical quantization in curved spacetimes

The discussion in the previous section on what canonical quantization of an elementary spin $\frac{3}{2}$ field means in flat spacetime can be directly taken over to general four-dimensional curved globally hyperbolic[3] spacetimes $(M, g_{\mu\nu})$. We consider a first order differential

[3]For a full definition of such spacetimes, see e.g. [BGP07, Wa84]. Loosely speaking, on such spacetimes there always exist unique solutions of hyperbolic partial differential equations with suitable initial data on equal-time surfaces.

operator $\mathcal{R}$ on smooth sections ψ^α of $RM = DM \otimes TM$, where TM and DM are the tangent and Dirac spinor bundles of $(M, g_{\mu\nu})$[4] and a supplementary constraint on the contraction $\not\psi = \gamma_\alpha \psi^\alpha$ of solutions ψ^α of $\mathcal{R}\psi^\alpha = 0$, i.e. ψ^α is (locally) a function on M carrying a tangent space index and a (suppressed) Dirac index. To canonically quantize the field theory related to $\mathcal{R}$ and the supplementary constraint, we impose canonical anticommutation relations

$$\{\boldsymbol{\psi}^\beta(x), \overline{\boldsymbol{\psi}}_\alpha(y)\} = iG^\beta_\alpha(x,y)\underline{1}$$

specified by the causal propagator $G^\alpha_\beta = G^{-\alpha}{}_\beta - G^{+\alpha}{}_\beta$ of $\mathcal{R}$, provided that unique advanced and retarded Green's operators $G^{\pm\alpha}{}_\beta$ exist for $\mathcal{R}$. Again, a quantum field $\boldsymbol{\psi}^\beta(x)$ at a point x is too singular to be a well-defined operator on some Hilbert space, and one rather has to consider fields integrated with test sections (i.e. compactly supported smooth sections)

$$\boldsymbol{\psi}^\alpha(\bar{f}_\alpha) := \int\limits_M d_g x\, \bar{f}_\alpha(x)\psi^\alpha(x) \qquad \overline{\boldsymbol{\psi}}_\alpha(f^\alpha) := \int\limits_M d_g x\, \overline{\boldsymbol{\psi}}_\alpha(x) f^\alpha(x) \qquad f^\alpha \in \Gamma_0(RM)\,,$$

where $d_g x$ denotes the metric-induced covariant measure on M, as candidates for proper Hilbert space operators. As the causal propagator $G^{\pm\alpha}{}_\beta$ of $\mathcal{R}$ maps test sections f^α to wave-packet solutions of $\mathcal{R}\psi^\alpha = 0$ in a surjective manner, $\boldsymbol{\psi}^\alpha(\bar{f}_\alpha)$, $\overline{\boldsymbol{\psi}}_\alpha(f^\alpha)$ can be considered as the quantum field operators corresponding to the classical wave packet $\psi^\alpha = G^\alpha_\beta f^\beta$. These operators should be interrelated by Hermitean conjugation † as

$$\left(\boldsymbol{\psi}^\alpha(\bar{f}_\alpha)\right)^* := \overline{\boldsymbol{\psi}}_\alpha(f^\alpha) \qquad \left(\overline{\boldsymbol{\psi}}_\alpha(f^\alpha)\right)^* := \boldsymbol{\psi}^\alpha(\bar{f}_\alpha)\,,$$

thus consistency requires that their anticommutator has a positive expectation value; this, i.e. "unitarity of the quantum theory", in turn is fulfilled only if

$$iG^\beta_\alpha(\bar{f}_\beta, f^\alpha) = i\langle f^\beta, G^\beta_\alpha f^\alpha\rangle = i\int\limits_M d_g x \bar{f}_\beta(x)\left[G^\beta_\alpha f\right](x) \geq 0.$$

This condition on $iG^\beta_\alpha(\bar{f}_\beta, f^\alpha)$ is, however, only required for f^α which are such that $G^\alpha_\beta f^\beta$ solves the supplementary constraint on $\gamma_\alpha G^\alpha_\beta f^\beta$, as we are only interested in quantizing the degrees of freedom corresponding to an elementary spin $\frac{3}{2}$ field, which in flat spacetimes corresponds to an irreducible representation of the Poincaré group specified by $\mathcal{R} = \mathcal{R}_0$ and the constraint $\not\psi = 0$. Note that $iG^\beta_\alpha(\bar{f}_\beta, g^\alpha) \geq 0$ on constrained test sections implies that $(\cdot,\cdot)$ defined as

$$(G^\alpha_\mu f^\mu, G^\beta_\nu g^\nu) := iG^\beta_\alpha(\bar{f}_\beta, g^\alpha)$$

gives a positive semidefinite product on constrained wave packet solutions of $\mathcal{R}\psi^\alpha = 0$.

[4]It is well-known that a spin structure exists on all four-dimensional, globally hyperbolic curved spacetimes [Ger68, Ger70], such that DM can always be constructed for such manifolds.

From the above outline of the canonical quantization of a free Rarita-Schwinger field on general curved spacetimes we can already read off a few necessary conditions for such quantization to be well-defined. In the following, we complete and/or transform these conditions to a – as we will argue – complete set of sufficient conditions. Note that, strictly speaking, we are only dealing with algebraic conditions on the quantum field theory at hand and it does not matter how the Hilbert space on which $\boldsymbol{\psi}^\alpha(\bar{f}_\alpha)$, $\overline{\boldsymbol{\psi}}_\alpha(f^\alpha)$ are supposed to act looks like. In fact, if $iG^\beta_\alpha(\bar{f}_\beta, f^\alpha)$ fails to be positive semidefinite, one can not consider $\boldsymbol{\psi}^\alpha(\bar{f}_\alpha)$, $\overline{\boldsymbol{\psi}}_\alpha(f^\alpha)$ as operators on *any* Hilbert space. Moreover, the question of selecting a Hilbert space in quantum field theory on curved spacetime is always tricky if the curved spacetime does not possess any symmetries, see e.g. [BiDa82, Wa95, Hac10] for a general discussion.

We now state the anticipated conditions and comment on them afterwards.

***Definition* 1** *Let $\mathcal{R}$ be a first order differential operator on smooth sections $\Gamma(RM) \ni \psi^\alpha$ of $RM = DM \otimes TM$. Moreover, let A_α be a differential operator which maps smooth sections of RM to smooth sections of DM and let $S(\mathcal{R}, M)$ and $\not{S}(\mathcal{R}, M)$ be defined as*

$$S(\mathcal{R}, M) := \{\psi^\alpha \in \Gamma(RM) \,|\, \mathcal{R}\psi^\alpha = 0 \textit{ and } \psi^\alpha \textit{ has compact support on any Cauchy surface}\}$$

$$\not{S}(\mathcal{R}, M) := \{\psi^\alpha \in S(\mathcal{R}, M) \,|\, \not{\psi} = A_\alpha \psi^\alpha\}\,.$$

We say that $\mathcal{R}$ and A_α lead to a consistent canonical quantum field theory of an elementary spin $\frac{3}{2}$ field on $(M, g_{\mu\nu})$, if the following conditions are satisfied.

(Irreducibility) *In Minkowski spacetime, $A_\alpha \equiv 0$ and $\not{S}(\mathcal{R}, \mathbb{M}) \subset \not{S}(\mathcal{R}_0, \mathbb{M})$.*

(Covariance) *$\not{S}(\mathcal{R}, M)$ is locally covariant[5], i.e. for every globally hyperbolic region $(M', g_{\mu\nu}|_{M'}) \subset (M, g_{\mu\nu})$ whose causal structure is independent of $(M, g_{\mu\nu})$ outside of $(M', g_{\mu\nu}|_{M'})$[6], $\not{S}(\mathcal{R}, M')$ is independent of $(M, g_{\mu\nu})$ outside of $(M', g_{\mu\nu}|_{M'})$, i.e. $\not{S}(\mathcal{R}, M') = \not{S}(\mathcal{R}, M)|_{M'}$. Moreover, either $A_\alpha \equiv 0$ on all spacetimes, or $\not{\psi} = A_\alpha \psi^\alpha$ is automatically fulfilled for all solutions, viz. $\not{S}(\mathcal{R}, M) = S(\mathcal{R}, M)$.*

(Causality) *$\mathcal{R}$ is strictly hyperbolic.*

(Selfadjointness) *$\mathcal{R}$ is formally selfadjoint w.r.t. $\langle \cdot, \cdot \rangle$, i.e. $\langle \mathcal{R}^\dagger g^\beta, f^\alpha \rangle = \langle \mathcal{R} g^\beta, f^\alpha \rangle$ with $\mathcal{R}^\dagger$ defined as*

$$\langle \mathcal{R}^\dagger g^\beta, f^\alpha \rangle = \int_M d_g x\, \overline{\mathcal{R}^\dagger g_\alpha(x)} f^\alpha(x) := \int_M d_g x\, \bar{g}_\alpha(x) \mathcal{R} f^\alpha(x) = \langle g^\beta, \mathcal{R} f^\alpha \rangle$$

for all f^α, g^β in $\Gamma_0(RM)$.

[5] This condition has been emphasised and elevated to a principle in [HoWa01, BFV03].

[6] By this we mean that every causal curve between two points in $(M', g_{\mu\nu}|_{M'})$ as a subspacetime of $(M, g_{\mu\nu})$ lies completely in M', such that the embedding of $(M', g_{\mu\nu}|_{M'})$ into $(M, g_{\mu\nu})$ does not add new causal relations between points in M'.

Let us first comment on *(Causality)*. This condition avoids that solutions of $\mathcal{R}$ propagate acausally or have no sensible propagation behaviour at all. Additionally, it assures the existence of unique advanced and retarded propagators ${G^{\pm}}^{\alpha}_{\beta}$ of $\mathcal{R}$, such that covariant canonical anticommutation relations can be formulated at all by means of $G^{\alpha}_{\beta} = {G^{-}}^{\alpha}_{\beta} - {G^{+}}^{\alpha}_{\beta}$. *(Causality)* is in particular fulfilled if there exists a fiducial operator $\tilde{\mathcal{R}}$ such that $\tilde{\mathcal{R}}\mathcal{R}$ is a normally hyperbolic operator, i.e. a wave operator [BGP07, Müh10], but the more general condition that $\mathcal{R}$ be strictly hyperbolic is sufficient for $\mathcal{R}$ to have a good solution theory, see e.g. [CoHi89, CDD96, Hör94]. Without going into details, we briefly mention that strict hyperbolicity is a condition on the principal symbol – the coefficient of the highest derivative – of a differential operator.

Additionally, the first two of the above conditions assure that the field theory defined by $\mathcal{R}$ and A_α is a covariant generalisation of the free Rarita-Schwinger field theory on Minkowski spacetime. Particularly, we want to assure that the considered field theory does not contain more (or less!) physical degrees of freedom than those possessed by a free Rarita-Schwinger field theory (with $m \geq 0$) on flat spacetime, and that we can analyse these physical degrees of freedom in an arbitrarily small region of a spacetime without knowing what the spacetime looks like far away from this arbitrarily small region[7]. One may think that simply writing down the Rarita-Schwinger equation and the constraint as covariant tensor equations is enough to achieve this. That this is not the case and what can go wrong will become clear when we discuss the minimally coupled Rarita-Schwinger equation in the next section.

The additional requirement posed in *(Covariance)*, namely, that $A_\alpha \equiv 0$ or $\not{S}(\mathcal{R}, M) = S(\mathcal{R}, M)$, is in principle stronger than the local covariance we would like to achieve, but we have not been able to prove that the constraint $\not{\psi} = A_\alpha \psi^\alpha$ fulfils local covariance except in these two special cases. Indeed, if $\mathcal{R}$ is a covariant differential operator and *(Causality)* holds, then $S(\mathcal{R}, M)$ is locally covariant, because initial conditions in $M' \subset M$ completely determine elements of $S(\mathcal{R}, M')$; given that $S(\mathcal{R}, M)$ is locally covariant, $\not{S}(\mathcal{R}, M)$ is locally covariant as well if one of the two required conditions on A_α is fulfilled, as both are trivially "spacetime-independent". Hence, with our current understanding, the additional conditions on A_α posed in *(Covariance)* are only sufficient, but not necessary.

One might hope that abandoning *(Irreducibility)* and allowing for a non-trivial coupling of the components with spin $\frac{3}{2}$ and spin $\frac{1}{2}$ of $\psi^\alpha \in \Gamma(DM \otimes TM)$ simplifies the situation. However, we will see in the next section that two examples of operators which fulfil *(Covariance)*, *(Causality)*, and *(Selfadjointness)*, but not *(Irreducibility)*, do not lead to a consistent quantum field theory. Of course, a composite field of spin $\frac{3}{2}$, e.g. a tensor product of a Dirac field and a vector field, can be quantized consistently. But this composite field fulfils a differential equation of mixed (second and first) order, and we are only considering first order operators on $\Gamma(DM \otimes TM)$ in this treatment (this also rules out a tensor product of three Dirac fields, which can certainly also be quantized in a consistent manner).

[7]See [BFV03] for an in-depth discussion of the physical idea behind local covariance.

We finally comment on *(Selfadjointness)*. As we have seen in the last subsection, the relation between equal-time and covariant CAR is essential for the proof that the free Rarita-Schwinger field in Minkowski spacetime can be canonically quantized in a consistent way. This relation in turn relies on the fact that $\mathcal{R}_0$ is formally selfadjoint. Indeed, as discussed in appendix D, formal selfadjointness of a first order differential operator $\mathcal{R}$ implies that the principal symbol σ^μ of $\mathcal{R}$ defines a covariantly conserved current

$$j^\mu\left[\psi_1^\alpha, \psi_2^\beta\right] = \overline{\psi}_{\alpha,1}\sigma^\mu\psi_2^\alpha$$

and that the smeared causal propagator $iG_\alpha^\beta(\bar{f}_\beta, g^\alpha)$ is just the "charge" corresponding to this current, i.e. the time-component of j^μ integrated over an equal-time surface. We are not aware of any way to assure the duality between equal-time and covariant CAR without using formal selfadjointness of the considered first order differential operator $\mathcal{R}$, but, although it may be awkward to abandon this duality, one might consider the possibility that only covariant CAR can be implemented and $iG_\alpha^\beta(\bar{f}_\beta, f^\alpha) \geq 0$ can be proven even if $\mathcal{R}$ is not formally selfadjoint. However, our results in appendix E demonstrate the tight relation between positivity of $iG_\alpha^\beta(\bar{f}_\beta, f^\alpha)$ and formal selfadjointness of $\mathcal{R}$. On the one hand, we are able to prove that selfadjointness implies positivity on curved spacetimes, if positivity on flat spacetime is already known. On the other hand, we prove a weak converse of this: if positivity holds for Majorana fields, then the current j^μ constructed out of the principal symbol of $\mathcal{R}$ must be conserved. Hence, although we don't have a full proof that selfadjointness of $\mathcal{R}$ is necessary for the unitarity of the canonical quantum theory associated to $\mathcal{R}$, we altogether consider selfadjointness to be an essential ingredient.

Note that, in principle, it is sufficient to prove selfadjointness of $\mathcal{R}$ only on test sections f^α which via G_β^β correspond to solutions ψ^α of $\mathcal{R}\psi^\alpha = 0$ that fulfil the constraint $\not{\psi}^\alpha = A_\alpha\psi^\alpha$. However, as we have not been able to characterise these constrained test sections explicitly and without knowing selfadjointness *a priori*, we require the sufficient condition that selfadjointness holds on all test sections. This holds for e.g. the Rarita-Schwinger-operator $\mathcal{R}_0$ on Minkowski spacetime and for the Dirac operator on all curved spacetimes. Moreover, if $\not{S}(\mathcal{R}, M) = S(\mathcal{R}, M)$, as optionally required in *(Covariance)*, then all test sections trivially belong to the class of constrained test sections.

4 (Modified) Rarita-Schwinger equations present in the literature and their drawbacks

We shall now comment on various versions of the Rarita-Schwinger equation on curved spacetimes present in the literature, and remark why they fail to satisfy one or several consistency conditions formulated in definition 1.

4.1 The original Rarita-Schwinger operator with $\not{\psi} = 0$

The minimally coupled original Rarita-Schwinger operator $\mathcal{R}_0$ on RM is defined as

$$\mathcal{R}_0 := -i\not{\nabla} + m\underline{1}$$

and supplemented by the constraint

$$\not{\psi} = 0\,,$$

where ∇ is the covariant derivative associated to the Levi-Civita connection on $DM \otimes TM$. $\mathcal{R}_0$ and $A_\alpha \equiv 0$ manifestly fulfils *(Irreducibility)*. Moreover, *(Causality)* is fulfilled as $(i\not{\nabla} + m\underline{1})\mathcal{R}_0$ is a wave operator, and one can easily verify that *(Selfadjointness)* holds as well. However, *(Covariance)* does not hold, as one can check by a direct computation. Namely, using $\not{\psi} = 0 \wedge \mathcal{R}_0\psi^\alpha = 0 \Rightarrow \nabla_\alpha\psi^\alpha = 0$ and the curvature identities listed in section B, we can compute

$$\nabla_\alpha(-i\not{\nabla} + m)\psi^\alpha = 0 \quad \Rightarrow \quad \nabla_\alpha\not{\nabla}\psi^\alpha = 0 \quad \Rightarrow \quad [\nabla_\alpha, \not{\nabla}]\psi^\alpha = 0 \quad \Rightarrow \quad R_{\alpha\beta}\gamma^\beta\psi^\alpha = 0\,.$$

Unless the spacetime $(M, g_{\mu\nu})$ is an Einstein manifold, i.e. unless $R_{\mu\nu} \equiv \frac{1}{4}Rg_{\mu\nu}$, the latter identity can only hold if $\psi^\alpha = 0$, hence, $\mathcal{R}_0\psi^\alpha = 0 \wedge \not{\psi} = 0$ has only the trivial solution on general spacetimes $(M, g_{\mu\nu})$. This phenomenon is well-known since the works of Buchdahl [Buc58] and usually called 'inconsistency of the classical field equation'. Moreover, this is in conflict with *(Covariance)* as one can see by considering the following *gedankenexperiment*: we image that we are living in a region of spacetime, where $R_{\mu\nu} = \frac{1}{4}Rg_{\mu\nu}$ holds, e.g. in a locally Ricci-flat region, and we ask ourselves if non-trivial solutions of $\mathcal{R}_0\psi^\alpha = 0 \wedge \not{\psi} = 0$ exist. There is no way to unambiguously answer this question by knowing only our local Ricci-flat spacetime region. If there is an arbitrarily small and arbitrarily distant region in spacetime, where $R_{\mu\nu} \neq \frac{1}{4}Rg_{\mu\nu}$, then $\not{S}(\mathcal{R}_0, M)$ contains only the trivial element, but if $R_{\mu\nu} \equiv \frac{1}{4}Rg_{\mu\nu}$ on the full spacetime we are living in, then $S(\mathcal{R}_0, M)$ is as large as it is suitable for the solution space of the equations describing an elementary field of spin $\frac{3}{2}$.

4.2 The original Rarita-Schwinger operator without $\not{\psi} = 0$

Since in the previous example, the constraint $\not{\psi} = 0$ has been used in the derivation of $R_{\alpha\beta}\gamma^\beta\psi^\alpha = 0$ for all solutions of $\mathcal{R}_0\psi^\alpha = 0$, abandoning this constraint immediately avoids this very strong restriction on solutions or on the background spacetime [Müh11]. Indeed this setup fulfils *(Covariance)*, *(Causality)*, and *(Selfadjointness)* [Müh11], but obviously fails to satisfy *(Irreducibility)*. Hence, looking at the proof of unitarity in Minkowski, one can easily find elements ψ^α of $S(\mathcal{R}_0, \mathbb{M})$ which fail to fulfil

$$-\int d\vec{x}\,\overline{\psi}_\alpha(t,\vec{x})\gamma^0\psi^\alpha(t,\vec{x}) \geq 0\,,$$

e.g. all ψ^α whose mode expansion coefficients $\hat{\psi}^\alpha_{i,\vec{k}}$ are timelike for all i, $\vec{k}$ would do.

4.3 The 'modern' Rarita-Schwinger operator

Let us now consider the minimally coupled version of $\mathcal{R}_1$, i.e. the equation

$$\mathcal{R}_1\psi^\mu := \left(i\varepsilon^{\mu\nu\rho\sigma}\gamma^5\gamma_\nu\nabla_\rho + m\gamma^{[\mu}\gamma^{\sigma]}\right)\psi_\sigma = 0 \qquad \psi^\alpha \in \Gamma(RM)\,.$$

Following the route taken in the analysis of this operator on Minkowski spacetime, one can contract the above equation with both ∇_μ and γ_μ to obtain derived identities satisfied on shell. Upon doing so, one finds that this differential equation can be equivalently expressed as

$$(-i\nabla\!\!\!/ + m)\,\psi^\alpha + \left(i\nabla^\alpha + \frac{m}{2}\gamma^\alpha\right)\psi\!\!\!/ = 0\,,$$

and that solutions of this equation satisfy

$$3m^2\psi\!\!\!/ = G_{\mu\nu}\gamma^\mu\psi^\nu \qquad 3m^2\nabla_\mu\psi^\mu = \left(\nabla\!\!\!/ - \frac{3im}{2}\right)G_{\mu\nu}\gamma^\mu\psi^\nu\,, \tag{8}$$

with $G_{\mu\nu}$ denoting the Einstein tensor $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. From this we can immediately infer that this equation does not satisfy *(Covariance)*, since $3m^2\psi\!\!\!/ = G_{\mu\nu}\gamma^\mu\psi^\nu$ is identically fulfilled if $G_{\mu\nu} \equiv 3m^2 g_{\mu\nu}$, but restricts the solution space otherwise [DeZu76, Tow77]. Furthermore, the above equation satisfies *(Irreducibility)* only if $m > 0$, and *(Selfadjointness)* only if $G_{\mu\nu} \equiv 3m^2 g_{\mu\nu}$ implies $\psi\!\!\!/ = 0$ and one inserts this into $\mathcal{R}_1$ and considers the resulting "effective differential operator"; this is possible only on Einstein manifolds which do not fulfil $G_{\mu\nu} \equiv 3m^2 g_{\mu\nu}$. Additionally, there are difficulties with *(Causality)*, as already pointed out by Velo and Zwanziger in [VeZw69]. They have analysed the equation $\mathcal{R}_1\psi^\mu = 0$ in the context of a classical electromagnetic background field rather than on a classical curved background, but their results also hold in the latter case. We will not perform the computations necessary to see the failure of *(Causality)* for $\mathcal{R}_1$ at this point since we will present the same calculations in a more general context in the next section. For the moment, we would like to point out that the notion of a "minimally coupled Rarita-Schwinger" field in curved spacetimes is ambiguous, as this can mean the minimally coupled version of either $\mathcal{R}_0$, or $\mathcal{R}_1$, whereas both equations have different drawbacks and are not equivalent on general curved spacetimes.

4.4 The projected Rarita-Schwinger operator on RM mod $\psi\!\!\!/ = 0$

This operator is the version of the Rarita-Schwinger operator which is usually discussed in the rather mathematically oriented literature, e.g. [BäGi11], see also [FrSp95] for a generalisation to arbitrary half-integral spin. It can be constructed by implementing the condition $\psi\!\!\!/ = 0$ directly into the underlying vector bundle. This is achieved by defining the (pointwise) projector P as

$$P\psi^\alpha = \psi^\alpha - \frac{1}{4}\gamma^\alpha\psi\!\!\!/\,,$$

and considering the (with obvious notation) bundle $PRM := P(DM \otimes TM)$ rather than $RM = DM \otimes TM$. In order to obtain a well-defined field theory on PRM, one has to make sure that the considered differential operator $\mathcal{R}$ maps $\Gamma(PRM)$ into itself rather than into $\Gamma(RM)$. The latter is indeed the case for $\mathcal{R}_0$, but this can be easily cured by considering the operator $\mathcal{R}_P := P\mathcal{R}_0$ instead. Hence,

$$\mathcal{R}_P\psi^\alpha = (-i\nabla\!\!\!/ + m\underline{1})\psi^\alpha + \frac{i}{2}\gamma^\alpha\nabla_\mu\psi^\mu = 0$$

is in principle a well-defined field equation for $\psi^\alpha \in \Gamma(PRM)$. Moreover, the authors of [BäGi11, FrSp95] verify that $\mathcal{R}_P$ fulfils *(Causality)* and in [BäGi11] it is remarked that $\mathcal{R}_P$ fulfils *(Selfadjointness)*, but that the current $j^\mu[\psi^\alpha, \psi^\alpha]$ built out of the principal symbol σ^μ of $\mathcal{R}_P$ does not give a positive result once integrated over a Cauchy surface. However, the authors of [BäGi11] require that such positivity holds for arbitrary $\psi^\alpha \in \Gamma(PRM)$, whereas for a unitary quantum theory it only has to hold for solutions of $\mathcal{R}_P\psi^\alpha = 0$. Notwithstanding, as $\mathcal{R}_P$ does not fulfil *(Irreducibility)*, one can not repeat the positivity proof obtained for $\mathcal{R}_0$ in Minkowski spacetime. Although one can very well repeat it for the solutions of $\mathcal{R}_P\psi^\alpha = 0$ which satisfy $\partial_\alpha\psi^\alpha = 0$ in addition, one can employ the mode expansion of solutions of the Dirac equation in Minkowski spacetime to explicitly construct a mode expansion for solutions of $\mathcal{R}_P\psi^\alpha = 0$ which do not satisfy $\partial_\alpha\psi^\alpha = 0$ and find examples for which $j^\mu[\psi^\alpha, \psi^\alpha]$ integrated over an equal-time surface gives a negative result.

4.5 The Buchdahl-Rarita-Schwinger operator with $\psi\!\!\!/ = 0$

This operator has been constructed by Buchdahl in [Buc82] and is in fact the first operator in the enumeration of this section which contains a non-minimal coupling to the background curvature. Buchdahl's idea was to modify the minimally coupled operator $\mathcal{R}_0$ in such a way that the equation $R_{\mu\nu}\gamma^\mu\psi^\nu = 0$ resulting from $\mathcal{R}_0\psi^\alpha = 0$ obtains a suitable 'right hand side' which assures that it is identically fulfilled on all curved spacetimes. To this avail, one starts with the *ansatz*

$$\mathcal{R}_B\psi^\alpha := (-i\nabla\!\!\!/ + m)\psi^\alpha + \widetilde{\psi}^\alpha = 0 \qquad \psi^\alpha \in \Gamma(RM)\,,$$

where $\widetilde{\psi}^\alpha$ is the sought-for non-minimal coupling term. Contracting this equation with ∇_α and γ_α, combining the results, and using the spin curvature identities in section B, one obtains the following differential equation for $\widetilde{\psi}^\alpha$

$$(-i\nabla\!\!\!/ + m)\widetilde{\psi\!\!\!/}^\alpha + 2i\nabla_\alpha\widetilde{\psi}^\alpha = R_{\mu\nu}\gamma^\mu\psi^\nu\,.$$

Realising that this is a DM-valued equation, one makes the *ansatz* that $\widetilde{\psi}^\alpha$ is completely determined by a Dirac spinor B, i.e.

$$\widetilde{\psi}^\alpha := (ai\nabla^\alpha + bi\gamma^\alpha\nabla\!\!\!/ + cm\gamma^\alpha)\,B\,.$$

Inserting this into the equation for $\widetilde{\psi}^\alpha$, one obtains

$$\left\{(2b-a)\Box + \frac{(2b+a)}{4}R + 4cm^2 + (a+4b-2c)im\nabla\!\!\!/\right\} B = R_{\mu\nu}\gamma^\mu\psi^\nu\,.$$

If this equation is not an algebraic equation for B, it will not assure that $B = 0$ in Minkowski spacetime, and the resulting operator would not fulfil *(Irreducibility)*. Hence, one chooses $a = 2b$, $c = 3b$ and finds that

$$\widetilde{\psi}^\alpha = (2i\nabla^\alpha + i\gamma^\alpha\nabla\!\!\!/ + 3m\gamma^\alpha)\,\frac{R_{\mu\nu}\gamma^\mu\psi^\nu}{12m^2+R}$$

is the only solution compatible with *(Irreducibility)*. Note that the resulting operator is well-defined also for $m = 0$, although the original derivation in [Buc82] has assumed $m > 0$. However, if $m = 0$, $\mathcal{R}_B$ is not analytic in the spacetime curvature. $\mathcal{R}_B$ fulfils *(Irreducibility)* and *(Covariance)* by construction, but one can compute that *(Selfadjointness)* and *(Causality)* do not hold on general spacetimes. Again, we refer the reader interested in computational details to the next section.

4.6 The Buchdahl-Wünsch operator on BM

The Buchdahl-Wünsch operator $\mathcal{R}_{BW}$ is constructed on a bundle differing from RM, the Buchdahl-bundle BM. We shall not go into details here, but only mention that this bundle corresponds to the representation $(\frac{3}{2},0)\oplus(1,\frac{1}{2})$ of $SL(2,\mathbb{C})$ – we refer the reader interested in details to [Buc82, Wün85] and [Müh07, Mak11, BäGi11] for a recent review. Much like in the case of the projected Rarita-Schwinger operator on RM mod $\psi\!\!\!/ = 0$, BM has an irreducibility condition similar to the constraint $\psi\!\!\!/ = 0$ built in and $\mathcal{R}_{BW}$ can be understood as the projection of a fiducial operator to $\Gamma(BM)$; this projection is only possible for $m > 0$. $\mathcal{R}_{BW}$ on $\Gamma(BM)$ fulfils *(Covariance)* and *(Irreducibility)* (as discussed above, in a generalised sense) and can be shown to fulfil *(Causality)* as well [IlSch99, Müh07, Müh10]. However, it does not fulfil *(Selfadjointness)* as observed in [Mak11]. The problem here is that any canonical product on $\Gamma_0(BM)$ which corresponds to

$$\langle f^\alpha, g^\beta\rangle = \int\limits_M d_g\; \bar{f}_\alpha g^\beta$$

on $\Gamma_0(RM)$ has to include two covariant derivatives in order to make sure that "all free indices are contracted in a covariant way". Consequently, there are three canonical products on $\Gamma_0(RM)$, corresponding to the three possibilites to distribute two covariant derivatives among two sections. All, this does not pose a problem in flat spacetime, where partial derivatives commute, hence, all canonical products on $\Gamma_0(BM)$ and the one on $\Gamma_0(RM)$ are equivalent in this case [Mak11]. However, in curved spacetimes, covariant derivatives do not commute, consequently, the operator $\mathcal{R}_{BW}$ is likely not to satisfy *(Selfadjointness)* on $\Gamma_0(BM)$ endowed with any of the canonical products. Indeed, in [Mak11] it has been shown that $\mathcal{R}_{BW}$ is not formally selfadjoint with respect to the canonical product where the field on the right is differntiated twice on $\Gamma_0(BM)$ by proving that a necessary condition for *(Selfadjointness)* to hold, the covariant conservation of the canonical current associated to $\mathcal{R}_{BW}$ and the chosen product on $\Gamma_0(BM)$, is only met in spacetimes of constant curvature.

5 A large class of modified Rarita-Schwinger equations and a no-go theorem for their consistent quantization

After posing sufficient conditions for a consistent canonical quantization in definition 1 and discussing several counterexamples, we proceed to the first main goal of this paper, i.e. proving that a large class of first order differential operators $\mathcal{R}$ on $\Gamma(RM)$ fails to satisfy all four conditions *(Covariance)*, *(Irreducibility)*, *(Selfadjointness)*, and *(Causality)*. In the course of proving this no-go theorem, it will become clear that the proof can be extended to any larger class of operators without much effort, such that the class we shall consider can be safely regarded as effectively exhausting all possible covariant first order differential operators on $\Gamma(RM)$. Our proof does not cover operators on the projected bundle PRM (cf. subsection 4.4), since any such operator must be equal to $\mathcal{R}_P$ on flat spacetime due to the requirement that it maps $\Gamma(RP\mathbb{M})$ to itself, and we have seen that already $\mathcal{R}_P$ itself does not fulfil *(Irreducibility)* in subsection 4.4.

Theorem 2 *Let $\mathcal{R}$ be a differential operator on $\Gamma(RM) \ni \psi^\alpha$ of the form*

$$\mathcal{R}\psi^\alpha := \left(-i\nabla\!\!\!/ + m\right)\psi^\alpha + a_0 m\gamma^\alpha \psi\!\!\!/ + a_1 i\nabla^\alpha \psi\!\!\!/ + a_2 i\gamma^\alpha \nabla_\mu \psi^\mu + a_3 i\gamma^\alpha \nabla\!\!\!/ \psi\!\!\!/ + \widetilde{\psi}^\alpha$$

where $a_i \in \mathbb{C}$ are arbitrary constants and $\widetilde{\psi}^\alpha$ contains the following explicit non-minimal curvature couplings[8]

$$\begin{aligned}
\widetilde{\psi}^\alpha &:= m\gamma^\alpha B + mC^\alpha + iD^\alpha + i\gamma^\alpha E\\
B &:= b_1 R_{\mu\nu}\gamma^\mu\psi^\nu + b_2 R\psi\!\!\!/\\
C^\alpha &:= c_1 R^\alpha{}_\nu\psi^\nu + c_2 R^\alpha{}_\nu\gamma^\nu\psi\!\!\!/ + c_3 R\psi^\alpha + c_4\mathfrak{R}^\alpha{}_\nu\psi^\nu\\
D^\alpha &:= d_1 R^\alpha{}_\nu \nabla\!\!\!/\psi^\nu + d_2\left(\nabla\!\!\!/ R^\alpha{}_\nu\right)\psi^\nu + d_3 R^\alpha{}_\nu\gamma^\nu\nabla\!\!\!/\psi\!\!\!/ + d_4\left(\nabla\!\!\!/ R^\alpha{}_\nu\right)\gamma^\nu\psi\!\!\!/ + d_5 R\nabla\!\!\!/\psi^\alpha + d_6\left(\nabla\!\!\!/ R\right)\psi^\alpha\\
&\quad + d_7 R^\alpha{}_\nu\nabla^\nu\psi\!\!\!/ + d_8\left(\nabla^\alpha R\right)\psi\!\!\!/ + d_9 R\nabla^\alpha\psi\!\!\!/ + d_{10}\mathfrak{R}^\alpha{}_\nu\nabla^\nu\psi\!\!\!/ + d_{11}\left(\nabla^\nu\mathfrak{R}^\alpha{}_\nu\right)\psi\!\!\!/ + d_{12}R_{\mu\nu}\nabla^\alpha\gamma^\mu\psi^\nu\\
&\quad + d_{13}\left(\nabla^\alpha R_{\mu\nu}\right)\gamma^\mu\psi^\nu + d_{14}\left(\nabla_\nu R^\alpha{}_\mu\right)\gamma^\mu\psi^\nu + d_{15}R^\alpha{}_\mu\gamma^\mu\nabla_\nu\psi^\nu\\
E &:= e_1 R_{\mu\nu}\gamma^\nu\nabla\!\!\!/\psi^\nu + e_2\left(\nabla\!\!\!/ R_{\mu\nu}\right)\gamma^\mu\psi^\nu + e_3 R\nabla\!\!\!/\psi\!\!\!/ + e_4\left(\nabla\!\!\!/ R\right)\psi\!\!\!/ + e_5\left(\nabla_\nu R\right)\psi^\nu + e_6 R\nabla_\nu\psi^\nu\\
&\quad + e_7\left(\nabla^\mu\mathfrak{R}_{\mu\nu}\right)\psi^\nu + e_8\mathfrak{R}_{\mu\nu}\nabla^\mu\psi^\nu + e_9 R_{\mu\nu}\nabla^\mu\psi^\nu\,.
\end{aligned}$$

Here, derivatives in brackets are meant to act only on the curvature tensors in the brackets, and b_i, c_i, d_i, e_i are arbitrary complex-valued functions of curvature invariants and m of

[8] Note that all couplings containing the Riemann tensor $R_{\alpha\beta\mu\nu}$ can be expressed via the spin curvature tensor $\mathfrak{R}_{\alpha\beta}$. Furthermore, we have omitted all couplings which would be linearly dependent by means of Bianchi identities.

mass dimension −2. No such $\mathcal{R}$ fulfils all four conditions (Irreducibility), (Covariance), (Causality), and (Selfadjointness).

Proof. We start by checking *(Selfadjointness)*, since this turns out to be the strongest condition. Indeed, as one can check by direct computation, *(Selfadjointness)* is fulfilled on arbitrary curved spacetimes iff the following conditions hold.

$$a_0^* = a_0 \qquad a_2 = a_1^* \qquad a_3^* = a_3 \qquad b_1 = c_2^* \qquad b_2^* = b_2 \qquad c_1^* = c_1 \qquad c_3^* = c_3 \qquad c_4^* = c_4$$

$$d_1 = d_3 = d_5 = d_7 = d_9 = d_{10} = d_{12} = d_{15} = e_1 = e_3 = e_6 = e_8 = e_9 = 0$$

$$d_2^* = d_2 \qquad d_4^* = e_2 \qquad d_6^* = d_6 \qquad d_8 = e_5^* \qquad d_{11} = e_7^* \qquad d_{13}^* = d_{14} \qquad e_4^* = e_4$$

Here, * denotes complex conjugation. In essence, requiring *(Selfadjointness)* rules out terms where a curvature tensor multiplies a derivative of ψ^α, because such terms generate derivatives of curvature tensors by the partial integration involved in the definition of the formal adjoint of $\mathcal{R}^\dagger$. These curvature tensor derivatives can not be cured by explicitly adding couplings of ψ^α to curvature derivatives, as such terms must be present both in $\mathcal{R}$ and in $\mathcal{R}^\dagger$. Hence, *(Selfadjointness)* rules out *arbitrary* terms where a curvature tensor multiplies a derivative of ψ^α, extending the validity of this proof to a larger class of $\mathcal{R}$ containing all possible such terms. Moreover, the remaining terms in $\mathcal{R}$ allowed by *(Selfadjointness)* must be either "symmetric" themselves or appear in a "symmetrised" fashion. Altogether one sees why the Buchdahl-Rarita-Schwinger-operator $\mathcal{R}_B$ and the minimally coupled "modern" Rarita-Schwinger-operator $\mathcal{R}_1$ are already ruled out by *(Selfadjointness)*.

We proceed by checking *(Causality)*. We have to check (see e.g. [BäGi11, remark 2.27]) that the principal symbol σ^μ of $\mathcal{R}$ is an invertible linear map if contracted with timelike or spacelike k^μ, but is non-invertible once contracted with a lightlike k_μ. Let k_μ be timelike or spacelike and let $\psi^\alpha \in \Gamma_0(RM)$ fulfil

$$ik_\mu \sigma^\mu \psi^\alpha = \not{k}\psi^\alpha - a_1 k^\alpha \not{\psi} - a_2 \gamma^\alpha k_\mu \psi^\mu - a_3 \gamma^\alpha \not{k}\not{\psi} = 0\,,$$

where we have already taken into account that the allowed principal symbols are reduced by *(Selfadjointness)*. We have to check for which a_i the above equation implies $\psi^\alpha \equiv 0$. By multiplying the above equation with $\not{k}$ and k^α, we can obtain the following derived equations

$$(1-a_2)\not{k} k_\mu \psi^\mu = (a_1 + a_3) k^2 \not{\psi} \qquad (1-3a_2)\not{k} k_\mu \psi^\mu = (1+3a_3) k^2 \not{\psi}\,,$$

which we can rewrite as

$$\begin{pmatrix} (1-a_2)\underline{1} & -(a_1+a_3)\underline{1} \\ (1-3a_2)\underline{1} & -(1+3a_3)\underline{1} \end{pmatrix} \begin{pmatrix} \not{k} k_\mu \psi^\mu \\ k^2 \not{\psi} \end{pmatrix} = 0\,.$$

As k_μ is timelike or spacelike, this equation together with $ik_\mu \sigma^\mu \psi^\alpha = 0$ implies $\psi^\alpha \equiv 0$ iff the determinant of the appearing 8×8 matrix is non-zero; this in turn is the case iff

$$-3a_1 a_2 + a_1 + a_2 - 2a_3 - 1 \neq 0\,. \tag{9}$$

We do not discuss lightlike k_μ, as (9) will be sufficient to prove the theorem.[9]

Finally, we check *(Covariance)* and *(Irreducibility)*. To this avail, we contract $\mathcal{R}\psi^\alpha = 0$ with both γ_α and ∇_α, combine the results and use the spin curvature identities listed in appendix B to obtain the following equation for $\not{\psi}$.

$$\begin{aligned} -\left(\frac{(a_2-1)(1+a_2+4a_3)}{2-4a_2}+a_1+a_3\right)\Box\not{\psi} \\ +\left(\frac{(a_2-1)(1+4a_0)}{2-4a_2}+\frac{1+a_1+4a_3}{2-4a_2}+a_0\right)im\not{\nabla}\not{\psi} \\ +\left(\frac{(a_2-1)(1+a_2+4a_3)}{2-4a_2}+a_3\right)\frac{R}{4}\not{\psi}+\frac{1+4a_0}{2-4a_2}m^2\not{\psi}-\frac{1}{2}R_{\mu\nu}\gamma^\mu\psi^\nu \\ +\frac{a_2-1}{2-4a_2}i\not{\nabla}\widetilde{\not{\psi}}+i\nabla_\mu\widetilde{\psi}^\mu+\frac{m}{2-4a_2}\widetilde{\not{\psi}}=0\,. \end{aligned} \tag{10}$$

Here, requiring *(Irreducibility)* assures that $2-4a_2 \neq 0$. To see this, note that contracting $\mathcal{R}\psi^\alpha = 0$ with γ_α yields an equation which can be rewritten as

$$(2-4a_2)i\nabla_\mu\psi^\mu = (1+a_1+4a_3)i\not{\nabla}\not{\psi}+(1+4a_0)m\not{\psi}+\widetilde{\not{\psi}}\,. \tag{11}$$

If $2-4a_2=0$, then $i\nabla_\mu\psi^\mu = 0$ would not follow from $\mathcal{R}\psi^\alpha = 0$ and $\not{\psi} = 0$ on Minkowski spacetime, hence $\not{S}(\mathcal{R},\mathbb{M}) \subset \not{S}(\mathcal{R}_0,\mathbb{M})$ would not hold.

To assure that *(Covariance)* holds, we have to either guarantee that $\not{\psi}^\alpha = A_\mu\psi^\mu$ holds automatically for solutions of $\mathcal{R}\psi^\alpha = 0$ or that $A_\alpha \equiv 0$ on all spacetimes. Let us check if the first of these conditions can be fulfilled. Without specifying A_μ explicitly, we know that, in Minkowski spacetime, $A_\mu \equiv 0$ must hold on account of *(Irreducibility)*. However, in flat spacetime, (10) is a hyperbolic partial differential equation for $\not{\psi}$, as the coefficient of $\Box\not{\psi}$ is non-zero if we apply the condition (9) derived from *(Causality)* and *(Selfadjointness)*. Such a differential equation has certainly more possible solutions than just $\not{\psi} \equiv 0$, hence, by combining *(Causality)*, *(Selfadjointness)*, and *(Irreducibility)*, we find that only the optional condition in *(Covariance)* that A_μ be identically vanishing on all spacetimes can be fulfilled. Inserting this into (10), we are left with

$$-\frac{1}{2}R_{\mu\nu}\gamma^\mu\psi^\nu-\frac{a_2-1}{2-4a_2}i\not{\nabla}\widetilde{\not{\psi}}+i\nabla_\mu\widetilde{\psi}^\mu+\frac{m}{2-4a_2}\widetilde{\not{\psi}}=0\,. \tag{12}$$

In Minkowski spacetime, this equation is identically fulfilled and, hence, poses no additional constraints on solutions of $\mathcal{R}\psi^\alpha = 0$ and $\not{\psi} = 0$. To check if *(Covariance)* holds, we have to make sure that (12) is identically fulfilled on *all* spacetimes once $\mathcal{R}\psi^\alpha = 0$

[9] From the mathematical point of view, strict hyperbolicity does not require that the notions of "timelike", "spacelike" and "lightlike" must be the ones inferred from $g_{\mu\nu}$, but they could be related to any Lorentzian metric $g'_{\mu\nu}$ on M. However, our discussion of $k_\mu\sigma^\mu$ for k_μ spacelike or timelike w.r.t. $g_{\mu\nu}$ implies that (9) is a necessary condition for $\mathcal{R}$ to be strictly hyperbolic w.r.t. to any Lorentzian metric on M.

and $\not\psi = 0$ hold. To this avail, we insert $\not\psi = 0$ into (11), and both $\not\psi = 0$ and (11) into $\mathcal{R}\psi^\alpha = 0$ to obtain

$$i\nabla_\mu \psi^\mu = \frac{1}{2-4a_2}\widetilde{\not\psi}\,, \qquad (-i\not\nabla + m)\psi^\alpha + \frac{a_2}{2-4a_2}\gamma^\alpha \widetilde{\not\psi} + \widetilde{\psi}^\alpha = 0\,.$$

These two equations are the only information on first derivatives of ψ^α which one can obtain from $\mathcal{R}\psi^\alpha = 0$ and $\not\psi = 0$. However, the summand $\nabla_\mu \widetilde{\psi}^\mu$ in (12) contains first derivatives of ψ^α also in terms like e.g. $R_{\mu\nu}\nabla^\mu\psi^\nu$, on which $\mathcal{R}\psi^\alpha = 0$ and $\not\psi = 0$ give no information in general curved spacetimes. Hence, these terms must identically vanish in $\nabla_\mu \widetilde{\psi}^\mu$, which implies that the coefficients of all terms in $\widetilde{\psi}^\alpha$ which survive after inserting $\not\psi = 0$ and whose free index $^\alpha$ does not belong to γ^α or ψ^α must vanish. Moreover the coefficients of all terms where γ^α appears followed by other γ-matrices must vanish as well, as these terms also give rise to terms like e.g. $R_{\mu\nu}\nabla^\mu\psi^\nu$ if one considers them in $\nabla_\mu \widetilde{\psi}^\mu$ and commutes the contracted covariant derivative $\not\nabla$ with the additional γ-matrices in order to use the available information on $\not\nabla\psi^\alpha$. Analogously, the terms in $\widetilde{\psi}^\alpha$ where the free index $^\alpha$ belongs to ψ^α but ψ^α is multiplied by γ-matrices are problematic in $\not\nabla\widetilde{\not\psi}$ and have to vanish identically. Altogether, avoiding the appearance of in general undetermined ψ^α-derivatives in (12) enforces

$$b_1 = c_1 = c_4 = d_2 = d_6 = d_{13} = d_{14} = e_2 = e_7 = 0\,,$$

hence, the remaining terms in $\widetilde{\psi}^\alpha$ not yet ruled out by *(Covariance)* are

$$\widetilde{\psi}^\alpha = mc_3 R\psi^\alpha + e_5\gamma^\alpha \left(\nabla_\nu R\right)\psi^\nu\,.$$

We can now explicitly compute the left hand side of (12) by inserting this expression for $\widetilde{\psi}^\alpha$ and the knowledge on $\nabla_\mu\psi^\mu$ and $\not\nabla\psi^\alpha$ obtained from $\mathcal{R}\psi^\alpha = 0$ and $\not\psi = 0$. The result does not contain any derivatives of ψ^α, but is a sum various curvature tensors multiplying ψ^α. In general spacetimes, some of these terms are linearly independent and, hence, have to vanish individually in order for (12) to be identically fulfilled on all spacetimes. Particularly, since the only term in the left hand side of (12) containing the Ricci tensor turns out to be the one explicitly visible in (12), we obtain

$$R_{\mu\nu}\gamma^\mu\psi^\nu = 0$$

as a necessary condition for (12) to hold on general spacetimes. However, this is in conflict with *(Covariance)*, which closes the proof. One can imagine that the steps taken in the last paragraph of this proof can be generalised to arbitrary couplings of the curvature to ψ^α, and we have argued in the discussion of *(Selfadjointness)* that the same holds for arbitrary couplings of the curvature to derivatives of ψ^α, hence, we presume that our proof effectively exhausts *all* possible covariant first order differential operators on $\Gamma(RM)$. Finally, we would like to emphasise that our proof covers both $m > 0$ and $m = 0$. □

6 Supergravity - a possible way out?

In the first part of this work, we have proven a no-go theorem for the consistent quantization of a free spin $\frac{3}{2}$ field on spacetimes whose curvature is not explicitly known *a priori*. To make sure that this lack of knowledge has no influence on the outcome of the quantization procedure, we have imposed the consistency condition *(Covariance)* (cf. section 3), i.e. we have demanded that the number of degrees of freedom of the field theory is the same in *all* curved spacetimes. As can be inferred from the previous section, the condition *(Covariance)* has indeed played a crucial role in proving that none of the considered free spin $\frac{3}{2}$ field equations leads to a quantum theory satisfying the consistency conditions in definition 1.

In supergravity theories, however, one is mostly concerned with a setup where the background spacetime does satisfy the Einstein equation, even more, it satisfies the Einstein equation with the energy momentum tensor determined by the field content of the considered supermultiplets. Indeed, in the proof of the consistency of classical, simple $\mathcal{N} = 1$ supergravity [DeZu76] it has become clear that supergravity can solve the usual problems related to higher spin equations [Buc58] by requiring that both the gravitino and the background metric are on shell. Hence, *(Covariance)* is probably not a good condition to ask for in supergravity (at least not without further modifications), since the class of possible spacetime backgrounds is limited *a priori*. Consequently, our no-go theorem in the last section is not directly applicable to supergravity theories.

In fact, it has been worked out that supergravity can be quantized consistently in Minkowski spacetime, see e.g. [VNi81], and (anti-)de Sitter spacetime, e.g. [Mal97], i.e. on locally supersymmetric solutions of the Einstein equations obtained from supergravity Lagrangeans [DeZu76, Tow77]. Hence, the prevailing belief is that supergravity is an internally consistent theory (up to possible UV issues at high loops [Kal09]). However, it seems that not many conceptual results on the status of supergravity theories on backgrounds which break local supersymmetry, e.g. general flat Robertson-Walker spacetimes of interest in cosmology, are available.

A detailed analysis in this direction is contained in a series of papers by Kallosh, Kofman, Linde, and Van Proyen [KKLP99, Kof99, KKLP00], where the authors analyse e.g. the production of gravitinos after inflation from first principles. In their model, as well as in all models where the gravitino has the standard kinetic term and the field equations are derived from a minimally coupled Lagrangean rather than imposed *ad hoc*, the free field equation for the gravitino is

$$(-i\nabla\!\!\!/ + m)\,\psi^\alpha + \left(i\nabla^\alpha + \frac{m}{2}\gamma^\alpha\right)\psi\!\!\!/ = 0\,. \tag{13}$$

More specifically, in the model of [KKLP99, Kof99, KKLP00], both the mass m of the gravitino and the background flat Robertson-Walker spacetime, i.e. the energy density and pressure, are completely specified in terms of a classical scalar field ϕ, which is assumed to depend only on time and whose potential $V(\phi)$ is of F-term type. We shall not go into details here, as the exact form of $V(\phi)$ is not important for our discussion, we shall just take into account that m is implicitly time-dependent. We have already discussed the properties of the field equation (13) in subsection 4.2, however, the derived equations obtained from (13) by contracting it with γ_α and ∇_α slightly differ from the case discussed there because m is not constant. They are,

$$i\nabla_\mu\psi^\mu = \left(i\nabla\!\!\!/ + \frac{3}{2}m\right)\psi\!\!\!/$$

$$3m^2\psi\!\!\!/ = G_{\mu\nu}\gamma^\mu\psi^\nu + 2(i\nabla\!\!\!/ m)\psi\!\!\!/ - 2(i\nabla_\alpha m)\psi^\alpha\,. \tag{14}$$

Notwithstanding, the kinetic term in this equation is independent of the mass term, and we have mentioned in subsection 4.2 that the kinetic term in (13) fails to satisfy *(Causality)*, i.e. the solutions of the field equation (13) do not propagate causally in general. Hence, the first non-trivial check of the consistency of (13) is to analyse whether this equation leads to causal propagation in all backgrounds determined by the supergravity model.

Indeed, Kofman verifies in [Kof99] that this is the case, and proves that the specific F-term form of the scalar potential $V(\phi)$ assures the validity of *(Causality)*. To achieve this result, one realises that, in flat Robertson-Walker spacetimes, in comoving coordinates, and under the assumption that m is at most time-dependent, (14) can be written as

$$\gamma_0\psi^0 = A\gamma_i\psi^i\,, \tag{15}$$

where A is some four-by-four matrix and the left hand side is implicitly summed over $i \in \{1,2,3\}$; in the case at hand, the detailed form of A depends on the specifics of the background scalar field ϕ. Note that, if local supersymmetry is unbroken, then (14) is identically satisfied for all ψ^α and does not give any information about $\gamma_0\psi^0$, thus, A is not defined in this case. Motivated by the high symmetry of flat Robertson-Walker spacetimes and by (15), the authors of [KKLP99, Kof99, KKLP00] split the classical gravitino field ψ^α into a transversal part ψ^α_T and a longitudinal part ψ^α_L, where ψ^α_L is defined by $\gamma_i\psi^i_L = 0$, and hence satisfies $\psi^0_L = 0$ due to (15). ψ^α_L corresponds to helicity $\frac{3}{2}$ degrees of freedom already present prior to local supersymmetry breaking, whereas ψ^α_T corresponds to helicity $\frac{1}{2}$ degrees of freedom induced by this breaking. Now one can observe that ψ^α_L satisfies a simple Dirac equation, whose kinetic term satisfies *(Causality)*, whereas one can use the specific form of A determined by the local symmetry of the model Lagrangean to show that ψ^α_T also propagates causally [Kof99].

Although local supersymmetry apparently solves the potential problem of acausal propagation in the *classical* theory, the authors of [KKLP99, Kof99, KKLP00] do not seem to check whether their model leads to a consistent *quantum* theory. We shall now prove that this is not the case, and our proof shall be valid for all models where the gravitino has the standard kinetic term and the background spacetime is not a constant curvature spacetime. We recall that, in a consistent, i.e. unitary, quantum theory of the the free gravitino, which is the building block of any perturbative quantization of the interacting gravitino, the anticommutator between a gravitino field and its adjoint has a positive or zero expectation value in all quantum states. As we have discussed in sections 2 and 3, a non-trivial prerequisite for this to hold is that i times the anticommutator function $G^{\alpha}_{\beta}(x,y)$ – the difference of the advanced ${G^{+}}^{\alpha}_{\beta}(x,y)$ and the retarded ${G^{-}}^{\alpha}_{\beta}(x,y)$ Green's function of the free gravitino field equation – gives a positive number when integrated with infinitely often differentiable functions of compact support. Moreover, as we prove in proposition 6 in appendix E, this in turn only holds if the canonical current constructed from the principal symbol σ^μ – the coefficient of the highest derivative – of the field equation is covariantly conserved. In the case at hand, the principal symbol σ^μ of the field equation (13) is

$$\sigma^\mu \psi^\alpha := -i\gamma^\mu\psi^\alpha + i g^{\mu\alpha}\not{\psi}$$

and the corresponding canonical current of two classical solutions ϕ^α_1 and ϕ^α_2 of the field equation reads

$$j^\mu[\psi^\alpha_1, \psi^\beta_2] := \overline{\psi}_{1,\alpha}\sigma^\mu\psi^\alpha_2\,.$$

Proposition 6 now implies that

$$\nabla_\mu j^\mu[\psi^\alpha_1, \psi^\beta_2] = 0$$

is a necessary requirement for the canonically quantized field theory corresponding to (13) to be unitary and, hence, consistent. Let us compare the assumptions of proposition 6 with the considered case. In this result, it is required that the principal symbol is covariantly conserved, such that $\nabla_\mu\sigma^\mu$ and $\sigma^\mu\nabla_\mu$ are the same differential operator. This requirement is certainly fulfilled, as both the gamma matrices γ^α and the metric $g^{\alpha\beta}$ are covariantly conserved. Furthermore, it is required that the field equation has Majorana solutions; this is the case if the mass m is real. Finally, proposition 6 is valid only if one considers Majorana spinors and states that the current $j^\mu[\psi^\alpha_1, \psi^\beta_2]$ must be conserved only for Majorana solutions which satisfy an additional irreducibility constraint like $\not{\psi} = 0$. This also applies here, as in supergravity one usually considers the gravitino to be a Majorana Fermion. Moreover, no additional constraint is imposed in the discussed model. Rather, the usual irreducibility constraint $\not{\psi} = 0$ is replaced by the derived identity (15) which automatically holds for all solutions of (13).

To sum up, a consistent canonical quantization of the classical field theory associated to (13) is possible if and only if $\nabla_\mu j^\mu[\psi^\alpha_1, \psi^\beta_2]$ vanishes for any pair of Majorana solutions ψ^α_1, ψ^α_2 of (13). Hence, one can prove a no-go theorem by giving examples for solutions whose classical current does not vanish.

Theorem 3 *Let the free gravitino on a flat Robertson-Walker spacetime be described by the field equation*

$$\left(-i\not{\nabla} + m\right)\psi^\alpha + \left(i\nabla^\alpha + \frac{m}{2}\gamma^\alpha\right)\not{\psi} = 0$$

where m is real and possibly time-dependent, and ψ^α is taken to be a Majorana field. Assume that the derived equation

$$3m^2\not{\psi} = G_{\mu\nu}\gamma^\mu\psi^\nu + 2(i\not{\nabla}m)\not{\psi} - 2(i\nabla_\alpha m)\psi^\alpha$$

is not fulfilled for all differentiable functions ψ^α, but only for solutions of the field equation; then this equation can be written as

$$\gamma_0 \psi^0 = A \gamma_i \psi^i \,.$$

Finally, assume that the background spacetime is not an (anti-)de Sitter spacetime or Minkowski spacetime; this implies $A \neq -\underline{1}$, such that $\not\psi = 0$ does not automatically hold for all solutions ψ^α of the field equation. Under all these assumptions, a consistent, i.e. unitary, canonical quantum theory of the free gravitino does not exist.

Proof. According to proposition 6 in appendix E, the theorem is proven if we show that there are solutions of the field equation whose current

$$j^\mu[\psi_1^\alpha, \psi_2^\beta] := \overline{\psi}_{1,\alpha} \sigma^\mu \psi_2^\alpha := \overline{\psi}_{1,\alpha} \gamma^\mu \psi_2^\alpha + \overline{\psi}_1^\mu \not\psi_2$$

is not covariantly conserved. A direct computation using the field equation yields

$$\nabla_\mu j^\mu[\psi_1^\alpha, \psi_2^\beta] = \overline{i \nabla_\alpha \not\psi_1} \psi_2^\alpha - \overline{i \nabla_\alpha \psi_1^\alpha} \not\psi_2 \,.$$

Let us now consider the case where ψ_2^α is of longitudinal type, i.e., in comoving coordinates of flat Robertson-Walker spacetimes, $\psi_2^0 = 0$ and $\not\psi_2 = 0$. We additionally require that $\psi_2^3 = 0$, such that $\gamma_1 \psi_2^1 = -\gamma_2 \psi_2^2$. Moreover, we require that ψ_1^α is of transversal type and does not satisfy $\not\psi_1 = 0$. The existence of such solutions follows by the assumptions of the theorem and the special symmetry of flat Robertson-Walker spacetimes. For the chosen solutions, one has

$$\nabla_\mu j^\mu[\psi_1^\alpha, \psi_2^\beta] = \overline{i \nabla_\alpha \not\psi_1} \psi_2^\alpha = \overline{i \gamma^1 \nabla_1 \not\psi_1} \gamma_1 \psi_2^1 + \overline{i \gamma^2 \nabla_2 \not\psi_1} \gamma_2 \psi_2^2 = (\overline{i \gamma^1 \nabla_1 \not\psi_1 - i \gamma^2 \nabla_2 \not\psi_1}) \gamma_1 \psi_2^1 \,,$$

where we have made use of that fact that $\gamma_1 \gamma^1 = \gamma_2 \gamma^2 = \underline{1}$. Let us assume that this is vanishing for all solutions of the type considered. By our assumption on ψ_2^α, ψ_2^1 is an arbitrary solution of the Dirac equation. The space spanned by the values of all solutions of the Dirac equation at a point x is equal to $\mathbb{C}^4$ (or $\mathbb{R}^4$ if one considers the Majorana representation of the Clifford algebra and takes into account that ψ^α are Majorana spinors); this holds because one can give arbitrary $\mathbb{C}^4$-initial conditions for the Dirac equation at one point. The vanishing of $\nabla_\mu j^\mu[\psi_1^\alpha, \psi_2^\beta]$ for all solutions of the considered type hence implies that $\gamma^1 \nabla_1 \not\psi_1 - \gamma^2 \nabla_2 \not\psi_1$ itself must be vanishing by the non-degeneracy of the $\mathbb{C}^4$ scalar product. If this is the case, the Fourier transform of this expression with respect to $\vec{x}$, i.e.

$$k_1 \gamma^1 \widehat{\not\psi_1} - k_2 \gamma^2 \widehat{\not\psi_1}$$

must vanish as well. Unless $k_1 = k_2 = 0$ (which we can rule out since there are solutions ψ_1 which are not constant in x_1 and x_2, and, hence, their Fourier transform will not be a δ-function in k_1 and k_2), the vanishing of the above expression implies that $\widehat{\not\psi_1}$ is an eigenvector of both $\gamma_1 \gamma_2$ and $\gamma_2 \gamma_1$. However, the only possible such eigenvector is the zero vector, hence, $\widehat{\not\psi_1}$ must be vanishing, which contradicts our assumptions. Consequently, there are solutions in the class we have considered whose current is *not* conserved. □

The above proof shows that the crucial obstacle for a consistent quantization is the additional first derivative term $i\nabla^\alpha \not\psi$ in the field equation. If the gravitino is massless and the background spacetime is (anti-)de Sitter or Minkowski, one can "gauge away" this term by the partial gauge fixing $\not\psi = 0$. Hence, no problems arise if local supersymmetry is unbroken.

Moreover, as the class of solutions we used as an example for a non-conserved current was of mixed longitudinal-transversal type, one could be tempted to think that the current is conserved with the class of longitudinal or transversal solutions, such that the helicity $\frac{3}{2}$ and $\frac{1}{2}$ degrees of freedom could be quantized individually, but not as a coherent superposition. This is certainly the case for the longitudinal degrees of freedom, but, without giving a proof, it seems very unlikely that this holds for the transversal degrees of freedom, since the individual spatial components of these solutions are in general *not* linked by algebraic relations, but only by differential relations, as one can see by writing down the field equation for α being a spatial index. However, it seems that only algebraic relations can help in assuring the conservation of the current for *all* solutions in a considered class.

7 Conclusions

In this work, we have been concerned with the general issue of whether spin $\frac{3}{2}$ theories can be consistently quantized on curved spacetimes. In the general framework of quantum field theory on curved spacetimes, where one usually requires that the construction of the quantum theory is independent of the background spacetime, we have been able to prove that a large class of non-minimally coupled Rarita-Schwinger field equations does not satisfy a certain set of sufficient conditions for a consistent canonical quantum theory. We have argued why we consider this class to effectively exhaust all possible non-minimally coupled Rarita-Schwinger field operators, and why it seems unlikely that the situation improves if one instead considers (modifications of) the Buchdahl-Wünsch operator, which is written down in the only alternative spin $\frac{3}{2}$ representation of $SL(2,\mathbb{C})$.

As the point of view in supergravity theories is different in that the background spacetime is assumed to be well-known and controlled by the Einstein equations, we had to analyse these separately. We have shown that there is a necessary condition for a spin $\frac{3}{2}$ quantum field theory to be consistent which does not depend on whether or not one assumes knowledge of the background curvature to enter the construction of the quantum theory. We have shown that this condition is violated in all supergravity theories where the gravitino has the standard kinetic term and the background spacetime is not eternal (anti-)de Sitter or Minkowski spacetime. Hence, our result is model independent.

Our result strongly contradicts the usual intuition that curvature effects are small and can be treated perturbatively, such that, in particle physics, one can mostly just compute on Minkowski spacetime and trust the results. Quite on the contrary, quantum supergravity is fine on Minkowski spacetime and (anti-)de Sitter spacetime, but fails to be consistent as soon as the smallest bit of non-constant spacetime curvature is around.

We stress that our proof implies that, in contrast to the usual failure of unitarity at high energies, the failure of unitarity we find here is independent of the energy scale.

To close, we point out potential loopholes of our no-go theorem. Among other things, our theorem is based on the assumptions that

- the gravitino is a Majorana particle (however, we presume that the crucial proposition 6 in appendix E also holds for Dirac spinors),
- the gravitino has the standard kinetic term,
- one wants to canonically quantize the classical field theory, i.e. by means of the canonical anticommutation relations.

There may certainly be (and there most likely are) more loopholes of our no-go theorem we are not aware of. In any case, if no good and convincing evasion is found, the status of supergravity as a theory describing (part of) nature is uncertain.

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A Dirac and Majorana spinors

By Pauli's theorem, two different representations of the Clifford algebra $\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha\beta}\underline{1}$ are related by a similarity transformation, hence, there are matrices β, C which fulfil

$$(\gamma^\alpha)^\dagger = \beta\gamma^\alpha\beta^{-1} \qquad (\gamma^\alpha)^T = -C\gamma^\alpha C^{-1}\,,$$

where † (^{T}) denotes the Hermitean adjoint (transpose) of a matrix. One can additionally fix $\beta^\dagger = \beta$, and, in a standard representation of the Clifford algebra (e.g. in the Dirac-, Majorana-, or Weyl-representation) with $(\gamma^0)^\dagger = \gamma^0$, $(\gamma^i)^\dagger = -\gamma^i$, set $\beta = \gamma^0 = \beta^{-1}$. We shall work with a standard representation throughout this work. Given β and C, one can define the Dirac adjoint $\bar\psi$ and charge conjugated ψ^c version of a Dirac spinor as[10]

$$\overline{\psi} := \psi^\dagger\beta \qquad \psi^c := \beta C^\dagger\psi^*\,,$$

where * denotes complex conjugation and define the same operations on cospinors in such a way that $\overline{\overline{\psi}} = \psi$, $(\psi^c)^c = \psi$, $(\overline{\psi})^c = -\overline{\psi^c}$ and the same identities hold for cospinors.

[10] One can define C via the complex-conjugated rather than transposed representation of the Clifford algebra and then define the charge-conjugated spinor in a different manner such that the overall definition is equivalent to the one used here.

The above definitions of γ^α, β, C and the associated conjugations can be extended from Minkowski spacetime to a curved spacetime M by employing a frame/tetrad/vielbein-basis of the tangent bundle TM and Dirac bundle DM of M; the resulting section of γ-matrices is covariantly constant with respect to the Levi-Civita covariant derivative, see e.g. [San09, Hac10, Müh11]. Furthermore, the definitions of the Dirac and charge conjugation can be straightforwardly extended to Rarita-Schwinger spinors by applying them to the Dirac factor of the tensor product $DM \otimes TM$.

A spinor is defined to be Majorana if $\psi^c = \psi$. In the (real-linear) Majorana representation, where all γ-matrices are imaginary[11], one can choose $C = \gamma^0$, such that the Majorana condition becomes $\psi^* = \psi$. Hence, one often says that Majorana spinors are real, but the Majorana condition and the statement that every Dirac spinor is a unique complex linear combination of two Majorana spinors is independent of the chosen representation of the Clifford algebra, see e.g. [San09].

B Spinor curvature tensor identities

The curvature tensor $\mathfrak{R}_{\mu\nu}$ of the Levi-Civita connection on DM fulfils the following identities, see e.g. [Hac10, sec. I.2.2] for a proof.

$$\mathfrak{R}_{\mu\nu} = \frac{1}{4} R_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma \qquad -\gamma^\mu \mathfrak{R}_{\mu\nu} = \mathfrak{R}_{\mu\nu}\gamma^\nu = \frac{1}{2} R_{\mu\nu}\gamma^\mu \qquad \gamma^\mu\gamma^\nu\mathfrak{R}_{\mu\nu} = \mathfrak{R}_{\nu\mu}\gamma^\mu\gamma^\nu = \frac{1}{2}R$$

Note that, defining the Riemann and Ricci tensor, as well as the Ricci scalar by the convention chosen in [Wa84], these identities are valid for both metric sign conventions.

C Mode solutions of the Dirac equation

A complete and orthonormal set of mode solutions of the Dirac equation on Minkowski spacetime and in the Dirac representation of the Clifford algebra is given by

$$v^1_{\vec{k}}(t,\vec{x}) = \frac{e^{i(\vec{k}\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega+m)}} M^+ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad v^2_{\vec{k}}(t,\vec{x}) = \frac{e^{i(\vec{k}\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega+m)}} M^+ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v^3_{\vec{k}}(t,\vec{x}) = \frac{e^{-i(\vec{k}\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega+m)}} M^- \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad v^4_{\vec{k}}(t,\vec{x}) = \frac{e^{-i(\vec{k}\vec{x} - \omega t)}}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega(\omega+m)}} M^- \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where

$$M^\pm := \begin{pmatrix} (\mp\omega - m)\underline{1} & \vec{k}\vec{\sigma} \\ \vec{k}\vec{\sigma} & (\mp\omega + m)\underline{1} \end{pmatrix},$$

[11] If one chooses signature (-,+,+,+), they are real.

$\omega = \sqrt{\vec{k}^2 + m^2}$, and $\vec{\sigma}$ is the vector of Pauli matrices.

D The relation between equal-time and covariant canonical anticommutation relations

Let $\mathcal{R}$ be a strictly hyperbolic first order differential operator on some Hermitian vector bundle VM over a globally hyperbolic spacetime $(M, g_{\mu\nu})$ with principal symbol σ^μ, retarded and advanced Green's operators $G^\pm$, and causal propagator $G = G^- - G^+$. One can define a Hermitean product $\langle\cdot,\cdot\rangle$ on test sections of VM via

$$\langle f, g\rangle := \int\limits_M d_g x\, \bar{f}(x) g(x)\,,$$

where $\bar{\cdot}$ denotes the adjoint with respect to the Hermitian product on VM. With this setup, one can prove (see e.g. [BäGi11]) that the smeared causal propagator is the "charge" of a conserved current built from the principal symbol of $\mathcal{R}$.

***Lemma* 4** *Let $\mathcal{R}$ be formally selfadjoint with respect to $\langle\cdot,\cdot\rangle$, i.e. $\mathcal{R}^\dagger = \mathcal{R}$ with $\mathcal{R}^\dagger$ defined as $\langle\mathcal{R}^\dagger f, g\rangle := \langle f, \mathcal{R} g\rangle$ and let Σ be an arbitrary (smooth) Cauchy surface of $(M, g_{\mu\nu})$ with volume measure $d\Sigma$ and forward-pointing normal N^μ. Then $G(\bar{f}, g_{\mu\nu})$ can be expressed as*

$$G(\bar{f}, g_{\mu\nu}) = -\int\limits_\Sigma d\Sigma\, \overline{[Gf]}\sigma^\mu N_\mu [Gg]\,.$$

If M is foliated as[12] *$\{t\}\times\Sigma$ and $\vec{x}$ are coordinates on Σ, this identity is formally equivalent to*

$$G(t_1, \vec{x}_1, t_2, \vec{x}_2)|_{t_1=t_2} = -\sigma^\mu N_\mu \delta(\vec{x}_1, \vec{x}_2)\,.$$

Proof. We split M into the future (Σ^+) and past (Σ^-) of Σ and compute

$$\begin{aligned} G(\bar{f}, g_{\mu\nu}) &= \int\limits_{\Sigma^+} d_g x\, \bar{f} G g + \int\limits_{\Sigma^-} d_g x\, \bar{f} G g = \int\limits_{\Sigma^+} d_g x\, \overline{[\mathcal{R} G^- f]} G g + \int\limits_{\Sigma^-} d_g x\, \overline{[\mathcal{R} G^+ f]} G g \\ &= \langle \mathcal{R} G^- f, Gg\rangle_{\Sigma^+} + \langle \mathcal{R} G^+ f, Gg\rangle_{\Sigma^-} \\ &= \langle \mathcal{R} G^- f, Gg\rangle_{\Sigma^+} - \langle G^- f, \mathcal{R} Gg\rangle_{\Sigma^+} + \langle \mathcal{R} G^+ f, Gg\rangle_{\Sigma^-} - \langle G^+ f, \mathcal{R} Gg\rangle_{\Sigma^-}\,, \end{aligned}$$

where the definitions and support properties of $G^\pm$ and G have been used and where the index $_{\Sigma^\pm}$ means that the integration in the product $\langle\cdot,\cdot\rangle$ is restricted to $\Sigma^\pm$. Since $\mathcal{R}$ is formally selfadjoint, we can use Green's identity

$$\langle \mathcal{R}^\dagger f, g\rangle_{\mathcal{M}} - \langle f, \mathcal{R} g\rangle_{\mathcal{M}} = \int\limits_{\partial\mathcal{M}} d\partial\mathcal{M}\, \bar{f} \mathcal{N}^\mu \sigma_\mu g$$

[12] This is always possible on account of the results of [BeSa05, BeSa06].

valid for a manifold $\mathcal{M}$ with smooth boundary $\partial\mathcal{M}$, boundary volume measure $d\partial\mathcal{M}$, and outwards pointing boundary normal $\mathcal{N}^\mu$. Due to the support properties of $G^\pm f$, the only relevant boundary of $\Sigma^\pm$ is Σ, with outwards pointing normal $\mp N^\mu$. Hence, applying Green's identity and considering $G = G^- - G^+$ concludes the proof. □

If one would like to associate to a hyperbolic first order differential operator $\mathcal{R}$ a Fermionic quantum field theory whose covariant CAR are specified by G, i.e.

$$\{\boldsymbol{\psi}(\bar{f}), \overline{\boldsymbol{\psi}}(g)\} = iG(\bar{f}, g)\underline{1} := i\langle f, Gg\rangle\underline{1}\,,$$

the above lemma implies that one can deduce equal-time CAR from these covariant CAR if the differential operator $\mathcal{R}$ is formally selfadjoint, and we are not aware of any other way to do so if $\mathcal{R}$ does not bear this property. On the other hand, even if one imposes equal-time CAR at one time although $\mathcal{R}$ is not formally selfadjoint, there is, to the knowledge of the authors, no way to prove that these equal-time CAR are in fact time-independent and, hence, covariant.

E On the relation between positivity/unitary of the quantum field theory related to $\mathcal{R}$ and the selfadjointness of $\mathcal{R}$

In this section we would like to point out the strong relationship between the positivity of the product defined by the anticommutator function/causal propagator G of a first order differential operator $\mathcal{R}$ and the selfadjointness of $\mathcal{R}$. The first of our results implies that positivity on general spacetimes follows from selfadjointness if positivity on Minkowski spacetime is already known.

***Proposition* 5** *With the notation and definitions of section D, let $\Gamma(VM)$ ($\Gamma_0(VM)$) denote the smooth sections (compactly supported smooth sections) of VM and let $S(\mathcal{R}, M)$ be the space of solutions ψ of $\mathcal{R}\psi = 0$ with compactly supported initial conditions. Moreover, let $C\psi = 0$ be a linear, local and covariant constraint on $\Gamma(VM)$ and let $\not S(\mathcal{R}, M)$ ($\not\Gamma_0(VM)$) be the subspace of $S(\mathcal{R}, M)$ ($\Gamma_0(VM)$) defined as*

$$\not S(\mathcal{R}, M) := \{\psi \in S(\mathcal{R}, M) \,|\, C\psi = 0\}$$

$$\not\Gamma_0(VM) := \{f \in \Gamma_0(VM) \,|\, Gf \in \not S(\mathcal{R}, M)\}\,.$$

If $\mathcal{R}^* f = \mathcal{R} f$ for all $f \in \not\Gamma_0(VM)$ on arbitrary spacetimes and $iG(\bar{f}, f) \geq 0$ for all $f \in \not\Gamma_0(VM)$ on Minkowski spacetime, then $iG(\bar{f}, f) \geq 0$ for all $f \in \not\Gamma_0(VM)$ on all globally hyperbolic spacetimes with the topology of $\mathbb{R}^4$.

Proof. Let Σ be a Cauchy surface of M. For definiteness, we can pick a Cauchy surface which lies to the past of the support of f. From lemma 4 we know that $iG(\bar{f}, f)$ can be computed on Σ and that the result is independent of Σ. We can now use a result of [FNW81] to "deform" the past of Σ in $(M, g_{\mu\nu})$ into a piece of Minkowski spacetime

$(\mathbb{M}, \eta_{\mu\nu})$. In more detail, the authors of [FNW81] show that there exists a fiducial globally hyperbolic spacetime $(M', g'_{\mu\nu})$ which contains two Cauchy surfaces Σ_1 and Σ_2 such that Σ_1 lies to the future of Σ_2, the future of Σ_1 in $(M', g'_{\mu\nu})$ is isometric to the future of Σ in $(M, g_{\mu\nu})$ and the past of Σ_2 in $(M', g'_{\mu\nu})$ is isometric to the past of a Cauchy surface Σ_0 in $(\mathbb{M}, \eta_{\mu\nu})$. The computation of $iG(\bar{f}, f) \geq 0$ on Σ in $(M, g_{\mu\nu})$ is equivalent to the same computation on Σ_1 in $(M', g'_{\mu\nu})$, which, by lemma 4, gives the same result as a computation on Σ_2 in $(M', g'_{\mu\nu})$ and, hence, on Σ_0 in $(\mathbb{M}, \eta_{\mu\nu})$. By assumption, the latter gives a positive result; this proves $iG(\bar{f}, f) \geq 0$. □

The above proof may seem awkward, and one might think that it depends on the chosen deformation. However, this is not the case, and the reason for this is the covariance of all objects as well as the deterministic nature of solutions to hyperbolic partial differential equations. Given sufficient initial data at one "time", the future and past of the solutions are completely determined, no matter how the background spacetime "looks like" at those times. The apparent strength of the above employed deformation argument is the reason for its ubiquity in modern works on quantum field theory in curved spacetimes, e.g. [FNW81, Köh95, San08, DHP09]. We presume that the proof can be extended to spacetimes of arbitrary topology by a separation of unity argument.

While proposition 5 gives a sufficient condition for the positivity of $iG(\bar{f}, f)$, we now prove a necessary condition. Unfortunately, we have not been able to prove that positivity implies selfadjointness, but only a weaker statement.

***Proposition* 6** *With the notation and definitions of section D and proposition 5, let $RM = DM \otimes TM$ and let the Majorana subspaces of $\not{V}_0(RM)$ and $\not{S}(\mathcal{R}, M)$ be defined as*

$$\not{S}^c(\mathcal{R}, M) := \{\psi^\alpha \in \not{S}(\mathcal{R}, M) \,|\, (\psi^\alpha)^c = \psi^\alpha\}$$

$$\not{V}_0^c(RM) := \{f^\alpha \in \not{V}_0(RM) \,|\, (f^\alpha)^c = f^\alpha\}\,.$$

Moreover, let $\mathcal{R}$ commute with charge conjugation (i.e. $\mathcal{R}$ has real coefficients in the Majorana representation), let the principal symbol of $\mathcal{R}$ be covariantly conserved, i.e. $\nabla_\mu \sigma^\mu = \sigma^\mu \nabla_\mu$, and let $iG^\beta_\alpha(\bar{f}_\beta, f^\alpha) \geq 0$ for all $f^\alpha \in \not{V}_0^c(RM)$. Then the current

$$j^\mu \left[\psi_1^\alpha, \psi_2^\beta\right] := \bar{\psi}_{1,\alpha} \sigma^\mu \psi_2^\alpha$$

is covariantly conserved on all constrained Majorana solutions, i.e.

$$\nabla_\mu j^\mu \left[\psi_1^\alpha, \psi_2^\beta\right] = 0 \qquad \forall \psi_1^\alpha, \psi_2^\alpha \in \not{S}^c(\mathcal{R}, M)$$

.

Proof. We define $(f^\alpha, g^\beta) := iG^\beta_\alpha(\bar{f}_\beta, g^\alpha)$. Since $(f^\alpha, f^\alpha) \geq 0$ for all $f^\alpha \in \not{V}_0^c(RM)$, (f^α, g^β) is a positive semidefinite sesquilinear form on $\not{V}_0^c(RM)$ by the polarisation identity. As $\mathcal{R}$ commutes with charge conjugation, the same holds for G^β_α. From this and the definitions of the Dirac adjoint and charge conjugation one can deduce that, in addition, $(f^\alpha, g^\beta) = ((f^\alpha)^c, (g^\beta)^c) = (g^\beta, f^\alpha)$ for all f^α, g^β in $\not{V}_0^c(RM)$, i.e. $(\cdot,\cdot)$ is symmetric, which in turn implies that iG^α_β is formally selfadjoint with respect to $\langle\cdot,\cdot\rangle$. As the formal adjoint of the causal propagator is the causal propagator of the formally adjoint differential operator, and as causal propagators map test sections to solutions in a surjective manner, we obtain

$$\not{S}^c(\mathcal{R}, M) \subset S(\mathcal{R}^\dagger, M)\,.$$

Let now σ_0 be defined as

$$\sigma_0 = \mathcal{R} - \sigma^\mu \nabla_\mu\,,$$

i.e. as the "zeroth order part" of $\mathcal{R}$. Note that σ_0 is not the subprincipal symbol of $\mathcal{R}$ in the mathematical sense, as the latter is not covariantly defined, but σ_0 is. One can check that $\mathcal{R}^\dagger$ can be expressed in terms of σ^μ and σ_0 as

$$\mathcal{R}^\dagger = -\beta\sigma^\mu\nabla_\mu\beta + \beta\sigma_0\beta\,.$$

Using this, $\not{S}^c(\mathcal{R}, M) \subset S(\mathcal{R}^\dagger, M)$, and $\nabla_\mu\sigma^\mu = \sigma^\mu\nabla_\mu$, one can straightforwardly compute that $\nabla_\mu(\bar{\psi}_{1,\alpha}\sigma^\mu\psi_2^\alpha) = 0$ for all $\psi_1^\alpha, \psi_2^\beta \in \not{S}^c(\mathcal{R}, M)$. □

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