# COUPLING OF GRAVITY TO MATTER, SPECTRAL ACTION AND COSMIC TOPOLOGY

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Abstract. We consider a model of modified gravity based on the spectral action functional, for a cosmic topology given by a spherical space form, and the associated slow-roll inflation scenario. We consider then the coupling of gravity to matter determined by an almost commutative geometry over the spherical space form. We show that this produces a multiplicative shift of the amplitude of the power spectra for the density fluctuations and the gravitational waves, by a multiplicative factor equal to the total number of fermions in the matter sector of the model. We obtain the result by an explicit nonperturbative computation, based on the Poisson summation formula and the spectra of twisted Dirac operators on spherical space forms, as well as by a heat-kernel computation.

Contents

## 1. Introduction

Models of gravity coupled to matter based on Noncommutative Geometry are usually obtained (see [2], [6], [9], [10]) by considering an underlying geometry given by a product $X \times F$ of an ordinary 4-dimensional (Riemannian compact) spacetime manifold and a finite noncommutative space $F$.

The main purpose of the paper is to show how the slow-roll inflation potential derived in [20], [21] is affected by the presence of the matter content and the almost commutative geometry. We first consider the Poisson summation formula technique and the nonperturbative calculation of the spectral action and then a heat kernel argument to show that the amplitude of the slow-roll potential is affected by a multiplicative factor $N$ equal to the dimension of the representation, that is, to the total number of fermions in the theory.

### 1.1. Spectral triples.

Noncommutative spaces are described, in this context, as a generalization of Riemannian manifolds, via the formalism of *spectral triples* introduced in [11]. An ordinary Riemannian spin manifold $X$ is identified with the spectral triple $(C^\infty(X), L^2(X,S), \not{D})$, with the algebra of smooth functions acting as multiplication operators on the Hilbert space of square integrable spinors and the Riemannian metric reconstructed from the Dirac operator $\not{D}$.

More generally, for a noncommutative space, a spectral triple is a similar set $(\mathcal{A}, \mathcal{H}, D)$ consisting of a $*$-algebra represented by bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator $D$ with compact resolvent acting on $\mathcal{H}$ with a dense domain and such that the commutators $[D, a]$ extend to bounded operators on all of $\mathcal{H}$. A *finite* noncommutative space is one for which the algebra $\mathcal{A}$ is finite dimensional.

A recent powerful reconstruction theorem ([12], see also [27]) shows that commutative spectral triples that satisfy certain natural axioms, related to properties such as orientability and Poincaré duality, are spectral triples of smooth Riemannian manifolds in the sense mentioned above.

In the models of gravity coupled to matter, the choice of the finite geometry $F = (\mathcal{A}_F, \mathcal{H}_F, D_F)$ determines the field content of the particle physics model. As shown in [9], the coordinates on the moduli space of possible Dirac operators $D_F$ on the finite geometry $(\mathcal{A}_F, \mathcal{H}_F)$ specify the Yukawa parameters (Dirac and Majorana masses and mixing angles) for the particles. A classification of the moduli spaces of Dirac operators on the finite geometries was given in [3].

### 1.2. The spectral action.

One obtains then a theory of (modified) gravity coupled to matter by taking as an action functional the spectral action on the noncommutative space $X \times F$, considered as the product of the spectral triples $(C^\infty(X), L^2(X,S), \not{D})$ and $(\mathcal{A}_F, \mathcal{H}_F, D_F)$.

The *spectral action functional* introduced in [6] is a function of the spectrum of the Dirac operator on a spectral triple, given by summing over the spectrum with a cutoff function. Namely, the spectral action functional is defined as $\mathrm{Tr}(f(D/\Lambda))$, where $\Lambda$ is an energy scale, $D$ is the Dirac operator of the spectral triple, and $f$ is a smooth approximation to a cutoff function. As shown in [6] this action functional has an asymptotic expansion at high energies $\Lambda$ of the form

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{k\in \mathrm{DimSp}} f_k \Lambda^k ⨍ |D|^{-k} + f(0)\zeta_D(0) + o(1), \tag{1.1}$$

where the $f_k$ are the momenta $f_k = \int_0^\infty f(v)v^{k-1}dv$ of the test function $f$, for $k$ a non-negative integer in the dimension spectrum of $D$ (the set of poles of the zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$) and the term $⨍|D|^{-k}$ given by the residue at $k$ of the zeta function $\zeta_D(s)$.

These terms in the asymptotic expansion of the spectral action can be computed explicitly: for a suitable choice of the finite geometry spectral triple $(\mathcal{A}_F, \mathcal{H}_F, D_F)$ as in [9], they recover all the bosonic terms in the Lagrangian of the Standard Model (with additional right handed neutrinos with Majorana mass terms) and gravitational terms including the Einstein–Hilbert action, a cosmological term, and conformal gravity terms, see also Chapter 1 of [13]. For a different choice of the finite geometry, one can obtain supersymmetric QCD, see [2].

The higher order terms in the spectral action, which appear with coefficients $f_{-2k} = (-1)^k k!/(2k)! f^{(2k)}(0)$ depending on the derivatives of the test function, and involve higher derivative terms in the fields, were considered explicitly recently, in work related to renormalization of the spectral action for gauge theories [31], [32], and also in [7]. In cases where the underlying geometry is very symmetric (space forms) and the Dirac spectrum is explicitly known, it is also possible to obtain explicit non-perturbative computations of the spectral action, computed directly as $\mathrm{Tr}(f(D/\Lambda))$, using Poisson summation formula techniques applied to the Dirac spectrum and its multiplicities, see [7], [20], [21], [30].

1.3. **Almost commutative geometries.** It is also natural to consider a generalization of the product geometry $X \times F$, where this type of *almost commutative geometry* is generalized to allow for nontrivial fibrations that are only locally, but not globally, products. This means considering almost commutative geometries that are fibrations over an ordinary manifold $X$, with fiber a finite noncommutative space $F$. A first instance where such topologically non-trivial cases where considered in the context of models of gravity coupled to matter was the Yang–Mills case considered in [1].

In the setting of [1], instead of a product geometry $X \times F$, one considers a noncommutative space obtained as an algebra bundle, namely where the algebra of the full space is isomorphic to sections $\Gamma(X, \mathcal{E})$ of a locally trivial $*$-algebra bundle whose fibers $\mathcal{E}_x$ are isomorphic to a fixed finite dimensional algebra $\mathcal{A}_F$. The spectral triple that replaces the product geometry is then of the form $(C^\infty(X,\mathcal{E}), L^2(X,\mathcal{E}\otimes S), \mathcal{D}_\mathcal{E})$, where the Dirac operator $\mathcal{D}_\mathcal{E} = c\circ(\nabla^\mathcal{E}\otimes 1 + 1\otimes\nabla^S)$ is defined using the spin connection and a hermitian connection on the algebra bundle $\mathcal{E}$ (with respect to an inner product obtained using a faithful tracial state $\tau_x$ on $\mathcal{E}_x$. The spectral triple obtained in this way can be endowed with a compatible grading and real structure and it is described in [1] in terms of unbounded Kasparov product of KK-cycles. In the Yang–Mills case, where the finite dimensional algebra is $\mathcal{A}_F = M_N(\mathbb{C})$, it is then shown in [1] that this type of spectral triples describes $PSU(N)$-gauge theory with a nontrivial principal bundle and the Yang–Mills action functional coupled to gravity is recovered from the asymptotic expansion of the spectral action.

A reconstruction theorem for almost commutative geometries (defined in this more general topologically nontrivial sense), was recently obtained in [4], as a consequence of the reconstruction theorem for commutative spectral triples of [12]. In this more general setting, an abstract class of almost commutative spectral triple $(\mathcal{A},\mathcal{H},D)$ with a commutative base is defined and proved to be equivalent to the concrete definition, where $\mathcal{H} = L^2(X,\mathcal{V})$, with $\mathcal{V}$ a self-adjoint Clifford module bundle, and where the algebra is given by sections $\mathcal{A} = C^\infty(X,\mathcal{E})$ with $\mathcal{E}$ a unital $*$-algebra sub-bundle of $\mathrm{End}^+_{Cl(X)}(\mathcal{V})$, and $D$ is a Dirac type operator on $\mathcal{V}$.

1.4. **Cosmic topology and inflation.** The asymptotic expansion of the spectral action provides naturally an action functional for (Euclidean) modified gravity, where in addition to the ordinary Einstein–Hilbert action with cosmological term one also has a topological term (Euler characteristic) and conformal gravity terms like the Weyl curvature and a conformal coupling of the Higgs field to gravity. It also produces the additional bosonic terms: the action for the Higgs with quartic potential and the Yang–Mills action for the gauge fields. Thus, it is natural to consider the spectral action as a candidate action functional for a modified gravity model and study its consequences for cosmology.

Cosmological implications of the spectral action, based on the asymptotic expansion, were considered in [17], [19], [22], [23], [24], [25]. For recent developments in the case of Robertson–Walker metrics see [8].

In [20] and [21] the nonperturbative spectral action was computed explicitly for the 3-dimensional spherical space forms and the flat 3-dimensional Bieberbach manifolds, via the same type of Poisson summation technique first used in [7] for the sphere case. In these computations one considers the spectral action as a pure gravity functional (that is, only on the manifold $X$, without the finite geometry $F$). It is shown that a perturbation $D^2+\phi^2$ of the Dirac operator produces in the nonperturbative spectral action a slow-roll potential $V(\phi)$ for the scalar field $\phi$, which can be used as a model for cosmic inflation.

It is shown in [20], [30] and [21] that nonperturbative spectral action for the spherical space forms $S^3/\Gamma$ is, up to an overall constant factor that depends on the order of the finite group $\Gamma$, the same as that of the sphere $S^3$, hence so is the slow-roll potential. Similarly, the spectral action and potentials for the flat Bieberbach manifolds are a multiple of those of the flat torus $T^3$. In particular, for each such manifold, although the spectra depend explicitly on the different spin structures, the spectral action does not. These results show that, in a model of gravity based on the spectral action functional, the amplitudes and slow-roll parameters in the power spectra for the scalar and tensor fluctuation would depend on the underlying cosmic topology, hence constraints on these quantities derived from cosmological data (see [18], [28], [29]) may, in principle, be able to distinguish between different topologies.

Here we discuss a natural question arising from the results of [20] and [21], namely how the presence of the finite geometry $F$ may affect the behavior of the slow-roll inflation potential. As a setting, we consider here the case of the spherical space forms $S^3/\Gamma$ as the commutative base of an almost commutative geometry in the sense of [4], where the Clifford module bundle $\mathcal{V}$ on $S^3/\Gamma$ is a flat bundle corresponding to a finite-dimensional representation $\alpha : \Gamma \to \mathrm{GL}_N(\mathbb{C})$ of the group $\Gamma$. As the Dirac operator on the almost commutative geometry we consider the corresponding twisted Dirac operator $D^\Gamma_\alpha$ on $S^3/\Gamma$. From the point of view of the physical model this means that we only focus on the gravity terms and we do not include the part of the Dirac operator $D_F$ that describes the matter content and which comes from a finite spectral triple in the fiber direction.

Our main result is that, for any such almost commutative geometry, the spectral action and the associated slow-roll potential only differ from those of the sphere $S^3$ by an overall multiplicative amplitude factor equal to $N/\#\Gamma$. Thus, the only modification to the amplitude factor in the power spectra is a correction, which appears uniformly for all topologies, by a multiplicative factor $N$ depending on the fiber of the almost commutative geometry. In terms of the physical model, this $N$ represents the number of fermions in the theory. We first compute the spectral action in its nonperturbative form, as in [7], [20], [21], [30], using the Poisson summation formula technique and the explicit form of the Dirac spectra derived in [5]; then we recover the same result via an argument based on the perturbative form of the spectral action, and heat kernel methods.

1.5. **Basic setup.** We recall the basic setting, following the notation of [5]. Let $\Gamma \subset SU(2)$ be a finite group acting by isometries on $S^3$, identified with the Lie group $SU(2)$ with the round metric. The spinor bundle on the spherical form $S^3/\Gamma$ is given by $S^3 \times_\sigma \mathbb{C}^2 \to S^3/\Gamma$, where $\sigma$ is the representation of $\Gamma$ defined by the standard representation of $SU(2)$ on $\mathbb{C}^2$.

A unitary representation $\alpha : \Gamma \to U(N)$ defines a flat bundle $\mathcal{V}_\alpha = S^3 \times_\alpha \mathbb{C}^N$ endowed with a canonical flat connection. By twisting the Dirac operator with the flat bundle, one obtains an operator $D^\Gamma_\alpha$ on the spherical form $S^3/\Gamma$ acting on the twisted spinors, that is, on the $\Gamma$-equivariant sections $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^\Gamma$, where $\Gamma$ acts by isometries on $S^3$ and by $\sigma \otimes \alpha$ on $\mathbb{C}^2 \otimes \mathbb{C}^N$. These are the sections of the twisted spinor bundle $S^3 \times_{\sigma\otimes\alpha} (\mathbb{C}^2 \otimes \mathbb{C}^N) \to S^3/\Gamma$. Thus, $D^\Gamma_\alpha$ is the restriction of the Dirac operator $D \otimes id_{\mathbb{C}^N}$ to the subspace $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^\Gamma \subset C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)$.

This setup gives rise to an almost commutative geometry in the sense of [4], where the twisted Dirac operator $D^\Gamma_\alpha$ represents the "pure gravity" part of the resulting model of gravity coupled to matter, while the fiber $\mathbb{C}^N = \mathcal{H}_F$ determines the fermion content of the matter part and can be chosen according to the type of particle physics model one wishes to consider (Standard Model with right handed neutrinos, supersymmetric QCD, for example, as in [9], [2], or other possibilities). Since we will only be focusing on the gravity terms, we do not need to specify in full the data of the almost commutative geometry, beyond assigning the flat bundle $\mathcal{V}_\alpha$ and the twisted Dirac operator $D^\Gamma_\alpha$, as the additional data would not enter directly in our computations.

## 2. Poisson summation formula

Following the method developed in [7] and [20], [30], we compute the spectral action of the quotient spaces $S^3/\Gamma$ equipped with the twisted Dirac operator corresponding to a finite-dimensional representation $\alpha$ of $\Gamma$ as follows. We define a finite set of polynomials labeled $P^+_m$, and $P^-_m$ which describe the multiplicities of, respectively, the positive and negative eigenvalues of the twisted Dirac operator, in the sense that $P^\pm_m(u)(\lambda)$ equals the multiplicity of the eigenvalue

$$\lambda = -1/2 \pm (k+1), \;\; k \geq 1 \tag{2.1}$$

whenever $k \equiv m \bmod \mathrm{c}_\Gamma$, where $c_\Gamma$ is the exponent of the group $\Gamma$, the least common multiple of the orders of the elements in $\Gamma$.

The main technical result we will prove is the following relation between these polynomials:

$$\sum_{m=1}^{c_\Gamma} P^+_m(u) = \sum_{m=0}^{c_\Gamma - 1} P^-_m(u) = \frac{N c_\Gamma}{\#\Gamma}\left(u^2 - \frac{1}{4}\right). \tag{2.2}$$

Since the polynomial on the right-hand-side is a multiple of the polynomial for the spectral multiplicities of the Dirac spectrum of the sphere $S^3$ (see [7]), we will obtain from this the relation between the non-perturbative spectral action of the twisted Dirac operator $D^\Gamma_\alpha$ on $S^3/\Gamma$ and the spectral action on the sphere, see Theorem 2.1 below.

Furthermore, we shall show that the polynomials $P^+_m(u)$ match up perfectly with the polynomials $P^-_m(u)$, so that the polynomials $P^+_m(u)$ alone describe the entire spectrum by allowing the parameter $k$ in equation 2.1 to run through all of $\mathbb{Z}$. Namely, what we need to show is that

$$P_m^+(u) = P_{m'}^-(u), \tag{2.3}$$

where for each $m$, $m'$ is the unique number between 0 and $c_\Gamma - 1$ such that $m + m' + 2$ is a multiple of $c_\Gamma$. To be more precise,

$$m' = \begin{cases} c_\Gamma - 2 - m, & \text{if } 1 \le m \le c_\Gamma - 2 \\ c_\Gamma - 1, & \text{if } m = c_\Gamma - 1 \\ c_\Gamma - 2 & \text{if } m = c_\Gamma \end{cases} \tag{2.4}$$

Define

$$g_m(u) = P_m^+(u) f(u/\Lambda). \tag{2.5}$$

Now, we apply the Poisson summation formula, to obtain,

$$\begin{aligned} \mathrm{Tr}(f(D/\Lambda)) &= \sum_m \sum_{l\in\mathbb{Z}} g_m(1/2 + c_\Gamma l + m + 1) \\ &= \frac{N}{\#\Gamma} \sum_m \widehat{g_m}(0) + O(\Lambda^{-\infty}) \\ &= \frac{N}{\#\Gamma} \left( \int_{\mathbb{R}} u^2 f(u/\Lambda) - \frac{1}{4} \int_{\mathbb{R}} f(u/\Lambda) \right) + O(\Lambda^{-\infty}) \\ &= \frac{N}{\#\Gamma} \left( \Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4} \Lambda \widehat{f}(0) \right) + O(\Lambda^{-\infty}), \end{aligned}$$

and so we have the main result.

**Theorem 2.1.** *Let $\Gamma$ be a finite subgroup of $S^3$, and let $\alpha$ be a $N$-dimensional representation of $\Gamma$. Then the spectral action of $S^3/\Gamma$ equipped with the twisted Dirac operator is*

$$\mathrm{Tr} f(D/\Lambda) = \frac{N}{|\Gamma|} \left( \Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4} \Lambda \widehat{f}(0) \right) + O(\Lambda^{-\infty}). \tag{2.6}$$

*Here $\widehat{f}^{(2)}$ denotes the Fourier transform of $u^2 f(u)$.*

Similar computations of the spectral action have also been performed in [20], [21], and [30]. In the sequel we describe how to obtain equation (2.2), by explicitly analyzing the cases of the various spherical space forms: lens spaces, dicyclic group, and binary tetrahedral, octahedral, and icosahedral groups. In all cases we compute explicitly the polynomials of the spectral multiplicities and check that (2.2) is satisfied. Our calculations are based on a result of Cisneros-Molina, [5], on the explicit form of the Dirac spectra of the twisted Dirac operators $D_\alpha^\Gamma$, which we recall here below.

2.1. **Twisted Dirac spectra of spherical space forms.** The spectra of the twisted Dirac operators on the quotient spaces are derived in [5]. Let us recall the notation and the main results.

Let $E_k$ denote the $k+1$-dimensional irreducible representation of $SU(2)$ on the space of homogeneous complex polynomials in two variables of degree $k$. By the Peter–Weyl theorem, one can decompose $C^\infty(S^3, \mathbb{C}) = \oplus_k E_k \otimes E_k^*$ as a sum of irreducible representations of $SU(2)$. This gives that, on $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N) = \oplus_k E_k \otimes E_k^* \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$, the Dirac operator $D \otimes id_{\mathbb{C}^N}$ decomposes as $\oplus_k \mathrm{id}_{E_k} \otimes D_k \otimes id_{\mathbb{C}^N}$, with $D_k : E_k^* \otimes \mathbb{C}^2 \to E_k^* \otimes \mathbb{C}^2$. Upon identifying $C^\infty(S^3, \mathbb{C}^2 \otimes \mathbb{C}^N)^\Gamma = \oplus_k E_k \otimes \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$, one sees that, as shown in [5], the multiplicities of the spectrum of the twisted Dirac operator $D^\Gamma_\alpha$ are given by the dimensions $\dim_\mathbb{C} \mathrm{Hom}_\Gamma(E_k, \mathbb{C}^2 \otimes \mathbb{C}^N)$, which in turn can be expressed in terms of the pairing of the characters of the corresponding $\Gamma$-representation, that is, as $\langle \chi_{E_k}, \chi_{\sigma\otimes\alpha}\rangle_\Gamma$. One then obtains the following:

**Theorem 2.2.** (Cisneros-Molina, [5]) *Let $\alpha : \Gamma \to GL_N(\mathbb{C})$ be a representation of $\Gamma$. Then the eigenvalues of the twisted Dirac operator $D^\Gamma_\alpha$ on $S^3/\Gamma$ are*

$$-\frac{1}{2} - (k+1) \text{ with multiplicity } \langle \chi_{E_{k+1}}, \chi_\alpha\rangle_\Gamma (k+1), \qquad k \geq 0,$$
$$-\frac{1}{2} + (k+1) \text{ with multiplicity } \langle \chi_{E_{k-1}}, \chi_\alpha\rangle_\Gamma (k+1), \qquad k \geq 1,$$

**Proposition 2.3.** (Cisneros-Molina, [5]) *Let $k = c_\Gamma l + m$ with $0 \leq m < c_\Gamma$.*

(1) *If $-1 \in \Gamma$, then*

$$\langle \chi_{E_k}, \chi_\alpha\rangle_\Gamma = \begin{cases} \frac{c_\Gamma l}{|\Gamma|}(\chi_\alpha(1) + \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha\rangle_\Gamma & \text{if } k \text{ is even} \\ \frac{c_\Gamma l}{|\Gamma|}(\chi_\alpha(1) - \chi_\alpha(-1)) + \langle \chi_{E_m}, \chi_\alpha\rangle_\Gamma & \text{if } k \text{ is odd} \end{cases}$$

(2) *If $-1 \notin \Gamma$, then*

$$\langle \chi_{E_k}, \chi_\alpha\rangle_\Gamma = \frac{N c_\Gamma l}{\#\Gamma} + \langle \chi_{E_m}, \chi_\alpha\rangle_\Gamma$$

2.2. **Lens spaces, odd order.** In this section we consider $\Gamma = \mathbb{Z}_n$, where $n$ is odd. When $n$ is odd, $-1 \notin \Gamma$, which affects the expression for the character inner products in Proposition 2.3.

For $m \in \{1, \dots, n\}$, we introduce the polynomials,

$$P^+_m(u) = \frac{N}{n}u^2 + (\beta^\alpha_m - \frac{mN}{n})u + \frac{\beta^\alpha_m}{2} - \frac{mN}{2n} - \frac{N}{4n},$$

where

$$\beta^\alpha_m = \langle \chi_{E_{m-1}}, \chi_\alpha\rangle_\Gamma,$$

and $m$ takes on values in $\{1, 2, \dots, n\}$

Using Theorem 2.2 and Proposition 2.3, it is easy to see that the polynomials $P_m^+(u)$ describe the spectrum on the positive side of the real line, in the sense that $P_m^+(u)(\lambda)$ equals the multiplicity of the eigenvalue

$$\lambda = -1/2 + (k+1),\ \ k \geq 1$$

whenever $k \equiv m$ mod n.

For the negative eigenvalues, the multiplicities are described by the polynomials

$$P_m^-(u) = \frac{N}{n}u^2 + \left(\frac{2N}{n} + \frac{mN}{n} - \gamma_m^\alpha\right)u + \frac{3N}{4n} + \frac{mN}{2n} - \frac{\gamma_m^\alpha}{2},$$

$m \in \{0, 1, \ldots n-1\}$, in the sense that $P_m^-(u)(\lambda)$ equals the multiplicity of the eigenvalue

$$\lambda = -1/2 - (k+1),\ \ k \geq 0$$

whenever $k \equiv m$ mod n. Here $\gamma_m^\alpha$ is defined by

$$\gamma_m^\alpha = \langle \chi_{E_{m+1}}, \chi_\alpha \rangle_\Gamma.$$

Let us denote the irreducible representations of $\mathbb{Z}_n$ by $\chi_t$, sending the generator to $\exp(\frac{2\pi i t}{N})$. Here $t$ is a residue class of integers modulo $n$.

For the sake of computation, we take $\mathbb{Z}_n$ to be the group generated by

$$B = \begin{bmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{bmatrix}.$$

Then in the representation $E_k$, $B$ acts on the basis polynomials $P_j(z_1, z_2)$, $j \in \{0, 1, \ldots k\}$ as follows.

$$\begin{aligned} B \cdot P_j(z_1, z_2) &= P_j\left((z_1, z_2)B\right) \\ &= P_j(e^{\frac{2\pi i}{n}} z_1, e^{-\frac{2\pi i}{n}} z_2) \\ &= (e^{\frac{2\pi i}{n}} z_1)^{k-j}(e^{-\frac{2\pi i}{n}} z_2)^j \\ &= e^{\frac{2\pi i}{n}(k-2j)} P_j(z_1, z_2). \end{aligned}$$

Hence, $B$ is represented by a diagonal matrix with respect to this basis, and we have

**Proposition 2.4.** *The irreducible characters $\chi_{E_k}$ of the irreducible representations of $SU(2)$ restricted to $\mathbb{Z}_n$, $n$ odd, are decomposed into the irreducible characters $\chi_{[t]}$ of $\mathbb{Z}_n$ by the equation*

$$\chi_{E_k} = \sum_{j=0}^{j=k} \chi_{[k-2j]}. \tag{2.7}$$

*Here, $[t]$ denotes the number from $0$ to $n-1$ to which $t$ is equivalent mod $n$.*

In the case where $-1 \notin \Gamma$, that is to say, when $\Gamma = \mathbb{Z}_n$ where $n$ is odd, by equating coefficients of the quadratic polynomials $P_m^+$ and $P_{m'}^-$, the condition 2.3 is replaced by one that may be simply checked.

**Lemma 2.5.** *Let $\Gamma$ be any finite subgroup of $SU(2)$ such that $-1 \notin \Gamma$ the condition 2.3 is equivalent to the condition*

$$\beta^\alpha_m + \gamma^\alpha_{m'} = \begin{cases} \chi_\alpha(1), & \text{if } 1 \leq m \leq c_\Gamma - 2 \\ 2\chi_\alpha(1), & \text{if } m = c_\Gamma - 1, c_\Gamma \end{cases}, \tag{2.8}$$

*where $\alpha$ is an irreducible representation of $\Gamma$. Furthermore this condition holds in all cases.*

Using proposition 2.4, it is a simple combinatorial matter to see that

$$\sum_{m=1}^{n} \langle \chi_{E_{m-1}}, \chi_\alpha \rangle_\Gamma = N\frac{n+1}{2}, \tag{2.9}$$

for any representation $\alpha$ of $\mathbb{Z}_n$

For the argument to go through, one also needs to check the special case

$$P^+_{c_\Gamma}(1/2) = 0.$$

By direct evaluation one can check that this indeed holds.

For the negative side, we see that

$$\sum_{m=1}^{n} \langle \chi_{E_{m+1}}, \chi_\alpha \rangle_\Gamma = N\frac{n+3}{2}, \tag{2.10}$$

for any representation $\alpha$ of $\mathbb{Z}_n$, and so

**Proposition 2.6.** *Let $\Gamma$ be cyclic with $\#\Gamma$ odd, and let $\alpha$ be a $N$-dimensional representation of $\Gamma$. Then*

$$\sum_{m=1}^{n} P^+_m(u) = \sum_{m=0}^{n-1} P^-_m(u) = Nu^2 - \frac{N}{4}.$$

Note that in the statement of theorem 2.2, the first line holds even if we take $k = -1$, since the multiplicity for this value evaluates to zero. Therefore, we automatically have

$$P^-_{c_\Gamma - 1}(-1/2) = 0,$$

which we still needed to check.

2.3. **Lens spaces, even order.** When $n$ is even, we have $-1 \in \mathbb{Z}_n$. When $-1 \in \Gamma$, from Theorems 2.2 and 2.3 it follows that the multiplicity of the eigenvalue

$$\lambda = 1/2 + lc_\Gamma + m, \ l \in \mathbb{N}$$

is given by, $P^+_m$, $m \in \{1, 2, \ldots, c_\Gamma\}$,

$$\begin{aligned}P_m^+(u) =&\frac{1}{|\Gamma|}(\chi_\alpha(1)+(-1)^{m+1}\chi_\alpha(-1))u^2+\\&\left(\beta_m^\alpha-\frac{1}{\#\Gamma}(m(\chi_\alpha(1)+(-1)^{m+1}\chi_\alpha(-1))\right)u\\&+\frac{1}{2}\beta_m^\alpha-\frac{1}{4\#\Gamma}(\chi_\alpha(1)+(-1)^{m+1}\chi_\alpha(-1))\\&-\frac{1}{2\#\Gamma}m(\chi_\alpha(1)+(-1)^{m+1}\chi_\alpha(-1))).\end{aligned}$$

The one case that is not clear is $\lambda = 1/2$. It is not an eigenvalue of the twisted Dirac operator. However, it is not clear from Theorems 2.2 and 2.3 that

$$P_{c_\Gamma}^+(1/2) = 0, \tag{2.11}$$

and this needs to hold in order for the argument using the Poisson summation formula to go through. Evaluating equation (2.11), we see that one needs to check that

$$\langle \chi_{E_{c_\Gamma -1}}, \chi_\alpha\rangle = \frac{c_\Gamma}{\#\Gamma}(\chi_\alpha(1)+(-1)^{c_\Gamma+1}\chi_\alpha(-1)), \tag{2.12}$$

and indeed it holds for each subgroup $\Gamma$ and irreducible representation $\alpha$.

**Proposition 2.7.** *For any subgroup $\Gamma\subset S^3$ of even order, the sum of the polynomials $P_m^+$ is*

$$\begin{aligned}\sum_{m=1}^{c_\Gamma} P_m^+(u) =&\frac{c_\Gamma}{\#\Gamma}\chi_\alpha(1)u^2\\&+\left(-\frac{c_\Gamma^2\chi_\alpha(1)}{2\#\Gamma}-\frac{c_\Gamma(\chi_\alpha(1)-\chi_\alpha(-1))}{2\#\Gamma}+\sum_{m=1}^{c_\Gamma}\beta_m^\alpha\right)u\\&-\frac{c_\Gamma\chi_\alpha(1)}{2\#\Gamma}-\frac{c_\Gamma^2\chi_\alpha(1)}{4\#\Gamma}+\frac{c_\Gamma}{4\#\Gamma}\chi_\alpha(-1)+\frac{1}{2}\sum_{m=1}^{c_\Gamma}\beta_m^\alpha\end{aligned}$$

Since the coefficients of the polynomial are additive with respect to direct sum, it suffices to consider only irreducible representations.

In the case of lens spaces, $c_\Gamma = \#\Gamma$, and $\chi_t(-1) = (-1)^t$. As a matter of counting, one can see that

**Proposition 2.8.**

$$\sum_{m=1}^{c_\Gamma}\beta_m^t = \begin{cases}\frac{n+2}{2} & \textit{if } t \textit{ is even}\\ \frac{n}{2} & \textit{if } t \textit{ is odd}\end{cases}$$

Putting this all into the expression of proposition 2.7, we have, for an $N$-dimensional representation, $\alpha$,

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = N\left(u^2 - \frac{1}{4}\right). \tag{2.13}$$

The negative eigenvalues are described by the polynomials

$$\begin{aligned} P_m^-(u) = & \frac{1}{\#\Gamma}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1))u^2 + \\ & \left(\frac{2+m}{\#\Gamma}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)) - \gamma_m^\alpha\right) u \\ & \frac{3+2m}{4|\Gamma|}(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)) - \frac{1}{2}\gamma_m^\alpha, \end{aligned}$$

$m \in \{0, 1, \ldots c_\Gamma - 1\}$. And so, we have the following proposition.

**Proposition 2.9.** *For any subgroup $\Gamma \subset S^3$ of even order, the sum of the polynomials $P_m^-$ is*

$$\begin{aligned} \sum_{m=1}^{c_\Gamma} P_m^-(u) = & \frac{c_\Gamma}{\#\Gamma}\chi_\alpha(1)u^2 + \\ & \left(\frac{\chi_\alpha(1)c_\Gamma^2}{2\#\Gamma} + \frac{3\chi_\alpha(1)c_\Gamma}{2\#\Gamma} + \frac{\chi_\alpha(-1)c_\Gamma}{2\#\Gamma} - \sum_{m=0}^{c_\Gamma-1}\gamma_m^\alpha\right) u + \\ & \frac{\chi_\alpha(1)c_\Gamma}{2\#\Gamma} + \frac{\chi_\alpha(1)c_\Gamma^2}{4\#\Gamma} + \frac{\chi_\alpha(-1)c_\Gamma}{4\#\Gamma} - \frac{1}{2}\sum_{m=0}^{c_\Gamma-1}\gamma_m^\alpha. \end{aligned}$$

By counting, one can see that

$$\sum_{m=0}^{c_\Gamma-1}\gamma_m^t = \begin{cases} \frac{n+4}{2} & \text{if } t \text{ is even} \\ \frac{n+2}{2} & \text{if } t \text{ is odd} \end{cases} \tag{2.14}$$

To complete the computation of the spectral action one still needs to verify the condition (2.3). We have the following lemma, which is obtained by equating the coefficients of $P_m^+$ and $P_{m'}^-$, and it covers the cases of the binary tetrahedral, octahedral and icosahedral groups as well.

**Lemma 2.10.** *Let $\Gamma$ be any finite subgroup of $SU(2)$ such that $-1 \in \Gamma$ the condition* (2.3) *is equivalent to the condition*

$$\beta_m^\alpha + \gamma_{m'}^\alpha = \begin{cases} \chi_\alpha(1)(\chi_\alpha(1) + (-1)^{m+1}\chi_\alpha(-1)), & \text{if } 1 \le m \le c_\Gamma - 2 \\ 2\chi_\alpha(1)(\chi_\alpha(1) + \chi_\alpha(-1)), & \text{if } m = c_\Gamma - 1 \\ 2\chi_\alpha(1)(\chi_\alpha(1) - \chi_\alpha(-1)), & \text{if } m = c_\Gamma \end{cases}, \tag{2.15}$$

*where $\alpha$ is an irreducible representation of $\Gamma$. Furthermore this condition holds in all cases.*

2.4. **Dicyclic group.** The character table for the dicyclic group of order $4r$ is, for $r$ odd,

| Class | $1_+$ | $1_-$ | $2_l$ | $r_0$ | $r_1$ |
|---|---|---|---|---|---|
| $\psi_t$ | 2 | $2(-1)^t$ | $\zeta_{2r}^{lt}+\zeta_{2r}^{-lt}$ | 0 | 0 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | $(-1)^l$ | $i$ | $-i$ |
| $\chi_3$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_4$ | 1 | $-1$ | $(-1)^l$ | $-i$ | $i$, |

and for $r$ even,

| Class | $1_+$ | $1_-$ | $2_l$ | $r_0$ | $r_1$ |
|---|---|---|---|---|---|
| $\psi_t$ | 2 | $2(-1)^t$ | $\zeta_{2r}^{lt}+\zeta_{2r}^{-lt}$ | 0 | 0 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | $(-1)^l$ | $i$ | $-i$ |
| $\chi_3$ | 1 | 1 | 1 | $-1$ | $-1$ |
| $\chi_4$ | 1 | $-1$ | $(-1)^l$ | $-i$ | $i$ |

Here $\zeta_{2r} = e^{\frac{\pi i}{r}}$, $1 \le t \le r-1$, $1 \le l \le r-1$. The notation for the different conjugacy classes can be understood as follows. The number indicates the order of the conjugacy class. A sign in the subscript indicates the sign of the traces of the elements in the conjugacy class as elements of $SU(2)$.

For the dicyclic group of order $4r$, the exponent of the group is

$$c_\Gamma = \begin{cases} 2r & \text{if } r \text{ is even} \\ 4r & \text{if } r \text{ is odd} \end{cases}$$

One can decompose the characters $\chi_{E_k}$ into the irreducible characters by inspection, and with some counting obtain the following propositions.

**Proposition 2.11.** *Let $\Gamma$ be the dicyclic group of order $4r$, where $r$ is even.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} \frac{r}{2} & \chi_\alpha \in \{\chi_1,\chi_2,\chi_3,\chi_4\} \\ r & \chi_\alpha = \psi_t,\ t \text{ is even} \\ r+1 & \chi_\alpha = \psi_t,\ t \text{ is odd} \end{cases}$$

$$\sum_{m=0}^{c_\Gamma-1} \gamma_m^\alpha = \begin{cases} \frac{r}{2}+1 & \chi_\alpha \in \{\chi_1,\chi_2,\chi_3,\chi_4\} \\ r+2 & \chi_\alpha = \psi_t,\ t \text{ is even} \\ r+1 & \chi_\alpha = \psi_t,\ t \text{ is odd} \end{cases}$$

**Proposition 2.12.** *Let $\Gamma$ be the dicyclic group of order $4r$, where $r$ is odd.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 2r & \chi_\alpha \in \{\chi_1,\chi_3\} \\ 2r+1 & \chi_\alpha \in \{\chi_2,\chi_4\} \\ 4r & \chi_\alpha = \psi_t,\ t \text{ is even} \\ 4r+2 & \chi_\alpha = \psi_t,\ t \text{ is odd} \end{cases}$$

$$\sum_{m=0}^{c_\Gamma-1} \gamma_m^\alpha = \begin{cases} 2r+2 & \chi_\alpha \in \{\chi_1,\chi_3\} \\ 2r+1 & \chi_\alpha \in \{\chi_2,\chi_4\} \\ 4r+4 & \chi_\alpha = \psi_t,\ t \text{ is even} \\ 4r+2 & \chi_\alpha = \psi_t,\ t \text{ is odd} \end{cases}$$

2.5. **Binary tetrahedral group.** The binary tetrahedral group has order 24 and exponent 12. The character table of the binary tetrahedral group is

| Class | $1_+$ | $1_-$ | $4_{a+}$ | $4_{b+}$ | $4_{a-}$ | $4_{b-}$ | 6 |
|---|---|---|---|---|---|---|---|
| Order | 1 | 2 | 6 | 6 | 3 | 3 | 4 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $\omega^2$ | $\omega$ | $\omega$ | $\omega^2$ | 1 |
| $\chi_3$ | 1 | 1 | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega$ | 1 |
| $\chi_4$ | 2 | $-2$ | 1 | 1 | $-1$ | $-1$ | 0 |
| $\chi_5$ | 2 | $-2$ | $\omega^2$ | $\omega$ | $-\omega$ | $-\omega^2$ | 0 |
| $\chi_6$ | 2 | $-2$ | $\omega$ | $\omega^2$ | $-\omega^2$ | $-\omega$ | 0 |
| $\chi_7$ | 3 | 3 | 0 | 0 | 0 | 0 | $-1$ |

Here, $\omega = e^{\frac{2\pi i}{3}}$.

For the remaining three groups, we can use matrix algebra to decompose the characters $\chi_{E_k}$.

Let $\chi_j$, $x_j$, $j = 1, 2 \ldots, d$ denote the irreducible characters, and representatives of the conjugacy classes of the group $\Gamma$. Then since every character decomposes uniquely into the irreducible ones, we have a unique expression for $\chi_{E_k}$ as the linear combination

$$\chi_{E_k} = \sum_{j=0}^{d} c_j^k \chi_j.$$

If we let $b = (b_j)$ $j = 1, \ldots, d$ be the column with $b_j = \chi_{E_k}(x_j)$, and let $A = (a_{ij})$ be the $d \times d$ matrix where $a_{ij} = \chi_j(x_i)$ and let $c = (c_j^k)$ $j = 1, \ldots d$ be another column. Then we have

$$b = Ac,$$

$A$ is necessarily invertible by the uniqueness of the coefficient column $c$, and so $c$ is given by

$$c = A^{-1}b.$$

By this method, we obtain the following proposition.

**Proposition 2.13.** *Let $\Gamma$ be the binary tetrahedral group.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 3, & \chi_\alpha \in \{\chi_1, \chi_2, \chi_3\} \\ 7, & \chi_\alpha \in \{\chi_4, \chi_5, \chi_6\} \\ 9, & \chi_\alpha = \chi_7 \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 4, & \chi_\alpha \in \{\chi_1, \chi_2, \chi_3\} \\ 7, & \chi_\alpha \in \{\chi_4, \chi_5, \chi_6\} \\ 12, & \chi_\alpha = \chi_7 \end{cases}$$

2.6. **Binary octahedral group.** The binary octahedral group has order 48 and exponent 24. The character table of the binary octahedral group is

| Class | $1_+$ | $1_-$ | $6_+$ | $6_0$ | $6_-$ | $8_+$ | $8_-$ | 12 |
|---|---|---|---|---|---|---|---|---|
| Order | 1 | 2 | 8 | 4 | 8 | 6 | 3 | 4 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\chi_3$ | 2 | 2 | 0 | 2 | 0 | $-1$ | $-1$ | 0 |
| $\chi_4$ | 2 | $-2$ | $\sqrt{2}$ | 0 | $-\sqrt{2}$ | 1 | $-1$ | 0 |
| $\chi_5$ | 2 | $-2$ | $-\sqrt{2}$ | 0 | $\sqrt{2}$ | 1 | $-1$ | 0 |
| $\chi_6$ | 3 | 3 | $-1$ | $-1$ | $-1$ | 0 | 0 | 1 |
| $\chi_7$ | 3 | 3 | 1 | $-1$ | 1 | 0 | 0 | $-1$ |
| $\chi_8$ | 4 | $-4$ | 0 | 0 | 0 | $-1$ | 1 | 0 |

**Proposition 2.14.** *Let* $\Gamma$ *be the binary octahedral group.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 6, & \chi_\alpha \in \{\chi_1, \chi_2\} \\ 12, & \chi_\alpha = \chi_3 \\ 13, & \chi_\alpha =\in \{\chi_4, \chi_5\} \\ 18, & \chi_\alpha \in \{\chi_6, \chi_7\} \\ 26, & \chi_\alpha = \chi_8 \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 7, & \chi_\alpha \in \{\chi_1, \chi_2\} \\ 14, & \chi_\alpha = \chi_3 \\ 13, & \chi_\alpha =\in \{\chi_4, \chi_5\} \\ 21, & \chi_\alpha \in \{\chi_6, \chi_7\} \\ 26, & \chi_\alpha = \chi_8 \end{cases}$$

2.7. **Binary icosahedral group.** The binary icosahedral group has order 120 and exponent 60. The character table of the binary icosahedral group is

| Class | $1_+$ | $1_-$ | 30 | $20_+$ | $20_-$ | $12_{a+}$ | $12_{b+}$ | $12_{a-}$ | $12_{b-}$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 2 | 4 | 6 | 3 | 10 | 5 | 5 | 10 |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 2 | $-2$ | 0 | 1 | $-1$ | $\mu$ | $\nu$ | $-\mu$ | $-\nu$ |
| $\chi_3$ | 2 | $-2$ | 0 | 1 | $-1$ | $-\nu$ | $-\mu$ | $\nu$ | $\mu$ |
| $\chi_4$ | 3 | 3 | $-1$ | 0 | 0 | $-\nu$ | $\mu$ | $-\nu$ | $\mu$ |
| $\chi_5$ | 3 | 3 | $-1$ | 0 | 0 | $\mu$ | $-\nu$ | $\mu$ | $-\nu$ |
| $\chi_6$ | 4 | 4 | 0 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $\chi_7$ | 4 | $-4$ | 0 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |
| $\chi_8$ | 5 | 5 | 1 | $-1$ | $-1$ | 0 | 0 | 0 | 0 |
| $\chi_9$ | 6 | $-6$ | 0 | 0 | 0 | $-1$ | 1 | 1 | $-1$ |

Here, $\mu = \frac{\sqrt{5}+1}{2}$, and $\nu = \frac{\sqrt{5}-1}{2}$.

**Proposition 2.15.** *Let $\Gamma$ be the binary icosahedral group.*

$$\sum_{m=1}^{c_\Gamma} \beta_m^\alpha = \begin{cases} 15, & \chi_\alpha = \chi_1 \\ 31, & \chi_\alpha \in \{\chi_2, \chi_3\} \\ 45, & \chi_\alpha \in \{\chi_4, \chi_5\} \\ 60, & \chi_\alpha = \chi_6 \\ 62, & \chi_\alpha = \chi_7 \\ 75, & \chi_\alpha = \chi_8 \\ 93, & \chi_\alpha = \chi_9 \end{cases}$$

$$\sum_{m=0}^{c_\Gamma - 1} \gamma_m^\alpha = \begin{cases} 16, & \chi_\alpha = \chi_1 \\ 31, & \chi_\alpha \in \{\chi_2, \chi_3\} \\ 48, & \chi_\alpha \in \{\chi_4, \chi_5\} \\ 64, & \chi_\alpha = \chi_6 \\ 62, & \chi_\alpha = \chi_7 \\ 80, & \chi_\alpha = \chi_8 \\ 93, & \chi_\alpha = \chi_9 \end{cases}$$

2.8. **Sums of polynomials.** If we input the results of propositions 2.8, 2.11, 2.12, 2.13, 2.14, 2.15 into propositions 2.7, 2.9 and also recalling proposition 2.6 we obtain the following.

**Proposition 2.16.** *Let $\Gamma$ be any finite subgroup of $SU(2)$ and let $\alpha$ be an $N$-dimensional representation of $\Gamma$. Then the sums of the polynomials $P_m^+$ and $P_m^-$ are given by*

$$\sum_{m=1}^{c_\Gamma} P_m^+(u) = \sum_{m=0}^{c_\Gamma - 1} P_m^-(u) = \frac{N c_\Gamma}{\#\Gamma} \left( u^2 - \frac{1}{4} \right)$$

## 3. A heat-kernel argument

It may at first seem surprising that, in the above calculation, using the Poisson summation formula and the explicit Dirac spectra, although the spectra themselves depend in a subtle way upon the representation theoretic data of the unitary representation $\alpha : \Gamma \to U(N)$, through the pairing of the characters of representations, the resulting spectral action only depends upon the dimension $N$ of the representation, the order of $\Gamma$, and the spectral action on $S^3$.

This phenomenon is parallel to the similar observation in the Poisson formula computation of the spectral action for the spherical space forms and the flat Bieberbach manifolds in the untwisted case [20], [21], [30], where one finds that, although the Dirac spectra are different for different spin structures, the resulting spectral action depends only on the order $\#\Gamma$ of the finite group and the spectral action on $S^3$ or $T^3$.

In this section, we give a justification for this phenomenon based on a heat-kernel computation that recovers the result of Theorem 2.1 and justifies the presence of the factor $N/\#\Gamma$.

3.1. **Generalities.** We begin with some background on the spectral action for almost-commutative spectral triples. In what follows, let $\mathcal{L}$ denote the Laplace transform, and let $\mathcal{S}(0,\infty) = \{\phi \in \mathcal{S}(\mathbb{R}) \mid \phi(x) = 0,\ x \leq 0\}$.

The following result establishes the basic properties of the spectral action for almost-commutative spectral triples:

**Theorem 3.1** ([26, Theorem 1])**.** *Let $\mathcal{V}$ be a self-adjoint Clifford module bundle on a compact oriented Riemannian manifold $M$, and let $D$ be a symmetric Dirac-type operator on $\mathcal{V}$. Let $f \in C^\infty(\mathbb{R})$ be of the form $f(x) = \mathcal{L}[\phi](x^2)$ for $\phi \in \mathcal{S}(0,\infty)$. Finally, let $\Lambda > 0$. Then $f(D/\Lambda)$ is trace-class with asymptotic expansion*

$$\operatorname{Tr}\left(f(D/\Lambda)\right) \sim \sum_{k=-\dim M}^{\infty} \Lambda^{-k}\phi_k \int_M a_{k+\dim M}(x, D^2) d\mathrm{Vol}(x), \tag{3.1}$$

*as $\Lambda \to +\infty$, where $a_n(x, D^2)$ is the $n$-th Seeley-DeWitt coefficient of the generalised Laplacian $D^2$, and the constants $\phi_n$ are given by*

$$\phi_n = \int_0^\infty \phi(s) s^{n/2} ds.$$

In particular, since $a_n(\cdot, D^2) = 0$ for $n$ odd [15, Lemma 1.7.4], one has that the asymptotic form of $\operatorname{Tr}\left(f(D/\Lambda)\right)$, as $\Lambda \to +\infty$, is given by

$$\begin{cases} \sum_{n=0}^{\infty} \Lambda^{2(m-n)}\phi_{2(m-n)} \int_M a_{2n}(x, D^2) d\mathrm{Vol}(x) & \text{if } \dim M = 2m, \\ \sum_{n=0}^{\infty} \Lambda^{2(m-n)+1}\phi_{2(m-n)+1} \int_M a_{2n}(x, D^2) d\mathrm{Vol}(x) & \text{if } \dim M = 2m+1. \end{cases}$$

Note also that for $n > 0$,

$$\phi_{-n} = \int_0^\infty \phi(s) s^{-n/2} ds = \frac{1}{\Gamma(n/2)} \int_0^\infty f(u) u^{n-1} du.$$

The following result guarantees that the $\phi_k$ can be chosen at will:

**Proposition 3.2.** *For any $(a_n) \in \mathbb{C}^{\mathbb{Z}}$ there exists some $\phi \in \mathcal{S}(0,\infty)$ such that*

$$a_n = \int_0^\infty s^{n/2}\phi(s) ds, \textit{for all } n \in \mathbb{Z}.$$

In fact, this turns out to be a simple consequence of the following result by Durán and Estrada, solving the strong moment problem for smooth functions of rapid decay:

**Theorem 3.3** (Durán–Estrada [14])**.** *For any $(a_n) \in \mathbb{C}^{\mathbb{Z}}$ there exists some $\phi \in \mathcal{S}(0,\infty)$ such that*

$$a_n = \int_0^\infty s^n \phi(s) ds, \quad \textit{for all } n \in \mathbb{Z}.$$

*Proof of Proposition 3.2.* By Theorem 3.3, let $\psi \in \mathcal{S}(0,\infty)$ be such that

$$a_n = 2\int_0^\infty s^{n+1}\psi(s) ds, \quad \text{for all } n \in \mathbb{Z}.$$

Then for $\phi(s) = \psi(\sqrt{s}) \in \mathcal{S}(0,\infty)$,

$$\int_0^\infty s^{n/2}\phi(s)ds = 2\int_0^\infty t^{n+1}\psi(t)dt = a_n, \quad \text{for all } n \in \mathbb{Z},$$

as required. □

We have already seen that the spectral action $\mathrm{Tr}\,(f(D/\Lambda))$ converges and admits an asymptotic expansion, which is derived from the heat kernel trace asymptotics of $D^2$. However, the exact spectral action itself can be rewritten in terms of the heat kernel trace of $D^2$, a result we shall use repeatedly in the sequel:

**Corollary 3.4.** *Under the hypotheses of Theorem 3.1, one has that*

$$\mathrm{Tr}\,(f(D/\Lambda)) = \int_0^\infty \left[\int_M \mathrm{Tr}\left(K(s/\Lambda^2,x,x)\right) d\mathrm{Vol}(x)\right]\phi(s)ds, \tag{3.2}$$

*where $K(t,x,y)$ is the heat kernel of $D^2$.*

*Proof.* Let $\mu_k$ denote the $k$-th eigenvalue of $D^2$ in increasing order, counted with multiplicity. Then, since $f(D/\Lambda) = \mathcal{L}[\phi](D^2/\Lambda^2)$ is trace-class,

$$\begin{aligned}
\mathrm{Tr}\,(f(D/\Lambda)) &= \mathrm{Tr}\left(\mathcal{L}[\phi](D^2/\Lambda^2)\right)\\
&= \sum_{k=1}^\infty \mathcal{L}[\phi](\mu_k^2/\Lambda^2)\\
&= \sum_{k=1}^\infty \int_0^\infty e^{-s\mu_k^2/\Lambda^2}\phi(s)ds\\
&= \int_0^\infty \left[\sum_{k=1}^\infty e^{-s\mu_k^2/\Lambda^2}\right]\phi(s)ds\\
&= \int_0^\infty \mathrm{Tr}\left(e^{-sD^2/\Lambda^2}\right)\phi(s)ds\\
&= \int_0^\infty \left[\int_M \mathrm{Tr}\left(K(s/\Lambda^2,x,x)\right) d\mathrm{Vol}(x)\right]\phi(s)ds,
\end{aligned}$$

as was claimed. □

3.2. **Non-perturbative results.** We now give a non-perturbative heat-kernel-theoretic analysis of the phenomenon mentioned above.

Let $\widetilde{M} \to M$ be a finite normal Riemannian covering with $\widetilde{M}$ and $M$ compact, connected and oriented, and let $\Gamma$ be the deck group of the covering. Let $\widetilde{\mathcal{V}} \to \widetilde{M}$ be a $\Gamma$-equivariant self-adjoint Clifford module bundle, and let $\widetilde{D}$ be a $\Gamma$-equivariant symmetric Dirac-type operator on $\widetilde{\mathcal{V}}$. We can therefore form the quotient self-adjoint Clifford module bundle $\mathcal{V} := \widetilde{\mathcal{V}}/\Gamma \to M = \widetilde{M}/\Gamma$, with $\widetilde{D}$ descending to a symmetric Dirac-type operator $D$ on $\mathcal{V}$; under the identification $L^2(M,\mathcal{V}) \cong L^2(\widetilde{M},\widetilde{\mathcal{V}})^\Gamma$, we can identify $D$ with the restriction of $\widetilde{D}$ to $C^\infty(\widetilde{M},\widetilde{\mathcal{V}})^\Gamma$, where the unitary action $U:\Gamma\to U(L^2(\widetilde{M},\widetilde{\mathcal{V}}))$ is given by $U(\gamma)\xi(\widetilde{x}) := \xi(\widetilde{x}\gamma^{-1})\gamma$.

Our first goal is to prove the following result, relating the spectral action of $D$ to the spectral action of $\widetilde{D}$ in the high energy limit:

**Theorem 3.5.** *Let $f\in C^\infty(\mathbb{R})$ be of the form $f(x)=\mathcal{L}[\phi](x^2)$ for $\phi\in\mathcal{S}(0,\infty)$. Then for $\Lambda>0$,*

$$\operatorname{Tr}(f(D/\Lambda)) = \frac{1}{\#\Gamma}\operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda\to+\infty.$$

**Remark 3.6.** *Theorem 3.5 continues to hold even when inner fluctuations of the metric are introduced, since for $A\in C^\infty(M,\operatorname{End}(\mathcal{V}))$ symmetric, $D+A$ on $\mathcal{V}$ lifts to $\widetilde{D}+\widetilde{A}$ on $\widetilde{\mathcal{V}}$, where $\widetilde{A}$ is the lift of $A$ to $\widetilde{\mathcal{V}}$.*

To prove this result, we will need a couple of lemmas. First, we have the following well-known general fact:

**Lemma 3.7.** *Let $G$ be a finite group acting unitarily on a Hilbert space $\mathcal{H}$, and let $A$ be a $G$-equivariant self-adjoint trace-class operator on $H$. Let $\mathcal{H}^G$ denote the subspace of $\mathcal{H}$ consisting of $G$-invariant vectors. Then the restriction $A\mid\mathcal{H}^G$ of $A$ to $\mathcal{H}^G$ is also trace-class, and*

$$\operatorname{Tr}\left(A\mid\mathcal{H}^G\right) = \frac{1}{\#G}\sum_{g\in G}\operatorname{Tr}(gA).$$

*Proof.* This immediately follows from the observation that $\frac{1}{\#G}\sum_{g\in G} g$ is the orthogonal projection onto $\mathcal{H}^G$. □

Now, we can compute the heat kernel trace of $D$ using the heat kernel for $\widetilde{D}$:

**Lemma 3.8.** *For $t>0$,*

(3.3)
$$\begin{aligned}\operatorname{Tr}\left(e^{-tD^2}\right) = {} & \frac{1}{\#\Gamma}\operatorname{Tr}\left(e^{-t\widetilde{D}^2}\right)\\ & + \frac{1}{\#\Gamma}\sum_{\gamma\in\Gamma\setminus\{e\}}\int_{\widetilde{M}}\operatorname{Tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\right)d\mathrm{Vol}(\widetilde{x}),\end{aligned}$$

*where $\widetilde{K}(t,\widetilde{x},\widetilde{y})$ denotes the heat kernel of $\widetilde{D}$, and $\rho$ denotes the right action of $\Gamma$ on the total space $\widetilde{\mathcal{V}}$.*

*Proof.* Let $\gamma\in\Gamma$. Then for any $\xi\in C^\infty(\widetilde{M},\widetilde{\mathcal{V}})$,

$$\begin{aligned}\left(U(\gamma)e^{-t\widetilde{D}^2}\right)\xi(\widetilde{x}) &= U(\gamma)\left(\int_{\widetilde{M}}\widetilde{K}(t,\widetilde{x},\widetilde{y})\xi(\widetilde{y})d\mathrm{Vol}(\widetilde{y})\right)\\ &= \rho(\gamma)(\widetilde{x}\gamma^{-1})\left(\int_{\widetilde{M}}(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{y})\xi(\widetilde{y})d\mathrm{Vol}(\widetilde{y})\right)\\ &= \int_{\widetilde{M}}\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{y})\xi(\widetilde{y})d\mathrm{Vol}(\widetilde{y})\end{aligned}$$

so that the operator $U(\gamma)e^{-t\widetilde{D}^2}$ has the integral kernel

$$(t,\widetilde{x},\widetilde{y}) \mapsto \rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{y}).$$

Since $L^2(M,\mathcal{V}) \cong L^2(\widetilde{M},\widetilde{\mathcal{V}})^\Gamma$, we can therefore apply Lemma 3.7 to obtain the desired result. □

Finally, we can proceed with our proof:

*Proof of Theorem 3.5.* By Corollary 3.4 and Lemma 3.8, it suffices to show that for $\gamma \in G \setminus \{e\}$,

$$\int_0^\infty \left[\int_{\widetilde{M}} \operatorname{Tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\right) d\mathrm{Vol}(\widetilde{x})\right] \phi(s)ds = O(\Lambda^{-\infty}),$$

as $\Lambda \to \infty$.

Now, since $\widetilde{M}$ is compact and since the finite group $\Gamma$ acts freely and properly,

$$\inf_{(\widetilde{x},\gamma)\in\widetilde{M}\times\Gamma} d(\widetilde{x}\gamma^{-1},\widetilde{x}) = \min_{(\widetilde{x},\gamma)\in\widetilde{M}\times\Gamma} d(\widetilde{x}\gamma^{-1},\widetilde{x}) > 0.$$

Hence, by [16, Proposition 3.24], there exist constants $C > 0$, $c > 0$ such that

$$\sup_{\widetilde{x}\in\widetilde{M}} \|\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\|_2 \leq Ce^{-c/t}, \quad t > 0,$$

for $\|\cdot\|_2$ the fibre-wise Hilbert-Schmidt norm, implying, in turn, that for every $n \in \mathbb{N}$ there exists a constant $C_n > 0$ such that

$$\sup_{\widetilde{x}\in\widetilde{M}} \|\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\|_2 \leq C_n t^n, \quad t > 0.$$

Hence, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\left|\int_0^\infty \left[\int_{\widetilde{M}} \operatorname{Tr}\left(\rho(\gamma)(\widetilde{x}\gamma^{-1})\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\right) d\mathrm{Vol}(\widetilde{x})\right] \phi(s)ds\right| \\
&\leq \int_0^\infty \mathrm{Vol}(M)\left(\sup_{\widetilde{x}\in\widetilde{M}} \|\rho(\gamma)(\widetilde{x})\|_2\right)\left(\sup_{\widetilde{x}\in\widetilde{M}} \|\widetilde{K}(t,\widetilde{x}\gamma^{-1},\widetilde{x})\|_2\right)|\phi(s)|ds \\
&\leq \mathrm{Vol}(M)\cdot\left(\sup_{\widetilde{x}\in\widetilde{M}} \|\rho(\gamma)(\widetilde{x})\|_2\right)\cdot C_n \int_0^\infty (s/\Lambda^2)^n|\phi(s)|ds \\
&= \left(\mathrm{Vol}(M)\cdot\left(\sup_{\widetilde{x}\in\widetilde{M}} \|\rho(\gamma)(\widetilde{x})\|_2\right)\cdot C_n \cdot \int_0^\infty s^n|\phi(s)|ds\right)\Lambda^{-2n},
\end{aligned}$$

yielding the desired result. □

Now, let $\alpha : \Gamma \to \mathrm{GL}_N(\mathbb{C})$ be a representation of $\Gamma$; by endowing $\mathbb{C}^N$ with a $\Gamma$-equivariant inner product, we take $\alpha : \Gamma \to U(N)$. Since $\widetilde{M} \to M$ is a principal $\Gamma$-bundle, we form the associated Hermitian vector bundle $\mathcal{F} := \widetilde{M} \times_\alpha \mathbb{C}^N \to M$; since $\Gamma$ is finite, we endow $\mathcal{F}$ with the trivial flat connection $d$. We can therefore form the self-adjoint Clifford module bundle

$\mathcal{V}\otimes\mathcal{F}\to M$, which admits the symmetric Dirac-type operator $D_\alpha$ obtained from $D$ by twisting by $d$, that is,

$$D_\alpha = D\otimes 1 + c(1\otimes d),$$

where $c$ denotes the Clifford action on $\mathcal{V}\otimes\mathcal{F}$.

We now obtain the following generalisation of Theorem 3.1, which explains the factor of $N/\#\Gamma$ appearing in Theorem 2.1 above:

**Theorem 3.9.** *Let $f\in C^\infty(\mathbb{R})$ be of the form $f(x)=\mathcal{L}[\phi](x^2)$ for $\phi\in\mathcal{S}(0,\infty)$. Then for $\Lambda>0$,*

$$\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) = \frac{N}{\#\Gamma}\operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda\to+\infty.$$

**Remark 3.10.** *This result is again compatible with inner fluctuations of the metric, insofar as if $A\in C^\infty(M,\operatorname{End}(\mathcal{V}))$ is symmetric, then $D_\alpha + A\otimes 1$ on $\mathcal{V}\otimes\mathcal{F}$ is induced from $\widetilde{D}+\widetilde{A}$ on $\widetilde{\mathcal{V}}$, where $\widetilde{A}$ is $A$ viewed as a $\Gamma$-equivariant element of $C^\infty(\widetilde{M},\operatorname{End}(\widetilde{\mathcal{V}}))$.*

*Proof of Theorem 3.9.* On the one hand, consider the trivial bundle $\widetilde{\mathcal{F}} := \widetilde{M}\times\mathbb{C}^N$ over $\widetilde{M}$, together with the trivial flat connection $d$. Then for the action $(\widetilde{x},v)\gamma := (\widetilde{x}\gamma,\alpha(\gamma^{-1})v)$, $\widetilde{\mathcal{F}}$ is a $\Gamma$-equivariant Hermitian vector bundle, and $d$ is a $\Gamma$-equivariant Hermitian connection on $\widetilde{\mathcal{F}}$. Then, by taking the tensor product of $\Gamma$-actions, we can endow $\widetilde{\mathcal{V}}\otimes\widetilde{\mathcal{F}}$ with the structure of a $\Gamma$-equivariant self-adjoint Clifford module bundle, admitting the $\Gamma$-equivariant symmetric Dirac-type operator $\widetilde{D_\alpha} = \widetilde{D}\otimes 1 + c(1\otimes d)$. As a vector bundle, however, we may simply identify $\widetilde{\mathcal{V}}\otimes\widetilde{\mathcal{F}}$ with $\widetilde{\mathcal{F}}^{\oplus N}$, in which case we may identify $\widetilde{D_\alpha}$ with $\widetilde{D}\otimes 1_N$.

On the other hand, by construction, the bundle $\mathcal{F}$ defined above is the quotient of $\widetilde{\mathcal{F}}$ by the action of $\Gamma$. Hence, under the action of $\Gamma$, the quotient of $\widetilde{\mathcal{V}}\otimes\widetilde{\mathcal{F}}$ is the the self-adjoint Clifford module bundle $\mathcal{V}\otimes\mathcal{F}$, with $\widetilde{D_\alpha}$ descending to the operator $D\otimes 1 + c(1\otimes d) = D_\alpha$.

Finally, by Theorem 3.5 and our observations above,

$$\begin{aligned}\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) &= \frac{1}{\#\Gamma}\operatorname{Tr}\left(f(\widetilde{D_\alpha}/\Lambda)\right) + O(\Lambda^{-\infty})\\ &= \frac{1}{\#\Gamma}\operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\otimes 1_N\right) + O(\Lambda^{-\infty})\\ &= \frac{N}{\#\Gamma}\operatorname{Tr}\left(f(\widetilde{D}/\Lambda)\right) + O(\Lambda^{-\infty}), \qquad \text{as } \Lambda\to+\infty,\end{aligned}$$

as was claimed. □

One can apply these results to give a quick second proof of Theorem 2.1. Let $\Gamma\subset SU(2)$ be a finite group acting by isometries on $S^3$, identified with $SU(2)$ endowed with the round metric, and let $\alpha:\Gamma\to U(N)$ be a representation. Since $S^3$ is parallelizable and $\Gamma$ acts by isometries, the spinor bundle $\mathbb{C}^2 \to \mathcal{S}_{S^3} \to S^3$ and the Dirac operator $\not{D}_{S^3}$ are trivially $\Gamma$-equivariant. Then, by construction, the Dirac-type operator $D^\Gamma_\alpha$ on $\mathcal{S}_{S^3} \otimes \mathcal{V}_\alpha$ is precisely the induced operator $D_\alpha$ corresponding to $\widetilde{D} = \not{D}_{S^3}$, so that by Theorem 3.9,

$$\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) = \frac{N}{\#\Gamma} \operatorname{Tr}\left(f(\not{D}_{S^3}/\Lambda)\right) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda \to +\infty.$$

However, by [7, §2.2], one has that

$$\operatorname{Tr}\left(f(\not{D}_{S^3}/\Lambda)\right) = \Lambda^3 \widehat{f^{(2)}}(0) - \frac{1}{4}\Lambda \widehat{f}(0) + O(\Lambda^{-\infty}),$$

where $\widehat{f^{(2)}}$ denotes the Fourier transform of $u^2 f(u)$. Hence,

$$\operatorname{Tr}\left(f(D_\alpha/\Lambda)\right) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f^{(2)}}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) + O(\Lambda^{-\infty}),$$

as required.

3.3. **Perturbative results.** Let us now turn to the perturbative picture. In light of Theorem 3.1, it suffices to compare the Seeley-DeWitt coefficients of $\widetilde{D}^2$ with those of $D^2$ and $D_\alpha^2$.

**Proposition 3.11.** *Let $\widetilde{D}$ and $D$ be as above. Let $\pi : \widetilde{M} \to M$ denote the quotient map. Then for all $n \in \mathbb{N}$,*

$$a_n(\pi(\widetilde{x}), D^2) = a_n(\widetilde{x}, \widetilde{D}^2), \quad \widetilde{x} \in \widetilde{M},$$

*and hence*

$$\int_M a_n(x, D^2) d\mathrm{Vol}(x) = \frac{1}{\#\Gamma} \int_{\widetilde{M}} a_n(\widetilde{x}, \widetilde{D}^2) d\mathrm{Vol}(\widetilde{x}).$$

*Proof.* By [15, Lemma 4.8.1], there exist a unique connection $\nabla$ and endomorphism $E$ on $\mathcal{V}$ such that $D^2 = \nabla^*\nabla - E$, and similarly a unique connection $\widetilde{\nabla}$ and endomorphism $\widetilde{E}$ on $\widetilde{\mathcal{V}}$ such that $\widetilde{D}^2 = \widetilde{\nabla}^*\widetilde{\nabla}$. Since $\widetilde{D}^2$ is the lift of $D^2$ to $\widetilde{\mathcal{V}}$, it follows by uniqueness that $\widetilde{\nabla}$ and $\widetilde{E}$ are the lifts of $\nabla$ and $E$, respectively, to $\widetilde{\mathcal{V}}$ as well.

Now, since the finite group $\Gamma$ acts freely and properly on $\widetilde{M}$, let $\{(U_\alpha, \Psi_\alpha)\}$ be an atlas for $\widetilde{M}$ such that for each $\alpha$, $\pi \mid U_\alpha : U_\alpha \to \pi(U_\alpha)$ is an isometry. Hence, the local data defining $a_n(\cdot, D^2)$ on $U_\alpha$ lifts to the local data defining $a_n(\cdot, \widetilde{D}^2)$ on $\pi^{-1}(U_\alpha)$, so that $a_n(\cdot, \widetilde{D}^2)$ is indeed the lift to $\widetilde{M}$ of $a_n(\cdot, D^2)$, as required. ∎

**Proposition 3.12.** *Let $\widetilde{D}$ and $D_\alpha$ be as above. Then for all $n \in \mathbb{N}$,*

$$a_n(\pi(\widetilde{x}), D_\alpha^2) = N a_n(\widetilde{x}, \widetilde{D}^2), \quad \widetilde{x} \in \widetilde{M},$$

*and hence*

$$\int_M a_n(x, D_\alpha^2) d\mathrm{Vol}(x) = \frac{N}{\#\Gamma} \int_M a_n(\widetilde{x}, \widetilde{D}^2) d\mathrm{Vol}(\widetilde{x}).$$

*Proof.* On the one hand, by [15, Lemma 1.7.5], $a_n(\cdot, \widetilde{D}^2 \otimes 1_N) = N a_n(\cdot, \widetilde{D}^2)$, for $\widetilde{D}^2 \otimes 1_N$ on $\widetilde{\mathcal{V}}^{\oplus N}$. On the other hand, $\widetilde{D}^2 \otimes 1_N$ is the lift to $\widetilde{\mathcal{V}}^{\oplus N}$ of $D_\alpha^2$ on $\mathcal{V} \otimes \mathcal{F}$, so that by Proposition 3.11, $a_n(\cdot, \widetilde{D}^2 \otimes 1_N)$ is the lift to $\widetilde{M}$ of $a_n(\cdot, D_\alpha^2)$. Hence, $N a_n(\cdot, \widetilde{D}^2)$ is the lift to $\widetilde{M}$ of $a_n(\cdot, D_\alpha^2)$, as required. □

Let us now apply these results to the Dirac operator $\not{D}_{S^3}$ on the round 3-sphere $S^3$, together with a finite subgroup $\Gamma$ of $SU(2)$ acting freely and properly on $S^3 \cong SU(2)$, and a representation $\alpha : \Gamma \to U(N)$. Since $\not{D}_{S^3}^2 = (\nabla^S)^* \nabla^S + \frac{3}{2}$ by the Lichnerowicz formula, it follows from [15, Theorem 4.8.16] that

$$\int_{S^3} a_0(x, \not{D}_{S^3}^2) d\mathrm{Vol}(x) = \int_{S^3} (4\pi)^{-3/2} \mathrm{Tr}(\mathrm{id}) d\mathrm{Vol}(x) = \frac{\sqrt{\pi}}{2},$$
$$\int_{S^3} a_2(x, \not{D}_{S^3}^2) d\mathrm{Vol}(x) = \int_{S^3} (4\pi)^{-3/2} \mathrm{Tr}\left(\frac{6}{6}\mathrm{id} - \frac{3}{2}\mathrm{id}\right) d\mathrm{Vol}(x) = -\frac{\sqrt{\pi}}{4}.$$

Since the operator $D_\alpha^\Gamma$ is precisely $D_\alpha$ as induced by $\widetilde{D} = \not{D}_{S^3}$, it therefore follows by Proposition 3.12 that

$$\int_{S^3/\Gamma} a_0(y, (D_\alpha^\Gamma)^2) d\mathrm{Vol}(y) = \frac{N}{\#\Gamma} \int_{S^3} a_0(x, \not{D}_{S^3}^2) d\mathrm{Vol}(x) = \frac{N\sqrt{\pi}}{(\#\Gamma)2},$$
$$\int_{S^3/\Gamma} a_2(y, (D_\alpha^\Gamma)^2) d\mathrm{Vol}(y) = \frac{N}{\#\Gamma} \int_{S^3} a_2(x, \not{D}_{S^3}^2) d\mathrm{Vol}(x) = -\frac{N\sqrt{\pi}}{(\#\Gamma)4}.$$

Finally, one has that

$$\phi_{-3} = \frac{2}{\Gamma(3/2)} \int_0^\infty f(u) u^2 du = \frac{2}{\sqrt{\pi}} \int_{-\infty}^\infty f(u) u^2 du = \frac{2}{\sqrt{\pi}} \widehat{f^{(2)}}(0),$$
$$\phi_{-1} = \frac{2}{\Gamma(1/2)} \int_0^\infty f(u) du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^\infty f(u) du = \frac{1}{\sqrt{\pi}} \widehat{f}(0),$$

where $\widehat{f^{(2)}}$ is the Fourier transform of $f(u)u^2$. Hence,

$$\begin{aligned} \mathrm{Tr}\left(f(\not{D}_{S^3}/\Lambda)\right) &\sim \Lambda^3 \phi_{-3} \int_{S^3} a_0(x, \not{D}_{S^3}^2) d\mathrm{Vol}(x) \\ &\quad + \Lambda \phi_{-1} \int_{S^3} a_2(x, \not{D}_{S^3}^2) d\mathrm{Vol}(x) + O(\Lambda^{-1}) \\ &= \Lambda^3 \widehat{f^{(2)}}(0) - \frac{1}{4} \Lambda \widehat{f}(0) + O(\Lambda^{-1}), \end{aligned}$$

and

$$\begin{aligned}\operatorname{Tr}\left(f(D_\alpha^\Gamma)\right) &\sim \Lambda^3\phi_{-3}\int_{S^3/\Gamma} a_0(y,(D_\alpha^\Gamma)^2)d\mathrm{Vol}(y)\\ &\quad+\Lambda\phi_{-1}\int_{S^3/\Gamma} a_2(y,(D_\alpha^\Gamma)^2)d\mathrm{Vol}(y)+O(\Lambda^{-1})\\ &=\frac{N}{\#\Gamma}\left(\Lambda^3\widehat{f^{(2)}}(0)-\frac{1}{4}\Lambda\widehat{f}(0)\right)+O(\Lambda^{-1}),\end{aligned}$$

which is indeed consistent with Theorem 2.1.

## 4. The inflation potential and the power spectra

It was shown in [20], [21] that for a 3-manifold $Y$ that is a spherical space form $S^3/\Gamma$ or a flat Biebarbach manifold (a quotient of the flat torus $T^3$ by a finite group action), the non-perturbative spectral action determines a slow-roll potential for a scalar field $\phi$ by setting

$$\operatorname{Tr}(h((D^2_{Y\times S^1}+\phi^2)/\Lambda^2))-\operatorname{Tr}(h(D^2_{Y\times S^1}/\Lambda^2))=V_Y(\phi),$$

up to terms of order $O(\Lambda^{-\infty})$, where, in the spherical space form case the potential is of the form

$$V_Y(\phi)=\pi\Lambda^4\beta a^3\mathcal{V}_Y(\frac{\phi^2}{\Lambda^2})+\frac{\pi}{2}\Lambda^2\beta a\mathcal{W}_Y(\frac{\phi^2}{\Lambda^2}),$$

where $h$ the test function for the computation of the spectral action on the 4-manifold $Y\times S^1$, $a>0$ is the radius of the sphere and $\beta>0$ is the size of the circle compactification $S^1$. The functions $\mathcal{V}_Y$ and $\mathcal{W}_Y$ are of the form

$$\mathcal{V}_Y(x)=\lambda_Y\,\mathcal{V}_{S^3}(x)\quad\text{and}\quad\mathcal{W}_Y(x)=\lambda_Y\,\mathcal{W}_{S^3}(x),\tag{4.1}$$

where, for $Y=S^3/\Gamma$, the factor $\lambda_Y=(\#\Gamma)^{-1}$, and

$$\mathcal{V}_{S^3}(x)=\int_0^\infty u\,(h(u+x)-h(u))\,du\quad\text{and}\quad\mathcal{W}_{S^3}(x)=\int_0^x h(u)\,du.\tag{4.2}$$

Thus, the potential satisfies

$$V_Y(\phi)=\lambda_Y\,V_{S^3}(\phi)=\frac{V_{S^3}(\phi)}{\#\Gamma}.\tag{4.3}$$

The slow-roll potential $V_Y(\phi)$ can be used as a model for cosmological inflation. As such, it determines the behavior of the power spectra $\mathcal{P}_{s,Y}(k)$ and $\mathcal{P}_{t,Y}(k)$ for the density fluctuations and the gravitational waves, respectively given in the form

$$\mathcal{P}_s(k)\sim\frac{1}{M_{Pl}^6}\frac{V^3}{(V')^2}\quad\text{and}\quad\mathcal{P}_t(k)\sim\frac{V}{M_{Pl}^4},\tag{4.4}$$

with $M_{Pl}$ the Planck mass, see [28] and [21] for more details. Including second order terms, these can be written also as power laws as in [28],

$$\begin{aligned} \mathcal{P}_s(k) \sim & \quad \mathcal{P}_s(k_0)\left(\frac{k}{k_0}\right)^{1-n_s+\frac{\alpha_s}{2}\log(k/k_0)} \\ \mathcal{P}_t(k) \sim & \quad \mathcal{P}_t(k_0)\left(\frac{k}{k_0}\right)^{n_t+\frac{\alpha_t}{2}\log(k/k_0)}, \end{aligned} \tag{4.5}$$

where the exponents also depend on the slow roll potentials through certain slow-roll parameters. Since, as already observed in [20], [21], the slow-roll parameters are not sensitive to an overall multiplicative scaling factor in the potential, we focus here only on the amplitude only, which, as shown in [21], correspondingly changes by a multiplicative factor. Namely, in the case of a spherical space form with the spectral action computed for the untwisted Dirac operator, one has

$$\begin{aligned} \mathcal{P}_{s,Y}(k) \sim & \quad \lambda_Y\, \mathcal{P}_s(k_0)\left(\frac{k}{k_0}\right)^{1-n_{s,S^3}+\frac{\alpha_{s,S3}}{2}\log(k/k_0)} \\ \mathcal{P}_{t,Y}(k) \sim & \quad \lambda_Y\, \mathcal{P}_t(k_0)\left(\frac{k}{k_0}\right)^{n_{t,S^3}+\frac{\alpha_{t,S3}}{2}\log(k/k_0)}, \end{aligned} \tag{4.6}$$

where, as above, $\lambda_Y = 1/\#\Gamma$.

The amplitude and the exponents of the power law are parameters subject to constraints coming from cosmological observational data, as discussed in [18], [28], [29], so that, in principle, such data may be able to constrain the possible cosmic topologies in a model of gravity based on the spectral action. To this purpose, it is important to understand how much the amplitude and the slow-roll parameter are determined by the model. A discussion of the role of the parameters $\Lambda$, $a$, and $\beta$ is included in [21], while here we focus on how the coupling of gravity to matter affects these parameters.

By directly comparing the argument given in [20] proving (4.6) with the result of Theorem 2.1 above, we see that, in our case, we obtain then the following version of (4.6), modified by an overall multiplicative factor $N$, the total number of fermions in the model of gravity coupled to matter.

**Proposition 4.1.** *For a spherical space form $Y = S^3/\Gamma$, consider the slow-roll potential $V_{Y,\alpha}(\phi)$ determined by the nonperturbative spectral action*

$$\mathrm{Tr}(h((D^2_{\alpha,Y\times S^1} + \phi^2)/\Lambda^2)) - \mathrm{Tr}(h(D^2_{\alpha,Y\times S^1}/\Lambda^2)) = V_{Y,\alpha}(\phi),$$

*where $D_{\alpha,Y\times S^1}$ is the Dirac operator induces on the product geometry $Y\times S^1$ by the twisted Dirac operator $D^\Gamma_\alpha$ on $Y$. Then the associated power spectra as in* (4.4), (4.5) *satisfy* (4.6), *with $\lambda_Y = N/\#\Gamma$.*

4.1. **Inflation potential in the heat kernel approach.** Let us now consider inflation potentials on space-times of the form $M \times S^1_\beta$ for $M$ compact oriented Riemannian and odd-dimensional, arising from general almost-commutative triples over $M$.

Let $D$ is a symmetric Dirac-type operator on a self-adjoint Clifford module bundle $\mathcal{V} \to M$, with $M$ compact oriented Riemannian and odd-dimensional, and let $\not{D}_\beta$ be the Dirac operator with simple spectrum $\frac{1}{\beta}(\mathbb{Z} + \frac{1}{2})$ on the trivial spinor bundle $\mathbb{C} \to \mathcal{S}_{S^1_\beta} \to S^1_\beta$. We may immediately generalise the construction of [7, §2.3] to obtain an odd symmetric Dirac-type operator $D_{M \times S^1_\beta}$ on the self-adjoint Clifford module bundle $\left(\mathcal{V} \boxtimes \mathcal{S}_{S^1_\beta}\right)^{\oplus 2} \to M \times S^1_\beta$. Hence, we may define an inflation potential $V_M : C^\infty\left(M \times S^1_\beta\right) \to \mathbb{R}$ by

$$V_M(\phi) := \operatorname{Tr}\left(h((D^2_{M \times S^1_\beta} + \phi^2)/\Lambda^2)\right) - \operatorname{Tr}\left(h(D^2_{M \times S^1_\beta}/\Lambda^2)\right),$$

where $h = \mathcal{L}[\psi]$ for $\psi \in \mathcal{S}(0, \infty)$; note that $D^2_{M \times S^1_\beta} + \phi^2$ has heat trace

$$\operatorname{Tr}\left(e^{-t\left(D^2_{M \times S^1_\beta} + \phi^2\right)}\right) = 2\operatorname{Tr}\left(e^{-t\not{D}^2_\beta}\right)\operatorname{Tr}\left(e^{-tD^2}\right)e^{-\phi^2 t}$$

for $\phi$ locally constant.

Let $\Gamma \to \widetilde{M} \to M$, $\widetilde{\mathcal{V}} \to \widetilde{M}$, $\mathcal{V} \to M$, $\widetilde{D}$, $D$, $\alpha$, $\mathcal{F} \to M$ and $D_\alpha$ be defined as in Subsection 3.2, with $M$ and $\widetilde{M}$ odd-dimensional, generalising the discussion above of $\Gamma \to S^3 \to Y$. We may then form odd Dirac-type operators $\widetilde{D}_{\tilde{M} \times S^1_\beta}$, $D_{M \times S^1_\beta}$, and $D_{\alpha, M \times S^1_\beta}$ from $\widetilde{D}$, $D$ and $D_\alpha$, respectively, as above. On the other hand, if one trivially extends the action of $\Gamma$ on $\widetilde{M}$ to $\widetilde{M} \times S^1_\beta$ and the action on$\widetilde{\mathcal{V}} \to \widetilde{M}$ to $\left(\mathcal{V} \boxtimes \mathcal{S}_{S^1_\beta}\right)^{\oplus 2} \to M \times S^1_\beta$, then $D_{\widetilde{M} \times S^1_\beta}$ becomes a $\Gamma$-equivariant Dirac-type operator on $\left(\mathcal{V} \boxtimes \mathcal{S}_{S^1_\beta}\right)^{\oplus 2}$, and the constructions of Subsection 3.2 applied to the $\Gamma$-equivariant Dirac-type operator $D_{\widetilde{M} \times S^1_\beta}$ reproduce precisely the Dirac-type operators $D_{M \times S^1_\beta}$, and $D_{\alpha, M \times S^1_\beta}$.

Now, let $V_{\widetilde{M}}$, $V_M$, and $V_{M,\alpha}$ denote the inflation potentials corresponding to $\tilde{D}$, $D$, and $D_\alpha$, respectively, which we all view as nonlinear functionals on $C^\infty(\widetilde{M} \times S^1_\beta, \mathbb{R})^\Gamma \cong C^\infty(M \times S^1_\beta, \mathbb{R})$. Then, since we also have that $D^2_{\widetilde{M} \times S^1_\beta} + \phi^2$ is the lift of $D^2_{M \times S^1_\beta} + \phi^2$ and $\left(D^2_{\widetilde{M} \times S^1_\beta} + \phi^2\right) \otimes 1_N$ is the lift of $D^2_{M \times S^1_\beta, \alpha} + \phi^2$, Theorems 3.5 and 3.5, *mutatis mutandis*, therefore imply that

$$V_M(\phi) = \frac{1}{\#\Gamma} V_{\widetilde{M}}(\phi) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda \to +\infty, \tag{4.7}$$

and

$$V_{M,\alpha}(\phi) = \frac{N}{\#\Gamma} V_{\widetilde{M}}(\phi) + O(\Lambda^{-\infty}), \quad \text{as } \Lambda \to +\infty, \tag{4.8}$$

thereby explaining the factor $\lambda_Y$ in Equations 4.3 and 4.5 and Proposition 4.1.

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