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SHARP ASYMPTOTICS FOR TOEPLITZ DETERMINANTS, FLUCTUATIONS AND THE GAUSSIAN FREE FIELD ON A RIEMANN SURFACE

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Abstract. We consider canonical determinantal point processes with N particles on a compact Riemann surface X defined with respect to the constant curvature metric and show that in the large $N-$limit these processes satisfy strong exponential concentration of measure type properties involving Dirichlet norms. In the higher genus (hyperbolic) cases the point processes may be defined in terms of modular forms equipped with the Petersson norm. As a consequence we obtain a sharp optimal Central Limit Theorem (CLT), saying that the fluctuations of the corresponding empirical measures converge in distibution towards the Laplacian of the Gaussian free field on X in the strongest possible sense. The CLT is shown to be equivalent to a new sharp Szegö type limit theorem for Toeplitz determinants in this context.

1. Introduction

This paper is one a series which deal with $N-$particle determinantal point processes on a polarized compact complex manifold X, i.e. associated to high powers of an ample line bundle $L \to X$. A general Large Deviation Principle (LDP) was established in the large $N-$limit in [4], showing that the empirical measure convergenes exponentially towards the deterministic pluripotential equilibrium measure. Moreover, in the paper [2] a Central Limit Theorem (CLT) was obtained, showing that the fluctuations in the "bulk" may be desribed by a Gaussian free field in the case of *smooth* test functions (linear statistics). In the present paper we specialize to the lowest dimensinal case when X is a Riemann surface and the corresponding $N-$particle point processes are the "canonical" ones, i.e. the they are induced by the Kähler-Einstein metric on X. In this setting we obtain sharp versions of the upper large deviation bound and show that the convergence towards the Gaussian free field holds in the strongest possible sense, i.e. for linear statistics with minimal regularity assumptions (finite Dirichlet norm). The CLT is also shown to be equivalent to a new sharp Szegö type limit theorem for Toeplitz determinants in this context. The results are obtained from new "determinantal" Moser-Trudinger type inequalities, which imply strong concentration of measures properties. The proof of these latter inequalities is based on a convexity argument in the space of all Kähler metrics, combined with Bergman kernel asymptotics and potential theory.

1.1. The general setup.

Let $L \to X$ be an ample holomorphic line bundle over a compact complex manifold X of dimension n. We will denote by $H^0(X, L)$ the $N-$dimensional vector space of all global holomorphic sections of L. Given the geometric data $(\nu, \|\cdot\|)$ consisting of a probability measure ν on X and a continuous Hermitian metric $\|\cdot\|$ on L one obtains an associated probability measure $\mu^{(N)}$ on the $N-$fold product X^N defined as

$$\mu^{(N)} := \frac{1}{\mathcal{Z}_N} \|\det \Psi\|^2 (x_1, ...x_N)\nu(x_1) \otimes \cdots \otimes \nu(x_N) \tag{1.1}$$

where $\det \Psi$ is a holomorphic section of the pulled-back line bundle $L^{\boxtimes N}$ over X^N representing the Nth (i.e. maximal) exterior power of $H^0(X,L)$ and $\mathcal{Z}_N$ is the normalizing constant. Concretely, fixing a base $(\Psi_i)_{i=1}^N$ in $H^0(X,L)$ we can take

$$(\det \Psi)(x_1,...,x_{N_k}) = \det(\Psi_i(x_j)) \tag{1.2}$$

We will denote $\frac{i}{2\pi}$ times the curvature two-form of the metric on L by ω (compared with mathematical physics notation $\omega = \frac{i}{2\pi}F_A$ where A is the Chern connection induced by the metric on L). It will be convenient to take the pair (ω,ν), which will refer to as a *weighted measure*, as the given geometric data. The *empirical measure* of the ensemble above is the following random measure:

$$(x_1,...,x_N) \mapsto \delta_N := \sum_{i=1}^N \delta_{x_i} \tag{1.3}$$

which associates to any $N-$particle configuration $(x_1,...,x_N)$ the sum of the delta measures on the corresponding points in X. In probabilistic terms this setting hence defines a *determinantal random point process on X with N particles* [19, 24].

If the correponding L^2-norm on $H^0(X,L)$

$$\|\Psi\|_X^2 = \langle \Psi,\Psi\rangle_X := \int_X \|\Psi(x)\|^2 \, d\nu(x)$$

is non-degenerate (which will always be the case in this paper) then the probability measure $\mu^{(N)}$ on X^N may be expressed as a determinant of the *Bergman kernel* of the Hilbert space $(H^0(X,L), \|\cdot\|_X)$, i.e. the integral kernel of the corresponding orthogonal projection. A central role in this paper will be played by the *logarithmic generating function* (or *free enegy*)

$$\log \mathbb{E}(e^{-(\sum_{i=1}^N(\phi(x_i)})$$

of the *linear statistic*

$$\sum_{i=1}^N \phi(x_i), \tag{1.4}$$

where $\mathbb{E}$ denotes the expectation wrt the ensemble $(X^N,\mu^{(N)})$, i.e. $\mathbb{E}(\cdot) = \int_{X^N}(\cdot)\mu^{(N)}$. By a well-known formula going back to the work of Heine in the theory of orthogonal polynomials the expectation above can also be writen as a *Toeplitz determinant* with symbol $e^{-\phi}$:

$$\mathbb{E}(e^{-(\sum_{i=1}^N(\phi(x_i))}) = \det(\left\langle e^{-\phi}\Psi_i,\Psi_j\right\rangle_X) \tag{1.5}$$

where $(\Psi_i)_{i=1}^N$ is an orthonormal base in the Hilbert space $(H^0(X,L), \|\cdot\|_X)$. Replacing L with its k th tensor power, which we will write in additive notation as kL, yields,a sequence of point processes on X of an increasing number N_k of particles. We will be concerned with the asymptotic situation when $k \to \infty$. This corresponds to a large $N-$limit of many particles, since

$$N_k := \dim H^0(X,kL) = Vk^n + o(k^n)$$

where the constant V is, by definition, the *volume* of L.

As shown in [4] the normalized empirical measure δ_N/N_k converges towards a pluripotential equlibrium measure μ_{eq}, exponentially in probability. In particular, letting

$$\epsilon_{N_k,\lambda}(\phi) := \text{Prob}\left\{\left|\frac{1}{N_k}(\phi(x_1)+....+\phi(x_{N_k})) - \int_X \mu_{eq}\phi\right| > \lambda\right\} \tag{1.6}$$

denote the *tail of the linear statistic* determined by ϕ, at level k, it was shown that $\epsilon_{N_k,\lambda}(\phi) \to 0$ as $k \to \infty$ for any $\lambda > 0$ at a rate of the order $e^{-k^{n+1}/C}$. In the case when X is a Riemann surface the curvature current ω of the metric on L is semi-positive (so that $\mu_{eq} = \omega$) the following more precise estimate was obtained:

$$(1.7) \qquad \epsilon_{N_k,\lambda}(\phi) \leq 2\exp(-N_k^2\left(\frac{2V\lambda^2}{\|d\phi\|_X^2}(1+o(1))\right))$$

where the error term $o(1)$ denotes a sequence tending to zero as $k \to \infty$ (but depending on ϕ).

1.2. **Statement of the main results.** Let now $L \to X$ be a line bundle of positive degree V over a Riemann surface X of genus g. It determines a particular sequence of determinantal point process that we will refer to as the *canonical detetermintal point process on X associated to kL*. These processes are obtained by letting $(\nu, \omega) = (\omega/V, \omega)$ for ω the the unique volume form on X of volume V such that Riemannian metric determined by ω has constant scalar curvature. By the Riemann-Roch theorem we have (for k sufficently large)

$$N_k = kV - (g-1)$$

giving a simple relation between the level k and the corresponding number of particles N_k. Accordingly, we it will be convenient to talk about the *canonical determinantal random point process on X with N particles* and use N as the asymptotic parameter. Strictly speaking $N(= N_k)$ only determines L up to twisting by a flat line bundle, but the results will be independant of the flat line bundle. Physically, the canonical processes associated to kL represents the groundstate of a gas of free ferrmions in the "uniform" magnetic field kF_A where $\omega = \frac{i}{2\pi}F_A$ (see [4] and references therein) and A is a unitary connection on L.

The simplest case of this setting occurs when $g = 0$, i.e. X is the Riemann sphere and then $H^0(X, kL)$ may be identified with the space of all polynomials on the affine piece $\mathbb{C}$ of degree at most $k = N - 1$ equipped with the usual $SU(2)-$invariant Hermitian product. Alternatively, embedding X as the unit-sphere in Euclidian $\mathbb{R}^3$ the $N-$point correlation function of the process, i.e. the density of the probability measure, may be explicitely expanded as

$$\rho^{(N)}(x_1, ..., x_N) := \Pi_{1\leq i<j\leq N} \|x_i - x_j\|^2 / Z_N$$

where $1/Z_N = N^N\binom{N-1}{0}...\binom{N-1}{N-1}/N!$. In the physics litterature this ensemble also appears as a *Coulomb gas* of N unit-charge particles (i.e a one component plasma) confined to the sphere in a neutralizing uniform background ω (see for example [7]). An interesting random matrix model for this process was recently given in [25]. In the higher genus case the role of polynomials are played by, theta functions and modular (automorphic) forms on the universal covers $\mathbb{C}$ and $\mathbb{H}$ of X (when $g = 1$ and $g > 1$ respectively) equipped with their standard Hermitian products. See for example [14] for the case $g = 1$ in connection to fermions and bosonization. When $g > 1$ the Riemann surface X may be represented as the quotient $\Gamma/\mathbb{H}$ of the upper half-plane with a suitable discrete subgroup Γ of $SL(2, \mathbb{R})$. Taking $L := \frac{1}{2}K_X$, where K_X denotes the canonical line bundle $K_X = T^*X$ (using the induced spin structure to take the square root) realizes $H^0(X, kL)$ as the Hilbert space of all modular forms of weight k, i.e. all holomorphic funtions on $\mathbb{H}$ satisfying $f((az + b)/(cz + d)) = (cz + d)^k f(z)$ equipped with the Petterson norm

$$\|f\|_X^2 := \int_{\Gamma/\mathbb{H}} |f|^2 y^k \frac{dx \wedge dy}{y^2},$$

integrating over a fundamental domain for Γ. In special arithmetic situation the base (Ψ_i) in 1.2 may be represented by Hecke eigenfunctions (but note that we have assumed that X is smooth and compact, so that there are no cusps)[26].

It will be convenient to use the following conformally invariant notation for the normalized Dirichlet norm of a function ϕ on X, i.e. the L^2-norm of its gradient times $1/4\pi$:

$$\|d\phi\|_X^2 := \int_X d\phi \wedge d^c\phi : \left(= \frac{i}{2\pi}\int_X \partial\phi \wedge \bar{\partial}\phi\right)$$

We will obtain a very useful Moser-Trudinger type inequality for the canonical determinantal point processes, which generalizes Onofri's sharp version of the Moser-Trudinger inequality [28] (obtained when X is the two-sphere and $N = 1$).

Theorem 1.1. *Let X be a genus g Riemann surface and consider the canonical determinantal point process on X with N particles. It satisfies the following Moser-Trudinger type inequality:*

$$\log \mathbb{E}(e^{-(\sum_{i=1}^N(\phi(x_i)-\int_X \phi\frac{\omega}{V}))}) \leq \left(\frac{1}{1+(1-g)/N}+\epsilon_N\right)\frac{1}{2}\|d\phi\|_X^2+\epsilon_N \tag{1.8}$$

where the error term ϵ_N stand for a rapidly decreasing sequence of numbers, i.e. $\epsilon_N \leq C_j/N^j$ for any $j > 0$ (where $C_j > 0$ depends on ω, but not on ϕ). Similarly,

$$\log \mathbb{E}(e^{-(\sum_{i=1}^N(\phi(x_i)-\mathbb{E}(\phi(x_i))}) \leq \left(\frac{1}{1+(1-g)/N}+\epsilon_N\right)\frac{1}{2}\|d\phi\|_X^2+\epsilon_N\|\phi\|_{L^1(X)/\mathbb{R}}+\epsilon_N \tag{1.9}$$

Moreover, when X is the Riemann sphere (i.e .$g = 0$) all the error terms above vanish identically.

An important ingredient in the previous proof is a convexity result of Berndtsson [8] which in this particular case amounts to the positivity of a certain determinant line bundle over the space of all Kähler metrics in the first Chern class of L.

As a simple consequence of the previous theorem we then obtain a sharp version of the tail estimate 1.7 for such canonical processes. The main point is that it shows that the error term $o(1)$ appearing in the estimate 1.7 can be taken to be independent of the function ϕ. As a consequence the estimate holds with minimal regularity assumptions on ϕ :

Corollary 1.2. *Let X be a genus g Riemann surface and consider the canonical determinantal point process on X with N particles.* Let ϕ be a function on X such that its differential $d\phi$ is in $L^2(X)$. Then the linear statistic defined by ϕ has an exponentially decaying tail:

$$\epsilon_{N,\lambda}(\phi) \leq 2\exp(-N^2\left(\frac{2\lambda^2}{\|d\phi\|_X^2(1+\frac{(1-g)}{N})+\epsilon_k}+\epsilon_N\right))$$

where the error terms ϵ_N are as in the previous theorem.

We will also show that the Moser-Trudinger inequality in Theorem 1.1 is in fact an asymptotic *equality* in the following sense:

Theorem 1.3. *(Szegö type strong limit theorem). Let X be a genus g Riemann surface and consider the canonical determinantal point process on X with N particles. Let ϕ be a complex valued function on X such that* its differential is in $L^2(X,\mathbb{C})$, i.e. ϕ has finite Dirichlet norm. Then

$$\log \mathbb{E}(e^{-(\sum_{i=1}^N(\phi(x_i)-\int_X \phi\omega))}) \to \frac{1}{2}\int_X d\phi \wedge d^c\phi$$

as $N \to \infty$ and the same convergence holds when the exponent above is replaced with the fluctuation of the linear statistic of ϕ.

In [2] it was shown that, as long as $\omega > 0$ and $\phi \in C^1(X)$ an analogue of the convergence above holds in any dimension n if the conformally invariant norm above is replaced by the Dirichlet norm wrt ω. But it should be emphasized that when $n > 1$ the convergence does *not* hold if one relaxes the smoothness assumption on ϕ (see section 2.4 for counter examples).

The previous theorem may be equivalently formulated as the following Central Limit Theorem (CLT), valid under minimal regularity assumptions:

Corollary 1.4. (CLT) The fluctuations $\delta_N - \mathbb{E}(\delta_N)$ of the empirical measure δ_N converge in distribution to the the Laplacian (or rather dd^c) of the Gaussian free field (GFF). In other words, for any $\phi \in L^1(X)$ with $d\phi \in L^2(X)$ the fluctuations

$$\sum_{i=1}^{N}(\phi(x_i) - \mathbb{E}(\phi(x_i))$$

of the corresponding linear statistics converge in distribution to a centered normal random variable with variance $\|d\phi\|_X^2$.

The GFF is also called the *massless bosonic free field* in the physics litterature. Heuristically, this is a random function wrt the Gaussian measure on the Hilbert space of all ϕ (mod $\mathbb{R}$) equipped with the Dirichlet norm $\|d\phi\|_X^2/2$. For the precise definition of the GFF and its Laplacian see [32] (Prop 2.13 and Remark 2.14) and for a comparison with the physics litterature on Coulomb gases see section 1.3 in [33].

1.3. **Relations to previous results.**

Exponential concentration. A determinantal Moser-Trudinger (M-T) inequality on S^2, but with non-optimal constants was first obtained by Fang [12] building on previous work by Gillet-Soulé concerning the S^1-invariant case [15], which in turn used the classical Moser-Truding (one-particle) inequality. The motivation came from arithmetic (Arakelov) geometry and spectral geometry. The optimal constants on S^2 were obtained by the author in [3] using methods further developed in the present paper. It would be interesting to know for which other (determinantal) random point processes similar inequalities hold, i.e. upper bounds on the logarithmic moment generating function of the linear statistic defined by $\phi(x)$ in terms of the Dirichlet norm $\|d\phi\|_X^2$. The only previously known case seems to be the case when the measure measure ν is the invariant measure on S^1 (and $\omega = 0$), corresponding to the standard unitary random matrix ensemble. Then the corresponding inequalities follow from a simple monotonicity argument going back to the classical work of Szegö (see for example [21] and references therein). Recently, several works have been concerned with a weaker form of such moment inequalities where the role of the Dirichlet norm is played by the Lipschitz norm. These inequalities fit into a circle of ideas sourrounding the "concentration of measure phenomena" in high dimensions. We refer to the survey [18] and the book [27] for precise references. Formulated in the present settings these latter inequalities hold for $\nu = 1_{\mathbb{R}} e^{-v(x)}dx$ with $v(x)$ strictly convex (satisfying $d^2v/d^2x > C$). As explained in [18], by the Bakry-Emery theorem and Klein's lemma, the corresponding point processes satisfy a log Sobolev inequality, which by Herbst's argument yields the desired moment inequality.

Szegö type limits and CLT:s. The convergence in Theorem 1.3 (and its Corollary) in the case when $X = S^2$ was first obtained by Ryder-Virag [30], using combinatorial (and diagrammatic) arguments to estimate the cumulants (i.e. the coefficients in the Taylor expansion of the log moment generating function), combined with estimates on the 2-point functions. They also obtained analagous results for the homogenous determinantal point processes on the other two simply connected Riemann surfaces, i.e on $\mathbb{C}$ and $\mathbb{H}$. However, in the latter cases the processes have an infinite number of particles and are hence different from the sequence of non-homogenous ones considered in the present paper on a *compact* Riemann surfaces of genus $g > 0$. In the circle case (refered to above) and assuming ϕ *smooth* the analogue of the convergence in Thm 1.3 is the celebrated Szegö strong limit theorem from 1952. In this case the Dirichlet norm of ϕ has to be replaced by the Dirichlet norm of the harmonic extension of ϕ to the unit-disc. The result of Szegö was motivated by Onsager's work on phase transitions for the 2D Ising model. The case of a general ϕ was eventually shown by Ibragimov [20]. A new proof was then given by Kurt Johansson [21], who also pointed out the relation to a CLT for the unitary random matrix ensemble. We refer to the survey [34] for an interesting account of the history of Szegö's theorem and elaborations. The proof in the Riemann surface cases in the present paper is partly inspired by the argument in [21], where the determinantal Moser-Trudinger inequalities on S^1 (refered to above) were used to reduce the *upper* bound in the convergence to the smooth case, also using analytic continuation. There are also similar convergence results for other weighted measures in the plane appearing in Random Matrix Theory, but rather strong regularity assumptions on ϕ are then imposed [21, 22].

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1.4. **Notation**[1]**.** Let $L \to X$ be a holomorphic line bundle over a compact complex manifold X.

1.4.1. *Metrics on L.* We will fix, once and for all, a Hermitian metric $\|\cdot\|$ on L. Its curvature form times the normalization factor $\frac{i}{2\pi}$ will be denoted by ω. The normalization is made so that $[\omega]$ defines an *integer* cohomology class, i.e. $[\omega] \in H^2(X, \mathbb{Z})$. The local description of $\|\cdot\|$ is as follows: let s be a trivializing local holomorphic section of L, i.e. s is non-vanishing an a given open set U in X. Then we define the local *weight* Φ of the metric $\|\cdot\|$ by the relation

$$\|s\|^2 = e^{-\Phi}$$

The (normalized) curvature current ω may now by defined by the following expression:

$$\omega = \frac{i}{2\pi}\partial\overline{\partial}\Phi := dd^c\Phi,$$

(where we, as usual, have introduced the real operator $d^c := i(-\partial+\overline{\partial})/4\pi$ to absorb the factor $\frac{i}{2\pi}$). The point is that, even though the function ϕ is merely locally well-defined the form ω is globally well-defined (as any two local weights differ by $\log |g|^2$ for g a non-vanishing holomorphic function). The current ω is said to be *positive* if the weight Φ is *plurisubharmonic (psh).* If Φ is smooth this simply means that the Hermitian matrix $\omega_{ij} = (\frac{\partial^2\Phi}{\partial z_i \partial \bar{z}_j})$ is positive definite (i.e. ω is a Kähler form) and in general it means that, locally, Φ can be written as a decreasing limit of such smooth functions.

[1]general references for this section are the books [16, 11]. See also [**?**] for the Riemann surface case.

1.4.2. *Holomorphic sections of* L. We will denote by $H^0(X,L)$ the space of all global holomorphic sections of L. In a local trivialization as above any element Ψ in $H^0(X,L)$ may be represented by a local holomorphic function f, i.e.

$$\Psi = fs$$

The squared point-wise norm $\|\Psi\|^2(x)$ of Ψ, which is a globally well-defined function on X, may hence be locally written as

$$\|\Psi\|^2(x) = (|f|^2 e^{-\Phi})(x)$$

It will be convenient to take the curvature current ω as our geometric data associated to the line bundle L. Strictly speaking, it only determines the metric $\|\cdot\|$ up to a multiplicative constant but all the geometric and probabilistic constructions that we will make are independent of the constant.

1.4.3. *Metrics and weights vs* $\omega-$ *psh functions.* Having fixed a continuous Hermitian metric $\|\cdot\|$ on L with (local) weight Φ_0 any other metric may be written as

$$\|\cdot\|^2_\phi := e^{-\phi}\|\cdot\|^2$$

for a continuous function ϕ on X, i.e. $\phi \in C^0(X)$. In other words, the local weight of the metric $\|\cdot\|_\phi$ may be written as $\Phi = \phi + \Phi_0$ and hence its curvature current may be written as

$$dd^c\Phi = \omega + dd^c\phi := \omega_\phi$$

This means that we have a correspondence between the space of all (singular) metrics on L with positive curvature current and the space $PSH(X,\omega)$ of all upper-semi continuous functions on X such that $\omega_\phi \geq 0$ in the sense of currents. Note for example, that if $\Psi \in H^0(X,L)$ then $\log\|\Psi\|^2 \in PSH(X,\omega)$. In particular, in the Riemann surface case $PSH(X,\omega)(= SH(X,\omega))$ is the space of all usc functions ϕ such that $\Delta_\omega\phi \geq -1$, where Δ_ω denotes the Laplacian wrt the Riemannian metric corresponding to ω, i.e.

$$\Delta_\omega\phi = (dd^c\phi)/\omega$$

2. Proofs of the main results

For a general Kähler manifold (X,ω) there is well-known energy type functional which may be written as

$$\mathcal{E}_\omega(\phi) := \frac{1}{(n+1)!V}\sum_{j=0}^{n}\int_X \omega_\phi^j \wedge (\omega)^{n-j} \tag{2.1}$$

Up to normalization it can be defined as the primitive of the Monge-Ampère operator seen as a one-form on the space of all Kähler potentials ϕ (and it was in this form it was first introduced by Mabuchi in Kähler geometry; see [4] and references therein).

We now turn to the case when X is a Riemann surface, i.e. $n = 1$. In particular, after an integration by parts $\mathcal{E}_\omega$ can then be expressed in terms of the usual Dirichlet energy on a Riemann surface:

$$V\mathcal{E}_\omega(\phi) = -\frac{1}{2}\int d\phi \wedge d^c\phi + \int \phi\omega \tag{2.2}$$

Following [3] it will also be convenient to consider a variant of the setting given in the introduction of the paper where the Hilbert space is the space $H^0(X, kL + K_X)$

of holomorphic one-form with values in L equipped with the canonical Hermitian product induced by the weight Φ on L : .

$$\langle \Psi, \Psi \rangle_X := i \int_X \Psi \wedge \bar{\Psi} e^{-k\Phi} \tag{2.3}$$

(equivalently, one picks a volume form μ on X and takes $1/\mu$ as the metric on K_X). We will call this the *adjoint setting* and the corresponding process on X *the adjoint determinantal point process at level k*. Note that if δ_N denotes the empirical measure for this latter process then

$$\mathbb{E}(\delta_N) = i \sum_{i=1}^{N} \Psi_i \wedge \bar{\Psi}_i e^{-\Phi}$$

(the measure above was called the Bergman measure in [3]). The following proposition is essentially well-known.

Proposition 2.1. *Let $L \to X$ be a line bundle over a Riemann surface equipped with a metric $e^{-\Phi}$ with strictly positive curvature form $\omega (= dd^c\Phi)$ such that the Riemannian metric on X defined by ω has constant scalar curvature. Then the canonical determinantal point processes associated to kL (with $N(= N_k)$ particles) satisfy*

$$\sup_X |\frac{\mathbb{E}_N(\delta_N/N)}{\omega/V} - 1| \leq \epsilon_N,$$

where $\epsilon_N = O(1/N^\infty)$ (as the error terms in Theorem 1.1)

Proof. Consider the Hilbert space $H^0(kL)$ with Hermitian product defined by

$$\langle s, s \rangle := \int_X |s|^2 e^{-k\Phi} (dd^c\Phi)$$

and fix an orthonormal base (s_i) (depending on k). To simplify the notation we assume that $V := \int_X \omega = 1$ (the proof in the general case is essentially the same). By Lu's theorem (see [36] and references therein) there is an asymptotic Bergman kernel expansion

$$(\sum_{i=1}^{N} |s_i|^2 e^{-k\Phi}) = k + \frac{R}{2} + a_1(x)k^{-1} + a_2(x)k^{-2} ...,$$

where R is the scalar curvature of ω and a_i can be expressed as a universal polynomials in the covariant derivatives of the curvature tensor of the Hermitian metric ω (more generally this is true in any dimension n if the rhs above is multiplied by k^{n-1}). Since we have assumed that $n = 1$ the coefficents a_i are in fact universal polynomials in differential operators applied to R. But since R is constant it follows that a_i must also be constant and we will conclude the argument by showing that, in general, $\int_X a_i \omega = 0$. To this end first note that by the Riemann-Roch theorem

$$N_k = \int_X (\sum_{i=1}^{N} |s_i|^2 e^{-k\Phi} \omega = k + \int R/2\omega$$

and thus comparing the previous two expansions forces $a_i = 0$. All in all this means that

$$\frac{1}{N} (\sum_{i=1}^{N} |s_i|^2 e^{-k\Phi}) = \frac{k + \frac{R}{2} + O(k^{-\infty})}{k + \frac{R}{2}} = 1 + O(k^{-\infty})$$

which concludes the proof with an $O(k^{-\infty})-$error term. It be pointed out that in the case when X is the sphere it follows immediately from the fact that X is homogenous (under the $SU(2)-$action) that the error term in the previous proposition vanishes.

More generally, in the case of stricly positive genus g, when L is the theta line bundle ($g = 1$) or a multiple of the canonical line bundle K_X, the error term can be estimated by Ce^{-Ck} for some positive constant C. This can be shown by lifting the problem to the universal cover of X where the full Bergman kernel is constant by homogenity. When $g = 1$ the calculation can then be done essentially explicetely using theta functions (see [13]) and when $g > 1$ the proof uses Selberg's trace formula (the author is grateful to Steve Zelditch for pointing this out). $\square$

2.1. **Proof of Theorem 1.1 (determinantal Moser-Trudinger inequality).** We start with the following non-asymptotic inequality.

Proposition 2.2. *Let $L \to X$ be a line bundle over a Riemann surface equipped with a smooth metric with strictly positive curvature form ω. Consider the corresponding adjoint determinantal point process. Then the following estimate holds*

$$\frac{1}{N}\log \mathbb{E}(e^{-\phi}) - \mathcal{E}_\omega(\phi) \leq \sup_X |\frac{\mathbb{E}(\delta/N)}{\omega/V} - 1|(-\mathcal{E}_\omega(\phi - \sup_X \phi))$$

for any smooth function ϕ satisfying $\omega_\phi := dd^c\phi + \omega \geq 0$, where $\delta(= \delta_N)$ denotes the empirical measure of the process.

Proof. The proof is a simple modification of the proof Theorem 33 in [3]. As a courtesy to the reader we will recall the main points the argument in [3]. An important ingredient in the proof is the notion of a C^0-geodesic segments in $C^0(X) \cap PSH(X,\omega)$. This may be defined as the continuous path ϕ_t connecting ϕ_0 and ϕ_1 in $C^0(X) \cap PSH(X,\omega)$ obtained as the upper envelope of all S^1- invariant $\pi^*\omega-$psh extensions to the $n+1-$dimensional complex manifold $X \times [0,1[\times S^1$ (where π denotes the projection from $X \times [0,1[\times S^1$ to X). In particular, ϕ_t is convex in the real paramter t. See [3] for the precise construction. Now consider the following functional on $\mathcal{C}^0(X)$, which is invariant under addition of constants.

$$\mathcal{F}_\omega(\phi) := \mathcal{E}_{\omega_0}(\phi) + \frac{1}{N}\log \mathbb{E}(e^{-\phi})$$

The following variational formulas hold:

$$(i)\ -\frac{1}{N}d(\log \mathbb{E}(e^{-\phi_t})/dt = \left\langle \mathbb{E}_{\omega_{\phi_t}}(\delta/N), d\phi_t/dt \right\rangle,\ \ (ii)\ d\mathcal{E}_{\omega_0}(\phi_t)/dt = \frac{1}{V}\left\langle \omega_{\phi_t}, d\phi_t/dt \right\rangle$$

Moreover, if ϕ_t is a C^0-geodesic in $\mathrm{Psh}(X,\omega)$ then

$$(i')\ \ \log \mathbb{E}(e^{-\phi_t})\ \text{is convex},\ \ (ii')\ \mathcal{E}_{\omega_0}(\phi_t)\ \text{is affine}$$

in the real parameter t (the item (i') above follows from the Toeplitz determinant representation 1.5 combined with the positivity results for direct image bundles in [8]; see also the appendix in [3] for another proof using the structure of determinantal point processes). Now, for any given $\phi \in C^0(X) \cap \mathrm{Psh}(X,\omega)$ we let ϕ_t be the C^0-geodesic such that $\phi_0 = 0$ and $\phi_1 = \phi$. By the concavity of $\mathcal{F}_\omega(\phi_t)$ (resulting from (i') combined with (ii')) and since $\mathcal{F}_\omega(\phi_0) = 0$ we have

$$\mathcal{F}_\omega(\phi) \leq d(\mathcal{F}_\omega(\phi_t))/dt_{t=0} = \int (V\mathbb{E}(\delta/N)/\omega - 1)\frac{1}{V}\omega(-d\phi_t/dt)_{t=0}$$

Next, note that, since the inequality in the theorem that we are about to prove is invariant under $\phi \to \phi + C$ we may as well assume that $\sup_X \phi = 0$. Since ϕ_t is convex in t we have $-d\phi_t/dt \leq \phi_1 - \phi_0 = \phi$ (we are using *right* derivatives, which always exist by convexity) and hence

$$\mathcal{F}_\omega(\phi) \leq \sup_X(V\mathbb{E}(\delta/N)/\omega - 1)\frac{1}{V}(\int \omega(-d\phi_t/dt)_{t=0}$$

Next, note that, combining (ii) and (ii') above gives

$$(\int \omega(-d\phi_t/dt)_{t=0} = d\mathcal{E}_\omega(\phi_t)/dt_{t=0} = -\mathcal{E}_\omega(\phi)$$

and hence

$$\mathcal{F}_\omega(\phi) \leq \sup_X (V\mathbb{E}(\delta/N)/\omega - 1)(-\mathcal{E}_\omega(\phi))$$

Finally, replacing ϕ with $\phi - \sup_X \phi$ finishes the proof of the lemma. □

To reduce the case of a general smooth function ϕ to an $\omega-$psh one we will make use of the psh-projection P_ω mapping smooth functions to $\omega-$psh ones:

$$(P_\omega \phi)(x) := \sup \{\psi(x) : \psi \in PSH(X,\omega), \psi \leq \phi \text{ on } X\} \tag{2.4}$$

It is not hard to see that $P_\omega\phi$ is continuous when ϕ is and moreover that the following "orthogonality relation" holds [5]

$$\int_X (\phi - P_\omega\phi) dd^c(P_\omega\phi) = 0 \tag{2.5}$$

Moreover, as shown in [1] if $\phi \in \mathcal{C}^{1,1}(X)$, i.e. the differential $d\phi$ is Lipshitz continuous then $\phi \in \mathcal{C}^\infty(X)$.

Proposition 2.3. *Let* (X, ω) *be a Riemann surface with a Kähler. Then*

$$(i)\, \mathcal{E}_\omega(\phi) \leq \mathcal{E}_\omega(P_\omega\phi), \quad (ii)\ \|d(P_\omega\phi)\|_X^2 \leq \|d\phi\|_X^2$$

for any $\phi \in C^\infty(X)$.

Proof. (i) was proved in [3] and (ii) is proved in a similar way, as we will next see. Integrating by parts gives

$$\|d(P_\omega\phi)\|_X^2 = \int (-P_\omega\phi) dd^c(P_\omega\phi) = \int (-P_\omega\phi)(dd^c P_\omega\phi + \omega) + \int (P_\omega\phi)\omega$$

Next, since $P_\omega\phi = \phi$ a.e. with respect to $(dd^c P_\omega\phi + \omega)$ (by formula 2.5) this means that

$$\|d(P_\omega\phi)\|_X^2 = \int (-\phi)(dd^c P_\omega\phi + \omega) + \int (P_\omega\phi)\omega = \int (-\phi)(dd^c P_\omega\phi) + \int (P_\omega\phi - \phi)\omega$$

But since $(P_\omega\phi - \phi) \leq 0$ and $\omega \geq 0$ the last term above is non-positive and hence

$$\|d(P_\omega\phi)\|_X^2 \leq \|d(P_\omega\phi)\|_X \|d\phi\|_X,$$

also using the Cauchy-Schwartz inequality for the first term above. Dividing out $\|d(P_\omega\phi)\|_X$ (which is always non-zero if ϕ is) proves Step 2. □

2.1.1. *End of proof of Theorem 1.1.* We start with the proof of the inequality 1.8. Consider the line bundle kL with Φ the weight of a metric on L with curvature $\omega := dd^c\Phi > 0$ and decompose

$$kL =: L_k + K_X, \quad k\Phi =: \Phi_k + \Phi_\omega$$

where $\Phi_\omega := \log(\frac{\omega}{V i dz \wedge d\bar{z}})$ defines the weight of a metric on on K_X. Then the Hilbert space $H^0(kL)$ associated to the weighted measure $(\frac{\omega}{V}, \omega)$ is naturally isomorphic to the Hilbert space $H^0(L_k + K_X)$ associated to the weight Φ_k in the adjoint setting, just using that, by definition,

$$e^{-k\Phi}\frac{\omega}{V} = e^{-\Phi_k} i dz \wedge d\bar{z}$$

We will write $\omega_k := dd^c\Phi_k$ (and we let N_k be the dimension of $H^0(kL)$ and V_k the volume (degree) of L_k. Then

$$\omega_k/V_k = \omega/V \tag{2.6}$$

and in particular $\omega_k > 0$. This follows immediately from the fact that the forms in rhs and the lhs above both integrate to one over X and moreover, by assumption, ω satisfies the Kähler-Einstein equation:

$$dd^c\phi_\omega (:= \text{-Ric}\omega) \ = \lambda\omega$$

for some constant λ, so that ω_k is proportinal to ω.

Step one: scaling by k and assuming $(\omega_k)_\phi (:= \omega_k + dd^c\phi) \geq 0$.

Applying Prop 2.2 and Prop 2.1 to (L_k, ω_k) and ϕ and using formula 2.2 gives, using 2.6,

$$\frac{1}{N_k}\log \mathbb{E}(e^{-\phi}) + \mathcal{E}_{\omega_k}(\phi) \leq \epsilon_k \left(\frac{1}{2V_k} \|d\phi\|_X^2 + \int (\sup_X \phi - \phi)\frac{\omega}{V} \right)$$

Next, we recall the following basic inequality: there is a constant C (only depending on ω) such that

$$\sup_X \psi \leq \int_X \psi\omega + C$$

for any ψ such that $\omega_\psi \geq 0$ (as follows immediately from Green's formula; see [17] for more general inequalities). Setting $\psi = \phi/k$ and applying the previous inequality to the rhs in the preceeding inequality gives, since $\omega_k/k \sim \omega$, that

$$\frac{1}{N_k}\log \mathbb{E}(e^{-\phi}) + \mathcal{E}_{\omega_k}(\phi) \leq \epsilon_k (\|d\phi\|_X^2 + kC)$$

Step two: using P_{ω_k}

Let now ϕ be a general smooth function. Since $P_{(\omega_k)}\phi \leq \phi$ we have $\frac{1}{N}\log \mathbb{E}(e^{-\phi}) \leq \frac{1}{N}\log \mathbb{E}(e^{-P_{(\omega_k)}\phi})$ and hence the previous step applied to $P_{\omega_k}\phi$ combined with (i) in the previous proposition and step one gives

$$\frac{1}{N_k}\log \mathbb{E}(e^{-\phi}) + \mathcal{E}_{\omega_k}(\phi) - \epsilon_k \leq \epsilon_k \|d(P_\omega\phi\|_X^2 \leq \epsilon_k \|d\phi\|_X^2$$

also using (ii) in the previous proposition in the last inequality. Finally,using the scaling property

$$\log \mathbb{E}(e^{-(\psi+c)})/N = -c + \log \mathbb{E}(e^{-\psi})/N \qquad (2.7)$$

together with formula 2.2 and the identity 2.6 we can rewrite

$$\frac{1}{N_k}\log \mathbb{E}(e^{-\phi}) + \mathcal{E}_{\omega_k}(\phi) = \frac{1}{N_k}\log \mathbb{E}(e^{-(\phi - \int_X \phi\frac{\omega}{V}}) - \frac{1}{V_k}\frac{1}{2}\|d\phi\|_X^2$$

All in all this means that

$$\log \mathbb{E}(e^{-\phi}) \leq (\frac{N_k}{V_k}\frac{1}{2} + \epsilon_k)\|d\phi\|_X^2 + \epsilon_k$$

Finally, by the Riemann-Roch theorem

$$\frac{N_k}{V_k} = \frac{k\deg(L) - \deg(K_X)/2}{k\deg(L) - \deg(K_X)} = \frac{N_k}{N_k - \deg(K_X)/2} = \frac{N_k}{N_k + (1-g)}$$

finishing the proof of the inequality 1.8.

To prove the second inequality 1.9 in the theorem we first note that

$$\int \phi(\omega/V - \mathbb{E}(\delta/N)) \leq \epsilon_N \|\phi\|_{L^1(X)/\mathbb{R}} (:= \epsilon_N \inf_{c\in\mathbb{R}} \|\phi + c\|_{L^1(X)})$$

Indeed, the lhs above is invariant under the action of $\mathbb{R}$, $\phi \to \phi + c$, and hence the inequality follows immediately from Prop 2.1. The inequality 1.8 then follows immediately from the fact that $\phi \to d\phi$ is invariant under the action of $\mathbb{R}$ combined with the scaling property 2.7 (just take $\psi = \phi - \int \phi\omega$ and $c = \int \phi(\omega/V - \mathbb{E}(\delta/N))$).

2.2. **Proof of Cor 1.2.** The proof is a standard application of Markov's inequality: for any given $t > 0$ we have

$$\mathrm{Prob}\{Y > 1\} = \mathrm{Prob}\{e^{tY} > e^t\} \leq e^{-t}\mathbb{E}(e^{tY}),$$

where in our case $Y = \frac{1}{N\epsilon}(\phi(x_1) + ... + \phi(x_N))$. By the previous theorem the rhs above is bounded by $e^{-t+ct^2/2}e^{\epsilon_N}$ for $c = (a_N + \epsilon_N)\left\|d(\frac{1}{N\epsilon}\phi)\right\|^2$. Taking $t = 1/c$ shows that the first factor may be estimated by $e^{-\frac{1}{2c}}$ which finishes the proof of the corollary.

2.3. **Proof of Theorem 1.3 (Sharp Szegö type limit theorem).** We will use the following notation for the fluctuation of the linear statistic determined by a function ϕ on X :

$$\tilde{\phi} := \sum_{i=1}^{N}(\phi(x_i) - \mathbb{E}(\phi(x_i))$$

We start by proving the following universal bound on the variance for the canonical processes, which is of independent interest.

Proposition 2.4. *For any given function ϕ on X the following upper bound on the variance of the corresponding linear statistic holds:*

$$\mathbb{E}(|\tilde{\phi}|^2)/4 \leq (1 + \epsilon_N)\left\|d\phi\right\|_X^2 + \epsilon_N \left\|\phi\right\|_{L^1(X)/\mathbb{R}}^2$$

where ϵ_N denotes a sequence, independent of ϕ, tending to zero. In particular, if $\phi \in L^1(X)$ and $d\phi \in L^2(X)$ then the variance is uniformly bounded from above by a constant independent of N.

Proof. We will denote by ϵ_N a sequence tending to zero, which may change from line to line. By the second inequality in Theorem 1.1 we have

$$\mathbb{E}(e^{-t\tilde{\phi}}) \leq e^{(1+\epsilon_N))\frac{1}{2}t^2\|d\phi\|_X^2 + \epsilon_N t\|\phi\|_{L^1(X)/\mathbb{R}}}e^{\epsilon_N}$$

Using $2ab \leq a^2 + b^2$ hence gives

$$\mathbb{E}(e^{-t\tilde{\phi}}) \leq e^{\frac{1}{2}t^2 f_N}e^{\epsilon_N}, \;\; f_N = \Big((1 + \epsilon_N)\left\|d\phi\right\|_X^2 + \epsilon_N \left\|\phi\right\|_{L^1(X)/\mathbb{R}}^2 + \epsilon_N\Big)$$

Repeating the argument in the proof of 1.2 (involving Markov's inequality) hence gives

$$\mathrm{Prob}\{(\tilde{\phi} > \lambda\} \leq e^{-\lambda^2\frac{1}{2}\frac{1}{f_N}}e^{\epsilon_N}$$

Now using the push-forward formula for the integral in $\mathbb{E}(|\tilde{\phi}|^2)$ we can write

$$\mathbb{E}(|\tilde{\phi}|^2) = \int_0^\infty (\mathrm{Prob}\{\tilde{\phi}^2 > \lambda\})d(\lambda^2) + \int_0^\infty (\mathrm{Prob}\{(-\tilde{\phi})^2 > \lambda\})d(\lambda^2)$$
$$\leq 2 \cdot 2f_N e^{\epsilon_N}$$

where we used that $\int_0^\infty e^{-\frac{1}{2}\frac{1}{a}s}ds = 2a$ in the last step, finishing the proof. □

As shown in [2] we have for any fixed *smooth* function ϕ and $t \in \mathbb{R}$

$$\mathbb{E}(e^{it\tilde{\phi}}) \to e^{-t^2\frac{1}{2}\int_X d\phi\wedge d^c\phi} \tag{2.8}$$

as $N \to \infty$. Using the variance estimate above we can extend the previous convergence to the case when we merely assume that $\left\|d\phi\right\|_X < \infty$ (and hence $\left\|\phi\right\|_{L^1(X)} < \infty$). To this end take a sequence $\phi_j \in \mathcal{C}^\infty(X)$ such that $\left\|d(\phi_j - \phi)\right\|_X \to 0$ and $\left\|\phi_j - \phi\right\|_{L^1(X)} \to 0$. Since

$$|\mathbb{E}(e^{it\tilde{\phi}_j}) - \mathbb{E}(e^{it\tilde{\phi}})|^2 \leq \mathbb{E}(|\tilde{u}|^2)$$

for $u = \phi_j - \phi$ (just using $1 - e^{is} \leq |s|$) we deduce that

$$|\mathbb{E}(e^{it\tilde{\phi}_j}) - \mathbb{E}(e^{it\tilde{\phi}})|^2 \leq C(\|d(\phi - \phi_j)\|_X^2 + \|\phi - \phi_j\|_{L^1(X)}^2$$

for $N >> 1$ and hence letting first N and then j tend to infinity proves the convergence 2.8 in the non-smooth case as well.

Next, we observe that the convergence 2.8 moreover holds for any $t \in \mathbb{C}$. Indeed,

$$f_k(t) := \mathbb{E}(e^{it\tilde{\phi}})$$

is a sequence of holomorphic functions on $\mathbb{C}$ such that for t in a fixed compact subset K of $\mathbb{C}$

$$|f_k(t)| \leq \mathbb{E}(e^{-(Im(t))\tilde{\phi}}) \leq C_K$$

using the second inequality in Theorem 1.1. Since f_k converges point-wise to the holomorphic function $f(t) = e^{-t^2 \int d\phi \wedge d^c\phi}$ for $t \in \mathbb{R}$ it hence follows (e.g. by Vitali's theorem) that f_k converges to f everywhere on $\mathbb{C}$. In other words we have now proved Theorem 1.3 for the case of real and imaginary ϕ. Finally, if ϕ is complex valued we consider $\phi_s = u + sv$ where $\phi = \phi_s$ for $s = i$. The previous convergence shows that $\mathbb{E}(e^{-\tilde{\phi}_s})$ converges to an explicit holomorphic function (as above) for $s \in \mathbb{R}$. Moreover, since the upper bound on $|f_k(s)|$ still holds (by the same argument) the previous argument also shows that the convergence holds for any $s \in \mathbb{C}$ and in particular for $s = i$.

2.4. **A brief acount of the higher dimensionsional case.** Let us now come back to the case when X is $n-$dimensional and fix a Kähler form ω on X. In [2] the analogue of the convergence in Theorem 1.3 was shown to hold as long as $\phi \in C^1(X)$. More precisely, in the convergence statement ϕ has to be replaced by $k^{-(n-1)/2}\phi$ and the norm $\|d\phi\|_X^2$ by

$$\|d\phi\|_{(X,\omega)}^2 = \int d\phi \wedge d^c\phi \wedge \frac{\omega^{n-1}}{(n-1)!} (= \int |\nabla\phi|^2 dV)).$$

However, when $n > 1$ there are integrable functions ϕ with $\int_X |\nabla\phi|^2\omega^n < \infty$, but $\int e^{-\phi} dV = \infty$ (as is well-known in the context of Sobolev inequalities). As a consequence, it is not hard to check that for such a function ϕ we have $\mathbb{E}(e^{-(\phi(x_1)+\cdots)}) = \infty$ and in particular the analogue of the convergence in Theorem 1.3 cannot hold (after perhaps scaling ϕ). Moreover, the corresponding analogue of the Moser-Trudinger inequality in Theorem 1.1 fails when $n > 1$ (as is seen by approximating ϕ as above with a monotone smooth sequence ϕ_j). Explicit counter-examples are obtained, already when $N = 1$, by letting $X = \mathbb{P}^n (\supseteq \mathbb{C}_z^n)$ and ω be the standard $SU(n+1)-$invariant metric on $\mathbb{P}^n$ and taking $\phi_j(z) := m \log(\frac{1/j+|z|^2}{1+|z|^2})$ (for a fixed $m \geq n$) decreasing to $\phi(z)$. Note that ϕ_j is even $\omega-$psh.

On the other hand, another variant of the determinantal Moser-Trudinger inequality in Theorem 1.1 does hold in higher dimensions. More precisely, $\frac{1}{2}\|d\phi\|_X^2$ has to be replaced by Aubin's $J-$functional (which is comparable to $\int d\phi \wedge d^c\phi \wedge (\omega_\phi)^n$. Moreover ϕ has to be assumed $\omega-$psh (i.e. $\omega_\phi \geq 0$) (otherwise there are counter-examples, as explained in [3]) When $X = \mathbb{P}^n$ (or more generally X is a rational homogenous manifold) the corresponding inequality is the content of Cor 2 in [3], with vanishing error terms ϵ_N. More generally, the arguments in Step one in the proof of Theorem 1.1 extend in a straight-forward manner to the higher-dimensional case when the Kähler metric ω has a constant scalar curvature (but then the error terms ϵ_{N_k} are then of the order $O(1/k)$).

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