arXiv:1106.0469v1 [math-ph] 2 Jun 2011

General formulation of the Moyal and Voros products and its physical interpretation

Laure Gouba[*], Domagoj Kovacevic[†], Stjepan Meljanac[‡]

June 3, 2011

Abstract

A unifying perspective on the Moyal and Voros products and their physical meanings has been recently presented in the literature, where the Voros formulation admits a consistent physical interpretation. In the present work, we define in terms of an antisymmetric fixed matrix Θ, and an arbitrary symmetric matrix Φ, a star product $\star$, that is a generalization of the formulation of the Moyal and the Voros products. We show the equivalence between the star products for arbitrary matrices Φ. We discuss the quantum mechanics and the physical meaning of the generalized star product.

1 Introduction

Quantum field theories on noncommutative spaces are an important area of research in high energy physics due to the fact that they can be used as a tool to detect aspects of Planck scale physics, where one expects the spacetime to show noncommutative behaviour, their emergence in string theory and also as a tool to regularize quantum field theories [1]. Studying quantum field on noncommutative spaces leads to better understanding of the structure and the setup of quantum field theory itself [2]. The starting point for a large part of what is now called noncommutative geometry is the commutator

$$x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \tag{1}$$

[*]The Abdus Salam International Centre for Theoretical Physics(ICTP) Strada Costiera 11, 34014, Trieste, Italy. Email: lgouba@ictp.it

[†]University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia. Email: domagoj.kovacevic@fer.hr

[‡]Ruder Boskovic Institute, Bijenicka c.54 HR-10002 Zagreb, Croatia. Email: meljanac@irb.hr

implemented via the Moyal product often written in the asymptotic form

$$f(x) \star_M g(x) = f(x) e^{\frac{i}{2}\theta^{ij}\overleftarrow{\partial}_i \overrightarrow{\partial}_j} g(x). \tag{2}$$

It is a noncommutative, associative product introduced originally in quantum mechanics. It comes from a Weyl map between functions and operators. The commutation relation (1) has been introduced in the spacetime context by Doplicher, Fredenhagen and Roberts [3]. The Moyal product is not the only product which gives the above commutation relation. There is also the Voros product. In fact, it has been shown that the two products can be cast in the same general framework in the sense that they are both coming from a "Weyl map". More precisely, it has been shown that the Moyal product comes from a map, called the Weyl map, which associates operators to functions with symmetric ordering, while the Voros one comes from a similar map, a weighted Weyl map, which associates operators to functions with normal ordering [4]. The Moyal and the Voros formulations of noncommutative field theory has been a point of controversy in the past. This issue has been recently addressed in the context of noncommutative non relativitistic quantum mechanics [7]. In particular, it has been shown that the two formulations simply correspond to two different representations associated with two different choices of basis on the quantum Hilbert space. In the present paper, we define a star product in $2+1$ spacetime, that generalize the formulation of the Moyal and the Voros star product and discuss its physical meaning. In section 2, we define the generalized star product, followed by the equivalence between the star products in section 3 and in section 4 we discuss quantum mechanics associated to this star product.

2 Generalized star product

We consider the 2+1 dimensional space-time operators where the operators $\{x_\mu\}_{\mu=0,1,2}$ satisfy the commutation relations

$$[x_\mu,\, x_\nu] = 0,\ \mu = 0,1,2,\ \nu = 0,1,2. \tag{3}$$

We define, new operators

$$\hat{x}_\mu = x_\mu + \frac{i}{2}\Theta_{\mu\alpha}\partial_\alpha + \frac{i}{2}\Phi_{\mu\alpha}\partial_\alpha, \tag{4}$$

where the matrix Θ is fixed antisymmetric and the matrix Φ, an arbitrary symmetric matrix defined respectively as follows

$$\Theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix}, \quad \Phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \varphi_{11} & \varphi_{12} \\ 0 & \varphi_{12} & \varphi_{22} \end{pmatrix}, \tag{5}$$

with the time taken to be an ordinary c-number. The operators defined in equation (4) satisfy the commutation relations

$$[\hat{x}_\mu,\ \hat{x}_\nu] = i\theta\epsilon_{\mu\nu},\ \mu = 1, 2,\ \nu = 1, 2 \quad \text{and} \quad [\hat{x}_0, \hat{x}_\mu] = 0,\ \mu = 0, 1, 2. \tag{6}$$

We define in 2 + 1 dimensional spacetime a star product denoted by $\star$ as follows

$$f(\mathbf{x}) \star g(\mathbf{x}) = (f \star g)(\mathbf{x}) = f(\mathbf{x}) \exp(\frac{i}{2}(\Phi + \Theta)_{\mu\nu} \overleftarrow{\partial}_\mu \overrightarrow{\partial}_\nu) g(\mathbf{x}) \tag{7}$$

For the particular case, where the matrix $\Phi \equiv 0$, then the product is simply

$$f(\mathbf{x}) \star g(\mathbf{x}) = f(\mathbf{x}) \exp(\frac{i\theta}{2}(\overleftarrow{\partial}_1 \overrightarrow{\partial}_2 - \overleftarrow{\partial}_2 \overrightarrow{\partial}_1)) g(\mathbf{x}), \tag{8}$$

and that is equivalent to the Moyal product denoted $\star_M$. In the particular case when the matrix Φ is a diagonal matrix

$$\Phi_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i\theta & 0 \\ 0 & 0 & -i\theta \end{pmatrix}, \tag{9}$$

then we have the star product

$$f(\mathbf{x}) \star g(\mathbf{x}) = f(\mathbf{x}) \exp(\frac{i\theta}{2}(\overleftarrow{\partial}_1 \overrightarrow{\partial}_2 - \overleftarrow{\partial}_2 \overrightarrow{\partial}_1) + \frac{\theta}{2}(\overleftarrow{\partial}_1 \overrightarrow{\partial}_1 + \overleftarrow{\partial}_2 \overrightarrow{\partial}_2)) g(\mathbf{x}), \tag{10}$$

that is equivalent to the Voros product denoted $\star_V$.

Let us now consider the star product defined in equation (7), then it is good to make sure that such definition is consistent, means satisfies all properties of a product. In fact, the product defined in the equation (7) is associative but not commutative. We consider the commuting operators in (3) and calculate the star product

$$x_1 \star x_2 = x_1 x_2 + \frac{i}{2}(\theta + \varphi_{12}), \tag{11}$$

$$x_2 \star x_1 = x_2 x_1 + \frac{i}{2}(-\theta + \varphi_{12}). \tag{12}$$

Then the bracket with respect to the star product (7) is

$$[x_1,\ x_2]_\star = i\theta; \quad [x_2,\ x_1]_\star = -i\theta; \quad [x_0,\ x_1]_\star = 0; \quad [x_0,\ x_2]_\star = 0, \tag{13}$$

that is in agreement with the noncommutative relation established in (6). In a general point of view, we define the commutator of two functions with respect to the product (7) as

$$[f, g]_\star = f \star g - g \star f. \tag{14}$$

The commutator defined in the equation (14) is bilinear, antisymmetric, satisfies the Jacobi identity and the Leibniz rule.

3 Equivalence relation between the star products

Since the matrix Φ is arbitrary symmetric, each matrix Φ determine a particular star product. In this section we show that the products are equivalent. As the matrix Θ is fixed, and the case $\Phi = 0$ corresponds to the Moyal product, we analyse the equivalence between the star product $\star$ for any Φ and the Moyal product $\star_M$. In order to do that, we briefly recall that two star products $\star$ and $\star'$ are equivalent if there is an invertible map [4, 5, 6]

$$T(f) = \sum_r \theta^r t_r(f), \tag{15}$$

where t_r are differential operators, such that

$$T(f \star g) = T(f) \star' T(g). \tag{16}$$

Here, we consider on the space of functions on the noncommutative plane the map

$$T = e^{\frac{i}{4}\Phi_{ij}\partial_i\partial_j}. \tag{17}$$

and it is equivalence if

$$T(f \star_M g) = T(f) \star T(g). \tag{18}$$

The convenient framework to show the equality (18) is the momentum space, where in the momentum representation

$$f(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\tilde{f}(p)e^{i\mathbf{p}\cdot\mathbf{x}}, \tag{19}$$

and in the Fourier representation, the momentum operator P acts as multiplication by the coordinates[1] $p = (p_1, p_2)$. We have

$$\begin{aligned} (f \star_M g)(\mathbf{x}) &= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{-\frac{i}{2}\theta_{ij}p_iq_j}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{-\frac{i}{2}\theta\mathbf{p}\wedge\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}, \end{aligned} \tag{20}$$

where the vector product is

$$\mathbf{p}\wedge\mathbf{q} = \epsilon^{ij}p_iq_j. \tag{21}$$

[1] we use bold notation for vector

Now, we start with

$$
\begin{aligned}
T(f\star_M g)(\mathbf{x}) &= e^{\frac{i}{4}\Phi_{ij}\partial_i\partial_j}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{-\frac{i}{2}\theta_{ij}p_iq_j}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{-\frac{1}{2}\theta_{ij}p_iq_j} \\
&\times\ e^{-\frac{i}{4}\left(\varphi_{11}(p_1+q_1)^2+\varphi_{22}(p_2+q_2)^2+2\varphi_{12}(p_1+q_1)(p_2+q_2)\right)}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}, \qquad (22)
\end{aligned}
$$

and

$$
\begin{aligned}
T(f\star_M g)(\mathbf{x}) &= \int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{\frac{-i\theta}{2}\mathbf{p}\wedge\mathbf{q}}e^{\frac{-i}{4}(\Phi_{ij}(p_i+q_j)^2\delta_{ij}+\Phi_{ij}(p_i+q_i)(p_j+q_j)_{\{i\neq j\}})} \\
&\times\ e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \qquad (23)
\end{aligned}
$$

Now let us compute

$$
\begin{aligned}
T(f)\star T(g) &= e^{\frac{i}{4}\Phi_{ij}\partial_i\partial_j}\int\frac{d^3p}{(2\pi)^3}\tilde{f}(p)e^{i\mathbf{p}\cdot\mathbf{x}}\ \star\ e^{\frac{i}{4}\Phi_{ij}\partial_i\partial_j}\int\frac{d^3q}{(2\pi)^3}\tilde{g}(q)e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= \int\frac{d^3p}{(2\pi)^3}\tilde{f}(p)e^{\frac{-1}{4}(\varphi_{11}p_1^2+\varphi_{22}p_2^2+2\varphi_{12}p_1p_2)}e^{i\mathbf{p}\cdot\mathbf{x}}\ e^{\frac{i}{2}(\Phi+\Theta)_{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j} \\
&\times\ \int\frac{d^3q}{(2\pi)^3}\tilde{g}(q)e^{\frac{-1}{4}(\varphi_{11}q_1^2+\varphi_{22}q_2^2+2\varphi_{12}q_1q_2)}e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= \int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{\frac{-1}{4}(\varphi_{11}(p_1^2+q_1^2)+\varphi_{22}(p_2^2+q_2^2)+2\varphi_{12}(p_1p_2+q_1q_2)} \\
&\times\ e^{i\mathbf{p}\cdot\mathbf{x}}\ e^{\frac{i}{2}(\Phi+\Theta)_{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j}\ e^{i\mathbf{q}\cdot\mathbf{x}}. \qquad (24)
\end{aligned}
$$

Since

$$
e^{i\mathbf{p}\cdot\mathbf{x}}\ e^{\frac{i}{2}(\Phi+\Theta)_{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j}\ e^{i\mathbf{q}\cdot\mathbf{x}} = e^{\frac{-i\theta}{2}\mathbf{p}\wedge\mathbf{q}}e^{\frac{-i}{2}(\varphi_{11}p_1q_1+\varphi_{22}p_2q_2+\varphi_{12}(p_1q_2+p_2q_1))}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \qquad (25)
$$

The equation (24) transforms to

$$
\begin{aligned}
T(f)\star T(g) &= \int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{\frac{-i\theta}{2}\mathbf{p}\wedge\mathbf{q}}e^{\frac{-i}{4}(\Phi_{ij}(p_i+q_j)^2\delta_{ij}+\Phi_{ij}(p_i+q_i)(p_j+q_j)_{\{i\neq j\}})} \\
&\times\ e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \qquad (26)
\end{aligned}
$$

The following equality holds

$$
T(f\star_M g) = T(f)\star T(g), \qquad (27)
$$

we can then conclude that the $\star$ product is equivalent to the Moyal product $\star_M$, for any matrix Φ as we have set above. For the special case, Φ_θ defined in (9), the star product $\star$ corresponds to the Voros star product $\star_V$ and the map is now $T = e^{\frac{\theta}{4}\nabla^2}$ and replacing Φ_θ in the equations (23) and (26), we have

$$T(f \star_M g) = \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{\frac{-\theta}{4}(p+q)^2}e^{\frac{-i}{2}\theta\mathbf{p}\wedge\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}, \tag{28}$$

and

$$T(f) \star_V T(g) = \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\tilde{f}(p)\tilde{g}(q)e^{\frac{-\theta}{4}(p+q)^2}e^{\frac{-i}{2}\theta\mathbf{p}\wedge\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \tag{29}$$

The equivalence $T(f \star_M g) = T(f) \star_V T(g)$ is indicated in [7] and shown in [4].

4 Quantum Mechanics with star product

In this section, we review the formalism of noncommutative quantum mechanics as in [8], where noncommutative quantum mechanics is formulated as a quantum system on the Hilbert space of Hilbert-Schmidt operators acting on classical configuration space. Here, we consider $2+1$ dimensional spacetime with only spacial noncommutativity. Restricting to two dimensions, the coordinates of the noncommutative configuration space satisfy the commutation relation

$$[\hat{x}_1,\ \hat{x}_2] = i\theta. \tag{30}$$

It is convenient to define the creation and annihilation operators

$$b = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2); \quad b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2) \tag{31}$$

that satisfy the Fock algebra $[b,\ b^\dagger] = 1$. The noncommutative configuration space is then isomorphic to boson Fock space

$$\mathcal{H}_c = \text{span}\left\{|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle\right\}_{n=0}^{n=\infty}, \tag{32}$$

where the span is taken over the field of complex numbers.

The quantum Hilbert space, in which the physical states of the system are represented, is identified with the set of Hilbert Schmidt operators acting on noncommutative configuration space

$$\mathcal{H}_q = \{\psi(\hat{x}_1,\hat{x}_2) : \psi(\hat{x}_1,\hat{x}_2) \in \mathcal{B}(\mathcal{H}_c), tr_c[\psi(\hat{x}_1,\hat{x}_2)^\dagger\psi(\hat{x}_1,\hat{x}_2)] < \infty\}. \tag{33}$$

Here tr_c denotes the trace over noncommutative configuration space and $\mathcal{B}(\mathcal{H}_c)$ the set of bounded operators on $\mathcal{H}_c$. This space has a natural inner product and norm

$$(\phi(\hat{x}_1,\hat{x}_2),\ \psi(\hat{x}_1,\hat{x}_2)) = tr_c[\phi(\hat{x}_1,\hat{x}_2)^\dagger\psi(\hat{x}_1,\hat{x}_2)] \tag{34}$$

and form an Hilbert Space. Let us recall for a trace class operator A the formula

$$trA = \sum_{n\geq 0} \langle n|A|n\rangle = \int \frac{d^2z}{\pi} \langle z|A|z\rangle. \tag{35}$$

To distinguish the noncommutative configuration space, which is also a Hilbert space, from the quantum Hilbert space above, we use the notation $|\cdot\rangle$ for elements of the noncommutative configuration space, while elements of the quantum Hilbert space are denoted by $\psi(\hat{x}_1,\ \hat{x}_2) \equiv |\psi)$. The elements of its dual (linear functionals) are as usual denoted by bras, $(\psi|$, which maps elements of $\mathcal{H}_q$ onto complex numbers by $(\phi|\psi) = (\phi, \psi) = tr_c[\phi(\hat{x}_1, \hat{x}_2)^\dagger \psi(\hat{x}_1, \hat{x}_2)]$. We also need to be careful when denoting Hermitian conjugation to distinguish these two spaces. We reserve the notation $\dagger$ for Hermitian conjugation on noncommutative configuration space and the notation $\ddagger$ for Hermitian conjugation on quantum Hilbert space. We use also capital letters to distinguish operators acting on quantum Hilbert space from those acting on noncommutative configuration space. The noncommutative Heisenberg algebra in two dimensions

$$[\hat{x}_1,\ \hat{x}_2] = i\theta, \tag{36}$$

$$[\hat{x}_1, \hat{p}_{x_1}] = [\hat{x}_2, \hat{p}_{x_2}] = i\hbar, \tag{37}$$

$$[\hat{p}_{x_1},\ \hat{p}_{x_2}] = [\hat{x}_1,\ \hat{p}_{x_2}] = [\hat{x}_2,\ \hat{p}_{x_2}] = 0, \tag{38}$$

is now represented in terms of operators $\hat{X}_1,\ \hat{X}_2$ and $\hat{P}_{x_1},\ \hat{P}_{x_2}$ acting on the quantum Hilbert space (33) with the inner product (34), which is the analog of the Schrödinger representation of the Heisenberg algebra. These operators are given by

$$\hat{X}_1\psi(\hat{x}_1,\ \hat{x}_2) = \hat{x}_1\psi(\hat{x}_1,\ \hat{x}_2), \quad \hat{X}_2\psi(\hat{x}_1,\ \hat{x}_2) = \hat{x}_2\psi(\hat{x}_1,\ \hat{x}_2), \tag{39}$$

$$\hat{P}_{x_1}\psi(\hat{x}_1,\ \hat{x}_2) = \frac{\hbar}{\theta}[\hat{x}_1,\ \psi(\hat{x}_1, \hat{x}_2)]; \quad \hat{P}_{x_2}\psi(\hat{x}_1,\ \hat{x}_2) = -\frac{\hbar}{\theta}[\hat{x}_2,\ \psi(\hat{x}_1, \hat{x}_2)]. \tag{40}$$

The position operators act by left multiplication and the momentum acts adjointly. It is also useful to introduce the following quantum operators

$$B = \frac{1}{\sqrt{2\theta}}(\hat{X}_1 + i\hat{X}_2); \quad B^\ddagger = \frac{1}{\sqrt{2\theta}}(\hat{X}_1 - i\hat{X}_2), \quad \hat{P} = \hat{P}_{x_1} + i\hat{P}_{x_2}, \quad \hat{P}^\ddagger = \hat{P}_{x_1} - i\hat{P}_{x_2}. \tag{41}$$

We note that $\hat{P}^2 = \hat{P}_{x_1}^2 + \hat{P}_{x_2}^2 = \hat{P}^\ddagger\hat{P} = \hat{P}\hat{P}^\ddagger$. These operators act as following

$$\begin{aligned} B\psi(\hat{x}_1, \hat{x}_2) &= b\psi(\hat{x}_1, \hat{x}_2), \quad B^\ddagger\psi(\hat{x}_1, \hat{x}_2) = b^\dagger\psi(\hat{x}_1, \hat{x}_2), \\ \hat{P}\psi(\hat{x}_1, \hat{x}_2) &= -i\hbar\sqrt{\frac{2}{\theta}}[b,\ \psi(\hat{x}_1, \hat{x}_2)], \quad \hat{P}^\ddagger\psi(\hat{x}_1, \hat{x}_2) = i\hbar\sqrt{\frac{2}{\theta}}[b^\dagger,\ \psi(\hat{x}_1, \hat{x}_2)]. \end{aligned} \tag{42}$$

The momentum eigenstates $|p)$ are given by

$$|p) = \sqrt{\frac{\theta}{2\pi}} e^{ip\cdot\hat{x}}, \quad \hat{P}_i|p) = p_i|p), \tag{43}$$

and they satisfy the usual resolution of identity and orthogonality condition

$$\int d^2p|p)(p| = 1, \quad (p|p') = \delta(p_1 - p_1^{'})\delta(p_2 - p_2^{'}). \tag{44}$$

We consider now the following states as in [7] obtained by expansion in terms of the momentum states as follows

$$|x) = \int \frac{d^2p}{2\pi} e^{-i\mathbf{p}\cdot\mathbf{x}}|p). \tag{45}$$

Let us recall the following useful formulae

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} dk e^{ik(x-y)} = \delta(x-y), \quad \int_{\mathbb{R}} f(x)\delta(x-a)dx = f(a). \tag{46}$$

$$e^A e^B = e^B e^A e^{[A,B]}, \quad e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}. \tag{47}$$

We have

$$(p|x) = \frac{1}{2\pi} e^{-i\mathbf{p}\cdot\mathbf{x}}, \qquad (x|x') = \delta(x_1 - x_1^{'})(x_2 - x_2^{'}), \tag{48}$$

that means that the states $|x)$ are orthogonal. Do these states resolve the identity with respect to the star product $\star$ as follows

$$\int d^2x|x) \star (x| = 1_d \quad ? \tag{49}$$

In order to respond to this question, we consider the equation (44), and we perform the following calculation

$$\begin{aligned}(p|\left(\int d^2x|x) \star (x|\right)|p') &= \int d^2x (p|x) \star (x|p') \\ &= \frac{1}{2\pi}\frac{1}{2\pi}\int d^2x e^{-i\mathbf{p}\cdot\mathbf{x}} \star e^{i\mathbf{p}'\cdot\mathbf{x}}, \end{aligned} \tag{50}$$

where $\star$ is the star product defined in (7) and

$$\begin{aligned}\frac{1}{2\pi}\frac{1}{2\pi}\int d^2x e^{-i\mathbf{p}\cdot\mathbf{x}} \star e^{i\mathbf{p}'\cdot\mathbf{x}} &= e^{\frac{i}{2}\left(\varphi_{11}p_1p_1^{'} + (\varphi_{12}+\theta)p_1p_2^{'} + (\varphi_{21}-\theta)p_2p_1^{'} + \varphi_{22}p_2p_2^{'}\right)} e^{\frac{i\theta}{2}(p_1p_2 + p_1^{'}p_2^{'})} \\ &\times \ e^{-i\theta p_2 p_1^{'}} \delta(p_1 - p_1^{'})\delta(p_2 - p_2^{'}). \end{aligned} \tag{51}$$

Regarding the equation (51), we have

$$(p|\left(\int d^2x|x)\star(x|\right)|p')\neq(p|p'). \tag{52}$$

However, setting the matrix $\Phi=0$, then $\star$ corresponds to $\star_M$ and

$$\begin{aligned}(p|\left(\int d^2x|x)\star_M(x|\right)|p') &= e^{\frac{i\theta}{2}(p_1p_2'-p_2p_1'+p_1p_2+p_1'p_2'-2p_2p_1')}\delta(p_1-p_1')\delta(p_2-p_2') \\ &= (p|p'). \end{aligned}\tag{53}$$

In that case the equation (49) is satisfied. The states $|x)$ are then orthogonal and resolve the identity with respect to the Moyal product, then constitute a basis of the Hilbert space. Although this provides a consistent interpretational framework, the measurement of position needs more careful consideration as the position operators do not commute and thus a precise measurement of one of these observables leads to total uncertainty in the other. In order to preserve the notion of position in the sense of a particle being localized around a certain point, the best is to construct a minimum uncertainty state in noncommutative configuration space and use that to give meaning to the notion of position.

The minimum uncertainty states on noncommutative configuration space, which are isomorphic to the boson Fock space, are well known to be the normalized coherent states

$$|z\rangle=e^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle, \tag{54}$$

where $z=\frac{1}{\sqrt{2\theta}}(x_1+ix_2)$ is a dimensionless complex number that satisfies the relation $b|z\rangle=z|z\rangle$. These states provide an overcomplete basis on the noncommutative configuration space. Corresponding to these states, we can construct a state (operator) in quantum Hilbert space as follows

$$|z,\bar{z})=|z\rangle\langle z|, \tag{55}$$

and these states satisfy

$$B|z,\bar{z})=z|z,\bar{z}), \tag{56}$$

and

$$(z',\bar{z}'|z,\bar{z})=tr_c[(|z'\rangle\langle z'|)^\dagger(|z\rangle\langle z|)]=e^{-|z-z'|^2} \tag{57}$$

Since we aim to perform the generalized star product, $\star$, defined in equation (7) in the present formalism of noncommutative quantum mechanics, it is convenient to write the star product in terms of complex numbers $z,\bar{z}$. Using

$$\frac{\partial}{\partial z}=\frac{\partial x_1}{\partial z}\frac{\partial}{\partial x_1}+\frac{\partial x_2}{\partial z}\frac{\partial}{\partial x_2}, \tag{58}$$

we have

$$\frac{\partial}{\partial z} = \frac{\sqrt{2\theta}}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{\sqrt{2\theta}}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right) \tag{59}$$

and inverting, we have

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{2\theta}}(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}); \quad \frac{\partial}{\partial y} = \frac{i}{\sqrt{2\theta}}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}), \tag{60}$$

then the star product defined in (7) can be expressed in terms of complex variables as

$$\star \equiv e^{\frac{i}{4\theta}\left[(\varphi_{11}-\varphi_{22}+2i\varphi_{12})\overleftarrow{\partial}_z\overrightarrow{\partial}_z+(\varphi_{11}+\varphi_{22}-2i\theta)\overleftarrow{\partial}_z\overrightarrow{\partial}_{\bar{z}}+(\varphi_{11}+\varphi_{22}+2i\theta)\overleftarrow{\partial}_{\bar{z}}\overrightarrow{\partial}_z+(\varphi_{11}-\varphi_{22}-2i\varphi_{12})\overleftarrow{\partial}_{\bar{z}}\overrightarrow{\partial}_{\bar{z}}\right]}. \tag{61}$$

In case when $\Phi \equiv 0$, we recognize the form of the Moyal product

$$f(z,\bar{z}) \star g(z,\bar{z}) = f(z,\bar{z}) e^{\frac{1}{2}(\overleftarrow{\partial}_z\overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}}\overrightarrow{\partial}_z)} g(z,\bar{z}) = f(z,\bar{z}) \star_M g(z,\bar{z}), \tag{62}$$

and for non trivial matrix Φ_θ defined in (9), we recognize the form of the Voros product.

$$f(z,\bar{z}) \star g(z,\bar{z}) = f(z,\bar{z}) e^{\overleftarrow{\partial}_z\overrightarrow{\partial}_{\bar{z}}} g(z,\bar{z}) = f(z,\bar{z}) \star_V g(z,\bar{z}). \tag{63}$$

The question is whether the states $|z,\bar{z})$ resolve the identity

$$\int \frac{dzd\bar{z}}{\pi}|z,\bar{z}) \star (z,\bar{z}| = 1_d. \tag{64}$$

We now introduce the momentum eigenstates

$$|p) = \sqrt{\frac{\theta}{2\pi}} e^{i\sqrt{\frac{\theta}{2}}(\bar{p}b + pb^\dagger)}, \quad \int d^2p|p)(p| = 1, \quad \hat{P}_i|p) = p_i|p), \tag{65}$$

normalised such that $(p|p') = \delta(p_1 - p_1')(p_2 - p_2')$. The overlap of this basis with the momentum eigenstate is given by

$$(z,\bar{z}|p) = \sqrt{\frac{\theta}{2\pi}} e^{-\frac{\theta|p|^2}{4}} e^{i\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}. \tag{66}$$

We can check if equation (64) is satisfied with respect to the star product $\star$ as follows

$$(p'|\int \frac{dzd\bar{z}}{\pi}|z,\bar{z}) \star (z,\bar{z}|p) = \int \frac{dzd\bar{z}}{\pi}(p'|z,\bar{z}) \star (z,\bar{z}|p), \tag{67}$$

and

$$\begin{aligned}
\int \frac{dzd\bar{z}}{\pi}(p'|z,\bar{z})\star(z,\bar{z}|p) &= e^{\frac{-\theta}{4}(|p|^2+|p'|^2)}\int \frac{\theta}{2\pi}\frac{dzd\bar{z}}{\pi}e^{i\sqrt{\frac{\theta}{2}}(p'\bar{z}+\bar{p}'z)}\star e^{-i\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)} \\
&= e^{\frac{-\theta}{4}(|p|^2+|p'|^2)}e^{\frac{i}{8}[(\varphi_{11}-\varphi_{22}+2i\varphi_{12})\bar{p}'\bar{p}+(\varphi_{11}+\varphi_{22}-2i\theta)\bar{p}p']} \\
&\times\ e^{\frac{i}{8}[(\varphi_{11}+\varphi_{22}+2i\theta)p\bar{p}'+(\varphi_{11}-\varphi_{22}-2i\varphi_{12})p'p]} \\
&\times \int \frac{\theta}{2\pi}\frac{dx_1dx_2}{2\pi\theta}e^{i\sqrt{\frac{\theta}{2}}(p'\bar{z}+\bar{p}'z)}e^{-i\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)}. \qquad (68)
\end{aligned}$$

since

$$\int \frac{dx_1}{2\pi}\frac{dx_2}{2\pi}e^{i\sqrt{\frac{\theta}{2}}(p'\bar{z}+\bar{p}'z)}e^{-i\sqrt{\frac{\theta}{2}}(p\bar{z}+\bar{p}z)} = \delta(p_1-p_1')\delta(p_2-p_2'). \qquad (69)$$

one obtains

$$\begin{aligned}
\int \frac{dzd\bar{z}}{\pi}(p'|z,\bar{z})\star(z,\bar{z}|p) &= e^{\frac{-\theta}{4}(|p|^2+|p'|^2)}e^{\frac{i}{8}[(\varphi_{11}-\varphi_{22}+2i\varphi_{12})\bar{p}'\bar{p}+(\varphi_{11}+\varphi_{22}-2i\theta)\bar{p}p']} \\
&\times\ e^{\frac{i}{8}[(\varphi_{11}+\varphi_{22}+2i\theta)p\bar{p}'+(\varphi_{11}-\varphi_{22}-2i\varphi_{12})p'p]}\delta(p_1-p_1')\delta(p_2-p_2') \\
&\neq (p'|p). \qquad (70)
\end{aligned}$$

That means that the states $|z,\bar{z})$ do not resolve the identity operator with respect to the star product $\star$. For the particular case of the matrix Φ defined in (9), the equation (68) becomes

$$\int \frac{dzd\bar{z}}{\pi}(p'|z,\bar{z})\star_V(z,\bar{z}|p) = e^{\frac{-\theta}{4}(|p|^2+|p'|^2)}e^{\frac{\theta}{2}\bar{p}p'}\delta(p_1-p_1')(p_2-p_2') = (p|p'), \qquad (71)$$

that implies the resolution of the identity

$$\int \frac{dzd\bar{z}}{\pi}|z,\bar{z})\star_V(z,\bar{z}| = 1_d. \qquad (72)$$

Only the Voros product can allow for the quantum states $|z,\bar{z})$ a consistent physical interpretation. The completeness relation obtained in (72) implies a consistent probability interpretation of finding the particle at position z. In fact, an interpretational framework for the measurement of position can be set up, by noting that the following operators

$$\pi_z = \frac{1}{2\pi\theta}|z)\star_V(z|, \quad \int dx_1dx_2\pi_z = 1_d \qquad (73)$$

provide an operator valued measure in the sense of [14]. The only difference from the standard PVM is the non-orthogonality of these operators, which requires a relaxation of von Neuman's projective assumption and changes the measurement into a weak measurement. A consistent probability interpretation can be given by assigning the probability of finding the particle at position (x_1, x_2), given that the system is described in the pure state density matrix $\rho = |\psi)(\psi|$, to be

$$P(x_1, x_2) = tr_q(\pi_z \rho) = (\psi|\pi_z|\psi) \geq 0, \tag{74}$$

and

$$\int dx_1 dx_2 P(x_1, x_2) = \int dx_1 dx_2 (\psi|\pi_z|\psi) = (\psi|\psi) = 1, \tag{75}$$

where we assumed the states $|\psi)$ to be normalized.

5 Conclusion

We define a star product that is a general formulation of the Moyal and the Voros products in terms of a fixed matrix Θ and an arbitrary matrix Φ. We obtain then a family of star products with respect to Φ that are all equivalent to the Moyal product in terms of formulation. In order to do some physical interpretation of this star product, we set the problem in a completely general and abstract operator formulation of non-commutative quantum field theory and quantum mechanics. Setting the states $|x)$ as expansion of the momentum states, they resolve the identity only for the Moyal product. In a coherent states framework, the states $|z, \bar{z})$ do not resolve the identity with respect to the star product that we have defined. Under some restrictions that correspond to the Voros product, these states are overcomplete and resolve the identity. It is known that a set of states that is overcomplete without having a resolution of identity is not practically useful. So for the physical interpretation with respect to our setting, only the Moyal product and the Voros product are useful. However, in order to preserve the notion of position in the sense of a particle being localized around a certain point, the coherent states framework turn out to be the best and then the Voros product is convenient for a best measurement and probability interpretation. The Voros product has been also widely used in the litterature [4, 7, 12, 11, 13, 6]. For the generalized star product to be practically useful, one has to redefine an overcomplete set of states that resolve the identity with respect to this star product, or one should set Θ and Φ in a more general way, this setting can turn out to be more complicated, but one may discover other star products than the Voros and the Moyal product, that allow a consistent physical interpretation.

Acknowledgments

The work of LG is supported by the Associate and Federation Scheme and the High Energy Section of ICTP. LG would like to thank the Ruder Boskovic Institute for her visit during which the work has been started. DK and SM were supported by the Ministry of Science and Technology of the Republic of Croatia under contract No. 098-0000000-2865.

References

[1] A. P. Balachandran, *Quantum Spacetimes in the Year 1* , arXiv: hep-th/0203259

[2] R. J. Szabo, Phys. Rep. 378(2003) **207**, arXiv:hep-th/0109162.

[3] S. Doplicher, K. Fredenhagen and J. E. Roberts, Phys. Lett. **B** 331, 39 (1994); S. Doplicher, arXiv: hep-th/0105251.

[4] Salvatore Galluccio, Fedele Lizzi and Patrizia Vitale, Phys. Rev. **D 78**, 085007(2008)

[5] C. K. Zachos, *Geometrical evaluation of star products*, J. Math. Phys. **41** (2000) 5129, arXiv:hep-th/0010187.

[6] G. Alexanian, A. Pinzul, A. Stern, *Generalized coherent state approach to star products and applications to the fuzzy sphere*, Nuclear Physics **B 600**[PM](2001) 531-547.

[7] Prasad Basu, Biswajit Chakraborty and Frederik G Scholtz, *A unifying perspective on the Moyal and Voros products and their physical meaning*, arXiv: hep-th/1101.2495

[8] F. G. Scholtz, L. Gouba, A. Hafver, C. M. Rohwer, J. Phys. **A 42**, 175303

[9] P. Nicolini, A. Smailagic, E. Spallucci, Phys. Lett. **B 632** (2006) 547, arXiv:gr-qc/0510112.

[10] S. Gangopadhyay, F. G. Scholtz, Phys. Rev. Lett. **102** (2009)241602.

[11] S. Gangopadhyay and al. 2011, J. Phys. **A**: Math. Theor. 44 175401.

[12] Rabin Banerjee, Sunandan Gangopadhyay, Sujoy Kumar Modak, *Voros product, noncommutative Schwarzchild black hole and corrected area law, Physcis letters* **B 686** (2010) 181-187

[13] M. Daoud, *Extended Voros product in the coherent states framework*, Physics Letters **A 309** (2003) 167-175

[14] J. A. Bergou 2007 J. Phys. Conf. Ser. 84 01 2001.