Similarity solutions of Fokker-Planck equation with time-dependent coefficients

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(Dated: Jun 15, 2011)

In this work, we consider the solvability of the Fokker-Planck equation with both time-dependent drift and diffusion coefficients by means of the similarity method. By the introduction of the similarity variable, the Fokker-Planck equation is reduced to an ordinary differential equation. Adopting the natural requirement that the probability current density vanishes at the boundary, the resulted ordinary differential equation turns out to be integrable, and the probability density function can be given in closed form. New examples of exactly solvable Fokker-Planck equations are presented, and their properties analyzed.

PACS numbers: 05.10.Gg; 52.65.Ff; 02.50.Ey
Keywords: Fokker-Planck equation, time-dependent drift and diffusion, similarity method

I. INTRODUCTION

The Fokker-Planck equation (FPE) is one of the basic tools which is widely used for studying the effect of fluctuations in macroscopic systems [1]. It has been employed in many areas: Physics, chemistry, hydrology, biology, even finance, and others. Because of its broad applicability, it is therefore of great interest to obtain solutions of the FPE for various physical situations. Many methods, including analytical, approximate and numerical ones, have been developed to solve the FPE.

Generally, it is not easy to find analytic solutions of the FPE. Exact analytical solutions of the FPE are known for only a few cases, such as linear drift and constant diffusion coefficients [1]. In most cases, one can only solve the equation approximately, or numerically. The well-known analytic methods for solving the FPE include a change of variables, perturbation expansion, eigenfunction expansion, variational approach, Green's function, path integral, moment method, and the continued-fraction method (for a review of these method, see eg. [1]). Symmetry methods have also been quite useful in solving the FPE [2]. Two interesting analytic approximation approaches are the WKB analysis [1] and the normal mode analysis [3]. The finite-difference [4] and finite-element methods [5] are among some useful numerical methods. Most of these methods, however, are concerned only with FPE's with time-independent diffusion and drift coefficients.

Solving the FPE's with time-dependent drift and/or diffusion coefficient is in general an even more difficult task. It is therefore not surprising that the number of papers on such kind of FPE is far less than that on FPE with time-independent coefficients. Some recent works on the FPE with time-dependent diffusion coefficients appear in [6–8], and works involving time-dependent drift coefficients can be found in [9–11]. Refs. [12–14] consider the FPE's with both time-dependent diffusion and drift coefficients. The symmetry properties of the one-dimensional FPE with arbitrary coefficients of drift and diffusion are investigated in [14]. Such properties may in some cases allow one to transform the FPE into one with constant coefficients. It is proved that symmetry group of these equations can be of one, two, four or six parameters and the criteria are also obtained.

In our previous work [11], we have considered, within the framework of a perturbative approach, the similarity solutions of a class of FPE which have constant diffusion coefficient and small time-dependent drift coefficient. Motivated by our previous work, in this paper we would like to study the solvability of the FPE with both time-dependent drift and diffusion coefficients by means of the similarity method.

One advantage of the similarity method is that it allows one to reduce the order of a partial differential equation [15]. Thus the FPE can be transformed into an ordinary differential equation which may be easier to solve. However, for the FPE to admit similarity solutions, it must possess proper scaling property under certain scaling transformation of the basic variables. This all boils down to the scaling behaviors of the drift and diffusion coefficients. To study these scaling behaviors is the main aim of the present work.

The plan of the paper is as follows. Sect. II discusses the scaling form of the FPE. Then in Sect. III the similarity method is applied to reduce the FPE into an ordinary differential equation, and the exact solution of the probability density expressed in closed form. Sect. IV and V present detailed analyses of the similarity solutions of the FPE which are related to the confluent hypergeometric equation. Those related to the hypergeometric equation are considered in Sect. VI. Sect. VII shows an example which is not related to (confluent) hypergeometric equation. Sect. VIII concludes the paper.

II. SCALING OF FOKKER-PLANCK EQUATION

The general form of the FPE in $(1+1)$-dimension is

$$\frac{\partial W(x,t)}{\partial t} = \Big[-\frac{\partial}{\partial x} D^{(1)}(x,t) + \frac{\partial^2}{\partial x^2} D^{(2)}(x,t) \Big] W(x,t) \ , \tag{1}$$

where $W(x,t)$ is the probability distribution function, $D^{(1)}(x,t)$ is the drift coefficient and $D^{(2)}(x,t)$ the diffusion coefficient. The drift coefficient represents the external force acting on the particle, while the diffusion coefficient accounts for the effect of fluctuation. $W(x,t)$ as a probability distribution function should be normalized, *i.e.*, $\int_{\text{domain}} W(x,t)\, dx = 1$ for $t \geq 0$.

As mentioned in the Introduction, it is difficult, if not impossible, to find exact solutions of the general FPE with time-dependent drift and diffusion coefficients. Here we shall be content with a more modest aim by seeking a special class of solutions, namely, the similarity solutions of the FPE. Such solutions are possible provided that the FPE possesses certain scaling symmetry. Below we shall show that, if both the drift and diffusion coefficients assume proper scaling forms, then the FPE can be solved with the similarity method.

Consider the scale transformation

$$\bar{x} = \varepsilon^a x \quad , \quad \bar{t} = \varepsilon^b t, \tag{2}$$

where ε, a and b are real parameters. Suppose under this transformation, the probability density function and the two coefficients scale as

$$\bar{W}(\bar{x},\bar{t}) = \varepsilon^c W(x,t), \quad \bar{D}^{(1)}(\bar{x},\bar{t}) = \varepsilon^d D^{(1)}(x,t) \quad , \quad \bar{D}^{(2)}(\bar{x},\bar{t}) = \varepsilon^e D^{(2)}(x,t). \tag{3}$$

Here c, d and e are also some real parameters. Written in the transformed variables, eq.(1) becomes

$$\varepsilon^{b-c} \frac{\partial \bar{W}}{\partial \bar{t}} = \Big[-\varepsilon^{a-c-d} \frac{\partial}{\partial \bar{x}} \bar{D}^{(1)}(\bar{x},\bar{t}) + \varepsilon^{2a-c-e} \frac{\partial^2}{\partial \bar{x}^2} \bar{D}^{(2)}(\bar{x},\bar{t}) \Big] \bar{W}(\bar{x},\bar{t}) \ . \tag{4}$$

One sees that if the scaling indices satisfy $b = a - d = 2a - e$, then eq.(4) has the same functional form as eq.(1). In this case, the FPE admits similarity solutions. We shall present such solutions below.

III. SIMILARITY METHOD

The similarity method is a very useful method for solving a partial differential equation which possesses proper scaling behavior. One advantage of the similarity method is to reduce the order of a partial differential equation through some new independent variables (called similarity variables), which are certain combinations of the old independent variables such that they are scaling invariant, i.e., no appearance of parameter ε, as a scaling transformation is performed.

In our case, the second order FPE can be transformed into an ordinary differential equation which may be easier to solve. Here there is only one similarity variable z, which can be defined as

$$z \equiv \frac{x}{t^\alpha} \ , \quad \text{where} \quad \alpha = \frac{a}{b} \quad \text{and} \quad a \ , b \neq 0 \ . \tag{5}$$

For $a \ , b \neq 0$, one has $\alpha \neq 0 \ , \infty$. In what follows we derive the scaling forms and closed form solutions of the probability and current density functions.

A. Probability distribution function

The general scaling form of the probability density function $W(x,t)$ is $W(x,t) = (t^{\delta_1}/x^{\lambda_1}) y(z)$, where λ_1 and δ_1 are two real parameters and $y(z)$ is a scale-invariant function under the sale transformation (2). From the assumed scaling behavior of $W(x,t)$ in (3), we have $-c/a = \lambda_1 - (\delta_1/\alpha)$. Without loss of generality and for clarity of presentation, we hereby assume the parameters $(\lambda_1, \delta_1) = (0, \alpha c/a)$. This gives

$$W(x,t) = t^{\alpha \frac{c}{a}} y(z) \ . \tag{6}$$

The normalization of the distribution function is

$$\int_{\text{domain}} W(x,t)\,dx = \int_{\text{domain}} \left[t^{\alpha(1+\frac{c}{a})}\,y(z)\right] dz = 1\ . \tag{7}$$

For the above relation to hold at all $t \geq 0$, the t-dependent term in the integrand must vanish, and so one must have $c = -a$, and thus

$$W(x,t) = t^{-\alpha} y(z). \tag{8}$$

Next we determine the scaling forms of the drift and diffusion coefficients $D^{(1)}(x,t)$ and $D^{(2)}(x,t)$. Following the same way of determining the transformation between $W(x,t)$ and $y(z)$, we can have $D^{(1)}(x,t) = (t^{\delta_2}/x^{\lambda_2})\rho_1(z)$. The scaling behavior for $D^{(1)}(x,t)$ leads to $d = b\delta_2 - a\lambda_2$. For simplicity, we set $\lambda_2 = 0$. This gives $\delta_2 = \alpha - 1$, where the relations between the scaling exponents, namely, $b = a - d$ and $\alpha = a/b$, have been used. A similar procedure is applied to determine $D^{(2)}(x,t)$ in terms of $\rho_2(z)$. To summarize, the scaling forms of the coefficients are

$$D^{(1)}(x,t) = t^{\alpha-1}\rho_1(z) \quad , \quad D^{(2)}(x,t) = t^{2\alpha-1}\rho_2(z)\ . \tag{9}$$

With eqs. (5), (6) and (9), the FPE is reduced to

$$\rho_2(z)\,y''(z) + \Big[2\rho_2'(z) - \rho_1(z) + \alpha z\Big]\,y'(z) + \Big[\rho_2''(z) - \rho_1'(z) + \alpha\Big]\,y(z) = 0\ , \tag{10}$$

where the prime denotes the derivative with respect to z. Thus it is seen that the solvability of the FPE, eq. (1), under the similarity method depends solely on that of eq. (10). An easy way to find exact solutions of eq. (10) is to relate it to either the hypergeometric equation or the confluent hypergeometric equation. This requires that $\rho_1(z)$ be a linear function of z, and $\rho_2(z)$ a (linear) quadratic function of z for the (confluent) hypergeometric case.

However, it turns out ρ_1 and ρ_2 need not be so restricted in order to make eq. (10) exactly solvable. This is because eq. (10) is exactly integrable. By integrating it once, we get

$$\rho_2 y'(z) + (\rho_2' - \rho_1)\,y(z) + \alpha z y(z) = C, \tag{11}$$

where C is an integration constant.

To proceed further we shall assume the following conditions on the constant C and the boundary condition on $W(x,t)$:

$$C = 0, \quad zy(z)|_{\text{boundary}} = xW(x,t)|_{\text{boundary}} = 0, \tag{12}$$

where $W(x,t) = t^{-\alpha}\,y(z)$ and $z = x/t^{\alpha}$ have been used. As will be seen in the next subsection, these conditions are equivalent to the requirement that the probability current density $J(x,t)$ be vanished at the boundary, i.e., $J(x,t)|_{\text{boundary}} = 0$. With these conditions, the FPE is reduced to

$$\rho_2 y'(z) + (\rho_2' - \rho_1)\,y(z) + \alpha z y(z) = 0. \tag{13}$$

This equation is easily solved to give

$$y(z) \propto \exp\left(\int^z dz\,\frac{\rho_1(z) - \rho_2'(z) - \alpha z}{\rho_2(z)}\right). \tag{14}$$

Thus the probability density function $W(x,t)$ is given by

$$W(x,t) = At^{-\alpha}\exp\left(\int^z dz\,\frac{\rho_1(z) - \rho_2'(z) - \alpha z}{\rho_2(z)}\right)_{z=\frac{x}{t^{\alpha}}}, \tag{15}$$

where A is the normalization constant. It is interesting to see that the similarity solution of the FPE can be given in an such analytic closed form. Exact similarity solutions of the FPE can be obtained as long as $\rho_1(z)$ and $\rho_2(z)$ are such that the integral in eq. (15) is integrable.

B. Probability current density

Now we would like to present a simple and interesting relation between the probability distribution function $W(x,t)$ and the associated current density $J(x,t)$. From the continuity equation

$$\frac{\partial}{\partial t}W(x,t) = -\frac{\partial}{\partial x}\,J(x,t)\ , \tag{16}$$

we have

$$J(x,t) = D^{(1)}(x,t)\,W(x,t) - \frac{\partial}{\partial x}\left[D^{(2)}(x,t)\,W(x,t)\right]\ . \tag{17}$$

Using eqs. (5), (8) and (9), we get

$$J(x,t) = t^{-1}\left[(\rho_1 - \rho_2')\,y(z) - \rho_2 y'(z)\right]. \tag{18}$$

From eq. (13), we can reduce the above equation to

$$J(x,t) = \frac{1}{t}\alpha\, z\, y(z) = \frac{\alpha x}{t}\,W(x,t). \tag{19}$$

Thus the conditions in (12) imply that $J(x,t)$ is proportional to $W(x,t)$ and x, and that $J(x,t)|_{\rm boundary} = 0$, as mentioned before.

Eq. (19) can also be obtained by scaling consideration as follows. Under the scale transformation, the scaling behavior of $J(x,t)$ take the form $\bar{J}(\bar{x},\bar{t}) = \varepsilon^h\,J(x,t)$, where $h = -b$ is determined from expression (17) with the help of (2), (3), the relation $b = a - d = 2a - e$ and $c = -a$. The general scaling form of the probability current is $J(x,t) = (t^{\delta_3}/x^{\lambda_3})\,\Sigma(z)$. Without loss of generality we choose the set of parameters $(\lambda_3,\delta_3) = (0,-1)$. This gives

$$J(x,t) = t^{-1}\,\Sigma(z)\ . \tag{20}$$

Inserting the similarity solution $W(x,t) = t^{-\alpha}\,y(z)$ and (20) into eq. (16), one finds $\alpha y(z) + \alpha z y'(z) = \Sigma'(z)$. This can be integrated to give $\Sigma(z) = \int\left[\alpha y(z) + \alpha z y'(z)\right]dz + \text{constant} = \alpha z y(z) + \text{constant}$. Adopting the conditions (12) then gives eq. (19)

As mentioned before, exact similarity solutions of the FPE can be obtained as long as $\rho_1(z)$ and $\rho_2(z)$ are such that the integral in eq. (15) is integrable. Below we present several examples of FPE with time-dependent coefficients which care exactly solvable by the similarity method.

IV. FPE WITH $\rho_1(z) = \mu_1 z + \mu_2$ AND $\rho_2(z) = \mu_4$

Let us take $\rho_1(z)$ and $\rho_2(z)$ as

$$\rho_1(z) = \mu_1 z + \mu_2 \quad , \quad \rho_2(z) = \mu_4\ , \tag{21}$$

where μ_1, μ_2 and μ_4 are real constants. This choice of ρ_1 and ρ_2 generate the following drift and diffusion coefficients:

$$D^{(1)}(x,t) = \mu_1\,\frac{x}{t} + \mu_2\,t^{\alpha-1} \quad , \quad D^{(2)}(x,t) = \mu_4\,t^{2\alpha-1}\ . \tag{22}$$

Eq. (10) in this case is related to the confluent hypergeometric equation.

From eq. (15), the function $y(z)$ is

$$y(z)\ \propto\ \begin{cases} \exp\Big\{\frac{1}{\mu_4}\left[(\mu_1-\alpha)\,\frac{z^2}{2} + \mu_2 z\right]\Big\}, & \mu_1 \neq \alpha \\ \exp\Big\{\frac{\mu_2}{\mu_4}z\Big\}, & \mu_1 = \alpha \end{cases} \tag{23}$$

We shall discuss these two cases separately.

A. $\mu_1 \neq \alpha$

For this case, the normalized solution, from eq. (15), are

$$W(x,t) = \sqrt{\frac{\alpha-\mu_1}{2\pi\mu_4 t^{2\alpha}}}\exp\Big\{-\frac{\alpha-\mu_1}{2\mu_4 t^{2\alpha}}\Big(x-\frac{\mu_2 t^{\alpha}}{\alpha-\mu_1}\Big)^2\Big\}\,, \tag{24}$$

where either ($\mu_4 > 0$, $\mu_1 < \alpha$) or ($\mu_4 < 0$, $\mu_1 > \alpha$) must be satisfied. The well-known diffusion equation is in this class with $\alpha = 1/2$, $\mu_1 = \mu_2 = 0$ and $\mu_4 > 0$.

The solution (24) is a Gaussian (or normal) distribution in x. The full width at half maximum (FWHM) of the solution is related to the parameters α, μ_1, μ_4 and time t as $C_1\sqrt{(\mu_4 t^{2\alpha})/(\alpha-\mu_1)}$, where $C_1 = 2\sqrt{2\ln 2}$. Hence, from eq. (22), one can see that the FWHM is affected by the coefficient μ_1 of term x/t of the drift coefficient and the coefficient μ_4 of the term $t^{2\alpha-1}$ of the diffusion coefficient respectively. It also can be seen that the location of the peak of the solution, $x = (\mu_2 t^{\alpha})/(\alpha-\mu_1)$, depends on the parameters α, μ_1, μ_2 and time t. From eq. (22), it is seen that the location of the peak of the probability density can only be influenced by the drift coefficient. The parameter μ_2 in the drift coefficient plays an important role in the determination of the location of the peak. For instance, if $\mu_2 = 0$, then the peak stays at the origin and will not change with time. The value of the peak is $\sqrt{(\alpha-\mu_1)/(2\pi\mu_4 t^{2\alpha})}$, and is dependent on the parameters α, μ_1, μ_4 and time t. Thus it is affected by both the drift and diffusion coefficients.

One notes that at fixed time t, $W(x,t)$ with μ_2 and $-\mu_2$ are the mirror images to each other with respect to the y-axis as the rest of the parameters are kept fixed. Furthermore, when the parameters $(2\alpha-\mu_1, -\mu_4)$ take the place of (μ_1, μ_4) with the rest unchanged, the corresponding two $W(x,t)$'s are the mirror images to each other. One then finds that $W(x,t)$ is invariant as the parameters (μ_1, μ_2, μ_4) are replaced by $(2\alpha-\mu_1, -\mu_2, -\mu_4)$.

As shown previously, the FWHM and the peak value are both related to $(\mu_4 t^{2\alpha})/(\alpha-\mu_1)$. The parameter α can be either positive or negative. When $\alpha > 0$, the FWHM of the solution (24) is getting broader, the peak is turning smaller and is moving away from the origin with time. On the other hand, if $\alpha < 0$, then the FWHM of the solution will shrink, the peak value will become higher and move toward the origin as time elapses. The situation where the FWHM is neither expanding nor contracting is impossible because $\alpha \neq 0$.

FIG. 1 shows the evolutions of the probability density in solution (24) for a set of the parameters, where $\mu_1 \neq \alpha$.

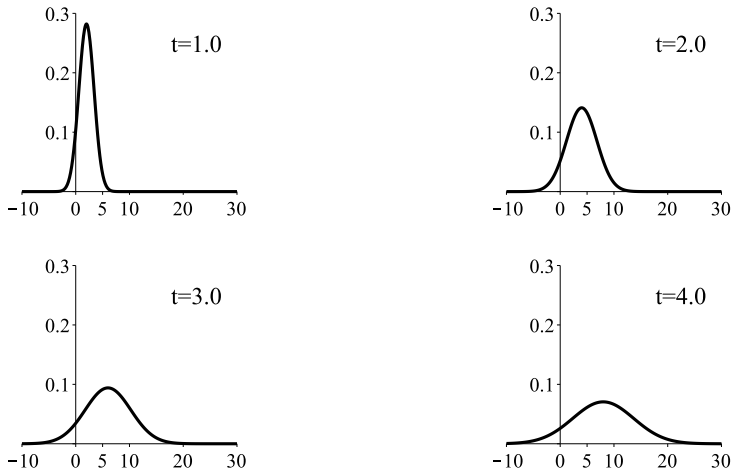

FIG. 1: Plot of $W(x,t)$ versus x for solution (24) with $\alpha = 1$, $\mu_1 = 1/2$, $\mu_2 = 1$, $\mu_4 = 1$ and time $t = 1.0$, 2.0, 3.0, 4.0.

For the set of parameters taken in FIG. 1, the drift and diffusion coefficient are $D^{(1)}(x,t) = x/(2t) + 1$ and $D^{(2)}(x,t) = t$, respectively. From the figures above and the analysis in Sect. IV. A, one sees that the peak of the probability distribution $W(x,t)$ moves to the right as time increases. This is due to the presence of the drift force. Furthermore, owing to the presence of μ_1 and μ_4 in the drift and diffusion coefficient, the FWHM of $W(x,t)$ is getting wider, and the peak value is getting smaller.

FIG. 1 shows the evolution of the solution of the FPE with a set of time-dependent drift and diffusion coefficient. In the next two subsections, we shall study the difference in behavior of the solutions as one of the coefficients changes from being time-independent to being time-dependent, while the other coefficient is being kept fixed.

B. $\mu_1 = \alpha$

In this case, the normalized solution is

$$W(x,t) = \left|\frac{\mu_2}{\mu_4\, t^{\alpha}}\right| \exp\Big\{\frac{\mu_2}{\mu_4\, t^{\alpha}}\, x\Big\}\,, \tag{25}$$

where it is valid in $x \geq 0$ for $(\mu_2/\mu_4) < 0$; $x \leq 0$ for $(\mu_2/\mu_4) > 0$.

Solution (25) is the time-dependent solution of the FPE with the time-dependent drift coefficient $D^{(1)}(x,t) = \alpha x/t + \mu_2 t^{\alpha-1}$ and diffusion coefficient $D^{(2)}(x,t) = \mu_4 t^{2\alpha-1}$. It possess certain symmetry properties. For instance, when the ratio of μ_2/μ_4 is taken to be the same for different sets of (μ_2, μ_4), the solution (25) is invariant. Also, when (μ_2, μ_4) are replaced with $(-\mu_2, \mu_4)$ or $(\mu_2, -\mu_4)$, the corresponding $W(x,t)$'s are the mirror images to each other with the other parameters fixed.

In the following, one example is given to illustrate the solution (25) and the probability current $J(x,t)$ of solution (25).

FIG. 2 displays the evolutions of the probability density in solution (25) for a set of the parameters, where $\mu_1 = \alpha$. FIG. 3 shows the probability current $J(x,t)$ corresponding to the solution (25) with the same set of the parameters as in FIG. 2.

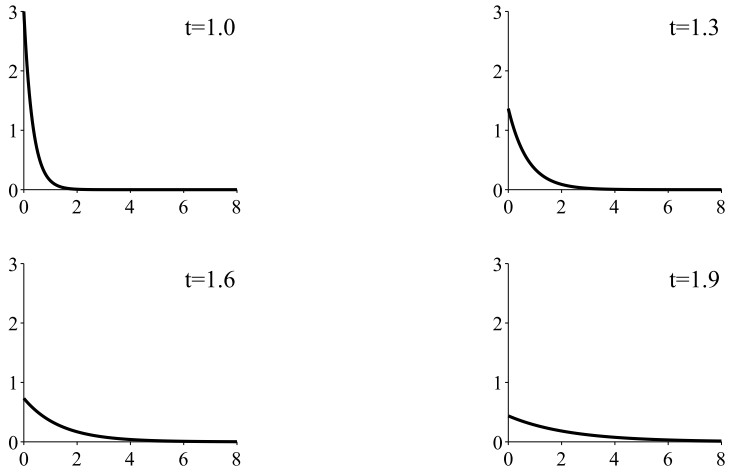

FIG. 2: Plot of $W(x,t)$ versus x for solution (25) with $\alpha = 3$, $\mu_1 = 3$, $\mu_2 = -6$, $\mu_4 = 2$ and time $t = 1.0$, 1.3, 1.6, 1.9.

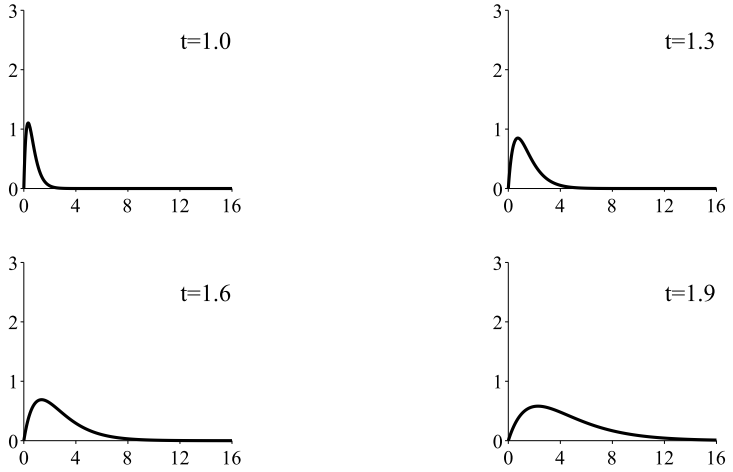

FIG. 3: Plot of $J(x,t)$ versus x for solution (25) with the same set of parameters as in FIG. 2.

The time evolution of $W(x,t)$ in FIG. 2 demonstrates the decreasing of the concentration near the origin and the spreading to the area away from the origin with time.

From (19) and (25), one can find that the maximum probability current $J(x,t)$ occurs at $x = -\mu_4 t^\alpha/\mu_2$, which moves with time.

The plots in FIG. 2 are the same for an infinite set of (μ_2, μ_4) with $\mu_2/\mu_4 = -3$ with other parameters remaining the same. It means that FPE's with drift coefficient $D^{(1)}(x,t) = 3x/t - 3\mu_4 t^2$ and diffusion coefficient $D^{(2)}(x,t) = \mu_4 t^5$ have the same solutions for arbitrary $\mu_4 \neq 0$.

C. Examples with the same drift but different diffusion coefficients

Let us take $\mu_2 = 0$, and consider two different values of α, namely, $\alpha = 1/2$ and $\alpha = 1$. The case with $\alpha = 1/2$ corresponds to $D^{(1)}(x,t) = \mu_1 x/t$ and $D^{(2)}(x,t) = \mu_4$, with the probability distribution

$$W(x,t) = \sqrt{\frac{1-2\mu_1}{4\pi\mu_4 t}}\exp\Big\{-\Big(\frac{1-2\mu_1}{4\mu_4 t}\Big)x^2\Big\}\,. \tag{26}$$

The other case, $\alpha = 1$, leads to $D^{(1)}(x,t) = \mu_1 x/t$, $D^{(2)}(x,t) = \mu_4 t$ and

$$W(x,t) = \sqrt{\frac{1-\mu_1}{2\pi\mu_4 t^2}}\exp\Big\{-\Big(\frac{1-\mu_1}{2\mu_4 t^2}\Big)x^2\Big\}\,. \tag{27}$$

These two examples have the same drift force, and differ only in the diffusion coefficients: constant in the first case, and linear in time in the second. One can thus study how the time-dependent diffusion modifies the behavior of the system with constant diffusion.

As $\mu_2 = 0$, the peak of the probability distribution will not move with time. Comparing eqs. (26) and (27), one sees that the FWHM is changed from $C_1\sqrt{(2\mu_4 t)/(1-2\mu_1)}$ to $C_1\sqrt{(\mu_4 t^2)/(1-\mu_1)}$, and the value of the peak is changed from $\sqrt{(1-2\mu_1)/(4\pi\mu_4 t)}$ to $\sqrt{(1-\mu_1)/(2\pi\mu_4 t^2)}$. The two distributions coincide only at time $t_c = 2(1-\mu_1)/(1-2\mu_1)$.

The evolution of the solutions (26) and (27) are plotted in FIG. 4 and FIG. 5 respectively with two sets of the parameters, which are the same except α.

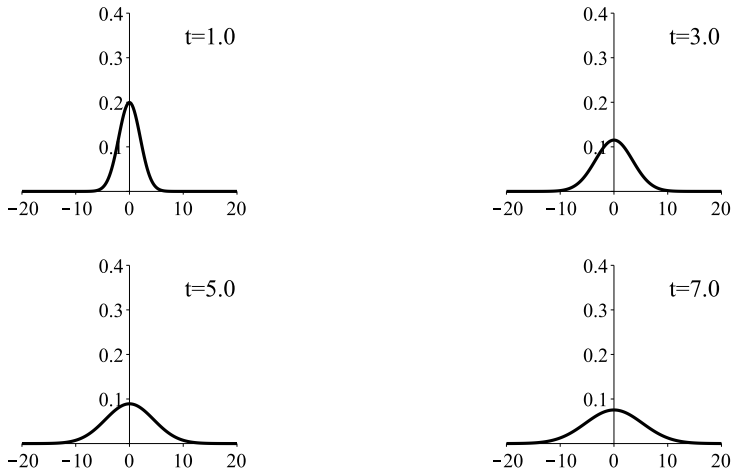

FIG. 4: Plot of $W(x,t)$ versus x for solution (26) with $\alpha = 1/2$, $\mu_1 = 1/4$, $\mu_2 = 0$, $\mu_4 = 1$, and time $t = 1.0$, 3.0, 5.0, 7.0.

Because of $\mu_2 = 0$, the peaks in both of the figures are not moving as expected. It is seen that before time $t_c = 3$, $W(x,t)$ in FIG. 4 has a larger FWHM and a smaller peak value than $W(x,t)$ in FIG. 5, and after $t_c = 3$, the situation reverses. That is because when $0 < t < 1$, the diffusion coefficient μ_4 causes more diffusion than $\mu_4 t$ does. It hence leads the peak value of $W(x,t)$ in FIG. 4 to be smaller than that in FIG. 5. However, when $t > 1$, the time-dependent diffusion coefficient $\mu_4 t$ starts to have more influence than the constant diffusion coefficient μ_4 does. When $t = 3$, both $W(x,t)$ has the same value for all x. After $t = 3$, the peak value of $W(x,t)$ in FIG. 5 is always smaller than $W(x,t)$ in FIG. 4.

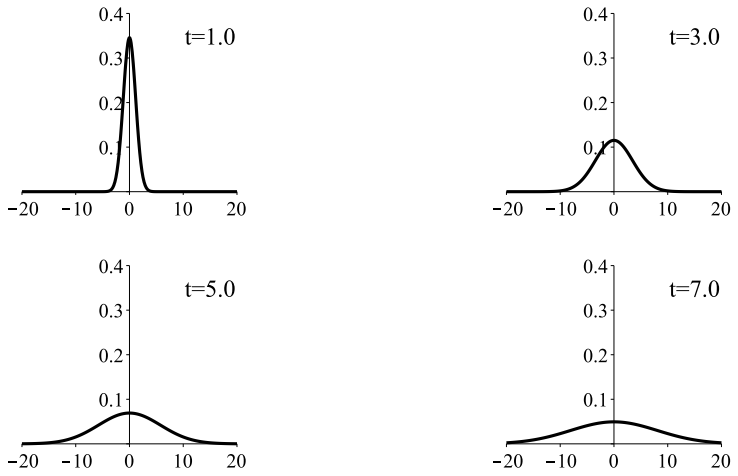

FIG. 5: Plot of $W(x,t)$ versus x for solution (27) with $\alpha = 1$, $\mu_1 = 1/4$, $\mu_2 = 0$, $\mu_4 = 1$, and time $t = 1.0$, 3.0, 5.0, 7.0

D. Examples with the same diffusion but different drift coefficients

When the set of parameters $\alpha = 1/2$, and $\mu_1 = \mu_2 = 0$ is taken, one has $D^{(1)}(x,t) = 0$, $D^{(2)}(x,t) = \mu_4$ and probability distribution

$$W(x,t) = \sqrt{\frac{1}{4\pi\mu_4 t}}\exp\Big\{-\frac{1}{4\mu_4 t}\,x^2\Big\} \tag{28}$$

The other set of parameter $\alpha = 1/2$ and $\mu_1 = 0$ leads to $D^{(1)}(x,t) = \mu_2/\sqrt{t}$, $D^{(2)}(x,t) = \mu_4$ and probability distribution

$$W(x,t) = \sqrt{\frac{1}{4\pi\mu_4 t}}\exp\Big\{-\frac{1}{4\mu_4 t}\Big(x - 2\mu_2\sqrt{t}\Big)^2\Big\} \tag{29}$$

In this case, (28) is the solution of the diffusion equation with a constant diffusion coefficient μ_4. The solution (28) demonstrates only the effect of the diffusion. Eq. (29) gives the solution with additional drift $\mu_2/\sqrt{t}$. The probability distribution (29) moves with time but keeps its FWHM at the value of $4\sqrt{(\ln 2)\,\mu_4 t}$ same as the solution (28). It is easy to see in (29) that when μ_2 takes the positive (negative) value, $W(x,t)$ will move in the $+x$ ($-x$) direction as time elapses, and the peak value $\sqrt{1/(4\pi\mu_4\, t)}$ gets smaller with time.

The evolution of these solutions (28) and (29) are plotted in FIG. 6 and FIG. 7 respectively with two sets of the parameters, which are the same except μ_2.

V. FPE WITH $\rho_1(z) = \mu_1 z + \mu_2$ AND $\rho_2(z) = \mu_3 z$

The next interesting exactly solvable example we shall discuss is an FPE with $\rho_1(z) = \mu_1 z + \mu_2$ and $\rho_2(z) = \mu_3 z$. Eq. (10) with this choice is related to the confluent hypergeometric equation. The corresponding drift and diffusion coefficients are

$$D^{(1)}(x,t) = \mu_1\,\frac{x}{t} + \mu_2\,t^{\alpha-1} \quad , \quad D^{(2)}(x,t) = \mu_3\,x\,t^{\alpha-1}\,. \tag{30}$$

Eq. (15) is integrable and gives

$$W(x,t) = \frac{\Big(\frac{\alpha-\mu_1}{\mu_3\,t^\alpha}\Big)^{\frac{\mu_2}{\mu_3}}}{\Gamma\Big(\frac{\mu_2}{\mu_3}\Big)}\,x^{\frac{\mu_2}{\mu_3}-1}\,\exp\Big\{-\frac{\alpha-\mu_1}{\mu_3\,t^\alpha}\,x\Big\}. \tag{31}$$

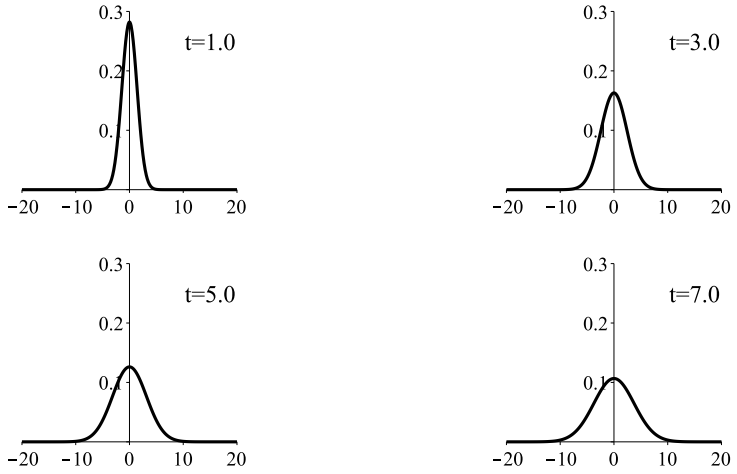

FIG. 6: Plot of $W(x,t)$ versus x for solution (28) with $\alpha = 1/2$, $\mu_1 = \mu_2 = 0$, $\mu_4 = 1$, and time $t = 1.0$, 3.0, 5.0, 7.0.

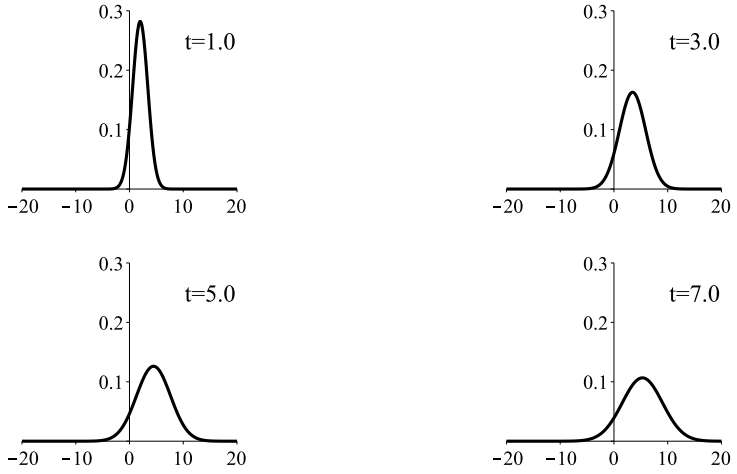

FIG. 7: Plot of $W(x,t)$ versus x for solution (29) with $\alpha = 1/2$, $\mu_1 = 0$, $\mu_2 = 1$, $\mu_4 = 1$, and time $t = 1.0$, 3.0, 5.0, 7.0.

The form of $W(x,t)$ implies that the domain of x is defined only on half-line. For definiteness we shall take $x \in [0, \infty)$. Normalizability of $W(x,t)$ then requires

$$\frac{\alpha - \mu_1}{\mu_3} > 0, \quad \frac{\mu_2}{\mu_3} \geq 1. \tag{32}$$

It is seen that the distribution $W(x,t)$ is invariant under the changes (μ_2, μ_3, x) and $(-\mu_2, -\mu_3, -x)$ (other parameters being the same). This means that the distributions $W(x,t)$ with the parameters (μ_2, μ_3) and $(-\mu_2, -\mu_3)$ are mirror images of each other with respect to the y-axis. It is also worth noticing that for a given α, one will have the same solutions for different drift and diffusion coefficients as long as the quantity $(\mu_1 - \alpha)/\mu_3$ remains unchanged. Furthermore, the system is invariant under the transformations $\alpha \to -\alpha,\ \mu_1 \to \mu_1 - 2\alpha,\ t \to 1/t$, and $\mu_2,\ \mu_3$ and μ_4 unchanged. Hence the distribution with (α, μ_1) at time t is the same as the distribution with $(-\alpha, \mu_1 - 2\alpha)$ at time $1/t$.

The properties of the solution (31) are analyzed in the following. For $\mu_2 = \mu_3$, $W(x,t)$ becomes the exponential function, whose peak is always located at the origin. Its peak value is $|(\mu_1 - \alpha)/(\mu_3\, t^\alpha)|$, which is dependent on the parameters α, μ_1 μ_3 and time t, and hence is affected by both the drift and diffusion coefficients. The peak at $x = 0$ is increasing (decreasing) as t increase (decreases) for $\alpha < 0$ ($\alpha > 0$). That means, by an appropriate choice of the drift and diffusion parameters, one can have a situation where the probability function is accumulating at the origin. For such situation, the effect of the drift force is stronger than that of the diffusion, causing the distribution to be

pushed toward the origin as time elapses. An example of such situation is depicted in FIG. 8, which demonstrates the evolution of solution (31) with $\alpha = -2$ and $\mu_2 = \mu_3$.

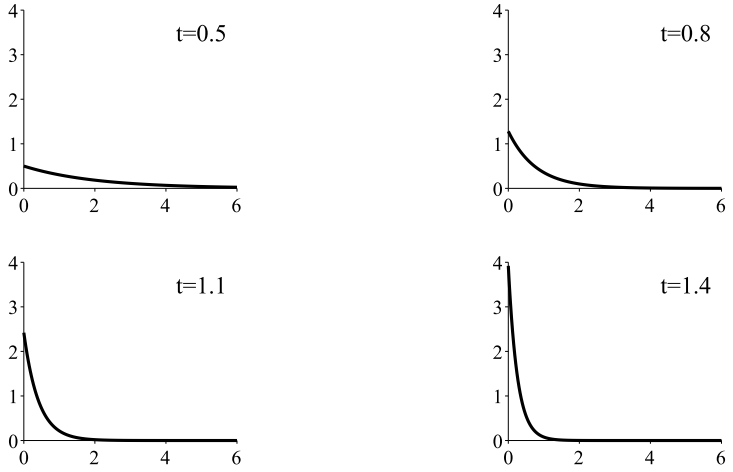

FIG. 8: Plot of $W(x,t)$ versus x for solution (31) with $\alpha = -2$, $\mu_1 = -3$, $\mu_2 = \mu_3 = 1/2$, and time $t = 0.5$, 0.8, 1.1, 1.4.

The case of $\mu_2 > \mu_3$ is depicted in FIG. 9, which shows the evolution of $W(x,t)$ under the influence of the time-dependent drift coefficient $D^{(1)}(x,t) = x/t + t$ and diffusion coefficient $D^{(2)}(x,t) = xt/2$. Since $\mu_3 = 1/2 > 0$, the location of the peak is at $x = t^2/2$.

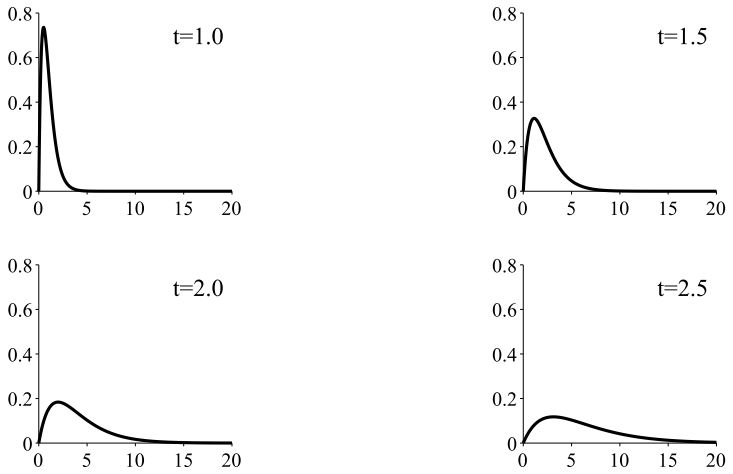

FIG. 9: Plot of $W(x,t)$ versus x for solution (31) with $\alpha = 2$, $\mu_1 = 1$, $\mu_2 = 2\mu_3 = 1$, and time $t = 1.0$, 1.5, 2.0, 2.5.

VI. FPE RELATED TO THE HYPERGEOMETRIC EQUATIONS

We now consider the cases related to the hypergeometric equation. This corresponds to the case where $\rho_1(z)$ and $\rho_2(z)$ are linear and quadratic function of z, respectively. Here we present two cases as examples.

A. $\rho_1(z) = \mu_1 z$ and $\rho_2(z) = z^2 + \mu_4$

When one takes $\rho_1(z) = \mu_1 z$ and $\rho_2(z) = z^2 + \mu_4$.

From eq. (15), the normalized solution of the FPE, with the time-dependent drift coefficient $D^{(1)}(x,t) = \mu_1 x/t$ and diffusion coefficient $D^{(2)}(x,t) = x^2/t + \mu_4\, t^{2\alpha-1}$, is

$$W(x,t) = \frac{\Gamma(\beta)}{\sqrt{\pi\mu_4}\,\Gamma(\beta-\frac{1}{2})}\,\frac{1}{t^\alpha}\left[\frac{1}{\mu_4}\Big(\frac{x}{t^\alpha}\Big)^2+1\right]^{-\beta}, \tag{33}$$

where $\beta = (\alpha + 2 - \mu_1)/2$. The conditions $\mu_1 < 1 + \alpha$ and $\mu_4 > 0$ must be satisfied.

The probability current is again given by eq. (19), i.e., $J(x,t) = (\alpha x/t)\, W(x,t)$.

Graphs showing the evolution of $W(x,t)$ and probability current $J(x,t)$ in solution (33) with a set of parameters are presented as follows.

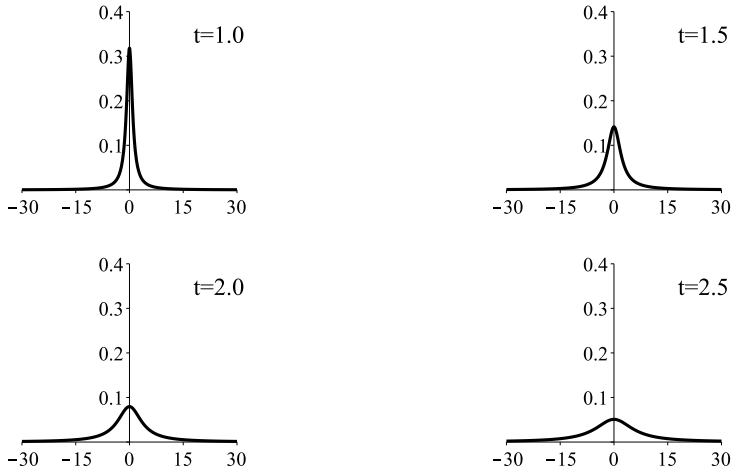

FIG. 10: Plot of $W(x,t)$ versus x for solution (33) with $\alpha = 2$, $\beta = 1$, $\mu_1 = 2$, $\mu_4 = 1$, and time $t = 1.0$, 1.5, 2.0, 2.5.

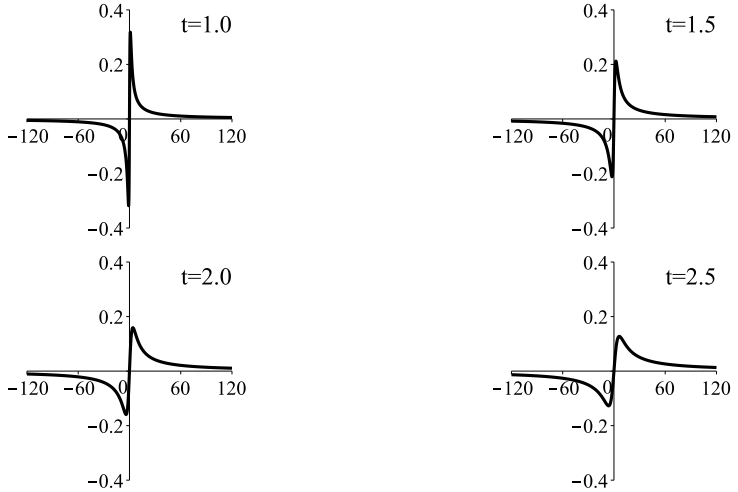

FIG. 11: Plot of $J(x,t)$ versus x for solution (33) with the same set of parameters as in FIG. 10.

In FIG. 10, the drift and diffusion coefficient are $D^{(1)} = 2x/t$ and $D^{(2)}(x,t) = x^2/t + t^3$. The location of the peak of the solution (33) does not change with time even there exits a drift coefficient $2x/t$. This is because the drift force is always pointing towards the origin. Diffusion causes the concentration around the origin to diffuse away and hence the peak value drops.

From the formula $J(x,t) = (\alpha x/t)\, W(x,t)$, it can be understood that in FIG. 11 the value of $J(x,t)$ at the origin is zero and for $x > 0$ ($x < 0$), its value is of positive(negative). It represents that the probability density in $x > 0$ ($x < 0$) region flows in the $+x$ ($-x$) direction with the origin as a separate point. The maximum probability current can also be found that it moves away from the origin with time.

B. $\rho_1(z) = (\alpha + 1/2)z + 1$ and $\rho_2(z) = z^2 - z$

Take $\rho_1(z) = (\alpha + 1/2)z + 1$ and $\rho_2(z) = z^2 - z$ and this case corresponds to the FPE with the drift coefficient $D^{(1)}(x,t) = (\alpha + 1/2)x/t + t^{\alpha-1}$ and the diffusion coefficient $D^{(2)}(x,t) = (x^2/t) - x\,t^{\alpha-1}$.

Eq. (14) gives

$$y(z) \propto \frac{\sqrt{z-1}}{z^2}\,. \tag{34}$$

The normalized solution of the FPE is

$$W(x,t) = \frac{2}{\pi}\,\frac{\sqrt{xt^{\alpha} - t^{2\alpha}}}{x^2}\,. \tag{35}$$

It should be noted that the leftmost point x_L at which $W(x,t) = 0$ is at $x_L = t^{\alpha}$, which shifts rightward away from the origin at time increases. For $t > 0$, the peak of $W(x,t)$ is located at $x = 4t^{\alpha}/3$ that can only in the region of $x > 0$ and is moving with time.

Graphs showing the evolution of $W(x,t)$ in solution (35) with a set of parameters is presented in FIG. 12.

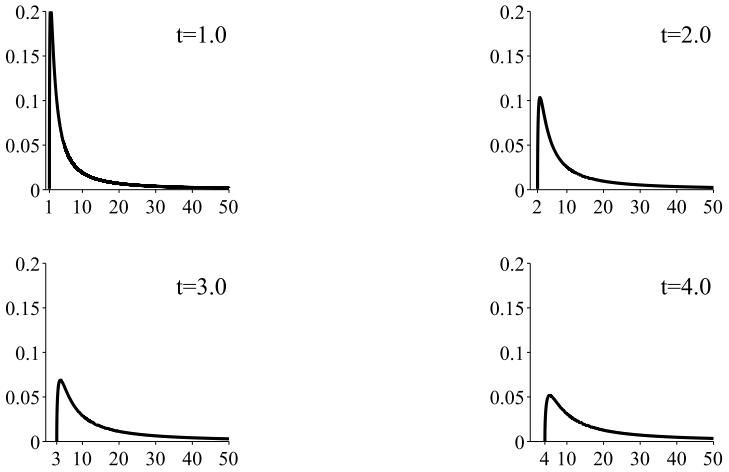

FIG. 12: Plot of $W(x,t)$ versus x for solution (35) with $\alpha = 1$ and time $t = 1.0$, 2.0, 3.0, 4.0.

VII. AN ADDITIONAL EXAMPLE: $\rho_1(z) = z^3$ AND $\rho_2(z) = -z^2$

As mentioned earlier, it is not necessary to restrict the forms of $\rho_1(z)$ and $\rho_2(z)$ so that eq. (10) is related to the confluent hypergeometric equation or the hypergeometric equation in order to obtain exactly solvable FPE. We now consider an example not related to those two types of equations. Take $\alpha = 2$, $\rho_1(z) = z^3$ and $\rho_2(z) = -z^2$, which correspond to the drift and diffusion coefficient $D^{(1)}(x,t) = x^3/t^5$ and $D^{(2)}(x,t) = -x^2/t$. Eq. (14) gives $y(z) \propto \exp\{-z^2/2\}$. The normalized solution of the corresponding FPE is

$$W(x,t) = \frac{1}{\sqrt{2\pi t^4}}\exp\Big\{-\frac{1}{2t^4}\,x^2\Big\}\,. \tag{36}$$

The solution (36) is the Gaussian distribution in the variable x. The location of the peak of the solution does not change with time, but the peak value varies as $1/\sqrt{2\pi t^4}$ that decreases with time. The FWHM also has the time-dependence as $2\sqrt{2\ln 2}\,t^2$, which is getting broader with time. Evolution of $W(x,t)$ is depicted in FIG. 13.

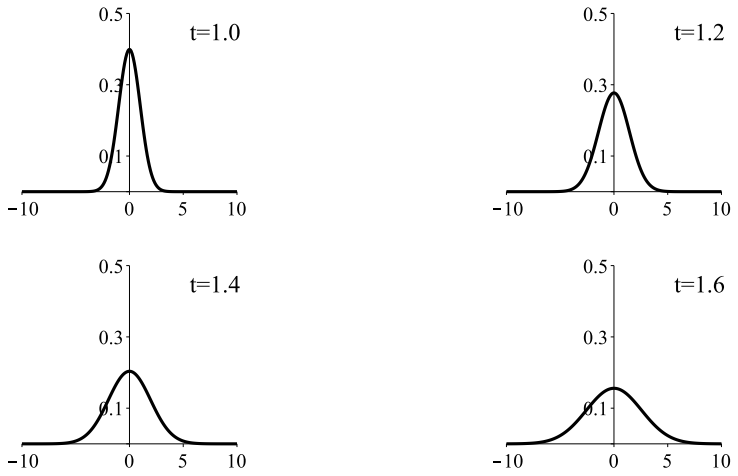

FIG. 13: Plot of $W(x,t)$ versus x for solution (36) for time $t = 1.0$, 1.2, 1.4, 1.6.

VIII. SUMMARY

The FPE is an important equation in many different areas. The general analytic solutions of the FPE's with both time-dependent drift and diffusion coefficients are generally difficult to obtain. In this paper, we present new exact similarity solutions of the FPE. Such similarity solutions exist when the FPE possesses proper scaling behavior.

The similarity method makes use of the scaling-invariant property of the FPE. By the introduction of the similarity variable, the FPE can be reduced to an ordinary differential equation, which may be easier to solve. The general expression of the ordinary differential equation corresponding to the FPE with time-dependent drift and diffusion coefficients is given in this paper. It is interesting to find, by the natural requirement that the probability current density vanishes at the boundary, that the resulted ordinary differential equation is integrable, and the probability density function can be given in closed form. We present several new examples of exactly solvable Fokker-Planck equations with time-dependent coefficients. Symmetry properties of the solutions are also discussed.

Acknowledgments

This work is supported in part by the National Science Council (NSC) of the Republic of China under Grants NSC-99-2112-M-032-002-MY3 and NSC-99-2811-M-032-012.

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