A short introduction to local fractional complex analysis

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This paper presents a short introduction to local fractional complex analysis. The generalized local fractional complex integral formulas, Yang-Taylor series and local fractional Laurent's series of complex functions in complex fractal space, and generalized residue theorems are investigated.

Key words：*Local fractional calculus, complex-valued functions, fractal, Yang-Taylor series, local fractional Laurent series, generalized residue theorems*

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1 Introduction

Local fractional calculus has played an important role in not only mathematics but also in physics and engineers [1-12]. There are many definitions of local fractional derivatives and local fractional integrals (also called fractal calculus). Hereby we write down local fractional derivative, given by [5-7]

$$f^{(\alpha)}(x_0)=\frac{d^\alpha f(x)}{dx^\alpha}\Big|_{x=x_0}=\lim_{x\to x_0}\frac{\Delta^\alpha\left(f(x)-f(x_0)\right)}{(x-x_0)^\alpha} \tag{1.1}$$

with $\Delta^\alpha\left(f(x)-f(x_0)\right)\cong\Gamma(1+\alpha)\Delta\left(f(x)-f(x_0)\right)$, and local fractional integral of $f(x)$, denoted by [5-6,8]

$${}_aI_b^{(\alpha)}f(x)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^\alpha=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_j)(\Delta t_j)^\alpha \tag{1.2}$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,...\}$, where for $j=0,...,N-1$, $[t_j,t_{j+1}]$ is a partition of the interval $[a,b]$ and $t_0=a, t_N=b$.

More recently, a motivation of local fractional derivative and local fractional integral of complex functions is given [11]. Our attempt, in the present paper, is to continue to study local fractional calculus of complex function. As well, a short outline of local fractional complex analysis will be established.

2 Local fractional calculus of the complex-variable functions

In this section we deduce fundamentals of local fractional calculus of the complex-valued functions. Here we start with local fractional continuity of complex functions.

2.1 Local fractional continuity of complex-variable functions

Definition 1

Given z_0 and $\left|z-z_0\right|<\delta$, then for any z we have [11]

$$\left|f\left(z\right)-f\left(z_0\right)\right|<\varepsilon^{\alpha}. \tag{2.1}$$

Here complex function $f\left(z\right)$ is called local fractional continuous at $z=z_0$, denoted by

$$\lim_{z\to z_0} f\left(z\right)=f\left(z_0\right). \tag{2.2}$$

A function $f\left(z\right)$ is called local fractional continuous on the region $\Re$, denoted by

$$f\left(z\right)\in C_{\alpha}\left(\Re\right).$$

As a direct result, we have the following results:

Suppose that $\lim_{z\to z_0} f\left(z\right)=f\left(z_0\right)$ and $\lim_{z\to z_0} g\left(z\right)=g\left(z_0\right)$, then we have that

$$\lim_{z\to z_0}\left[f\left(z\right)\pm g\left(z\right)\right]=f\left(z_0\right)\pm g\left(z_0\right), \tag{2.3}$$

$$\lim_{z\to z_0}\left[f\left(z\right)g\left(z\right)\right]=f\left(z_0\right)g\left(z_0\right), \tag{2.4}$$

and

$$\lim_{z\to z_0}\left[f\left(z\right)/g\left(z\right)\right]=f\left(z_0\right)/g\left(z_0\right), \tag{2.5}$$

the last only if $g\left(z_0\right)\neq 0$.

2.2 Local fractional derivatives of complex function

Definition 2

Let the complex function $f\left(z\right)$ be defined in a neighborhood of a point z_0. The local fractional derivative of $f\left(z\right)$ at z_0 is defined by the expression [11]

$${}_{z_0}D_z^{\alpha} f\left(z\right)=:\lim_{z\to z_0}\frac{\Gamma\left(1+\alpha\right)\left[f\left(z\right)-f\left(z_0\right)\right]}{\left(z-z_0\right)^{\alpha}}, 0<\alpha\leq 1. \tag{2.6}$$

If this limit exists, then the function $f\left(z\right)$ is called to be local fractional analytic at z_0, denoted by

$${}_{z_0}D_z^{\alpha} f\left(z\right),\ \frac{d^{\alpha}}{dz^{\alpha}}f\left(z\right)\Big|_{z=z_0} \text{ or } f^{(\alpha)}\left(z_0\right).$$

Remark 1. If the limits exist for all z_0 in a region $\Re$, then $f\left(z\right)$ is said to be local fractional analytic in a region $\Re$, denoted by

$$f\left(z\right)\in D\left(\Re\right).$$

Suppose that $f\left(z\right)$ and $g\left(z\right)$ are local fractional analytic functions, the following rules are valid [11].

$$\frac{d^{\alpha}\left(f\left(z\right)\pm g\left(z\right)\right)}{dz^{\alpha}}=\frac{d^{\alpha}f\left(z\right)}{dz^{\alpha}}\pm\frac{d^{\alpha}g\left(z\right)}{dz^{\alpha}}; \tag{2.7}$$

$$\frac{d^{\alpha}\left(f\left(z\right)g\left(z\right)\right)}{dz^{\alpha}}=g\left(z\right)\frac{d^{\alpha}f\left(z\right)}{dz^{\alpha}}+f\left(z\right)\frac{d^{\alpha}g\left(z\right)}{dz^{\alpha}}; \tag{2.8}$$

$$\frac{d^{\alpha}\left(\frac{f\left(z\right)}{g\left(z\right)}\right)}{dz^{\alpha}}=\frac{g\left(z\right)\frac{d^{\alpha}f\left(z\right)}{dz^{\alpha}}+f\left(z\right)\frac{d^{\alpha}g\left(z\right)}{dz^{\alpha}}}{g\left(z\right)^{2}} \tag{2.9}$$

if $g\left(z\right)\neq 0$;

$$\frac{d^{\alpha}\left(Cf\left(z\right)\right)}{dz^{\alpha}}=C\frac{d^{\alpha}f\left(z\right)}{dz^{\alpha}}, \tag{2.10}$$

where C is a constant;

If $y\left(z\right)=\left(f\circ u\right)\left(z\right)$ where $u\left(z\right)=g\left(z\right)$, then

$$\frac{d^{\alpha}y(z)}{dz^{\alpha}}=f^{(\alpha)}\left(g(z)\right)\left(g^{(1)}\left(z\right)\right)^{\alpha}. \tag{2.11}$$

2.3 Local fractional Cauchy-Riemann equations

Definition 3
If there exists a function

$$f\left(z\right)=u\left(x,y\right)+i^{\alpha}v\left(x,y\right), \tag{2.12}$$

where u and v are real functions of x and y . The local fractional complex differential equations

$$\frac{\partial^{\alpha}u\left(x,y\right)}{\partial x^{\alpha}}-\frac{\partial^{\alpha}v\left(x,y\right)}{\partial y^{\alpha}}=0 \tag{2.13}$$

and

$$\frac{\partial^{\alpha}u\left(x,y\right)}{\partial y^{\alpha}}+\frac{\partial^{\alpha}v\left(x,y\right)}{\partial x^{\alpha}}=0 \tag{2.14}$$

are called local fractional Cauchy-Riemann Equations.

Theorem 1
Suppose that the function

$$f\left(z\right)=u\left(x,y\right)+i^{\alpha}v\left(x,y\right) \tag{2.15}$$

is local fractional analytic in a region $\Re$. Then we have

$$\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}-\frac{\partial^{\alpha}v(x,y)}{\partial y^{\alpha}}=0 \quad (2.16)$$

and

$$\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}+\frac{\partial^{\alpha}v(x,y)}{\partial x^{\alpha}}=0. \quad (2.17)$$

Proof. Since $f(z)=u(x,y)+i^{\alpha}v(x,y)$, we have the following identity

$$f^{(\alpha)}(z_0)=\lim_{z\to z_0}\frac{\Gamma(1+\alpha)\left[f(z)-f(z_0)\right]}{(z-z_0)^{\alpha}}. \quad (2.18)$$

Consequently, the formula (2.18) implies that

$$\lim_{\Delta z\to 0}\frac{\Gamma(1+\alpha)\left[f(z+\Delta z)-f(z)\right]}{\Delta z^{\alpha}}$$
$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Gamma(1+\alpha)\left[u(x+\Delta x,y+\Delta y)-u(x,y)+i^{\alpha}\left(v(x+\Delta x,y+\Delta y)-v(x,y)\right)\right]}{\Delta x^{\alpha}+i^{\alpha}\Delta y^{\alpha}}. \quad (2.19)$$

In a similar manner, setting $\Delta y\to 0$ and taking into account the formula (2.19), we have $(\Delta y)^{\alpha}\to 0$ such that

$$f^{(\alpha)}(z_0)=\lim_{\Delta y\to 0}\frac{\Gamma(1+\alpha)\left[u(x,y+\Delta y)-u(x,y)+i^{\alpha}\left(v(x,y+\Delta y)-v(x,y)\right)\right]}{i^{\alpha}\Delta y^{\alpha}}. \quad (2.20)$$

Hence

$$f^{(\alpha)}(z_0)=-i^{\alpha}\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}+\frac{\partial^{\alpha}v(x,y)}{\partial y^{\alpha}} \quad (2.21)$$

If $\Delta x\to 0$, from (2.19) we have $(\Delta x)^{\alpha}\to 0$ such that

$$f^{(\alpha)}(z_0)=\lim_{\Delta x\to 0}\frac{\Gamma(1+\alpha)\left[u(x+\Delta x,y)-u(x,y)+i^{\alpha}\left(v(x+\Delta x,y)-v(x,y)\right)\right]}{\Delta x^{\alpha}} \quad (2.22)$$

Thus we get the identity

$$f^{(\alpha)}(z_0)=\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}+i^{\alpha}\frac{\partial^{\alpha}v(x,y)}{\partial x^{\alpha}}. \quad (2.24)$$

Since $f(z)=u(x,y)+i^{\alpha}v(x,y)$ is local fractional analytic in a region $\Re$, we have the following formula

$$f^{(\alpha)}(z_0)=\frac{\partial^{\alpha}u(x,y)}{\partial x^{\alpha}}+i^{\alpha}\frac{\partial^{\alpha}v(x,y)}{\partial x^{\alpha}}=-i^{\alpha}\frac{\partial^{\alpha}u(x,y)}{\partial y^{\alpha}}+\frac{\partial^{\alpha}v(x,y)}{\partial y^{\alpha}}. \quad (2.25)$$

Hence, from (2.25) , we arrive at the following identity

$$\frac{\partial^{\alpha} u(x,y)}{\partial x^{\alpha}} - \frac{\partial^{\alpha} v(x,y)}{\partial y^{\alpha}} = 0 \tag{2.26}$$

and

$$\frac{\partial^{\alpha} u(x,y)}{\partial y^{\alpha}} + \frac{\partial^{\alpha} v(x,y)}{\partial x^{\alpha}} = 0. \tag{2.27}$$

This completes the proof of Theorem 1.

Remark 2. Local fractional C-R equations are sufficient conditions that $f(z)$ is local fractional analytic in $\Re$.

The local fractional partial equations

$$\frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} u(x,y)}{\partial y^{2\alpha}} = 0 \tag{2.28}$$

and

$$\frac{\partial^{2\alpha} v(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} v(x,y)}{\partial y^{2\alpha}} = 0 \tag{2.29}$$

are called local fractional Laplace equations, denoted by

$$\nabla^{\alpha} u(x,y) = 0 \tag{2.30}$$

and

$$\nabla^{\alpha} v(x,y) = 0, \tag{2.31}$$

where

$$\nabla^{\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} \tag{2.32}$$

is called local fractional Laplace operator.

Remark 3. Suppose that $\nabla^{\alpha} u(x,y) = 0$, $u(x,y)$ is a local fractional harmonic function in $\Re$.

2.4 Local fractional integrals of complex function

Definition 4

Let $f(z)$ be defined, single-valued and local fractional continuous in a region $\Re$. The local fractional integral of $f(z)$ along the contour C in $\Re$ from point z_p to point z_q, is defined as [11]

$$\begin{aligned} &I_C^{\alpha} f(z) \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta z \to 0} \sum_{i=0}^{n-1} f(z_i)(\Delta z)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \int_C f(z)(dz)^{\alpha} \end{aligned} \tag{2.33}$$

where for $i = 0,1,...,n$ $\Delta z_i = z_i - z_{i-1}$, $z_0 = z_p$ and $z_n = z_q$.

For convenience, we assume that

$$ {}_{z_0}I_{z_0}^{(\alpha)} f(z) = 0 \tag{2.34} $$

if $z = z_0$.

The rules for complex integration are similar to those for real integrals. Some important results are as follows [11]:

Suppose that $f(z)$ and $g(z)$ be local fractional continuous along the contour C in $\Re$.

$$ \frac{1}{\Gamma(1+\alpha)}\int_C (f(z)+g(z))(dz)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_C f(z)(dz)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_C g(z)(dz)^\alpha ; \tag{2.35} $$

$$ \frac{1}{\Gamma(1+\alpha)}\int_C kf(z)(dz)^\alpha = \frac{k}{\Gamma(1+\alpha)}\int_C f(z)(dz)^\alpha , \tag{2.36} $$

for a constant k;

$$ \frac{1}{\Gamma(1+\alpha)}\int_C f(z)(dz)^\alpha = \frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^\alpha , \tag{2.37} $$

where $C = C_1 + C_2$;

$$ \frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^\alpha = -\frac{1}{\Gamma(1+\alpha)}\int_{-C_1} f(z)(dz)^\alpha ; \tag{2.38} $$

$$ \left|\frac{1}{\Gamma(1+\alpha)}\int_C f(z)(dz)^\alpha\right| \le \frac{1}{\Gamma(1+\alpha)}\int_C |f(z)|\left|(dz)^\alpha\right| \le ML , \tag{2.39} $$

where M is an upper bound of $f(z)$ on C and $L = \frac{1}{\Gamma(1+\alpha)}\int_C \left|(dz)^\alpha\right|$.

Theorem 2

If the contour C has end points z_p and z_q with orientation z_p to z_q, and if function $f(z)$ has the primitive $F(z)$ on C, then we have

$$ \frac{1}{\Gamma(1+\alpha)}\int_C f(z)(dz)^\alpha = F(z_q) - F(z_p). \tag{2.40} $$

Remark 4. *Suppose that* $f(z) \in D(\Re)$. For $k = 0,1,...,n$ and $0 < \alpha \le 1$ there exists a local fractional series

$$ f(z) = \sum_{k=0}^{\infty} \frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)}(z-z_0)^{k\alpha} \tag{2.41} $$

with $f^{(k\alpha)}(z) \in D(\Re)$, where $f^{(k\alpha)}(z) = \overbrace{D_z^{(\alpha)}...D_z^{(\alpha)}}^{k\ times} f(z)$.

This series is called Yang-Taylor series of local fractional analytic function (for real function case, see [12].)

Theorem 3

If C is a simple closed contour, and if function $f(z)$ has a primitive on C, then [11]

$$\frac{1}{\Gamma(1+\alpha)}\oint_C f(z)(dz)^{\alpha}=0. \tag{2.42}$$

Corollary 4

If the closed contours C_1 , C_2 is such that C_2 lies inside C_1, and if $f(z)$ is local fractional analytic on C_1 , C_2 and between them, then we have [11]

$$\frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha}. \tag{4.43}$$

Theorem 5

Suppose that the closed contours C_1, C_2 is such that C_2 lies inside C_1, and if $f(z)$ is local fractional analytic on C_1, C_2 and between them, then we have[11]

$$\frac{1}{\Gamma(1+\alpha)}\int_{C_1} f(z)(dz)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_{C_2} f(z)(dz)^{\alpha}. \tag{2.44}$$

3 Generalized local fractional integral formulas of complex functions

In this section we start with generalized local fractional integral formulas of complex functions and deduce some useful results.

Theorem 6

Suppose that $f(z)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to C. Then we have

$$\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-z_0)^{\alpha}}(dz)^{\alpha}=f(z_0). \tag{3.1}$$

Proof. From(2.44), we arrive at the formula

$$\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-z_0)^{\alpha}}(dz)^{\alpha}=\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_{C_1} \frac{f(z)}{(z-z_0)^{\alpha}}(dz)^{\alpha}, \tag{3.2}$$

where $C_1:\left|(z-z_0)^{\alpha}\right|=\varepsilon^{\alpha}$.

Setting $\left|(z-z_0)^{\alpha}\right|=\varepsilon^{\alpha}$ implies that

$$z^{\alpha}-{z_0}^{\alpha}=\varepsilon^{\alpha}E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right) \tag{3.3}$$

and

$$\left(dz\right)^{\alpha}=i^{\alpha}\varepsilon^{\alpha}E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right)\left(d\theta\right)^{\alpha}. \tag{3.4}$$

Taking (3.3) and (3.4), it follows from (3.2) that

$$\begin{aligned}&\frac{1}{\left(2\pi\right)^{\alpha}i^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{2\pi}\frac{f\left(z_{0}+\varepsilon E\left(i\theta\right)\right)}{\varepsilon^{\alpha}E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right)}i^{\alpha}\varepsilon^{\alpha}E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right)\left(d\theta\right)^{\alpha}\\&=\lim_{\varepsilon\to0}\frac{1}{\left(2\pi\right)^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{2\pi}f\left(z_{0}+\varepsilon E\left(i\theta\right)\right)\left(d\theta\right)^{\alpha}\end{aligned} \tag{3.5}$$

From (3.5) , we get

$$\frac{1}{\left(2\pi\right)^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{2\pi}\left(\lim_{\varepsilon\to0}f\left(z_{0}+\varepsilon E\left(i\theta\right)\right)\right)\left(d\theta\right)^{\alpha}=\frac{f\left(z_{0}\right)}{\left(2\pi\right)^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{2\pi}\left(d\theta\right)^{\alpha}. \tag{3.6}$$

Furthermore

$$\frac{f\left(z_{0}\right)}{\left(2\pi\right)^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\int_{0}^{2\pi}\left(d\theta\right)^{\alpha}=f\left(z_{0}\right). \tag{3.7}$$

Substituting (3.7) into (3.6) and (3.3) implies that

$$\frac{1}{\left(2\pi\right)^{\alpha}i^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\oint_{C}\frac{f\left(z\right)}{\left(z-z_{0}\right)^{\alpha}}\left(dz\right)^{\alpha}=f\left(z_{0}\right). \tag{3.8}$$

The proof of the theorem is completed.

Likewise, we have the following corollary:

Corollary 7

Suppose that $f\left(z\right)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to C . Then we have

$$\frac{1}{\left(2\pi\right)^{\alpha}i^{\alpha}}\cdot\frac{1}{\Gamma\left(1+\alpha\right)}\oint_{C}\frac{f\left(z\right)}{\left(z-z_{0}\right)^{\left(n+1\right)\alpha}}\left(dz\right)^{\alpha}=f^{\left(n\alpha\right)}\left(z_{0}\right). \tag{3.9}$$

Proof. Taking into account formula (3.1), we arrive at the identity.

Theorem 8

Suppose that $f\left(z\right)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to C . Then we have

$$\frac{1}{\Gamma\left(1+\alpha\right)}\oint_{C}\frac{\left(dz\right)^{\alpha}}{\left(z-z_{0}\right)^{\alpha}}=\left(2\pi\right)^{\alpha}i^{\alpha}. \tag{3.9}$$

Proof. Taking $f\left(z\right)=1$, from (3.9) we deduce the result.

Theorem 9

Suppose that $f\left(z\right)$ is local fractional analytic within and on a simple closed contour C and z_0 is any point interior to C . Then we have

$$\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{(dz)^\alpha}{(z-z_0)^{n\alpha}}=0 \text{, for } n>1. \tag{3.10}$$

Proof. Taking $f(z)=1$, from (3.9) we deduce the result.

4 Complex Yang-Taylor's series and local fractional Laurent's series

In this section we start with a Yang-Taylor's expansion formula of complex functions and deduce local fractional Laurent series of complex functions.

4.1 Complex Yang-Taylor's expansion formula

Definition 5

Let $f(z)$ be local fractional analytic inside and on a simple closed contour C having its center at $z=z_0$. Then for all points z in the circle we have the Yang-Taylor series representation of $f(z)$, given by

$$\begin{aligned} &f(z)\\ &=f(z_0)+\frac{f^{(\alpha)}(z_0)}{\Gamma(1+\alpha)}(z-z_0)^\alpha+\\ &\frac{f^{(2\alpha)}(z_0)}{\Gamma(1+2\alpha)}(z-z_0)^{2\alpha}+....+\frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)}(z-z_0)^{k\alpha}+... \end{aligned} \tag{4.1}$$

For $C:|z-z_0|^\alpha \le R^\alpha$, we have the complex Yang-Taylor series

$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^{k\alpha}. \tag{4.2}$$

From (3.44) the above expression implies

$$a_k=\frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-z_0)^{(k+1)\alpha}}(dz)^\alpha=\frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)}, \tag{4.3}$$

for $c:|z-z_0|^\alpha \le R^\alpha$.

Successively, it follows from (4.3) that

$$f(z)=\sum_{k=0}^{\infty}a_k(z-z_0)^{k\alpha}, \tag{4.4}$$

where

$$a_k=\frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-z_0)^{(k+1)\alpha}}(dz)^\alpha=\frac{f^{(k\alpha)}(z_0)}{\Gamma(1+k\alpha)}, \tag{4.5}$$

for $C:\left|z-z_0\right|^{\alpha} \le R^{\alpha}$.

Hence, the above formula implies the relation (4.2).

Theorem 10

Suppose that complex function $f(z)$ is local fractional analytic inside and on a simple closed contour C having its center at $z=z_0$. There exist all points z in the circle such that we have the Yang-Taylor's series of $f(z)$

$$f(z)=\sum_{k=0}^{\infty} a_k \left(z-z_0\right)^{k\alpha}, \tag{4.5}$$

where

$$a_k=\frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{\left(z-z_0\right)^{(k+1)\alpha}} (dz)^{\alpha} = \frac{f^{(k\alpha)}\left(z_0\right)}{\Gamma(1+k\alpha)},$$

for $C:\left|z-z_0\right|^{\alpha} \le R^{\alpha}$.

Proof. Setting $C_1:\left|z-z_0\right|^{\alpha}=R^{\alpha}$ and using (3.1) , we have

$$f(z)=\frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \frac{f(\xi)}{(\xi-z)^{\alpha}} (d\xi)^{\alpha}. \tag{4.6}$$

Taking $\xi \in C_1$, we get

$$\frac{\left|z-z_0\right|^{\alpha}}{\left|\xi-z_0\right|^{\alpha}}=q^{\alpha}<1 \tag{4.7}$$

and

$$\begin{aligned} &\frac{1}{(\xi-z)^{\alpha}} \\ &=\frac{1}{\left(\xi-z_0\right)^{\alpha}} \frac{1}{1-\frac{\left(z-z_0\right)^{\alpha}}{\left(\xi-z_0\right)^{\alpha}}} \\ &=\frac{1}{\left(\xi-z_0\right)^{\alpha}} \frac{1}{1-\left(\frac{z-z_0}{\xi-z_0}\right)^{\alpha}} \\ &=\sum_{n=1}^{\infty} \frac{1}{\left(\xi-z_0\right)^{(n+1)\alpha}}\left(z-z_0\right)^{n\alpha}. \end{aligned} \tag{4.8}$$

Substituting (4.8) into (4.6) implies that

$$
\begin{aligned}
&f(z) \\
&= \sum_{n=1}^{N}\left[\frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \frac{f(\xi)(d\xi)^{\alpha}}{(\xi - z_0)^{(n+1)\alpha}}\right](z - z_0)^{n\alpha} \\
&+ \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \sum_{n=N}^{\infty}\left[\frac{f(\xi)(z - z_0)^{n\alpha}}{(\xi - z_0)^{(n+1)\alpha}}\right](d\xi)^{\alpha}.
\end{aligned} \tag{4.9}
$$

Taking the Yang-Taylor formula of analytic function into account, we have the following relation

$$
f(z) = \sum_{n=0}^{N-1} \frac{f^{(n\alpha)}(z_0)(z - z_0)^{n\alpha}}{\Gamma(1+n\alpha)} + R_N, \tag{4.10}
$$

where R_N is reminder in the form

$$
R_N = \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \sum_{n=N}^{\infty}\left[\frac{f(\xi)(z - z_0)^{n\alpha}}{(\xi - z_0)^{(n+1)\alpha}}\right](d\xi)^{\alpha}. \tag{4.11}
$$

There exists a Yang-Taylor series

$$
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n\alpha)}(z_0)(z - z_0)^{n\alpha}}{\Gamma(1+n\alpha)} \tag{4.12}
$$

where is $f(z_0)$ is local fractional analytic at $z = z_0$.

Taking into account the relation $\left|\frac{(z - z_0)^{n\alpha}}{(\xi - z_0)^{n\alpha}}\right| = q^{n\alpha} < 1$ and $|f(z)| \le M$, from (4.11) we get

$$
\begin{aligned}
&|R_N| \\
&= \left|\frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \sum_{n=N}^{\infty}\left[\frac{f(\xi)(z - z_0)^{n\alpha}}{(\xi - z_0)^{(n+1)\alpha}}\right](d\xi)^{\alpha}\right| \\
&\le \frac{1}{(2\pi)^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \left|\sum_{n=N}^{\infty} \frac{|f(\xi)|\left|(z - z_0)^{n\alpha}\right|}{\left|(\xi - z_0)^{(n+1)\alpha}\right|}\right|(d\xi)^{\alpha} \\
&\le \frac{1}{(2\pi)^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \left|\sum_{n=N}^{\infty} \frac{|M|}{\left|(\xi - z_0)^{\alpha}\right|} \frac{\left|(z - z_0)^{n\alpha}\right|}{\left|(\xi - z_0)^{n\alpha}\right|}\right|(d\xi)^{\alpha} \\
&\le \frac{(2\pi)^{\alpha} R^{\alpha}}{(2\pi)^{\alpha}} \cdot \frac{|M|}{\Gamma(1+\alpha)} \frac{q^{n\alpha}}{1-q^{\alpha}} \\
&\le \frac{|M| R^{\alpha}}{\Gamma(1+\alpha)} \frac{q^{n\alpha}}{1-q^{\alpha}}
\end{aligned} \tag{4.13}
$$

Furthermore

$$
\lim_{N\to\infty} R_N = 0.
$$

From (4.9), we have

$$f(z)=\sum_{n=1}^{\infty}\left[\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_{C_1}\frac{f(\xi)(d\xi)^{\alpha}}{(\xi-z_0)^{(n+1)\alpha}}\right](z-z_0)^{n\alpha}. \tag{4.14}$$

Hence

$$a_n=\frac{1}{(2\pi)^{\alpha} i^{\alpha}}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_{C_1}\frac{f(\xi)(d\xi)^{\alpha}}{(\xi-z_0)^{(n+1)\alpha}}. \tag{4.15}$$

Hence the proof of the theorem is completed.

4.2 Singular point and poles

Definition 6

A singular point of a function $f(z)$ is a value of z at which $f(z)$ fails to be local fractional analytic. If $f(z)$ is local fractional analytic everywhere in some region except at an interior point $z=z_0$, we call $f(z)$ an isolated singularity.

If

$$f(z)=\frac{\phi(z)}{(z-z_0)^{n\alpha}} \tag{4.16}$$

and

$$\phi(z)\neq 0 \tag{4.17}$$

where $\phi(z)$ is local fractional analytic everywhere in a region including $z=z_0$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z=z_0$, which is called a pole of order n.

If $n=1$, the pole is often called a simple pole;

if $n=2$, it is called a double pole, and so on.

4.3 Local fractional Laurent's series

Definition 7

If $f(z)$ has a pole of order n at $z=z_0$ but is local fractional analytic at every other point inside and on a contour C with center at z_0, then

$$\phi(z)=(z-z_0)^{n\alpha} f(z) \tag{4.18}$$

is local fractional analytic at all points inside and on C and has a Yang-Taylor series about $z=z_0$ so that

$$f(z) = \frac{a_{-n}}{(z-z_0)^{n\alpha}} + \frac{a_{-n+1}}{(z-z_0)^{(n-1)\alpha}} + \ldots + \frac{a_{-1}}{(z-z_0)^{\alpha}} + a_0 + a_1 (z-z_0)^{\alpha} + \ldots. + a_n (z-z_0)^{n\alpha} + \ldots \tag{4.19}$$

This is called a local fractional Laurent series for $f(z)$.

More generally, it follows that

$$f(z) = \sum_{i=-\infty}^{\infty} a_k (z-z_0)^{k\alpha} \tag{4.20}$$

as a local fractional Laurent series.

For $C: r^{\alpha} < |z-z_0|^{\alpha} < R^{\alpha}$ we have a local fractional Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k\alpha} . \tag{4.21}$$

From (3.44), the above expression implies that

$$a_k = \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^{(k+1)\alpha}} (dz)^{\alpha} , \tag{4.22}$$

where $C: r^{\alpha} < |z-z_0|^{\alpha} < R^{\alpha}$.

Setting $C_1 : |z-z_0|^{\alpha} = r^{\alpha}$ and $C_2 : |z-z_0|^{\alpha} = R^{\alpha}$, from (2.44) we have

$$f(z) = \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_2} \frac{f(z)}{(z-z_0)^{(k+1)\alpha}} (dz)^{\alpha} - \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \frac{f(z)}{(z-z_0)^{(k+1)\alpha}} (dz)^{\alpha}$$

Successively, it follows from the above that

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^{k\alpha} , \tag{4.23}$$

where

$$a_k = \frac{1}{(2\pi)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^{(k+1)\alpha}} (dz)^{\alpha} , \tag{4.24}$$

for $C: r^{\alpha} \le |z-z_0|^{\alpha} \le R^{\alpha}$.

Theorem 11

If $f(z)$ has local fractional analytic at every other point inside a contour C with center at z_0, then $f(z)$ has a local fractional Laurent series about $z = z_0$ so that

$$f(z) = \sum_{i=-\infty}^{\infty} a_k (z-z_0)^{k\alpha} , 0 < \alpha \le 1, \tag{4.25}$$

where for $C: r^{\alpha} < |z-z_0|^{\alpha} < R^{\alpha}$ we have

$$a_k = \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_C \frac{f(z)}{(z-z_0)^{(k+1)\alpha}} (dz)^\alpha . \tag{4.26}$$

Proof. Setting $C_1 : |z-z_0|^\alpha = r^\alpha$ and $C_2 : |z-z_0|^\alpha = R^\alpha$, from (2.44) we have that

$$f(z) = \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_2} \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^\alpha - \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^\alpha . \tag{4.27}$$

Taking the right side of (4.27) into account implies that for $\xi \in C_2$

$$\left| \frac{(\xi-z_0)^\alpha}{(z-z_0)^\alpha} \right| = \frac{|\xi-z_0|^\alpha}{R^\alpha} = q^\alpha < 1 \tag{4.28}$$

and

$$|f(\xi)| \le M . \tag{4.29}$$

By using (4.29) it follows from (4.27) that

$$\begin{aligned} &\frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_2} \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^\alpha \\ &= \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{\infty} \left[\oint_{C_2} \frac{f(\xi)}{(\xi-z_0)^{(n+1)\alpha}} (d\xi)^\alpha \right] (z-z_0)^{n\alpha}. \end{aligned} \tag{4.30}$$

From (4.27) we get

$$\begin{aligned} &-\frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \oint_{C_1} \frac{f(\xi)}{(\xi-z_0)^\alpha} (d\xi)^\alpha \\ &= \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=0}^{N-1} \left[\oint_{C_1} \frac{f(\xi)}{(\xi-z_0)^{(-n+1)\alpha}} (d\xi)^\alpha \right] (z-z_0)^{-n\alpha} + R_N \end{aligned} \tag{4.31}$$

where

$$\lim_{N\to\infty} R_N = \lim_{N\to\infty} \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=N}^{\infty} \left[\oint_{C_1} \frac{f(\xi)}{(\xi-z_0)^{(-n+1)\alpha}} (d\xi)^\alpha \right] (z-z_0)^{-n\alpha}$$

is reminder.

Since $|f(\xi)| \le M_1$, taking $\left| \frac{\xi-z_0}{z-z_0} \right|^{n\alpha} = q^{n\alpha} < 1$, we have

$$\begin{aligned} &|R_N| \\ &= \left| \frac{1}{(2\pi)^\alpha i^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=N}^{\infty} \left[\oint_{C_1} \frac{f(\xi)}{(\xi-z_0)^{(-n+1)\alpha}} (d\xi)^\alpha \right] (z-z_0)^{-n\alpha} \right| \\ &\le \frac{1}{(2\pi)^\alpha} \cdot \frac{1}{\Gamma(1+\alpha)} \sum_{n=N}^{\infty} \left[\oint_{C_1} \frac{|f(\xi)|}{|(\xi-z_0)^\alpha|} \left| \frac{\xi-z_0}{z-z_0} \right|^{n\alpha} (d\xi)^\alpha \right] \end{aligned}$$

$$\le \frac{1}{(2\pi)^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\sum_{n=N}^{\infty}\left[\oint_{C_1}\frac{|M_1|}{\left|(\xi-z_0)^\alpha\right|}\left|\frac{\xi-z_0}{z-z_0}\right|^{n\alpha}(d\xi)^\alpha\right]$$

$$\le \frac{1}{(2\pi)^\alpha}\cdot\frac{|M_1|}{\Gamma(1+\alpha)}\sum_{n=N}^{\infty}\left[\oint_{C_1}\frac{1}{\left|(\xi-z_0)^\alpha\right|}\left|\frac{\xi-z_0}{z-z_0}\right|^{n\alpha}(d\xi)^\alpha\right]$$

$$\le \frac{1}{(2\pi)^\alpha}\cdot\frac{|M_1|}{\Gamma(1+\alpha)}\sum_{n=N}^{\infty}\left[\oint_{C_1}\frac{1}{\left|(\xi-z_0)^\alpha\right|}q^{n\alpha}(d\xi)^\alpha\right]$$

$$\le \frac{1}{(2\pi)^\alpha}\cdot\frac{|M_1|}{\Gamma(1+\alpha)}\sum_{n=N}^{\infty}\left[(2\pi)^\alpha q^{n\alpha}\right] \tag{4.32}$$

$$\le \frac{|M_1|}{\Gamma(1+\alpha)}\frac{q^{n\alpha}}{1-q^\alpha}.$$

Furthermore

$$\lim_{N\to\infty} R_N = 0 .$$

Hence

$$\begin{aligned}&-\frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_{C_1}\frac{f(\xi)}{(\xi-z_0)^\alpha}(d\xi)^\alpha\\ &=\frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\sum_{n=0}^{\infty}\left[\oint_{C_1}\frac{f(\xi)}{(\xi-z_0)^{(-n+1)\alpha}}(d\xi)^\alpha\right](z-z_0)^{-n\alpha}.\end{aligned} \tag{4.33}$$

Combing the formulas (4.30) and (4.33), we have the result.

Hence, the proof of the theorem is finished.

5 Generalized residue theorems

In this section we start with a local fractional Laurent series and study generalized residue theorems.

Definition 8

Suppose that z_0 is an isolated singular point of $f(z)$. Then there is a local fractional Laurent series

$$f(z)=\sum_{i=-\infty}^{\infty}a_k(z-z_0)^{k\alpha} \tag{5.1}$$

valid for $|z-z_0|^\alpha \le R^\alpha$. The coefficient a_{-1} of $(z-z_0)^{-\alpha}$ is called the generalized residue of $f(z)$ at $z=z_0$, and is frequently written as

$$\operatorname*{Res}_{z=z_0} f(z). \tag{5.2}$$

One of the coefficients for the Yang-Taylor series corresponding to

$$\phi(z)=(z-z_0)^{n\alpha} f(z), \tag{5.3}$$

the coefficient a_{-1} is the residue of $f(z)$ at the pole $z=z_0$. It can be found from the formula

$$\operatorname*{Res}_{z=z_0} f(z)=a_{-1}=\lim_{z\to z_0}\frac{1}{\Gamma(1+n\alpha)}\frac{d^{(n-1)\alpha}}{dz^{(n-1)\alpha}}\left\{(z-z_0)^{n\alpha} f(z)\right\} \tag{5.4}$$

where n is the order of the pole.

Setting $f(z)=\sum_{i=-\infty}^{\infty}a_k(z-z_0)^{k\alpha}$, the expression (5.3) yields

$$\begin{aligned}&\phi(z)\\&=(z-z_0)^{n\alpha}\sum_{i=-\infty}^{\infty}a_k(z-z_0)^{k\alpha}\\&=a_{-n}+a_{-n+1}(z-z_0)^{\alpha}+a_{-1}(z-z_0)^{(n-1)\alpha}+....\end{aligned} \tag{5.5}$$

We know that this is

$$a_{-1}=\frac{\phi^{(n-1)\alpha}(z_0)}{\Gamma(1+n\alpha)}, \tag{5.6}$$

which is the coefficient of $(z-z_0)^{(n-1)\alpha}$.

The generalized residue is thus

$$\operatorname*{Res}_{z=z_0} f(z)=a_{-1}=\frac{\phi^{(n-1)\alpha}(z_0)}{\Gamma(1+n\alpha)}, \tag{5.7}$$

where $\phi(z)=(z-z_0)^{n\alpha} f(z)$.

Corollary 12

If $f(z)$ is local fractional analytic within and on the boundary C of a region $\Re$ except at a number of poles a within $\Re$, having a residue a_{-1}, then

$$\frac{1}{(2\pi)^{\alpha} i^{\alpha}\Gamma(1+\alpha)}\oint_C f(z)(dz)^{\alpha}=\operatorname*{Res}_{z=z_0} f(z). \tag{5.8}$$

Proof. Taking into account the definitions of local fractional analytic function and the pole we have local fractional Laurent's series

$$f(z)=\sum_{i=-\infty}^{\infty}a_k(z-z_0)^{k\alpha} \tag{5.9}$$

and therefore

$$f(z)=\cdots+a_{-n}(z-z_0)^{-n\alpha}+\cdots+a_{-1}(z-z_0)^{-\alpha}+a_0+\cdots+a_n(z-z_0)^{n\alpha}+\cdots. \tag{5.10}$$

Hence we have the following relation

$$\frac{1}{\Gamma(1+\alpha)}\oint_C f(z)(dz)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\oint_C\left(\sum_{i=-\infty}^{\infty}a_k(z-z_0)^{k\alpha}\right)(dz)^{\alpha} . \tag{5.11}$$

furthermore

$$\frac{1}{\Gamma(1+\alpha)}\oint_C f(z)(dz)^\alpha = \frac{1}{\Gamma(1+\alpha)}\oint_C \frac{a_{-1}}{(z-z_0)^\alpha}(dz)^\alpha . \tag{5.12}$$

From (3.9), it is shown that

$$\frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C f(z)(dz)^\alpha = \frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{a_{-1}}{(z-z_0)^\alpha}(dz)^\alpha = a_{-1}. \tag{5.13}$$

Hence we have the formula

$$\frac{1}{\Gamma(1+\alpha)}\oint_C f(z)(dz)^\alpha = (2\pi)^\alpha i^\alpha a_{-1}. \tag{5.14}$$

Taking into account the definition of generalized residue, we have the result.

This proof of the theorem is completed.

From (5.8) , we deduce the following corollary:

Corollary 13

If $f(z)$ is local fractional analytic within and on the boundary C of a region $\Re^\alpha$ except at a finite number of poles $z_0, z_1, z_2 ...$ within $\Re^\alpha$, having residues $a_{-1}, b_{-1}, c_{-1} ...$ respectively, then

$$\frac{1}{(2\pi)^\alpha i^\alpha \Gamma(1+\alpha)}\oint_C f(z)(dz)^\alpha = \sum_{i=0}^{n} \underset{z=z_k}{\mathrm{Re}\,s}\, f(z) = a_{-1}+b_{-1}+c_{-1}+\ldots . \tag{5.15}$$

It says that the local fractional integral of $f(z)$ is simply $(2\pi)^\alpha i^\alpha$ times the sum of the residues at the singular points enclosed by the contour C .

6 Applications: Gauss formula of complex function

Theorem 14

Suppose that $f(z)$ is local fractional analytic and ω is any point, then for the circle

$$|z-\omega|^\alpha = \left|R^\alpha E_\alpha\left(i^\alpha \theta^\alpha\right)\right|$$

we have

$$f(\omega) = \frac{1}{(2\pi)^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\int_0^{2\pi} f\left(\omega + RE(i\theta)\right)(d\theta)^\alpha . \tag{6.1}$$

Proof. By using (3.1) there exists a simple closed contour C and z_0 is any point interior to C such that

$$f(\omega) = \frac{1}{(2\pi)^\alpha i^\alpha}\cdot\frac{1}{\Gamma(1+\alpha)}\oint_C \frac{f(z)}{(z-\omega)^\alpha}(dz)^\alpha . \tag{6.2}$$

When C can been taken to be $\omega^\alpha + R^\alpha E_\alpha\left(i^\alpha\theta^\alpha\right)$ for $\theta\in[0,2\pi]$, substituting the relations

$$(z-\omega)^\alpha = R^\alpha E_\alpha\left(i^\alpha\theta^\alpha\right) \tag{6.3}$$

and

$$\left(dz\right)^{\alpha} = i^{\alpha} R^{\alpha} E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right)\left(d\theta\right)^{\alpha} \quad , \tag{6.4}$$

in (6.2) implies that

$$f\left(\omega\right) = \frac{1}{\left(2\pi\right)^{\alpha} i^{\alpha}} \cdot \frac{1}{\Gamma\left(1+\alpha\right)} \oint_{C} \frac{f\left(\omega + RE\left(i\theta\right)\right) i^{\alpha} R^{\alpha} E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right)\left(d\theta\right)^{\alpha}}{R^{\alpha} E_{\alpha}\left(i^{\alpha}\theta^{\alpha}\right)} \tag{6.5}$$

and some cancelling gives the result.

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