TOWARDS THERMODYNAMICS OF THE QUARK QUASI-PARTICLES

S. V. Molodtsov[1,2a], G. M. Zinovjev[3b]

[1]Joint Institute for Nuclear Research, Dubna, Moscow region, RUSSIA
[2]Institute of Theoretical and Experimental Physics, Moscow, RUSSIA
[3]Bogolyubov Institute for Theoretical Physics, National Academy of Sciences of Ukraine, Kiev, UKRAINE

Some features of hot and dense gas of quarks which are considered as the quasi-particles of the model Hamiltonian with four-fermion interaction are studied. Being adapted to the Nambu-Jona-Lasinio model this approach allows us to accommodate a phase transition similar to the nuclear liquid-gas one at the proper scale and to argue an existence of the mixed phase of vacuum and normal baryonic matter as a plausible scenario of chiral symmetry (partial) restoration. Analyzing the transition layer between two phases we estimate the surface tension coefficient and discuss the possibility of quark droplet formation.

Understanding in full and describing dependably the critical phenomena (chiral and deconfinement phase transitions) in QCD is still elusive because of a necessity to have the corresponding efficient non-perturbative methods for strongly coupled regime. For the time being such studies are pursued by invoking diverse effective models. The Nambu-Jona-Lasinio(NJL)-type models are certainly playing the most advanced role in this analysis [1]. This approach deals with the multi-fermion interactions in lieu of a gluon field QCD dynamics and does not incorporate the property of confinement. At the same time it is quite successful in realizing the spontaneous breakdown of chiral symmetry and its restoration at nonzero temperatures or quark densities.

These and some related items are discussed in this paper inspired by well known and fruitful idea about the specific role of surface degrees of freedom in the finite fermi-liquid systems and in considerable extent by our previous works [2] and [3] in which the quarks were treated as the quasi-particles of the model Hamiltonian and the problem of filling up the Fermi sphere was studied in detail. Under such a treatment an unexpected singularity (discontinuity) of the mean energy functional as a function of the current quark mass was found. In the particular case of the NJL model the existence of new solution branches of the equation for dynamical quark mass as a function of chemical potential have been demonstrated and the appearance of state filled up with quarks which is almost degenerate with the vacuum state both in the quasi-particle chemical potential and in the ensemble pressure has been discovered.

Here we are going to study the quark ensemble features at finite temperature and fixed baryonic chemical potential and to analyse the first order phase transition which takes place in such a system of free quasi-particles. Analysis is performed within the framework of two approaches which are supplementary, in a sense, albeit giving the identical results. One of those approaches, based on the Bogolyubov transformations, is especially informative to study the process of filling the Fermi sphere up because at this point the density of quark ensemble develops a continuous dependence on the Fermi momentum. It allows us to reveal an additional structure in the solution of gap equation for dynamical quark mass just in the proper interval of parameters characteristic for phase transition and to trace its evolution. The result is that a quark ensemble might be found in two aggregate states, gas and liquid, and the chiral condensate is partially restored in a liquid phase. In order to make these conclusions easily perceptible we deal with the simplest version of the NJL model (with one flavor and one of the standard parameter sets) and, actually, do not aim to adjust the result obtained with well-known nuclear liquid-gas phase transition. Besides, it seems our approach might be treated as a sort of microscopic ground of the conventional bag model and those states filled up with quarks are conceivable as a natural 'construction material' for baryons.

Now as an input to start with we remind the key elements of approach developed. The corresponding Hamiltonian includes the interaction term taken in the form of a product of two coloured currents located in the spatial points **x** and **y** which are connected by a form-factor and its density reads as

$$\mathcal{H} = -\bar{q}(i\gamma\nabla + im)q - \bar{q}t^a\gamma_\mu q \int d\mathbf{y}\bar{q}'t^b\gamma_\nu q'\langle A_\mu^a A_\nu'^b\rangle, \tag{1}$$

e-mail: [a]molodtsov@itep.ru, [b]Gennady.Zinovjev@cern.ch

where $q = q(\mathbf{x})$, $\bar{q} = \bar{q}(\mathbf{x})$, $q' = q(\mathbf{y})$, $\bar{q}' = \bar{q}(\mathbf{y})$ are the quark and anti-quark operators,

$$q_{\alpha i}(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} \left[a(\mathbf{p}, s, c)u_{\alpha i}(\mathbf{p}, s, c)e^{i\mathbf{p}\mathbf{x}} + b^+(\mathbf{p}, s, c)v_{\alpha i}(\mathbf{p}, s, c)e^{-i\mathbf{p}\mathbf{x}} \right], \tag{2}$$

$p_4^2 = -\mathbf{p}^2 - m^2$, i–is the colour index, α is the spinor index in the coordinate space, a^+, a and b^+, b are the creation and annihilation operators of quarks and anti-quarks, $a|0\rangle = 0$, $b|0\rangle = 0$, $|0\rangle$ is the vacuum state of free Hamiltonian and m is a current quark mass. The summation over indices s and c is meant everywhere, the index s describes two spin polarizations of quark and the index c plays the similar role for a colour. $t^a = \lambda^a/2$ are the generators of $SU(N_c)$ colour gauge group, the Hamiltonian density is considered in the Euclidean space and γ_μ denote the Hermitian Dirac matrices, $\mu, \nu = 1, 2, 3, 4$. $\langle A_\mu^a A_\nu'^b \rangle$ stands for the form-factor of the following form $\langle A_\mu^a A_\nu'^b \rangle = \delta^{ab} \frac{2 \widetilde{G}}{N_c^2 - 1} [I(\mathbf{x} - \mathbf{y})\delta_{\mu\nu} - J_{\mu\nu}(\mathbf{x} - \mathbf{y})]$, where the second term is spanned by the relative distance vector and the gluon field primed denotes that in the spatial point $\mathbf{y}$. The effective Hamiltonian density (1) results from averaging the ensemble of quarks influenced by intensive stochastic gluon field A_μ^a, see Ref. [2]. For the sake of simplicity in what follows we neglect the contribution of the second term of Eq.(1). The ground state of the system is searched as the Bogolyubov trial function composed of the quark-anti-quark pairs with opposite momenta and with vacuum quantum numbers, i.e.

$$|\sigma\rangle = \mathcal{T} |0\rangle , \quad \mathcal{T} = \Pi_{p,s} \exp\{\varphi \left[a^+(\mathbf{p}, s)b^+(-\mathbf{p}, s) + a(\mathbf{p}, s)b(-\mathbf{p}, s) \right]\}. \tag{3}$$

In this formula and below, in order to simplify the notations we refer to one compound index only which means both the spin and colour polarizations. The parameter $\varphi(\mathbf{p})$ which describes the pairing strength is determined by the minimum of mean energy $E = \langle\sigma|H|\sigma\rangle$. By introducing the 'dressing transformation' we define the creation and annihilation operators of quasi-particles as $A = \mathcal{T}a\mathcal{T}^{-1}$, $B^+ = \mathcal{T}b^+\mathcal{T}^{-1}$ and for fermions $\mathcal{T}^{-1} = \mathcal{T}^\dagger$. Then the quark field operators are presented as

$$q(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} \left[A(\mathbf{p}, s) U(\mathbf{p}, s) e^{i\mathbf{p}\mathbf{x}} + B^+(\mathbf{p}, s) V(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}} \right] ,$$

$$\bar{q}(\mathbf{x}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{(2|p_4|)^{1/2}} \left[A^+(\mathbf{p}, s) \overline{U}(\mathbf{p}, s) e^{-i\mathbf{p}\mathbf{x}} + B(\mathbf{p}, s) \overline{V}(\mathbf{p}, s) e^{i\mathbf{p}\mathbf{x}} \right] ,$$

and the transformed spinors U and V are given by the following forms $U(\mathbf{p}, s) = \cos(\varphi)u(\mathbf{p}, s) - \sin(\varphi)v(-\mathbf{p}, s)$, $V(\mathbf{p}, s) = \sin(\varphi)u(-\mathbf{p}, s) + \cos(\varphi)v(\mathbf{p}, s)$ where $\overline{U}(\mathbf{p}, s) = U^+(\mathbf{p}, s)\gamma_4$, $\overline{V}(\mathbf{p}, s) = V^+(\mathbf{p}, s)\gamma_4$ are the Dirac conjugated spinors.

In Ref. [3] the process of filling in the Fermi sphere with the quasi-particles of quarks was studied by constructing the state of the Sletter determinant type $|N\rangle = \prod_{|\mathbf{P}|<P_F;S} A^+(\mathbf{P}; S)|\sigma\rangle$ which possesses the minimal mean energy over the state $|N\rangle$. The polarization indices run through all permissible values here and the quark momenta are bounded by the limiting Fermi momentum P_F. The momenta and polarizations of states forming the quasi-particle gas are marked by the capital letters similar to above formula and the small letters are used in all other cases.

As it is known the ensemble state at finite temperature T is described by the equilibrium statistical operator ρ. Here we use the Bogolyubov-Hartree-Fock approximation in which the corresponding statistical operator is presented by the following form

$$\rho = \frac{e^{-\beta \hat{H}_{\text{app}}}}{Z_0} , \quad Z_0 = \text{Tr} \{e^{-\beta \hat{H}_{\text{app}}}\} , \tag{4}$$

where an approximating effective Hamiltonian H_{app} is quadratic in the creation and annihilation operators of quark and anti-quark quasi-particles A^+, A, B^+, B and is defined in the corresponding Fock space with the vacuum state $|\sigma\rangle$ and $\beta = T^{-1}$. There is no need to know the exact form of this operator henceforth because all the quantities of our interest in the Bogolyubov-Hartree-Fock approximation are expressed by the corresponding averages (a density matrix) $n(P) = \text{Tr}\{\rho A^+(\mathbf{P}; S)A(\mathbf{P}; S)\}$, $\bar{n}(Q) = \text{Tr}\{\rho B^+(\mathbf{Q}; T)B(\mathbf{Q}; T)\}$, which are found by solving the following variational problem. The statistical operator ρ is determined in such a form in order to have at the fixed mean charge

$$\bar{Q}_4 = \text{Tr}\{\rho Q_4\} = V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} [n(p) - \bar{n}(p)] , \tag{5}$$

where $Q_4 = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{-ip_4}{|p_4|} [A^+(p)A(p) + B(p)B^+(p)]$ (for the diagonal component of our interest here, $Q_4 = -\int d\mathbf{x}\bar{q}i\gamma_4 q$) and fixed mean entropy

$$\bar{S} = -\text{Tr}\{\rho \ln \rho\} =$$
$$= -V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} [n(p) \ln n(p) + (1 - n(p)) \ln(1 - n(p)) + \bar{n}(p) \ln \bar{n}(p) + (1 - \bar{n}(p)) \ln(1 - \bar{n}(p))] , \tag{6}$$

($S = -\ln\rho$) the minimal value of mean energy of quark ensemble $E = \text{Tr}\{\rho H\}$. The definition of mean charge is given here up to the unessential (infinite) constant coming from permuting the operators BB^+ in the charge operator Q_4. It is reasonable to remind that the mean charge should be treated in some statistical sense because it characterizes quark ensemble density and has no colour indices.

Calculating the corresponding matrix elements leads to the following result for mean energy density per one quark degree of freedom (the details can be found in [4]) $w = \mathcal{E}/2N_c$, $\mathcal{E} = E/V$ where E is a total ensemble energy

$$
\begin{aligned}
w = {} & \int \frac{d\mathbf{p}}{(2\pi)^3}|p_4| + \int \frac{d\mathbf{p}}{(2\pi)^3}|p_4|\cos\theta[n(p) + \bar{n}(p) - 1] - \\
& - G\int \frac{d\mathbf{p}}{(2\pi)^3}\sin\left(\theta - \theta_m\right)[n(p) + \bar{n}(p) - 1]\int \frac{d\mathbf{q}}{(2\pi)^3}\sin\left(\theta' - \theta'_m\right)[n(q) + \bar{n}(q) - 1]\,I\,.
\end{aligned}
\tag{7}
$$

(up to the constant unessential for our consideration here). Here the following denotes are used $p = |\mathbf{p}|$, $q = |\mathbf{q}|$, $\theta = 2\varphi$, $\theta' = \theta(q)$, $I = I(\mathbf{p} + \mathbf{q})$ and the angle $\theta_m(p)$ is determined by $\sin\theta_m = m/|p_4|$. It was quite practical to single out the colour factor in the four-fermion coupling constant as $G = 2\widetilde{G}/N_c$. It is of important to notice that the existence of such an angle stipulates the discontinuity of mean energy functional mentioned above and found out in [2]

We are interested in minimizing the following functional $\Omega = E - \mu\,\bar{Q}_4 - T\,\bar{S}$ where μ and T are the Lagrange factors for the chemical potential and temperature respectively. The approximating Hamiltonian $\hat{H}_{\text{app}}$ is constructed simply by using the information on $E - \mu\,\bar{Q}_4$ of presented functional (see, also below). For the specific contribution per one quark degree of freedom $f = F/2N_c$, $F = \Omega/V$ we receive

$$
\begin{aligned}
f = {} & \int \frac{d\mathbf{p}}{(2\pi)^3}\left[|p_4|\cos\theta(n + \bar{n} - 1) - \mu(n - \bar{n})\right] + \int \frac{d\mathbf{p}}{(2\pi)^3}|p_4| - G\int \frac{d\mathbf{p}}{(2\pi)^3}\sin\left(\theta - \theta_m\right)(n + \bar{n} - 1)\times \\
& \times\int \frac{d\mathbf{q}}{(2\pi)^3}\sin\left(\theta' - \theta'_m\right)(n' + \bar{n}' - 1)I + T\int \frac{d\mathbf{p}}{(2\pi)^3}[n\ln n + (1 - n)\ln(1 - n) + \bar{n}\ln\bar{n} + (1 - \bar{n})\ln(1 - \bar{n})]\,.
\end{aligned}
\tag{8}
$$

Here the primed variables correspond to the momentum q. The optimal values of parameters are determined by solving the following system of equations ($df/d\theta = 0$, $df/dn = 0$, $df/d\bar{n} = 0$)

$$
\begin{aligned}
& |p_4|\,\sin\theta - M\cos\left(\theta - \theta_m\right) = 0\,, \\
& |p_4|\,\cos\theta - \mu + M\,\sin\left(\theta - \theta_m\right) - T\,\ln\left(n^{-1} - 1\right) = 0\,, \\
& |p_4|\,\cos\theta + \mu + M\,\sin\left(\theta - \theta_m\right) - T\,\ln\left(\bar{n}^{-1} - 1\right) = 0
\end{aligned}
\tag{9}
$$

where we denoted the induced quark mass as

$$
M(\mathbf{p}) = 2G\int \frac{d\mathbf{q}}{(2\pi)^3}(1 - n' - \bar{n}')\sin\left(\theta' - \theta'_m\right)I(\mathbf{p} + \mathbf{q})\,.
\tag{10}
$$

Turning to the presentation of obtained results in the form customary for mean field approximation we introduce a dynamical quark mass M_q parameterized as $\sin\left(\theta - \theta_m\right) = \frac{M_q}{|P_4|}$, $|P_4| = (\mathbf{p}^2 + M_q^2(\mathbf{p}))^{1/2}$ and ascertain the interrelation between induced and dynamical quark masses. From the first equation of system (9) we fix the pairing angle as $\sin\theta = pM/(|p_4||P_4|)$ and making use of the identity

$$
(|p_4|^2 - M\,m)^2 + M^2 p^2 = [p^2 + (M - m)^2]\,|p_4|^2
$$

find out that $\cos\theta = \pm\frac{|p_4|^2 - m\,M}{|p_4||P_4|}$. For clarity we choose the upper sign 'plus'. Then, as an analysis of the NJL model teaches, the branch of equation solution for negative dynamical quark mass is the most stable one. Let us remember here that we are dealing with the Euclidean metrics (though it is not a principal point) and a quark mass appears in the corresponding expressions as an imaginary quantity. Now substituting the calculated expressions for the pairing angle into the trigonometrical factor expression $\sin\left(\theta - \theta_m\right) = \sin\theta\frac{p}{|p_4|} - \cos\theta\frac{m}{|p_4|}$ and performing some algebraic transformations of both parts of equation we define $M_q(\mathbf{p}) = M(\mathbf{p}) - m$. And then the equation for dynamical quark mass (10) is getting the form characteristic for the mean field approximation

$$
M = 2G\int \frac{d\mathbf{q}}{(2\pi)^3}(1 - n' - \bar{n}')\frac{M'_q}{|P'_4|}\,I(\mathbf{p} + \mathbf{q})\,.
$$

The second and third equations of system (9) allow us to find for the equilibrium densities of quarks and anti-quarks as $n = [e^{\beta\,(|P_4| - \mu)} + 1]^{-1}$, $\bar{n} = [e^{\beta\,(|P_4| + \mu)} + 1]^{-1}$ and, hence, the thermodynamic properties of our system as well and, in particular, the pressure of quark ensemble $P = -dE/dV$. By definition we should
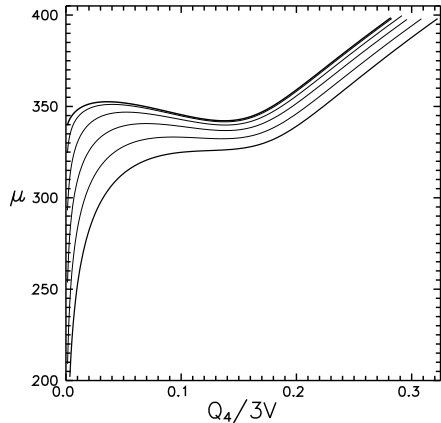
Figure 1. The chemical potential μ (MeV) as a function of charge density $\mathcal{Q}_4 = Q_4/(3V)$ (in the units of charge/fm^3). The factor 3 relates the densities of quark and baryon matter. The top curve corresponds to the zero temperature. The curves following down correspond to the temperature values $T = 10$ MeV, ... , $T = 50$ MeV with spacing $T = 10$ MeV.
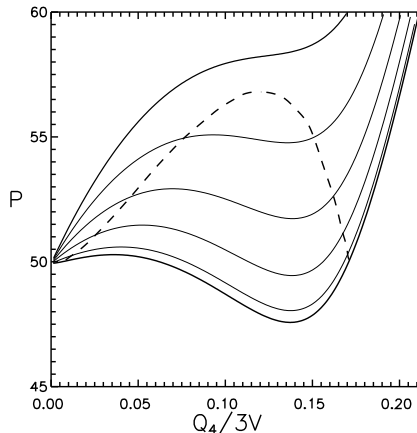
Figure 2. The ensemble pressure P (MeV/fm^3) as a function of charge density $\mathcal{Q}_4$ at temperatures $T = 0$ MeV, ... , $T = 50$ MeV with spacing $T = 10$ MeV. The lowest curve corresponds to zero temperature. The dashed curve shows the boundary of phase transition liquid–gas, see the text.

calculate this derivative at constant mean entropy $d\bar{S}/dV = 0$. This condition allows us, for example, to calculate the derivative $d\mu/dV$. However, this way is not reliable because then the mean charge $\bar{Q}_4$ might change, and it is more practical to introduce two independent chemical potentials — for quarks μ and for anti-quarks $\bar{\mu}$ (following formula for $\bar{n}$ with an opposite sign). In fact, it is the only possibility to obey both conditions simultaneously. It leads to the following definitions of corresponding densities $n = [e^{\beta \, (|P_4|-\mu)} + 1]^{-1}$, $\bar{n} = [e^{\beta \, (|P_4|+\bar{\mu})} + 1]^{-1}$. On this way of description we are able even to treat some non-equilibrium states of quark ensemble (albeit with losing a covariance similar to the situation which takes place in electrodynamics while one deals with electron-positron gas). But here we are interested in the particular case of $\bar{\mu} = \mu$. Then the corresponding derivative of specific energy dw/dV might be presented as

$$\frac{dw}{dV} = \int \frac{d\mathbf{p}}{(2\pi)^3} \left(\frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) \left[|p_4| \cos\theta - 2G \sin(\theta - \theta_m) \int \frac{d\mathbf{q}}{(2\pi)^3} \sin(\theta' - \theta'_m)(n' + \bar{n}' - 1)I \right]. \quad (11)$$

Now expressing the trigonometric factors via dynamical quark mass and exploiting Eq.(10) we obtain the ensemble pressure as $P = -\frac{E}{V} - V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} \left(\frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) |P_4|$. The requirement for mean charge conservation $\frac{d\bar{Q}_4}{dV} = \frac{\bar{Q}_4}{V} + V 2N_c \int \frac{d\mathbf{p}}{(2\pi)^3} \left(\frac{dn}{d\mu} \frac{d\mu}{dV} - \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) = 0$ provides us with an equation which interrelates the derivatives $d\mu/dV$, $d\bar{\mu}/dV$. Apparently, the regularized expressions for mean charge of quarks and anti-quarks are meant here. Dealing in a similar way with the requirement of mean entropy conservation, $d\bar{S}/dV = 0$, we receive another equation as $\frac{\bar{S}}{2N_c \, V^2} = - \int \frac{d\mathbf{p}}{(2\pi)^3} \ln(n^{-1} - 1) \frac{dn}{d\mu} \frac{d\mu}{dV} + \int \frac{d\mathbf{p}}{(2\pi)^3} \ln(\bar{n}^{-1} - 1) \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV}$. Substituting here $T \ln(n^{-1} - 1) = -\mu + |P_4|$ and $T \ln(\bar{n}^{-1} - 1) = \bar{\mu} + |P_4|$ after simple calculations (keeping in mind that $\bar{\mu} = \mu$ and the charge conservation) we have that $\int \frac{d\mathbf{p}}{(2\pi)^3} \left(\frac{dn}{d\mu} \frac{d\mu}{dV} + \frac{d\bar{n}}{d\bar{\mu}} \frac{d\bar{\mu}}{dV} \right) |P_4| = -\frac{\bar{S}T}{2N_c V^2} - \frac{\bar{Q}_4 \mu}{2N_c V^2}$. Finally it leads for the pressure to the following expression $P = -\frac{E}{V} + \frac{\bar{S}\,T}{V} + \frac{\bar{Q}_4 \, \mu}{V}$ (of course, the thermodynamic potential is $\Omega = -PV$). At small temperatures the anti-quark contribution is negligible and thermodynamic description can be grounded on utilizing one chemical potential μ only. If the anti-quark contribution is getting intrinsic the thermodynamic picture becomes complicated due to the presence of chemical potential $\bar{\mu}$ with the condition $\bar{\mu} = \mu$ imposed. In particular, at zero temperature the anti-quark contribution is absent and we might receive $P = -\mathcal{E} + \mu \, \rho_q$ where $\mu = [P_F^2 + M_q^2(P_F)]^{1/2}$, P_F is the Fermi momentum and $\rho_q = N/V$ is the quark ensemble density.

For clarity, we consider the NJL model in this paper, i.e. the correlation function behaves as the δ-function in coordinate space. It is a well known fact that in order to have an intelligent result in this model one needs to use a regularization cutting of momentum integration in Eq. (8). We adjust the standard set of parameters [5] here with $|\mathbf{p}| < \Lambda$, $\Lambda = 631$ MeV, $m = 5.5$ MeV and $G\Lambda^2/(2\pi^2) = 1.3$. This set of parameters at $n = 0$, $\bar{n} = 0$,
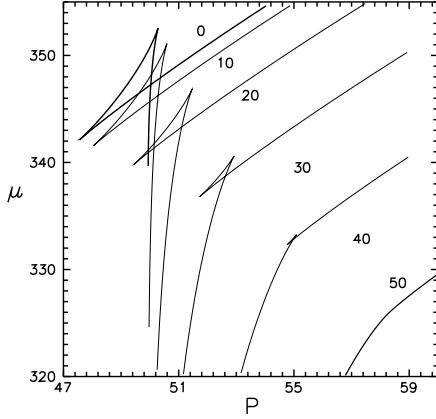
Figure 3. The fragments of isotherms in Fig. 1, see text. Chemical potential μ (MeV) as a function of pressure P (MeV/fm³). The top curve corresponds to the zero isotherm and following down with spacing 10 MeV till the isotherm 50 MeV (the lowest curve).
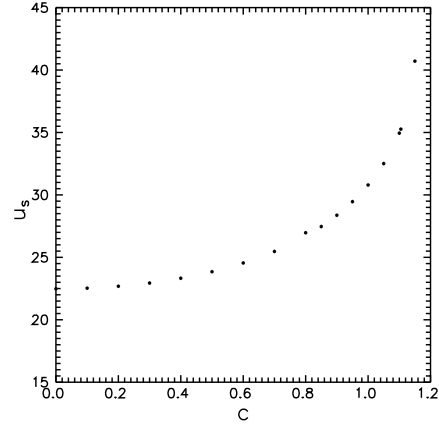
Figure 4. The surface tension coefficient u_s in MeV as a function of parameter c ($\zeta = c\,\eta$) for the curve of stable kinks (with $\eta \leq 1.2$).

$T = 0$ gives for the dynamical quark mass $M_q = 335$ MeV. Besides, it may be shown that the following form of ensemble energy is valid at the extremals of functional (8)

$$E = E_{vac} + 2N_c V \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3}|P_4|(n+\bar{n}) \ , \quad E_{vac} = 2N_c V \int^{\Lambda} \frac{d\mathbf{p}}{(2\pi)^3}(|p_4|-|P_4|) + 2N_c V \frac{M^2}{4G} \ .$$

It is easy to understand that this expression with the vacuum contribution subtracted looks like an energy of a gas of relativistic particles and anti-particles with the mass M_q and coincides identically with that calculated in the mean field approximation.

Thus, we determine the density of quark n and anti-quark $\bar{n}$ quasi- particles at given parameters μ and T from the second and third equations of system (9). From the first equation we receive the angle of quark and anti-quark pairing θ as a function of dynamical quark mass M_q which is handled as a parameter. The evolution of chemical potential as a function of charge density $\mathcal{Q}_4 = Q_4/(3V)$ (in the units of charge/fm^3) with the temperature increasing is depicted in Fig. 1 (factor 3 connects the quark and baryon matter densities). The top curve corresponds to the zero temperature. The other curves following down have been calculated for the temperatures $T = 10$ MeV, ... , $T = 50$ MeV with spacing $T = 10$ MeV. As it was found in Ref. [3] the chemical potential at zero temperature increases first with the charge density increasing, reaches its maximal value, then decreases and at the densities of order of normal nuclear matter density[1], $\rho_q \sim 0.16/fm^3$, becomes almost equal to its vacuum value. Such a behaviour of chemical potential results from the fast decrease of dynamical quark mass with the Fermi momentum increasing. It is clear from Fig. 1 that the charge density is still a multivalued function of chemical potential at the temperature slightly below 50 MeV. Fig. 2 shows the ensemble pressure P (MeV/fm³) as the function of charge density $\mathcal{Q}_4$ at several temperatures. The lowest curve corresponds to the zero temperature. The other curves following up correspond to the temperatures $T = 10$ MeV, ... , $T = 50$ MeV (the top curve) with spacing $T = 10$ MeV. It is interesting to remember now that in Ref. [3] the vacuum pressure estimate for the NJL model was received as 40—50 MeV/fm³ which is entirely compatible with the results of the conventional bag model. Besides, some hints at instability presence (rooted in the anomalous behavior of pressure $dP/dn < 0$) in an interval of Fermi momenta has been found. Fig. 3 shows the fragments of isotherms of Fig. 1, 2 but in the different coordinates (chemical potential — ensemble pressure). The top curve is calculated at the zero temperature, the other isotherms following down correspond to the temperatures increasing with spacing 10 MeV. The lowest curve is calculated at the temperature 50 MeV. The Fig. 3 obviously demonstrates a presence of the states on isotherm which are thermodynamically equilibrated and have equal pressure and chemical potential (see the characteristic Van der Waals triangle with the crossing curves). The calculated equilibrium points are shown in Fig. 2 by the dashed curve. The intersection points of dashed curve with an isotherm are fixing the boundary of gas —liquid phase transition. The corresponding

---

[1]At the Fermi momenta of dynamical quark mass order.

straight line $P = $ const which obeys the Maxwell rule separates the non-equilibrium and unstable fragments of isotherm and describes a mixed phase and appropriate critical temperature for the parameter we are using in this paper turns out to be $T_c \sim 45.7$ MeV with the critical charge density as $\bar{Q}_4 \sim 0.12$ charge/fm^3. Usually the thermodynamic description is grounded on the mean energy functional which is the homogeneous function of particle number like $E = N \ f(S/N, V/N)$ (without vacuum contribution). It is clear that such a description requires the corresponding subtractions to be introduced, however, this operation does not change the final results considerably. It was argued in Ref. [3] that the states filled up with quarks and separated from the instability region look like 'natural construction material' to compose the baryons and to understand the existing fact of equilibrium between vacuum and octet of stable (in strong interaction) baryons[2].

Apparently, our study of the quark ensemble thermodynamics produces quite reasonable arguments to propound the hypothesis that the phase transition of chiral symmetry (partial) restoration has already been realized as the mixed phase of physical vacuum and baryonic matter[3]. However, it is clear our quantitative estimates should not be taken as ones to be compared with, for example, the critical temperature of nuclear matter which has been experimentally measured and equals to 15 – 20 MeV. Besides, the gas component (at $T = 0$) has nonzero density (as 0.01 of the normal nuclear density) but in reality this branch should correspond to the physical vacuum, i.e. zero baryonic density[4]. In principle, an idea of global equilibrium of gas and liquid phases makes it possible to formulate the adequate boundary conditions at describing the transitional layer arising between the vacuum and filled state and to calculate the surface tension effects.

The idea advanced would obtain substantial confirmation if it becomes possible to claim an evidence of existing the transition layer at which the ensemble transformation from one aggregate state to another takes place. As it was argued above the practical parameter for describing an uniform phase (at a given temperature) is the mean charge (density) of ensemble. Thus, one can reconstruct all other characteristics, for example, a chiral condensate, dynamical quark mass, etc. Analyzing the transition layer at zero temperature we assume the parameters in the gas phase are approximately the same as at zero charge $\rho_g = 0$, i.e. as in the vacuum (ignoring the negligible distinctions in the pressure, chemical potential and quark condensate). Then dynamical quark mass obtained has maximal value and for the parameter choice of the NJL model it is $M = 335$ MeV. From the Van der Waals diagram one may draw out that the second (liquid) phase being in equilibrium with the gas phase develops the density $\rho_l = 3 \times 0.185$ ch/fm^3. The detached factor 3 here relates the magnitudes of quark and baryon densities. The quark mass in this phase is approximately $\overset{*}{M} \approx 70$ MeV (we are dealing further with the simple one-dimensional picture).

Usually an adequate description of heterogeneous states can be developed basing on the mean field approximation [7], specifically for our case, by dealing with the corresponding effective quark-meson Lagrangian (a sort of the Ginzburg-Landau functional)

$$\mathcal{L} = -\bar{q} \ (\hat{\partial} + M) \ q - \frac{1}{2} \ (\partial_\mu \sigma)^2 - U(\sigma) - \frac{1}{4} \ F_{\mu\nu} F_{\mu\nu} - \frac{m_v^2}{2} \ V_\mu V_\mu - g_\sigma \ \bar{q}q \ \sigma + i g_v \ \bar{q} \ \gamma_\mu \ q \ V_\mu \ , \qquad (12)$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$, $U(\sigma) = \frac{m_\sigma^2}{2}\sigma^2 + \frac{b}{3}\sigma^3 + \frac{c}{4}\sigma^4$, σ is the scalar field, V_μ is the field of vector mesons, m_σ, m_v are the masses of scalar and vector mesons and g_σ, g_v are the coupling constants of quark-meson interaction. The $U(\sigma)$ potential includes the nonlinear terms of sigma-field interactions up to the fourth order, for example. For the sake of simplicity we do not include the contribution coming from the pseudoscalar and axial-vector mesons.

We are not going beyond well elaborated (and quite reliable) one loop approximation (12), although recently the considerable progress was reached in scrutinizing the non uniform quark condensates by utilizing the powerful methods of exact integration [8]. We believe it is more practical to adjust phenomenologically the parameters of effective Lagrangian being guided also by transparent physical picture. It is easy to see that handling one loop approximation actually we have the Walecka model [9] but applied for the quarks. In what follows we are working with the notations of that model hoping it does not lead to the misunderstandings. In the context of our deliberation Eq. (12) can be interpreted in the following way. Each phase, in some extent, might be considered as an excited state as to its relation with another phase which requires an additional (besides a charge density) set of parameters just as the meson fields for describing and those fields characterize the measure of deviation from the equilibrium state. Then the key question becomes whether it is possible to adjust the effective Lagrangian parameters of (12) in order to obtain the solutions in which the quark field interpolates between the quasi-particles in the gas (vacuum) phase and in the quasi-particles of the filled up state. The density of ensemble of the filled up states should asymptotically approach an equilibrium value of ρ_l and turn to zero value in the gas phase (vacuum).

---

[2]The chiral quark condensate for the filled up state discussed develops the quantity about $(100 \text{ MeV})^3$ (at $T = 0$), see [3], that demonstrates the obvious tendency of restoring a chiral symmetry.

[3]Indirect confirmation of this hypothesis one could see, for example, in the existing degeneracy of excited baryon states Ref. [6].

[4]Similar uncertainty is present in the other predictions of chiral symmetry restoration scenarios, for example, it stretches from 2 to 6 units of normal nuclear density.

Taking the parameterization of the potential $U(\sigma)$ as $b_\sigma = 1.5 \, m_\sigma^2 \, (g_\sigma/M)$, $c_\sigma = 0.5 \, m_\sigma^2 \, (g_\sigma/M)^2$ we come to the sigma model and the choice $b = 0$, $c = 0$ results in the Walecka model. As to the application for nuclear matter the parameters b and c demonstrate essentially the model dependent character and are different from the parameter values of the sigma model. They are phenomenologically adjusted with requiring an accurate description of the saturation property. On the contrary, for the quark Lagrangian (12) we could intuitively anticipate some resemblance with the sigma model and, hence, introduce two dimensionless parameters η, ζ as $b = \eta \, b_\sigma$, $c = \zeta^2 \, c_\sigma$ which characterize some fluctuations of the effective potential. Then the scalar field potential is presented by the following form $U(\sigma) = \frac{m_\sigma^2}{8} \frac{g_\sigma^2}{M^2} \left(4 \frac{M^2}{g_\sigma^2} + 4 \frac{M}{g_\sigma} \eta \sigma + \zeta^2 \sigma^2 \right) \sigma^2$. The meson and quark fields are defined by the following system of the stationary equations

$$\Delta \, \sigma - m_\sigma^2 \, \sigma = b \, \sigma^2 + c \, \sigma^3 + g_\sigma \, \rho_s \, ,$$
$$\Delta \, V - m_v^2 \, V = -g_v \, \rho \, , \tag{13}$$
$$(\hat{\nabla} + \overset{*}{M}) \, q = (E - g_v \, V) \, q$$

where $\overset{*}{M} = M + g_\sigma \sigma$ is the running value of dynamical quark mass, E stands for the quark energy and $V = -iV_4$. The density matrix describing the quark ensemble at $T = 0$ has the form $\xi(x) = \int^{P_F} \frac{d\mathbf{p}}{(2\pi)^3} q_\mathbf{p}(x) \bar{q}_\mathbf{p}(x)$ where $\mathbf{p}$ is the quasi-particle momentum and the Fermi momentum P_F is defined by the ensemble chemical potential. The densities ρ_s, ρ in the right hand sides of equations (13) equal (by definition) to $\rho_s(x) = Tr\{\xi(x), 1\}$, $\rho(x) = Tr\{\xi(x), \gamma_4\}$. Here we confine ourselves to the Thomas–Fermi approximation while describing the quark ensemble. Then the densities in which we are interested in are given with some local Fermi momentum $P_F(x)$ as $\rho = \gamma \int^{P_F} \frac{d\mathbf{p}}{(2\pi)^3} = \frac{\gamma}{6\pi^2} P_F^3$ $\rho_s = \gamma \int^{P_F} \frac{d\mathbf{p}}{(2\pi)^3} \frac{\overset{*}{M}}{E}$ where γ is the quark gamma factor ($\gamma = 2 N_c N_f$, N_c is the number of colours, N_f number of flavours), $E = (\mathbf{p}^2 + \overset{*}{M}{}^2)^{1/2}$. By definition the ensemble chemical potential does not change and it leads to the situation in which the local value of Fermi momentum is defined by the running value of dynamical quark mass and vector field as $\mu = M = g_v \, V + (P_F^2 + \overset{*}{M}{}^2)^{1/2}$. The details of tuning the Lagrangian parameters (12) can be found in [10]. The point of our attraction here is the surface tension coefficient [11] $u_s = 4\pi \, r_o^2 \int_{-\infty}^{\infty} dx \left[\mathcal{E}(x) - \frac{\mathcal{E}_l}{\rho_l} \rho(x) \right]$, here $\mathcal{E}_l$ is the energy density in the liquid phase. The parameter r_o is discussed below. In the Thomas–Fermi approximation $\mathcal{E}(x) = \gamma \int^{P_F(x)} \frac{d\mathbf{p}}{(2\pi)^3} [\mathbf{p}^2 + \overset{*}{M}(x)]^{1/2} + \frac{1}{2} g_v \rho(x) V(x) - \frac{1}{2} g_\sigma \rho_s(x) \sigma(x)$. The surface tension coefficient u_s in MeV for the curve of stable kinks (see the details in Ref. [10]) with parameter $\eta \leq 1.2$ as the function of another parameter c ($\zeta = c \, \eta$) is depicted in Fig. 4.

Above results lead us to put the challenging question about the properties of finite quark systems or droplets of quark liquid which are in equilibrium with the vacuum state. As a droplet here we imply the spherically-symmetric solution of the equation system (13) for $\sigma(r)$ and $V(r)$ with the obvious boundary conditions $\sigma'(0) = 0$ and $V'(0) = 0$ in the origin (the primed variables denote the first derivatives over r) and rapidly decreasing at the large distances $\sigma \to 0$, $V \to 0$ when $r \to \infty$. Fig. 5 shows the set of solutions (σ-field (MeV)) of the equation system (13) at number of flavors $N_f = 1$. Fig. 6 presents the corresponding distributions of ensemble density ρ (ch/fm^3). The Table 1 exhibits the results of fitting the density $\rho(r)$ with the Fermi distribution $\rho_F(r) = \frac{\widetilde{\rho}_0}{1 + e^{(R_0 - r)/b}}$ where $\widetilde{\rho}_0$ is the density at the origin, R_0 is the mean size of the droplet and the parameter b determines the thickness of surface layer $t = 4\ln(3)b$. Besides, the coefficient r_0 which is included in the definition of surface tension coefficient, $R_0 = r_0 N_q^{1/3}$ is also presented together with the characteristic values of the σ-meson mass and the coefficient η at which all this values were obtained.

<u>Table 1</u>. Results of fitting by the Fermi distribution ($N_f = 1$).

N_q	$\widetilde{\rho}_0$ (ch/fm^3)	R_0 (fm)	b (fm^{-1})	t (fm)	r_0 (fm)	m_σ (MeV)	η
15	0.34	1.84	0.51	2.24	0.74	351	0.65
43	0.43	2.19	0.52	2.28	0.75	384	0.73
159	0.46	4.19	0.52	2.29	0.77	409	0.78
303	0.47	5.23	0.52	2.29	0.78	417	0.795
529	0.47	6.37	0.52	2.27	0.79	423	0.805
742	0.47	7.15	0.52	2.27	0.79	426	0.81

The curves plotted in the Fig. 5 and results of Table 1 allows us to conclude that the density distributions at $N_q \geq 50$ correspond quite well to the data typical for the nuclear medium. The thicknesses of transition layers are also similar. The coefficient r_0 with the factor $3^{1/3}$ included is in full correspondence with nuclear one. The values of the σ-meson mass turn out to be quite reasonable as well. Although at small quark numbers in the droplet the corresponding behaviors become essentially different. We know experimentally that in the nuclear
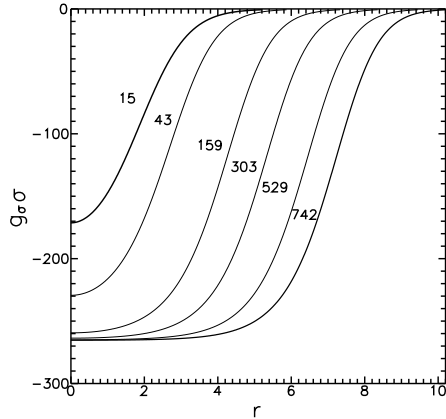
Figure 5. σ-field (MeV) as a function of the distance r (fm) for several solutions of the equation system (13) which are characterized by the net quark number N_q written to the left of each curve.
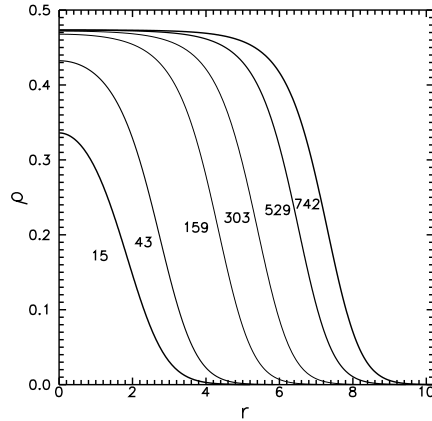
Figure 6.   Distribution of the quark density ρ (ch/fm^3) for the corresponding solutions presented in the Fig. 5.

matter one can observe some increase of the ensemble density which is quite considerable for the Helium and is much higher than the normal nuclear matter density for the Hydrogen. One may criticize us in this point because working within the Thomas–Fermi approximations becomes hardly justified at the small number of quarks and it is necessary to handle the solution of equation system (13). However, fortunately, the exploration we are interested in has been performed in the chiral soliton model of nucleon [12]. It has been demonstrated there that adding the contributions of pseudo-scalar and axial-vector fields to the Lagrangian (12) leads to reasonably good description of nucleon and Δ. The interesting remark here is that the soliton solutions obtained in [12] could be interpreted as a confluence of two kinks. Each of those kinks develops the restoration of chiral symmetry in a sense that the scalar field is approaching its zero value at the distance ~ 0.5 fm from the kink center. Actually, one branch corresponds to the solution with the positive value of the dynamical quark mass and another branch presents the solution with negative dynamical quark mass (in three-dimensional picture the pseudo-scalar fields appears just as a phase of chiral rotation from positive to negative value of quark mass).

References

[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961);
     M. Buballa, Phys. Rep. **407**, 205 (2005).

[2] S. V. Molodtsov and G. M. Zinovjev, Teor. Mat. Fiz. **160**, 244 (2009); Phys. Rev. D **80**, 076001 (2009).

[3] S. V. Molodtsov, A. N. Sissakian and G. M. Zinovjev, Europhys. Lett. **87**, 61001 (2009);
     Phys. of Atomic Nucl. **73** 1245 (2010).

[4] S. V. Molodtsov and G. M. Zinovjev, arXiv:1008.2319 [hep-ph].

[5] T. Hatsuda and T. Kunihiro, Phys. Rep. **247**, 221 (1994).

[6] L. Ya. Glozman, Phys. Rep. **444**, 1 (2007).

[7] A. I. Larkin and Yu. N. Ovchinnikov, JETF, **47**, 1136 (1964).

[8] T. Kunihiro, Y. Minami and Z. Zhang, arXiv:1009.4534 [nucl-th];
     S. Carignano, D. Nickel and M. Buballa, arXiv:1007.1397 [hep-ph];
     D. Nickel, Phys. Rev. Lett. **103**, 072301 (2009);
     G. Basar and G. V. Dunne, Phys. Rev. Lett. **100**, 200404 (2008).

[9] J. D. Walecka, Annals of Phys. **83**, 491 (1974).

[10] S. V. Molodtsov and G. M. Zinovjev, arXiv:1103.3351 [hep-ph].

[11] J. Boguta and A. Bodmer, Nucl. Phys. A **292**, 413 (1977).

[12] W. Broniowski and M. K. Banerjee, Phys. Lett. B **158**, 335 (1978).