

PATH PROPERTIES AND REGULARITY OF AFFINE PROCESSES ON GENERAL STATE SPACES

CHRISTA CUCHIERO AND JOSEF TEICHMANN

ABSTRACT. We provide a new proof for regularity of affine processes on general state spaces by methods from the theory of Markovian semimartingales. On the way to this result we also show that the definition of an affine process, namely as stochastically continuous time-homogeneous Markov process with exponential affine Fourier-Laplace transform, already implies the existence of a càdlàg version. This was one of the last open issues in the fundaments of affine processes.

1. INTRODUCTION

In the last twenty years affine processes have been of great interest in mathematical finance to model phenomena like stochastic volatility, stochastic interest rates, heavy tails, credit default, etc. Pars pro toto we mention here the one-dimensional short-rate model of Cox, Ingersoll, and Ross (1985) and the stochastic volatility model of Heston (1993). In order to accommodate the more and more complex structures in finance, these simple models have progressively been extended to higher dimensional affine jump diffusions with values in the so-called canonical state space $\mathbb{R}_+^m \times \mathbb{R}^{n-m}$, or in the cone of positive semidefinite matrices, see, e.g., Dai and Singleton (2000); Duffie and Kan (1996); Duffie, Pan, and Singleton (2000) for affine models on the canonical state space and Gourieroux and Sufana (2003); Da Fonseca, Grasselli, and Tebaldi (2008); Leippold and Trojani (2008) for interest rate and multivariate stochastic volatility models based on matrix-valued affine processes.

Axiomatically speaking affine processes are stochastically continuous Markov processes on some state space $D \subseteq V$, where V is a finite-dimensional Euclidean vector space with scalar product $\langle \cdot, \cdot \rangle$, such that the Fourier-Laplace transform is of exponential affine form in the initial values. More precisely, this means that there exist functions Φ and ψ such that

$$\mathbb{E}_x \left[e^{\langle u, X_t \rangle} \right] = \Phi(t, u) e^{\langle \psi(t, u), x \rangle},$$

for all $(t, x) \in \mathbb{R}_+ \times D$ and $u \in V + iV$, for which $x \mapsto e^{\langle u, x \rangle}$ is a bounded function on D . Since we do not know about the Feller property at this stage, it is delicate to make conclusions on the path and analytic properties of the affine process.

The reasons for the strong interest in affine processes are twofold: first, affine processes are a rich and flexible class of Markov processes containing Lévy processes,

Key words and phrases. affine processes, path properties, regularity, Markov semimartingales.
MSC 2000: Primary: 60J25; Secondary: 60G17.

We thank Enno Veerman for discussions and helpful comments.
 Both authors gratefully acknowledge the financial support by the ETH Foundation.

Ornstein-Uhlenbeck processes, squared Bessel processes and aggregates of them. Second, affine processes are analytically tractable in the sense that the Fourier-Laplace transform, which is a solution of the backward Kolmogorov equation, a PIDE with affine coefficients, can be calculated by solving a system of ODEs for Φ and ψ , the so-called generalized Riccati equations. Having the Fourier-Laplace transform at hand then means that real-time-calibration is at reach from a numerical point of view. The crucial property which actually allows to relate the generalized Riccati equations with an affine process is the differentiability of its Fourier-Laplace transform with respect to time, a concept called *regularity* (see Duffie, Filipović, and Schachermayer (2003, Definition 2.5)).

The theory of affine processes has been developed in several steps: in Kawazu and Watanabe (1971) the full classification on the state space $\mathbb{R}_+$ was proved, introducing already the generalized Riccati equations and the related affine technology. A key step in this article is to establish the aforementioned differentiability of the functions Φ and ψ with respect to time. After several seminal papers in finance the classification of affine processes for the so-called canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ was done in Duffie et al. (2003), although under the standing assumption of regularity. It remained open whether there are affine processes on the canonical state space which are not regular, or if regularity follows in fact from stochastic continuity and the property that the Fourier-Laplace transform is of exponential affine form. Indeed, in Keller-Ressel, Schachermayer, and Teichmann (2010) it is shown that affine processes on the canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ are regular, a reasoning motivated by insights from the solution of Hilbert's fifth problem, see Keller-Ressel et al. (2010) for details. However, this solution depends on the full solution of Duffie et al. (2003) and thus on the particular polyhedral nature of the canonical state space. It remained open if regularity holds on other “non-polyhedral” state spaces, for instance on sets whose boundary is described by a parabola or on (subsets of) the cone of positive semidefinite $d \times d$ matrices, denoted by S_d^+ .

The following example of a possible state space illustrates that affine processes can take values in various types of sets and that particular geometric properties of the state space cannot be taken for granted. Consider the subsets of the cone of positive semidefinite $d \times d$ matrices of the form

$$D_n = \{x \in S_d^+ \mid \text{rank}(x) \leq n\}, \quad n \in \{1, \dots, d\}.$$

In particular, if $n \in \{1, \dots, d-2\}$, these sets constitute *non-convex* state spaces of affine processes which are maximal in a sense made clear in the sequel. To illustrate this phenomenon by an example, let $\langle x, y \rangle := \text{tr}(xy)$ denote the scalar product on S_d , the vector space of $d \times d$ symmetric matrices, and let $d > 2$ and $n \in \{1, \dots, d-2\}$. Consider a $n \times d$ matrix of independent Brownian motions $(W_t)_{t \geq 0}$ with initial value $W_0 = y \in \mathbb{R}^{n \times d}$ and define the following process

$$X_t = W_t^\top W_t, \quad X_0 = x := y^\top y. \quad (1.1)$$

Then the distribution of X_t corresponds to the *non-central Wishart distribution* with shape parameter $\frac{n}{2}$, scale parameter $2tI$ and non-centrality parameter x (see, e.g., Letac and Massam (2008)). Its Fourier-Laplace transform is given by

$$\mathbb{E}_x \left[e^{\langle u, X_t \rangle} \right] = \det(I - 2tu)^{-\frac{n}{2}} e^{\langle (u^{-1} - 2tI)^{-1}, x \rangle}, \quad u \in -S_d^+ + iS_d, \quad (1.2)$$

and therefore of exponential affine form in all initial values x with $\text{rank}(x) \leq n$. This implies in particular that (1.1) is an affine process with state space $D_n = \{x \in$

$S_d^+ \mid \text{rank}(x) \leq n\}$ and functions Φ and ψ given by

$$\begin{aligned}\Phi(t, u) &= \det(I - 2tu)^{-\frac{n}{2}}, \\ \psi(t, u) &= (u^{-1} - 2tI)^{-1}.\end{aligned}$$

Note here that the set $\mathcal{U} := \{u \in S_d + iS_d \mid x \mapsto e^{\langle u, x \rangle} \text{ is bounded on } D_n\}$ corresponds to $-S_d^+ + iS_d$. By differentiating Φ and ψ it is easily seen that these functions are solutions of the following system of Riccati ODEs

$$\begin{aligned}\partial_t \Phi(t, u) &= n\Phi(t, u)\langle I, \psi(t, u) \rangle, & \Phi(0, u) &= 1, \\ \partial_t \psi(t, u) &= 2\psi(t, u)^2, & \psi(0, u) &= u.\end{aligned}$$

From the characterization of affine processes on S_d^+ via the Riccati equations and the corresponding admissible parameters (see Cuchiero, Filipović, Mayerhofer, and Teichmann (2011, Theorem 2.4 and Condition (2.4))), it then follows that (1.2) is the Fourier-Laplace transform of an affine process with state space S_d^{+1} , if and only if $n \geq d-1$. Hence, for $n \in \{1, \dots, d-2\}$, the state space D_n cannot be enlarged to its convex hull S_d^+ such that the constructed affine process on D_n can also be extended to an affine process on S_d^+ . Further affine processes with state space D_n can be obtained from squares of Ornstein-Uhlenbeck processes (see Bru (1991) or Mayerhofer (2011, Appendix A)).

The aim is thus to find a unified treatment which allows to prove regularity for all possible state spaces without relying on particular properties of them. In Keller-Ressel, Schachermayer, and Teichmann (2011) this general question has been solved: it is shown that affine processes are regular on general state spaces D , however, under the assumption that the affine process admits a càdlàg version. The method of proof is probabilistic in the sense that the “absence of regularity” leads – in a probabilistic way – to a contradiction.

This article now provides a new proof inspired by the theory of Markovian semimartingales as laid down in Çinlar, Jacod, Protter, and Sharpe (1980). In order to apply these reasonings, we first prove one of the last open issues in the basics of affine processes, namely that stochastic continuity and the affine property are already sufficient for the existence of a version with càdlàg trajectories, which can then be defined on the canonical probability space of càdlàg paths with a filtration satisfying the usual conditions for any initial value. Having achieved this, we provide a *full* and *complete* class in the sense of Çinlar et al. (1980) by using the process’ own harmonic analysis. More precisely, we use the fact that

$$x \mapsto \int_0^\eta \mathbb{E}_x \left[e^{\langle u, X_s \rangle} \right] ds, \quad \eta > 0$$

always lies in the domain of the extended infinitesimal generator of any time-homogeneous Markov process X . The particular form of $\mathbb{E}_x \left[e^{\langle u, X_s \rangle} \right]$ in the case of affine processes then allows to show that the domain of the extended generator actually contains a full and complete class. This in turn implies on the one hand the semimartingale property (up to the lifetime of the affine process) and on the other hand the absolute continuity of the involved characteristics with respect to the Lebesgue measure. The final proof of regularity then builds to a large extent on these results.

¹This means in particular that every starting value in S_d^+ is possible.

The remainder of the article is organized as follows. In Section 2 we define affine processes on general state spaces and derive some fundamental properties of the functions Φ and ψ . Section 3 and 4 are devoted to show the existence of a càdlàg version and the right-continuity of the appropriately augmented filtration. The results on the semimartingale nature of affine process are established in Section 5 and are used in Section 6 for the proof of regularity.

2. AFFINE PROCESSES ON GENERAL STATE SPACES

We define affine processes as a particular class of time-homogeneous Markov processes with state space $D \subseteq V$, some closed, non-empty subset of an n -dimensional real vector space V with scalar product $\langle \cdot, \cdot \rangle$. Symmetric matrices and the positive semidefinite matrices on V are denoted by $S(V)$ and $S_+(V)$, respectively. We write $\mathbb{R}_+$ for $[0, \infty)$, $\mathbb{R}_{++}$ for $(0, \infty)$ and $\mathbb{Q}_+$ for nonnegative rational numbers. For the stochastic background and notation we refer to standard text books such as Jacod and Shiryaev (2003) and Revuz and Yor (1999).

To further clarify notation, we find it useful to recall in this section the basic ingredients of the theory of time-homogeneous Markov processes and the particular conventions being made in this article (compare Blumenthal and Gettoor (1968, Chapter 1.3), Chung and Walsh (2005, Chapter 1.2), Ethier and Kurtz (1986, Chapter 4), Rogers and Williams (1994, Chapter 3, Definition 1.1)). Throughout D denotes a closed subset of V and $\mathcal{D}$ its Borel σ -algebra. Since we shall not assume the process to be conservative, we adjoin to the state space D a point $\Delta \notin D$, called cemetery state, and set $D_\Delta = D \cup \{\Delta\}$ as well as $\mathcal{D}_\Delta = \sigma(\mathcal{D}, \{\Delta\})$. We make the convention that $\|\Delta\| := \infty$, where $\|\cdot\|$ denotes the norm induced by the scalar product $\langle \cdot, \cdot \rangle$, and we set $f(\Delta) = 0$ for any other function f on D .

Consider the following objects on a space Ω :

- (i) a stochastic process $X = (X_t)_{t \geq 0}$ taking values in D_Δ such that

$$\text{if } X_s(\omega) = \Delta, \text{ then } X_t(\omega) = \Delta \text{ for all } t \geq s \text{ and all } \omega \in \Omega; \quad (2.1)$$
- (ii) the filtration generated by X , that is, $\mathcal{F}_t^0 = \sigma(X_s, s \leq t)$, where we set $\mathcal{F}^0 = \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t^0$;
- (iii) a family of probability measures $(\mathbb{P}_x)_{x \in D_\Delta}$ on $(\Omega, \mathcal{F}^0)$.

Definition 2.1 (Markov process). A time-homogeneous Markov process

$$X = (\Omega, (\mathcal{F}_t^0)_{t \geq 0}, (X_t)_{t \geq 0}, (p_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D_\Delta})$$

with state space $(D, \mathcal{D})$ (augmented by Δ) is a D_Δ -valued stochastic process such that, for all $s, t \geq 0$, $x \in D_\Delta$ and all bounded $\mathcal{D}_\Delta$ -measurable functions $f : D_\Delta \rightarrow \mathbb{R}$,

$$\mathbb{E}_x [f(X_{t+s}) | \mathcal{F}_s^0] = \mathbb{E}_{X_s} [f(X_t)] = \int_D f(\xi) p_t(X_s, d\xi), \quad \mathbb{P}_x\text{-a.s.} \quad (2.2)$$

Here, $\mathbb{E}_x$ denotes the expectation with respect to $\mathbb{P}_x$ and $(p_t)_{t \geq 0}$ is a transition function on $(D_\Delta, \mathcal{D}_\Delta)$. A transition function is a family of kernels $p_t : D_\Delta \times \mathcal{D}_\Delta \rightarrow [0, 1]$ such that

- (i) for all $t \geq 0$ and $x \in D_\Delta$, $p_t(x, \cdot)$ is a measure on $\mathcal{D}_\Delta$ with $p_t(x, D) \leq 1$, $p_t(x, \{\Delta\}) = 1 - p_t(x, D)$ and $p_t(\Delta, \{\Delta\}) = 1$;
- (ii) for all $x \in D_\Delta$, $p_0(x, \cdot) = \delta_x(\cdot)$, where $\delta_x(\cdot)$ denotes the Dirac measure;
- (iii) for all $t \geq 0$ and $\Gamma \in \mathcal{D}_\Delta$, $x \mapsto p_t(x, \Gamma)$ is $\mathcal{D}_\Delta$ -measurable;

- (iv) for all $s, t \geq 0$, $x \in D_\Delta$ and $\Gamma \in \mathcal{D}_\Delta$, the Chapman-Kolmogorov equation holds, that is,

$$p_{t+s}(x, \Gamma) = \int_{D_\Delta} p_s(x, d\xi) p_t(\xi, \Gamma).$$

If $(\mathcal{F}_t)_{t \geq 0}$ is a filtration with $\mathcal{F}_t^0 \subset \mathcal{F}_t$, $t \geq 0$, then X is a time-homogeneous Markov process relative to $(\mathcal{F}_t)$ if (2.2) holds with $\mathcal{F}_s^0$ replaced by $\mathcal{F}_s$.

We can alternatively think of the transition function as inducing a measurable contraction semigroup $(P_t)_{t \geq 0}$ defined by

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \int_D f(\xi) p_t(x, d\xi), \quad x \in D_\Delta,$$

for all bounded $\mathcal{D}_\Delta$ -measurable functions $f : D_\Delta \rightarrow \mathbb{R}$.

- Remark 2.2.** (i) Note that, in contrast to Duffie et al. (2003), we do not assume Ω to be the canonical space of all functions $\omega : \mathbb{R}_+ \rightarrow D_\Delta$, but work on some general probability space.
(ii) Since we have $p_t(x, \Gamma) = \mathbb{P}_x[X_t \in \Gamma]$ for all $t \geq 0$, $x \in D_\Delta$ and $\Gamma \in \mathcal{D}_\Delta$, property (ii) and (iii) of the transition function, imply $\mathbb{P}_x[X_0 = x] = 1$ for all $x \in D_\Delta$ and measurability of the map $x \mapsto \mathbb{P}_x[X_t \in \Gamma]$ for all $t \geq 0$ and $\Gamma \in \mathcal{D}_\Delta$.

For the following definition of affine processes, let us introduce the set $\mathcal{U}$ defined by

$$\mathcal{U} = \left\{ u \in V + iV \mid e^{\langle u, \cdot \rangle} \text{ is a bounded function on } D \right\}. \quad (2.3)$$

Clearly $iV \subseteq \mathcal{U}$. Here, the set iV stands for purely imaginary elements and $\langle \cdot, \cdot \rangle$ is the extension of the real scalar product to $V + iV$, but without complex conjugation. Moreover, we denote by p the dimension of $\text{Re}\mathcal{U}$ and write $\langle \text{Re}\mathcal{U} \rangle$ for its (real) linear hull and $\langle \text{Re}\mathcal{U} \rangle^\perp$ for its orthogonal complement in V . The set $i\langle \text{Re}\mathcal{U} \rangle^\perp \subset \mathcal{U}$ are the purely imaginary direction of $\mathcal{U}$. Finally, for some linear subspace $W \subset V$, $\Pi_W : V \rightarrow W$ denotes the orthogonal projection on W , which is extended to $V + iV$ by linearity, i.e., $\Pi_W(v_1 + i v_2) := \Pi_W v_1 + i \Pi_W v_2$.

Assumption 2.3. Recall that $\dim V = n$. We require that the state space D contains at least $n + 1$ affinely independent elements $x_1, \dots, x_{n+1}$, that is, the n vectors $(x_1 - x_j, \dots, x_{j-1} - x_j, x_{j+1} - x_j, \dots, x_{n+1} - x_j)$ are linearly independent for every $j \in \{1, \dots, n + 1\}$.

We are now prepared to give our main definition:

Definition 2.4 (Affine process). A time-homogeneous Markov process X relative to some filtration $(\mathcal{F}_t)$ and with state space $(D, \mathcal{D})$ (augmented by Δ) is called affine if

- (i) it is stochastically continuous, that is, $\lim_{s \rightarrow t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly on D for every $t \geq 0$ and $x \in D$, and
(ii) its Fourier-Laplace transform has exponential affine dependence on the initial state. This means that there exist functions $\Phi : \mathbb{R}_+ \times \mathcal{U} \rightarrow \mathbb{C}$ and $\psi : \mathbb{R}_+ \times \mathcal{U} \rightarrow V + iV$ such that

$$\mathbb{E}_x \left[e^{\langle u, X_t \rangle} \right] = P_t e^{\langle u, \cdot \rangle} = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = \Phi(t, u) e^{\langle \psi(t, u), x \rangle}, \quad (2.4)$$

for all $x \in D$ and $(t, u) \in \mathbb{R}_+ \times \mathcal{U}$.

Remark 2.5. (i) The above definition differs in three crucial details from the definitions given in Duffie et al. (2003, Definition 2.1, Definition 12.1).²

First, therein the right hand side of (2.4) is defined in terms of a function $\phi(t, u)$, namely as $e^{\phi(t, u) + \langle \psi(t, u), x \rangle}$, such that the function $\Phi(t, u)$ in our definition corresponds to $e^{\phi(t, u)}$. Our definition is in line with the one given in Kawazu and Watanabe (1971) and Keller-Ressel et al. (2010, 2011) and differs from the one in Duffie et al. (2003), as we do not require $\Phi(t, u) \neq 0$ a priori. However, since all affine processes on $D = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ are infinitely divisible (see Duffie et al. (2003, Theorem 2.15)), it turns out with hindsight that setting $\Phi(t, u) = e^{\phi(t, u)}$ is actually no restriction.

Second, we assume that the affine property holds for all $u \in \mathcal{U}$, whereas on the canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ it is restricted to $i\mathbb{R}^n$ (see Duffie et al. (2003)). This however turns out to imply the affine property also on $\mathcal{U}$.

Third, in contrast to Duffie et al. (2003), we take stochastic continuity as part of the definition of an affine process. We remark that there are simple examples of Markov processes which satisfy Definition 2.4 (ii), but are not stochastically continuous (see Duffie et al. (2003, Remark 2.11)).

- (ii) Let us remark that the assumption of a closed state space is no restriction. Indeed, if an affine process is defined on some state space D , which is only supposed to be an arbitrary Borel subset of V as done in Keller-Ressel et al. (2011), then the affine property (2.4) extends automatically to $\overline{D}$: Let $(x_k)_{k \in \mathbb{N}}$ be a sequence in D converging to some $x \in \overline{D}$. Due to the exponential affine form of the characteristic function, we have for all $t \in \mathbb{R}_+$ and $u \in iV$

$$\mathbb{E}_{x_n} \left[e^{\langle u, X_t \rangle} \right] = \Phi(t, u) e^{\langle \psi(t, u), x_n \rangle} \rightarrow \Phi(t, u) e^{\langle \psi(t, u), x \rangle}.$$

Since the left hand side is continuous in u , the same holds true for the right hand side. Whence Lévy's continuity theorem implies that the right hand side is a characteristic function of some substochastic measure $p_t(x, \cdot)$ on $\overline{D}$, which is the weak limit of $p_t(x_n, \cdot)$. As stochastic continuity and the Chapman-Kolmogorov equations extend to $\overline{D}$, and since weak convergence implies the convergence of the Fourier-Laplace transforms on $\mathcal{U}$, we thus have constructed an affine process with state space $\overline{D}$.

- (iii) Note furthermore that Assumption 2.3 is no restriction, since we can always pass to a lower dimensional ambient vector space if D does not contain $n + 1$ affinely independent elements.
- (iv) We finally remark that in Section 3 we consider affine processes on the filtered space $(\Omega, \mathcal{F}^0, \mathcal{F}_t^0)$, where $\mathcal{F}_t^0$ denotes the natural filtration and $\mathcal{F}^0 = \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t^0$, as introduced above. However, we shall progressively enlarge the filtration by augmenting with the respective null-sets.

²In Definition 2.1 affine processes on the canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$ are considered, whereas in Definition 12.1 the state space D can be an arbitrary subset of $\mathbb{R}^n$.

Before deducing the first properties of Φ and ψ from the above definition, let us introduce the sets

$$\mathcal{U}^m = \left\{ u \in V + iV \mid \sup_{x \in D} e^{\langle \text{Re} u, x \rangle} \leq m \right\}, \quad m \geq 1,$$

and note that $\mathcal{U} = \bigcup_{m \geq 1} \mathcal{U}^m$ and $iV \subseteq \mathcal{U}^m$ for all $m \geq 1$.

Proposition 2.6. *Let X be an affine process relative to some filtration $(\mathcal{F}_t)$. Then the functions Φ and ψ have the following properties:*

- (i) *For every $m \geq 1$, Φ and ψ can be chosen to be jointly continuous on $\mathcal{Q}^m = \{(t, u) \in \mathbb{R}_+ \times \mathcal{U}^m \mid \Phi(s, u) \neq 0, \text{ for all } s \in [0, t]\}$. This then yields a unique specification of Φ and ψ on $\mathcal{Q} = \{(t, u) \in \mathbb{R}_+ \times \mathcal{U} \mid \Phi(s, u) \neq 0, \text{ for all } s \in [0, t]\}$.*
- (ii) *ψ maps the set $\mathcal{O} = \{(t, u) \in \mathbb{R}_+ \times \mathcal{U} \mid \Phi(t, u) \neq 0\}$ to $\mathcal{U}$.*
- (iii) *$\Phi(0, u) = 1$ and $\psi(0, u) = u$ for all $u \in \mathcal{U}$.*
- (iv) *The functions Φ and ψ satisfy the semiflow property: Let $u \in \mathcal{U}$ and $t, s \geq 0$. Suppose that $\Phi(t + s, u) \neq 0$, then also $\Phi(t, u) \neq 0$ and $\Phi(s, \psi(t, u)) \neq 0$ and we have*

$$\begin{aligned} \Phi(t + s, u) &= \Phi(t, u)\Phi(s, \psi(t, u)), \\ \psi(t + s, u) &= \psi(s, \psi(t, u)). \end{aligned} \tag{2.5}$$

Proof. It follows e.g. from Bauer (1996, Lemma 23.7) that stochastic continuity of X implies joint continuity of $(t, u) \mapsto P_t e^{\langle u, x \rangle}$ on $\mathbb{R}_+ \times \mathcal{U}^m$ for all $x \in D$. Hence $(t, u) \mapsto \Phi(t, u)e^{\langle \psi(t, u), x \rangle}$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{U}^m$ for every $x \in D$. By Assumption 2.3 on the state space D , this in turn yields a unique continuous choice of the functions $(t, u) \mapsto \Phi(t, u)$ and $(t, u) \mapsto \psi(t, u)$ on $\mathcal{Q}^m$. (compare Keller-Ressel et al. (2011, Proposition 2.4) for details).

Concerning (ii), let $(t, u) \in \mathcal{O} = \{(t, u) \in \mathbb{R}_+ \times \mathcal{U} \mid \Phi(t, u) \neq 0\}$. Since

$$\left| \Phi(t, u)e^{\langle \psi(t, u), x \rangle} \right| = \left| \mathbb{E}_x \left[e^{\langle u, X_t \rangle} \right] \right| \leq \mathbb{E}_x \left[\left| e^{\langle u, X_t \rangle} \right| \right]$$

is bounded on D and as $\Phi(t, u) \neq 0$, we conclude that $\psi(t, u) \in \mathcal{U}$.

Assertion (iii) follows simply from

$$e^{\langle u, x \rangle} = \mathbb{E}_x \left[e^{\langle u, X_0 \rangle} \right] = \Phi(0, u)e^{\langle \psi(0, u), x \rangle}.$$

Assumption $\Phi(t + s, u) \neq 0$ in (iv) implies

$$\mathbb{E}_x \left[e^{\langle u, X_{t+s} \rangle} \right] = \Phi(t + s, u)e^{\langle \psi(t+s, u), x \rangle} \neq 0. \tag{2.6}$$

By the law of iterated expectations and the Markov property, we thus have

$$\mathbb{E}_x \left[e^{\langle u, X_{t+s} \rangle} \right] = \mathbb{E}_x \left[\mathbb{E}_x \left[e^{\langle u, X_{t+s} \rangle} \mid \mathcal{F}_s \right] \right] = \mathbb{E}_x \left[\mathbb{E}_{X_s} \left[e^{\langle u, X_t \rangle} \right] \right]. \tag{2.7}$$

If $\Phi(t, u) = 0$ or $\Phi(s, \psi(t, u)) = 0$, then the inner or the outer expectation evaluates to 0. This implies that the whole expression is 0, which contradicts (2.6). Hence $\Phi(t, u) \neq 0$ and $\Phi(s, \psi(t, u)) \neq 0$ and we can write (2.7) as

$$\mathbb{E}_x \left[e^{\langle u, X_{t+s} \rangle} \right] = \mathbb{E}_x \left[\Phi(t, u)e^{\langle \psi(t, u), X_s \rangle} \right] = \Phi(t, u)\Phi(s, \psi(t, u))e^{\langle \psi(s, \psi(t, u)), x \rangle}.$$

Comparing with (2.6) yields the claim. $\square$

Remark 2.7. *Henceforth, the symbols Φ and ψ always correspond to the unique continuous choice established in Proposition 2.6.*

3. CÀDLÀG VERSION

The aim of this section is to show that the definition of an affine process already implies the existence of a càdlàg version. This is the core section of this article and of a remarkable subtlety, which is maybe less surprising if one considers the generality of the question: so far we do not know that general affine processes are Feller processes and we can not apply the most general standard criteria for the existence of càdlàg versions, as for instance described in (Gihman and Skorohod, 1983, Theorem I.6.2).

Our approach to the problem is inspired by martingale regularization for a lot of “test martingales”, from which we want to conclude path properties of the original stochastic process. The main difficulty here is that explosions and/or killing might appear.

Indeed, for every fixed $x \in D$, we first establish that for $\mathbb{P}_x$ -almost every ω

$$t \mapsto M_t^{T,u}(\omega) := \Phi(T-t, u) e^{\langle \psi(T-t, u), X_t(\omega) \rangle}, \quad t \in [0, T],$$

is the restriction to $\mathbb{Q}_+ \cap [0, T]$ of a càdlàg function for almost all $(T, u) \in (0, \infty) \times \mathcal{U}$, in the sense that $M_t^{T,u} = 0$ if $\Phi(T-t, u) = 0$. This is an application of Doob’s regularity theorem for supermartingales, where we can conclude – using Fubini’s theorem – that there exists a $\mathbb{P}_x$ -null-set outside of which we observe appropriately regular trajectories for almost all (T, u) .

Proposition 3.1. *Let $x \in D$ be fixed and let X be an affine process relative to $(\mathcal{F}_t^0)$. Then*

$$\lim_{\substack{q \in \mathbb{Q}_+ \\ q \downarrow t}} M_q^{T,u} = \lim_{\substack{q \in \mathbb{Q}_+ \\ q \downarrow t}} \Phi(T-q, u) e^{\langle \psi(T-q, u), X_q \rangle}, \quad t \in [0, T],$$

exists $\mathbb{P}_x$ -a.s. for almost all $(T, u) \in (0, \infty) \times \mathcal{U}$ and defines a càdlàg function in t .

Proof. In order to prove this result, we adapt parts of the proof of Protter (2005, Theorem I.4.30) to our setting. Due to the law of iterated expectations

$$M_t^{T,u} = \Phi(T-t, u) e^{\langle \psi(T-t, u), X_t \rangle} = \mathbb{E}_x \left[e^{\langle u, X_T \rangle} \middle| \mathcal{F}_t^0 \right], \quad t \in [0, T],$$

is a (complex-valued) $(\mathcal{F}_t^0, \mathbb{P}_x)$ -martingale for every $u \in \mathcal{U}$ and every $T > 0$. From Doob’s regularity theorem (see, e.g., Rogers and Williams (1994, Theorem II.65.1)) it then follows that, for any fixed (T, u) , the function $t \mapsto M_t^{T,u}(\omega)$, with $t \in \mathbb{Q}_+ \cap [0, T]$, is the restriction to $\mathbb{Q}_+ \cap [0, T]$ of a càdlàg function for $\mathbb{P}_x$ -almost every ω . Define now the set

$$\Gamma = \{(\omega, T, u) \in \Omega \times (0, \infty) \times \mathcal{U} \mid t \mapsto M_t^{T,u}(\omega), t \in \mathbb{Q}_+ \cap [0, T], \\ \text{is not the restriction of a càdlàg function}\}. \quad (3.1)$$

Then Γ is a $\mathcal{F}^0 \otimes \mathcal{B}((0, \infty) \times \mathcal{U})$ -measurable set. Due to the above argument concerning regular versions of (super-)martingales, $\int_{\Omega} 1_{\Gamma}(\omega, T, u) \mathbb{P}_x(d\omega) = 0$ for any $(T, u) \in (0, \infty) \times \mathcal{U}$. By Fubini’s theorem, we therefore have

$$\int_{\Omega} \int_{(0, \infty) \times \mathcal{U}} 1_{\Gamma}(\omega, T, u) d\lambda \mathbb{P}_x(d\omega) = \int_{(0, \infty) \times \mathcal{U}} \int_{\Omega} 1_{\Gamma}(\omega, T, u) \mathbb{P}_x(d\omega) d\lambda = 0,$$

where λ denotes the Lebesgue measure. Hence, for $\mathbb{P}_x$ -almost every ω , $t \mapsto M_t^{T,u}(\omega)$, the map $t \in \mathbb{Q}_+ \cap [0, T]$, is the restriction of a càdlàg function for λ -almost all $(T, u) \in (0, \infty) \times \mathcal{U}$, which proves the result. $\square$

Having established path regularity of the martingales $M^{T,u}$, we want to deduce the same for the affine process X . This is the purpose of the subsequent lemmas and propositions, for which we need to introduce the following sets $\tilde{\Omega} \subseteq \Omega$, $\mathcal{T} \subseteq (0, \infty)$ and $\mathcal{V} \subseteq \mathfrak{i}V$:

$$\tilde{\Omega} \text{ is the projection of } \{\Omega \times (0, \infty) \times \mathfrak{i}V\} \setminus \Gamma \text{ onto } \Omega, \quad (3.2)$$

$$\mathcal{T} \text{ is the projection of } \{\Omega \times (0, \infty) \times \mathfrak{i}V\} \setminus \Gamma \text{ onto } (0, \infty), \quad (3.3)$$

$$\mathcal{V} \text{ is the projection of } \{\Omega \times (0, \infty) \times \mathcal{U}\} \setminus \Gamma \text{ onto } \mathcal{U}, \quad (3.4)$$

where Γ is given in (3.1). Denoting by $\mathcal{F}^x$ the completion of $\mathcal{F}^0$ with respect to $\mathbb{P}_x$, let us remark that the measurable projection theorem implies that $\tilde{\Omega} \in \mathcal{F}^x$ and by the above proposition we have $\mathbb{P}_x[\tilde{\Omega}] = 1$. Finally, for some $r > 0$, we denote by K the intersection of $\mathcal{V}$ with the closed ball with center 0 and radius r , that is,

$$K := \overline{B}(0, r) \cap \mathcal{V} := \{u \in \mathcal{U} \mid \|\operatorname{Re} u\|^2 + \|\operatorname{Im} u\|^2 \leq r^2\} \cap \mathcal{V}. \quad (3.5)$$

Lemma 3.2. *Consider the set K defined in (3.5) and the function ψ given in (2.4) with the properties of Proposition 2.6. Denote by p the dimension of $\operatorname{Re} \mathcal{U}$. Let $(u_1, \dots, u_p)$ be linearly independent vectors in $K \cap \operatorname{Re} \mathcal{U}$ and let $(u_{p+1}, \dots, u_n)$ be linearly independent vectors in $\Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} K$. Then there exists some $\delta > 0$ such that for every $t \in [0, \delta)$*

$$(\psi(t, u_1), \dots, \psi(t, u_p))$$

and

$$(\Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} \psi(t, u_{p+1}), \dots, \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} \psi(t, u_n))$$

are linearly independent.

Proof. This is simply a consequence of the fact that $\psi(0, u) = u$ for all $u \in \mathcal{U} \supset K$ and the continuity of $t \mapsto \psi(t, u)$. $\square$

The following lemma is needed to prove Proposition 3.4 below which is essential for establishing the existence of a càdlàg version of X .

Lemma 3.3. *Let ψ be given by (2.4) and assume that there exists some D -valued sequence $(x_k)_{k \in \mathbb{N}}$ such that*

$$\lim_{k \rightarrow \infty} \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle} x_k =: \lim_{k \rightarrow \infty} y_k \quad (3.6)$$

exists finitely valued and

$$\limsup_{k \rightarrow \infty} \|\Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} x_k\| = \infty. \quad (3.7)$$

Then we can choose a subsequence of (x_k) denoted again by (x_k) : along this sequence there exist a finite number of mutually orthogonal directions $g_i \in \langle \operatorname{Re} \mathcal{U} \rangle^\perp$ of length 1 such that

$$x_k - \sum_i \langle x_k, g_i \rangle g_i$$

converges as $k \rightarrow \infty$, $\langle x_k, g_i \rangle$ diverges as $k \rightarrow \infty$, where the rates of divergence are decreasing in i (see the proof for the precise statement). Furthermore, there exist continuous functions $R : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}$ and $\lambda_i : \mathbb{R}_+ \rightarrow \langle \operatorname{Re} \mathcal{U} \rangle^\perp$ such that

$$\langle \psi(t, u), g_i \rangle = \langle \lambda_i(t), u \rangle$$

for all $u \in \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} \mathcal{U}$ with $\|\operatorname{Im} u\| < R(t)$.

Proof. Concerning the first assertion, we define – by choosing appropriate subsequences, still denoted by (x_k) – the directions of divergence in $\langle \text{Re } \mathcal{U} \rangle^\perp$ inductively by

$$g_r = \lim_{k \rightarrow \infty} \frac{x_k - \sum_{i=1}^{r-1} \langle x_k, g_i \rangle g_i}{\|x_k - \sum_{i=1}^{r-1} \langle x_k, g_i \rangle g_i\|} \quad (3.8)$$

as long as $\limsup_{k \rightarrow \infty} \|x_k - \sum_{i=1}^{r-1} \langle x_k, g_i \rangle g_i\| = \infty$. Notice that we can choose the directions g_i mutually orthogonal and the rates of divergence $\langle g_i, x_k \rangle$ decreasing in i .

For the second part of the statement, we adapt the proof of Keller-Ressel et al. (2010, Lemma 3.1) to our situation, using in particular the existence of a sequence in D with the properties (3.6) and (3.7). As characteristic function, the map $\mathfrak{i} V \ni u \mapsto \mathbb{E}_x[e^{\langle u, X_t \rangle}]$ is positive definite for any $x \in D$ and $t \geq 0$. Define now for every $u \in \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \mathcal{U} \subseteq \mathfrak{i} V$, $x \in D$ and $t \geq 0$ the function

$$\Theta(u, t, x) = \frac{\mathbb{E}_x[e^{\langle u, X_t \rangle}]}{\Phi(t, 0)e^{\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle} \psi(t, 0), \Pi_{\langle \text{Re } \mathcal{U} \rangle} x \rangle}} = \frac{\Phi(t, u)e^{\langle \psi(t, u), x \rangle}}{\Phi(t, 0)e^{\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle} \psi(t, 0), \Pi_{\langle \text{Re } \mathcal{U} \rangle} x \rangle}}. \quad (3.9)$$

As $\mathbb{E}_x[e^{\langle 0, X_t \rangle}] = \Phi(t, 0)e^{\langle \psi(t, 0), x \rangle}$ is real-valued and positive for all $t \geq 0$, we conclude – due to Assumption 2.3 and the continuity of the functions $t \mapsto \Phi(t, 0)$ and $t \mapsto \psi(t, 0)$ – that $\text{Im} \Phi(t, 0) = 0$ and $\text{Im} \psi(t, 0) = 0$ for all $t \geq 0$. In particular, the denominator in (3.9) is positive, which implies that $\mathfrak{i} V \supseteq \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \mathcal{U} \ni u \mapsto \Theta(u, t, x)$ is a positive definite function for all $t \geq 0$ and $x \in D$. Moreover, since $\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(t, 0)$ is purely imaginary and thus in particular 0 for all $t \geq 0$, it follows that

$$\Theta(0, t, x) = \exp \left(\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(t, 0), \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x \rangle \right) = 1$$

for all $t \geq 0$ and $x \in D$. This together with the positive definiteness of $u \mapsto \Theta(u, t, x)$ yields

$$|\Theta(u + v, t, x) - \Theta(u, t, x)\Theta(v, t, x)|^2 \leq 1, \quad u, v \in \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \mathcal{U}, \quad t \geq 0, \quad x \in D, \quad (3.10)$$

(compare, e.g., Keller-Ressel et al. (2010, Lemma 3.2)). Let us now define $y := \Pi_{\langle \text{Re } \mathcal{U} \rangle} x$ and

$$\begin{aligned}
Z_1(u, v, y, t) &= \frac{\Phi(t, u+v) e^{\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle} \psi(t, u+v), y \rangle}}{\Phi(t, 0) e^{\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle} \psi(t, 0), y \rangle}}, \\
Z_2(u, v, y, t) &= \frac{\Phi(t, u) \Phi(t, v) e^{\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle} (\psi(t, u) + \psi(t, v)), y \rangle}}{\Phi(t, 0)^2 e^{2 \langle \Pi_{\langle \text{Re } \mathcal{U} \rangle} \psi(t, 0), y \rangle}}, \\
\beta_1(u, v, t) &= \text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp \psi(t, u+v)), \\
\beta_2(u, v, t) &= \text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp \psi(t, u)) + \text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp \psi(t, v)), \\
r_1(u, v, y, t) &= |Z_1| = \left| \frac{\Phi(t, u+v)}{\Phi(t, 0)} \right| e^{\langle \text{Re}(\Pi_{\langle \text{Re } \mathcal{U} \rangle} (\psi(t, u+v) - \psi(t, 0))), y \rangle}, \\
r_2(u, v, y, t) &= |Z_2| = \left| \frac{\Phi(t, u) \Phi(t, v)}{\Phi(t, 0)^2} \right| e^{\langle \text{Re}(\Pi_{\langle \text{Re } \mathcal{U} \rangle} (\psi(t, u) + \psi(t, v) - 2\psi(t, 0))), y \rangle}, \\
\alpha_1(u, v, y, t) &= \arg(Z_1) = \arg\left(\frac{\Phi(t, u+v)}{\Phi(t, 0)}\right) \\
&\quad + \langle \text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle} \psi(t, u+v)), y \rangle, \\
\alpha_2(u, v, y, t) &= \arg(Z_2) = \arg\left(\frac{\Phi(t, u) \Phi(t, v)}{\Phi(t, 0)^2}\right) \\
&\quad + \langle \text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle} (\psi(t, u) + \psi(t, v))), y \rangle.
\end{aligned}$$

Using (3.10) and the same arguments as in Keller-Ressel et al. (2010, Lemma 3.1), we then obtain

$$\begin{aligned}
1 &\geq \left| r_1 e^{i(\alpha_1 + \langle \beta_1, \Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp x \rangle)} - r_2 e^{i(\alpha_2 + \langle \beta_2, \Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp x \rangle)} \right|^2 \\
&= r_1^2 + r_2^2 - 2r_1 r_2 \cos(\alpha_1 - \alpha_2 + \langle \beta_1 - \beta_2, \Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp x \rangle) \\
&\geq 2r_1 r_2 (1 - \cos(\alpha_1 - \alpha_2 + \langle \beta_1 - \beta_2, \Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp x \rangle)),
\end{aligned}$$

whence

$$\begin{aligned}
&r_1(u, v, y, t) r_2(u, v, y, t) \\
&\times (1 - \cos(\alpha_1(u, v, y, t) - \alpha_2(u, v, y, t) + \langle \beta_1(u, v, t) - \beta_2(u, v, t), \Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp x \rangle)) \leq \frac{1}{2}.
\end{aligned} \tag{3.11}$$

Define now

$$\begin{aligned}
R(t, y) &= \sup \left\{ \rho \geq 0 \mid r_1(u, v, y, t) r_2(u, v, y, t) > \frac{3}{4} \text{ for } u, v \in \Pi_{\langle \text{Re } \mathcal{U} \rangle}^\perp \mathcal{U} \right. \\
&\quad \left. \text{with } \|\text{Im} u\| \leq \rho \text{ and } \|\text{Im} v\| \leq \rho \right\}.
\end{aligned}$$

Note that $R(t, y) > 0$ for all $(t, y) \in \mathbb{R}_+ \times \Pi_{\langle \text{Re } \mathcal{U} \rangle} D$, which follows from the fact that $r_1(0, 0, y, t) = r_2(0, 0, y, t) = 1$ and the continuity of $(u, v) \mapsto r_1(u, v, y, t) r_2(u, v, y, t)$. Continuity of $(t, y) \mapsto r_1(u, v, y, t) r_2(u, v, y, t)$ also implies that $(t, y) \mapsto R(t, y)$ is continuous. Set now $R(t) := \inf_k R(t, y_k)$ where $y_k = \Pi_{\langle \text{Re } \mathcal{U} \rangle} x_k$. Then (3.6) implies that $R(t) > 0$ for all $t \geq 0$.

Let now t be fixed and g_1 given by (3.8). Suppose that

$$\langle \beta_1(u^*, v^*, t) - \beta_2(u^*, v^*, t), g_1 \rangle \neq 0$$

for some $u^*, v^* \in \Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} \mathcal{U}$ with $\|\text{Im} u^*\| < R(t)$ and $\|\text{Im} v^*\| < R(t)$. Then due to the continuity of β_1 and β_2 , there exists some $\delta > 0$ such that for all u, v in a neighborhood O_δ of (u^*, v^*) defined by

$$O_\delta = \left\{ (u, v) \in (\Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} \mathcal{U})^2 \mid \|\text{Im}(u - u^*)\| < \delta, \|\text{Im}(v - v^*)\| < \delta \text{ and } \|\text{Im} u\| < R(t), \|\text{Im} v\| < R(t) \right\},$$

we also have

$$\langle \beta_1(u, v, t) - \beta_2(u, v, t), g_1 \rangle \neq 0. \quad (3.12)$$

Moreover, there exist some $(u, v) \in O_\delta$ and some $k \in \mathbb{N}$ such that

$$\begin{aligned} & \cos(\alpha_1(u, v, y_k, t) - \alpha_2(u, v, y_k, t) + \langle \beta_1(u, v, t) - \beta_2(u, v, t), \Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} x_k \rangle) \\ &= \cos \left(\arg \left(\frac{\Phi(t, u+v)}{\Phi(t, 0)} \right) - \arg \left(\frac{\Phi(t, u)\Phi(t, v)}{\Phi(t, 0)^2} \right) \right. \\ & \quad + \langle \text{Im}(\Pi_{\langle \text{Re} \mathcal{U} \rangle}(\psi(t, u+v) - \text{Im}\psi(t, u) - \text{Im}\psi(t, v))), y_k \rangle \\ & \quad \left. + \langle \beta_1(u, v, t) - \beta_2(u, v, t), \Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} x_k \rangle \right) \\ & \leq \frac{1}{3}, \end{aligned} \quad (3.13)$$

since y_k stays in a bounded set and $\Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} x_k$ explodes with highest divergence rate in direction g_1 .

However, inequality (3.13) now implies that

$$\begin{aligned} & r_1(u, v, y_k, t) r_2(u, v, y_k, t) \\ & \times (1 - \cos(\alpha_1(u, v, y_k, t) - \alpha_2(u, v, y_k, t) + \langle \beta_1(u, v, t) - \beta_2(u, v, t), \Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} x_k \rangle)) > \frac{1}{2}, \end{aligned}$$

which contradicts (3.11). Since g_1 corresponds to the direction of the highest divergence rate, we thus conclude that

$$\langle \beta_1(u, v, t) - \beta_2(u, v, t), g_1 \rangle = \text{Im}(\langle \psi(t, u+v) - \psi(t, u) - \psi(t, v), g_1 \rangle) = 0$$

for all u, v with $\|\text{Im} u\| < R(t)$ and $\|\text{Im} v\| < R(t)$. Continuity of $u \mapsto \psi(t, u)$ therefore implies that $u \mapsto \langle \psi(t, u), g \rangle$ is a linear function. Hence there exists a continuous curve of (real) vectors $\lambda_1(t) \in \langle \text{Re} \mathcal{U} \rangle^\perp$ such that

$$\langle \psi(t, u), g_1 \rangle = \langle \lambda_1(t), u \rangle$$

for all $u \in \Pi_{\langle \text{Re} \mathcal{U} \rangle^\perp} \mathcal{U}$ with $\|\text{Im} u\| < R(t)$.

We can now proceed inductively for the remaining directions of divergence g_i . Indeed, assume that $\langle \beta_1(u, v, t) - \beta_2(u, v, t), g_i \rangle = 0$ for all $i \leq r-1$ and all u, v with $\|\text{Im} u\| < R(t)$ and $\|\text{Im} v\| < R(t)$. Then repeating the above steps allows us to conclude that $\langle \beta_1(u, v, t) - \beta_2(u, v, t), g_r \rangle = 0$ for all u, v with $\|\text{Im} u\| < R(t)$ and $\|\text{Im} v\| < R(t)$ as well, which yields the assertion. $\square$

Consider now the set K defined in (3.5). Since $(t, u) \mapsto \Phi(t, u)e^{\langle \psi(t, u), x \rangle}$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{U}^m$ for every $m \geq 1$, with $\Phi(0, u) = 1$ and $\psi(0, u) = u$ for all

$u \in \mathcal{U}$ (see Proposition 2.6), it follows that there exists some $\eta > 0$ such that for all $t \in [0, \eta]$

$$\inf_{u \in K} |\Phi(t, u)| > c \text{ and } \sup_{u \in K} \|\operatorname{Re}\psi(t, u)\|^2 + \|\operatorname{Im}\psi(t, u)\|^2 < C, \quad (3.14)$$

with some constants c and C . By fixing these constants and some linearly independent vectors in K as described in Lemma 3.2, we define

$$\varepsilon := \min(\eta, \delta), \quad (3.15)$$

where $\delta > 0$ is given in Lemma 3.2.

Moreover, let $t \geq 0$ be fixed. Then we denote by $I_{t,\varepsilon}^T$ the set

$$I_{t,\varepsilon}^T := (t, t + \varepsilon) \cap \mathcal{T}, \quad (3.16)$$

where $\mathcal{T}$ is defined in (3.3).

Proposition 3.4. *Let K and $I_{t,\varepsilon}^T$ be the sets defined in (3.5) and (3.16). Consider the function ψ given in (2.4) with the properties of Proposition 2.6. Let $t \geq 0$ be fixed and consider a sequence $(q_k)_{k \in \mathbb{N}}$ with values in $\mathbb{Q}_+$ such that $q_k \uparrow t$. Moreover, let $(x_{q_k})_{k \in \mathbb{N}}$ be a sequence with values in $D_\Delta \cup \{\infty\}$. Here, ∞ corresponds to a “point at infinity” and $D_\Delta \cup \{\infty\}$ is the one-point compactification of D_Δ .³ Then the following assertions hold:*

- (i) *If for all $(T, u) \in I_{t,\varepsilon}^T \times K$*

$$\lim_{k \rightarrow \infty} N_{q_k}^{T,u} := \lim_{k \rightarrow \infty} e^{\langle \psi(T - q_k, u), x_{q_k} \rangle} \quad (3.17)$$

exists finitely valued and does not vanish, then also $\lim_{k \rightarrow \infty} x_{q_k}$ exists finitely valued.

- (ii) *If there exist some $(T, u) \in I_{t,\varepsilon}^T \times K$ such that*

$$\lim_{k \rightarrow \infty} N_{q_k}^{T,u} := \lim_{k \rightarrow \infty} e^{\langle \psi(T - q_k, u), x_{q_k} \rangle} = 0,$$

then we have $\lim_{k \rightarrow \infty} \|x_{q_k}\| = \infty$.

Moreover, let $(q_k^T)_{k \in \mathbb{N}, T \in I_{t,\varepsilon}^T}$ be a family of sequences with values in $\mathbb{Q}_+ \cap [t, T]$ such that $q_k^T \downarrow t$ for every $T \in I_{t,\varepsilon}^T$ and the additional property that for every $S, T \in I_{t,\varepsilon}^T$, with $S < T$, there exists some index $N \in \mathbb{N}$ such that, for all $k \geq N$, $q_{k-N}^S = q_k^T$. Then the above assertions hold true for these right limits with q_k replaced by q_k^T .

Remark 3.5. Concerning assertion (ii) of Proposition 3.4, note that, e.g. in the case $q_k \uparrow t$, $\lim_{k \rightarrow \infty} \|x_{q_k}\| = \infty$ corresponds either to explosion or to the possibility that there exists some index $N \in \mathbb{N}$ such that $x_{q_k} = \Delta$ for all $k \geq N$. In the latter case we also have, due to the convention $\|\Delta\| = \infty$, $\lim_{k \rightarrow \infty} \|x_{q_k}\| = \infty$.

Proof. We start by proving the first assertion (i). Let $T \in I_{t,\varepsilon}^T$ be fixed and define for all $u \in K$

$$A(u) := \limsup_{k \rightarrow \infty} \langle \operatorname{Re}\psi(T - q_k, u), x_{q_k} \rangle, \quad a(u) := \liminf_{k \rightarrow \infty} \langle \operatorname{Re}\psi(T - q_k, u), x_{q_k} \rangle.$$

³If the state space D is compact, we do not adjoin $\{\infty\}$ and only consider a sequence with values in D_Δ .

Then there exist subsequences $(x_{q_{k_m}})$ and $(x_{q_{k_l}})$ such that⁴

$$\begin{aligned} A(u) &= \lim_{m \rightarrow \infty} \langle \operatorname{Re} \psi(T - q_{k_m}, u), x_{q_{k_m}} \rangle, \\ a(u) &= \lim_{l \rightarrow \infty} \langle \operatorname{Re} \psi(T - q_{k_l}, u), x_{q_{k_l}} \rangle. \end{aligned}$$

First note that $A(u)$ and $a(u)$ exist finitely valued for all $u \in K$. Indeed, if there is some $u \in K$ such that $A(u) = \pm\infty$ or $a(u) = \pm\infty$, then the limit of $N_{q_k}^{T,u}$ does not exist, or $\lim_{k \rightarrow \infty} N_{q_k}^{T,u}$ is either 0 or $+\infty$, which contradicts assumption (3.17). We now define

$$\begin{aligned} r_1(u) &= \lim_{m \rightarrow \infty} \exp \left(\langle \operatorname{Re} \psi(T - q_{k_m}, u), x_{q_{k_m}} \rangle \right), \\ r_2(u) &= \lim_{l \rightarrow \infty} \exp \left(\langle \operatorname{Re} \psi(T - q_{k_l}, u), x_{q_{k_l}} \rangle \right), \\ \varphi_m(u) &= \langle \operatorname{Im} \psi(T - q_{k_m}, u), x_{q_{k_m}} \rangle, \\ \varphi_l(u) &= \langle \operatorname{Im} \psi(T - q_{k_l}, u), x_{q_{k_l}} \rangle. \end{aligned}$$

Then the limits of $\cos(\varphi_m(u))$, $\cos(\varphi_l(u))$, $\sin(\varphi_m(u))$ and $\sin(\varphi_l(u))$ necessarily exist and

$$\begin{aligned} r_1(u) \lim_{m \rightarrow \infty} \cos(\varphi_m(u)) &= r_2(u) \lim_{l \rightarrow \infty} \cos(\varphi_l(u)), \\ r_1(u) \lim_{m \rightarrow \infty} \sin(\varphi_m(u)) &= r_2(u) \lim_{l \rightarrow \infty} \sin(\varphi_l(u)). \end{aligned}$$

This yields $r_1(u) = r_2(u)$ for all $u \in K$, since

$$\lim_{m \rightarrow \infty} (\cos^2(\varphi_m(u)) + \sin^2(\varphi_m(u))) = \lim_{l \rightarrow \infty} (\cos^2(\varphi_l(u)) + \sin^2(\varphi_l(u))) = 1.$$

In particular, we have proved that

$$\lim_{k \rightarrow \infty} \langle \operatorname{Re} \psi(T - q_k, u), x_{q_k} \rangle \quad (3.18)$$

exists finitely valued and does not vanish for all $(T, u) \in I_{t,\varepsilon}^T \times K$. Choosing linear independent vectors $(u_1, \dots, u_p) \in K \cap \operatorname{Re} \mathcal{U}$ thus implies that

$$\lim_{k \rightarrow \infty} \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} x_{q_k}$$

exists finitely valued.

Therefore it only remains to focus on $\Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} x_{q_k}$. From the above, we know in particular that for all $(T, u) \in I_{t,\varepsilon}^T \times K$

$$\lim_{k \rightarrow \infty} e^{\langle \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} \psi(T - q_k, u), \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} x_{q_k} \rangle} \quad (3.19)$$

exists finitely valued and does not vanish. This implies that for all $(T, u) \in I_{t,\varepsilon}^T \times K$

$$\operatorname{Im} \left\langle \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} \psi(T - q_k, u), \Pi_{\langle \operatorname{Re} \mathcal{U} \rangle^\perp} x_{q_k} \right\rangle = \alpha_k(T, u) + 2\pi z_k(T, u), \quad (3.20)$$

where $\alpha_k(T, u) \in [-\pi, \pi)$, $\alpha(T, u) := \lim_{k \rightarrow \infty} \alpha_k(T, u)$ exists finitely valued and $(z_k(T, u))_{k \in \mathbb{N}}$ is a sequence with values in $\mathbb{Z}$, which a priori does not necessarily have a limit and/or $\lim_{k \rightarrow \infty} z_k(T, u) = \pm\infty$.

⁴Note that these subsequences depend on u . For notational convenience we however suppress the dependence on u .

Let us first assume that

$$\limsup_{k \rightarrow \infty} \|\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k}\| = \infty. \quad (3.21)$$

Then we are exactly in the situation of Lemma 3.3 and the above limit (3.19) can be written as

$$\lim_{k \rightarrow \infty} e^{(\langle \sum_i \lambda_i(T-q_k), u \rangle \langle g_i, x_{q_k} \rangle + \langle \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T-q_k, u), x_{q_k} - \sum_i g_i \langle g_i, x_{q_k} \rangle \rangle)}$$

for all $u \in \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} K$ with $\|\text{Im } u\| < P(T)$, where $P(T)$ is defined by $P(T) := \inf_k R(T - q_k)$ and R and the directions g_i are given in Lemma 3.3 after possibly selecting a subsequence such that $x_{q_k} - \sum_i g_i \langle g_i, x_{q_k} \rangle$ converges as $k \rightarrow \infty$. Note that due to the strict positivity and continuity of R , $P(T)$ is strictly positive as well. Furthermore, there exists some $T^* \in I_{t, \varepsilon}^T$ and some set $M_{T^*} \subseteq \{u \in \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} K \mid \|\text{Im } u\| < P(T^*), \sum_i \langle \lambda_i(T^* - t), u \rangle \neq 0\}$ of positive finite measure such that

$$\lim_{k \rightarrow \infty} \int_{M_{T^*}} e^{\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T^* - q_k, u), x_{q_k} - \sum_i g_i \langle g_i, x_{q_k} \rangle \rangle} e^{(\langle \sum_i \lambda_i(T^* - q_k), u \rangle \langle g_i, x_{q_k} \rangle)} du \neq 0. \quad (3.22)$$

However, it follows from the Riemann-Lebesgue Lemma that the previous limit is zero, whence contradicting (3.22). We therefore conclude that

$$\limsup_{k \rightarrow \infty} \|\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k}\| < \infty.$$

This in turn implies that there exists some $(T^*, u^*) \in I_{t, \varepsilon}^T \times K$ and $N \in \mathbb{N}$ such that for all $k \geq N$

$$\text{Im} \left\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T^* - q_k, u^*), \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k} \right\rangle \in (-\pi, \pi).$$

Indeed, this follows from the fact that for every $u \in K$ and $\eta > 0$ there exists some $T^* \in I_{t, \varepsilon}^T$ and $N \in \mathbb{N}$ such that for all $k \geq N$

$$\|\text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T^* - q_k, u) - \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} u)\| \leq \eta. \quad (3.23)$$

For u^* with $\|\text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} u^*)\|$ sufficiently small and k sufficiently large, we thus have

$$\begin{aligned} & \left| \left\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T^* - q_k, u^*), \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k} \right\rangle \right| \\ & \leq (\|\text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} u^*)\| + \|\text{Im}(\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T^* - q_k, u^*) - \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} u^*)\|) \\ & \quad \times (\limsup_{k \rightarrow \infty} \|\Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k}\| + 1) \\ & < \pi. \end{aligned}$$

Hence,

$$\lim_{k \rightarrow \infty} \text{Im} \left\langle \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} \psi(T^* - q_k, u^*), \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k} \right\rangle = \alpha(T^*, u^*). \quad (3.24)$$

As we can find $n - p$ linear independent vectors $u_{p+1}, \dots, u_n$ such that (3.24) is satisfied, we conclude using Lemma 3.2 that

$$\lim_{k \rightarrow \infty} \Pi_{\langle \text{Re } \mathcal{U} \rangle^\perp} x_{q_k}$$

exists finitely valued. This proves assertion (i).

Concerning the second statement, observe that we have

$$\lim_{k \rightarrow \infty} e^{\langle \psi(T-q_k, u), x_{q_k} \rangle} = 0, \quad (3.25)$$

if either explosion occurs or if x_{q_N} jumps to Δ for some $N \in \mathbb{N}$ and $x_{q_k} = \Delta$ for all $k \geq N$. (This happens when the corresponding process is killed.) Indeed, since (3.25) is equivalent to $\lim_{k \rightarrow \infty} e^{\langle \text{Re}\psi(T-q_k, u), x_{q_k} \rangle} = 0$ and as $\psi(T-t, u)$ is bounded on K due to the definition of $I_{t, \varepsilon}^T$, we necessarily have

$$\lim_{k \rightarrow \infty} \|x_{q_k}\| = \infty.$$

In the case of a jump to Δ , this is implied by the conventions $\|\Delta\| = \infty$ and $f(\Delta) = 0$ for any other function.

Similar arguments yield the assertion concerning right limits. $\square$

Using Proposition 3.1 and Proposition 3.4 above, we are now prepared to prove Theorem 3.7 below, which asserts the existence of a càdlàg version of X . Before stating this result, let us recall the notion of the (usual) augmentation of $(\mathcal{F}_t^0)$ with respect to $\mathbb{P}_x$, which guarantees the càdlàg version to be adapted.

Definition 3.6 (Usual augmentation). *We denote by $\mathcal{F}^x$ the completion of $\mathcal{F}^0$ with respect to $\mathbb{P}_x$. A sub- σ -algebra $\mathcal{G} \subset \mathcal{F}^x$ is called augmented with respect to $\mathbb{P}_x$ if $\mathcal{G}$ contains all $\mathbb{P}_x$ -null-sets in $\mathcal{F}^x$. The augmentation of $\mathcal{F}_t^0$ with respect to $\mathbb{P}_x$ is denoted by $\mathcal{F}_t^x$, that is, $\mathcal{F}_t^x = \sigma(\mathcal{F}_t^0, \mathcal{N}(\mathcal{F}^x))$, where $\mathcal{N}(\mathcal{F}^x)$ denotes all $\mathbb{P}_x$ -null-sets in $\mathcal{F}^x$.*

Theorem 3.7. *Let X be an affine process relative to $(\mathcal{F}_t^0)$. Then there exists a process $\tilde{X}$ such that, for each $x \in D_\Delta$, $\tilde{X}$ is a $\mathbb{P}_x$ -version of X , which is càdlàg in $D_\Delta \cup \{\infty\}$ (in D_Δ respectively if D is compact) and an affine process relative to $(\mathcal{F}_t^x)$. As before, ∞ corresponds to a “point at infinity” and $D_\Delta \cup \{\infty\}$ is the one-point compactification of D_Δ , if D is non-compact.*

Remark 3.8. *We here establish the existence of a càdlàg version $\tilde{X}$ whose sample paths may take ∞ as left limiting value if D is non-compact. A priori, we cannot identify $\tilde{X}_{s-}(\omega)$ with Δ , whenever $\|\tilde{X}_{s-}(\omega)\| = \infty$. Indeed, $\tilde{X}_t(\omega)$ might become finitely valued for some $t \geq s$. This issue is clarified in Theorem 3.10 below, where we prove that $\mathbb{P}_x$ -a.s. $\|\tilde{X}_t\| = \infty$ for all $t \geq s$ and all $s > 0$ if $\|\tilde{X}_{s-}\| = \infty$. In particular, this allows us to identify ∞ with Δ .*

In the case $\tilde{X}_s = \Delta$, which happens when the process is killed, Assumption (2.1) guarantees that $\tilde{X}_t = \Delta$ for all $t \geq s$ and all $s > 0$.

Proof. It follows from Proposition 3.1 that for every $\omega \in \tilde{\Omega}^5$, where $\mathbb{P}_x[\tilde{\Omega}] = 1$,

$$t \mapsto M_t^{T, u}(\omega) := \Phi(T-t, u) e^{\langle \psi(T-t, u), X_t(\omega) \rangle}, \quad t \in [0, T],$$

is the restriction to $\mathbb{Q}_+ \cap [0, T]$ of a càdlàg function for all $(T, u) \in \mathcal{T} \times \mathcal{V}$. Here, $\tilde{\Omega}$, $\mathcal{T}$ and $\mathcal{V}$ are defined in (3.2), (3.3) and (3.4). Hence, for every $\omega \in \tilde{\Omega}$ and all $(T, u) \in \mathcal{T} \times \mathcal{V}$, the limits

$$\lim_{\substack{q \in \mathbb{Q}_+ \\ q \uparrow t}} M_q^{T, u}(\omega), \quad \lim_{\substack{q \in \mathbb{Q}_+ \\ q \downarrow t}} M_q^{T, u}(\omega) \quad (3.26)$$

exist finitely valued for all $t \in [0, T]$.

⁵Note that due to the measurable projection theorem, $\tilde{\Omega} \in \mathcal{F}^x$.

Let us now show that the same holds true for X . For notational convenience we first focus on left limits. Consider the sets K and $I_{t,\varepsilon}^\mathcal{T}$ defined in (3.5) and (3.16) and let $t \geq 0$ be fixed. Take some sequence $(q_k)_{k \in \mathbb{N}}$, as specified in Proposition 3.4, such that $q_k \uparrow t$. Then there exists some $N \in \mathbb{N}$ such that, for all $k \geq N$ and $(T, u) \in I_{t,\varepsilon}^\mathcal{T} \times K$, $\Phi(T - q_k, u) \neq 0$. This is a consequence of the definition of ε (see (3.15)). Thus we can divide $M_{q_k}^{T,u}(\omega)$ by $\Phi(T - q_k, u)$ for all $k \geq N$ and $(T, u) \in I_{t,\varepsilon}^\mathcal{T} \times K$. By the continuity of $t \mapsto \Phi(t, u)$ and (3.26), it follows that, for every $\omega \in \tilde{\Omega}$, the limit

$$\lim_{k \rightarrow \infty} N_{q_k}^{T,u}(\omega) := \lim_{k \rightarrow \infty} e^{\langle \psi(T - q_k, u), X_{q_k}(\omega) \rangle}$$

exists finitely valued for all $(T, u) \in I_{t,\varepsilon}^\mathcal{T} \times K$. From Proposition 3.4 we thus deduce that, for every $\omega \in \tilde{\Omega}$, the limit

$$\lim_{k \rightarrow \infty} X_{q_k}(\omega)$$

exists either finitely valued or $\lim_{k \rightarrow \infty} \|X_{q_k}(\omega)\| = \infty$. Using similar arguments yields the same assertion for right limits. Hence we can conclude that $\mathbb{P}_x$ -a.s.

$$\tilde{X}_t = \lim_{\substack{q \in \mathbb{Q}_+ \\ q \downarrow t}} X_q \quad (3.27)$$

exists for all $t \geq 0$ and defines a càdlàg function in t .

Let now Ω_0 be the set of $\omega \in \Omega$ for which the limit $\tilde{X}_t(\omega)$ exists for every t and defines a càdlàg function in t . Then, as a consequence of Rogers and Williams (1994, Theorem II.62.7, Corollary II.62.12), $\Omega_0 \in \mathcal{F}^0$ and $\mathbb{P}_x[\Omega_0] = 1$ for all $x \in D_\Delta$. For $\omega \in \Omega \setminus \Omega_0$, we set $\tilde{X}_t(\omega) = \Delta$ for all t . Then $\tilde{X}$ is a càdlàg process and $\tilde{X}_t$ is $\mathcal{F}^0$ -measurable for every $t \geq 0$. Since X is assumed to be stochastically continuous, we have $X_s \rightarrow X_t$ in probability as $s \rightarrow t$. Using the fact that convergence in probability implies almost sure convergence along a subsequence, we have

$$\mathbb{P}_x \left[\lim_{\substack{q \in \mathbb{Q}_+ \\ q \downarrow t}} X_q = X_t \right] = 1. \quad (3.28)$$

By our definition of $\tilde{X}_t$, the limit in (3.28) is equal to $\tilde{X}_t$ on Ω_0 . Hence, for all $x \in D_\Delta$, we have $\mathbb{P}_x[\tilde{X}_t = X_t]$ for each t , implying that $\tilde{X}$ is a version of X . This then also yields

$$\mathbb{E}_x \left[e^{\langle u, \tilde{X}_t \rangle} \right] = \mathbb{E}_x \left[e^{\langle u, X_t \rangle} \right]$$

and augmentation of $(\mathcal{F}_t^0)$ with respect to $\mathbb{P}_x$ ensures that $\tilde{X}_t \in \mathcal{F}_t^x$ for each t . We therefore conclude that $\tilde{X}$ is an affine process with respect to $(\mathcal{F}_t^x)$. $\square$

If D is non-compact, the càdlàg version (3.27) on $D_\Delta \cup \{\infty\}$, still denoted by X , can be realized on the space $\Omega' = \mathbb{D}'(D_\Delta \cup \{\infty\})$ of càdlàg paths $\omega : \mathbb{R}_+ \rightarrow D_\Delta \cup \{\infty\}$ with $\omega(t) = \Delta$ for $t \geq s$, whenever $\omega(s) = \Delta$. However, we still have to prove that we can identify ∞ with Δ , as mentioned in Remark 3.8. In other words, we have to show that $\|\omega(t)\| = \infty$ for all $t \geq s$ if explosion occurs for some $s > 0$, that is, $\|\omega(s-)\| = \infty$. This is stated in the Theorem 3.10 below. For its proof let us introduce the following notations:

Due to the convention $\|\Delta\| = \infty$, we define the *explosion time* by (see Cheridito, Filipović, and Yor (2005) for a similar definition)

$$T_{\text{expl}} := \begin{cases} T_{\Delta}, & \text{if } T'_k < T_{\Delta} \text{ for all } k, \\ \infty, & \text{if } T'_k = T_{\Delta} \text{ for some } k, \end{cases}$$

where the stopping times T_{Δ} and T'_k are given by

$$\begin{aligned} T_{\Delta} &:= \inf\{t > 0 \mid \|X_{t-}\| = \infty \text{ or } \|X_t\| = \infty\}, \\ T'_k &:= \inf\{t \mid \|X_{t-}\| \geq k \text{ or } \|X_t\| \geq k\}, \quad k \geq 1. \end{aligned}$$

Moreover, we denote by $\text{relint}(C)$ the *relative interior* of a set C defined by

$$\text{relint}(C) = \{x \in C \mid \overline{B}(x, r) \cap \text{aff}(C) \subseteq C \text{ for some } r > 0\},$$

where $\text{aff}(C)$ denotes the affine hull of C .

Lemma 3.9. *Let X be an affine process with càdlàg paths in $D_{\Delta} \cup \{\infty\}$ and let $x \in D$ be fixed. If*

$$\mathbb{P}_x[T_{\text{expl}} < \infty] > 0, \quad (3.29)$$

then $\text{relint}(\text{Re}\mathcal{U}) \neq \emptyset$ and we have $\mathbb{P}_x$ -a.s.

$$\lim_{t \uparrow T_{\text{expl}}} e^{\langle u, X_t \rangle} = 0$$

for all $u \in \text{relint}(\text{Re}\mathcal{U})$.

Proof. Let us first establish that under Assumption (3.29), $\text{relint} \text{Re}\mathcal{U} \neq \emptyset$. To this end, we denote by Ω_{expl} the set

$$\Omega_{\text{expl}} = \{\omega \in \Omega' \mid T_{\text{expl}}(\omega) < \infty\}.$$

Then it follows from Proposition 3.1 and 3.4 that, for $\mathbb{P}_x$ -almost every $\omega \in \Omega_{\text{expl}}$, there exist some $(T(\omega), v(\omega)) \in (T_{\text{expl}}(\omega), \infty) \times \mathfrak{i}V$ such that

$$\lim_{t \uparrow T_{\text{expl}}(\omega)} \Phi(T(\omega) - t, v(\omega)) \neq 0$$

and

$$\lim_{t \uparrow T_{\text{expl}}(\omega)} N_t^{T(\omega), v(\omega)}(\omega) = \lim_{t \uparrow T_{\text{expl}}(\omega)} e^{\langle \psi(T(\omega) - t, v(\omega)), X_t(\omega) \rangle} = 0. \quad (3.30)$$

This implies that

$$\lim_{t \uparrow T_{\text{expl}}(\omega)} \langle \text{Re}\psi(T(\omega) - t, v(\omega)), X_t(\omega) \rangle = -\infty, \quad (3.31)$$

and in particular that $\mathcal{U} \ni \text{Re}\psi(T(\omega) - T_{\text{expl}}(\omega), v(\omega)) \neq 0$, which proves the claim, since $\text{Re}\mathcal{U} \subseteq \overline{\text{relint}(\text{Re}\mathcal{U})}$.

Furthermore, by (3.31) we have

$$\lim_{t \uparrow T_{\text{expl}}(\omega)} \|\Pi(X_t(\omega))\| = \infty, \quad (3.32)$$

where Π denotes the projection on the linear hull of $\text{Re}\mathcal{U}$, denoted by $\langle \text{Re}\mathcal{U} \rangle$. Define the vector space W by

$$W = \text{Re}\mathcal{U} \cap (-\text{Re}\mathcal{U}).$$

By the definition of $\mathcal{U}$, $|\langle w, X_t(\omega) \rangle|$ is bounded for all $t \leq T_{\text{expl}}(\omega)$ and $w \in W$, which implies that

$$\lim_{t \uparrow T_{\text{expl}}(\omega)} \|\Pi_1(X_t(\omega))\| = \infty, \quad (3.33)$$

where Π_1 denotes the projection on the orthogonal complement of W in $\langle \text{Re}\mathcal{U} \rangle$. As $\Pi_1(\text{Re}\mathcal{U})$ is a proper convex cone⁶ (see, e.g., Bruns and Gubeladze (2009, Proposition 1.18)), we thus have for all $u \in \text{relint}(\Pi_1(\text{Re}\mathcal{U}))$

$$\lim_{t \uparrow T_{\text{expl}}(\omega)} \langle u, X_t(\omega) \rangle = -\infty.$$

Writing $\text{relint}(\text{Re}\mathcal{U})$ as

$$\text{relint}(\text{Re}\mathcal{U}) = W + \text{relint}(\Pi_1(\text{Re}\mathcal{U})),$$

then yields the assertion, since $|\langle w, X_t(\omega) \rangle|$ is bounded for all $t \leq T_{\text{expl}}(\omega)$ and $w \in W$. $\square$

Theorem 3.10. *Let X be an affine process with càdlàg paths in $D_\Delta \cup \{\infty\}$. Then, for every $x \in D$, the following assertion holds $\mathbb{P}_x$ -a.s.: If*

$$\|X_{s-}\| = \infty, \tag{3.34}$$

then $\|X_t\| = \infty$ for all $t \geq s$ and $s \geq 0$. Identifying ∞ with Δ , then yields $X_t = \Delta$ for all $t \geq s$.

Proof. Let $x \in D$ be fixed and let $u \in \text{relint}(\text{Re}\mathcal{U})$. Note that by Lemma 3.9 $\text{relint}(\text{Re}\mathcal{U}) \neq \emptyset$ and that $\Phi(t, u)$ and $\psi(t, u)$ are real-valued functions with values in $\mathbb{R}_{++}$ and $\text{Re}\mathcal{U}$, respectively. Take now some $T > 0$ and $\delta > 0$ such that

$$\mathbb{P}_x [T - \delta < T_{\text{expl}} \leq T] > 0,$$

and $\psi(t, u) \in \text{relint}(\text{Re}\mathcal{U})$ for all $t < \delta$. Consider the martingale

$$M_t^{T,u} = \Phi(T - t, u) e^{\langle \psi(T-t, u), X_t \rangle}, \quad t \leq T,$$

which is clearly nonnegative and has càdlàg paths. Moreover, by the choice of δ , it follows from Lemma 3.9 and the conventions $\|\Delta\| = \infty$ and $f(\Delta) = 0$ for any other function that $\mathbb{P}_x$ -a.s.

$$M_{s-}^{T,u} = 0, \quad s \in (T - \delta, T], \tag{3.35}$$

if and only if $\|X_{s-}\| = \infty$ for $s \in (T - \delta, T]$. We thus conclude using Rogers and Williams (1994, Theorem II.78.1) that $\mathbb{P}_x$ -a.s. $M_t^{T,u} = 0$ for all $t \geq s$, which in turn implies that $\|X_t\| = \infty$ for all $t \geq s$. This allows us to identify ∞ with Δ and we obtain $X_t = \Delta$ for all $t \geq s$. Since T was chosen arbitrarily, the assertion follows. $\square$

Combining Theorem 3.7 and Theorem 3.10 and using Assumption (2.1), we thus obtain the following statement:

Corollary 3.11. *Let X be an affine process relative to $(\mathcal{F}_t^0)$. Then there exists a process $\tilde{X}$ such that, for each $x \in D_\Delta$, $\tilde{X}$ is a $\mathbb{P}_x$ -version of X , which is an affine process relative to $(\mathcal{F}_t^x)$, whose paths are càdlàg and satisfy $\mathbb{P}_x$ -a.s. $\tilde{X}_t = \Delta$ for $t \geq s$, whenever $\|\tilde{X}_{s-}\| = \infty$ or $\|\tilde{X}_s\| = \infty$.*

Remark 3.12. *We will henceforth always assume that we are using the càdlàg version of an affine process, given in Corollary 3.11, which we still denote by X . Under this assumption X can now be realized on the space $\Omega = \mathbb{D}(D_\Delta)$ of càdlàg paths $\omega : \mathbb{R}_+ \rightarrow D_\Delta$ with $\omega(t) = \Delta$ for $t \geq s$, whenever $\|\omega(s-)\| = \infty$ or $\|\omega(s)\| = \infty$. The canonical realization of an affine process X is then defined by $X_t(\omega) = \omega(t)$. Moreover, we make the convention that $X_\infty = \Delta$, which allows us to write certain formulas without restriction.*

⁶A cone is called *proper* if $K \cap (-K) = \{0\}$.

4. RIGHT-CONTINUITY OF THE FILTRATION AND STRONG MARKOV PROPERTY

Using the existence of a right-continuous version of an affine process, we can now show that $(\mathcal{F}_t^x)$, that is, the augmentation of $(\mathcal{F}_t^0)$ with respect to $\mathbb{P}_x$, is right-continuous.

Theorem 4.1. *Let $x \in D$ be fixed and let X be an affine process relative to $(\mathcal{F}_t^x)$ with càdlàg paths. Then $(\mathcal{F}_t^x)$ is right-continuous.*

Proof. We adapt the proof of Protter (2005, Theorem I.4.31) to our setting. We have to show that for every $t \geq 0$, $\mathcal{F}_{t+}^x = \mathcal{F}_t^x$, where $\mathcal{F}_{t+}^x = \bigcap_{s>t} \mathcal{F}_s^x$. Since the filtration is increasing, it suffices to show that $\mathcal{F}_t^x = \bigcap_{n \geq 1} \mathcal{F}_{t+\frac{1}{n}}^x$. In particular, we only need to prove that

$$\mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \dots + \langle u_k, X_{t_k} \rangle} \middle| \mathcal{F}_t^x \right] = \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \dots + \langle u_k, X_{t_n} \rangle} \middle| \mathcal{F}_{t+}^x \right] \quad (4.1)$$

for all $(t_1, \dots, t_k)$ and all $(u_1, \dots, u_k)$ with $t_i \in \mathbb{R}_+$ and $u_i \in \mathcal{U}$, since this implies $\mathbb{E}_x[Z | \mathcal{F}_t^x] = \mathbb{E}_x[Z | \mathcal{F}_{t+}^x]$ for every bounded $Z \in \mathcal{F}^x$. As both $\mathcal{F}_{t+}^x$ and $\mathcal{F}_t^x$ contain the nullsets $\mathcal{N}(\mathcal{F}^x)$, this then already yields $\mathcal{F}_{t+}^x = \mathcal{F}_t^x$ for all $t \geq 0$.

In order to prove (4.1), let $t \geq 0$ be fixed and take first $t_1 \leq t_2 \leq \dots \leq t_k \leq t$. Then we have for all $(u_1, \dots, u_k)$

$$\begin{aligned} \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \dots + \langle u_k, X_{t_k} \rangle} \middle| \mathcal{F}_t^x \right] &= \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \dots + \langle u_k, X_{t_k} \rangle} \middle| \mathcal{F}_{t+}^x \right] \\ &= e^{\langle u_1, X_{t_1} \rangle + \dots + \langle u_k, X_{t_k} \rangle}. \end{aligned}$$

In the case $t_k > t_{k-1} > \dots > t_1 > t$, we give the proof for $k = 2$ for notational convenience. Let $t_2 > t_1 > t$ and fix $u_1, u_2 \in \mathcal{U}$. Then we have by the affine property

$$\begin{aligned} \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \langle u_2, X_{t_2} \rangle} \middle| \mathcal{F}_{t+}^x \right] &= \lim_{s \downarrow t} \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \langle u_2, X_{t_2} \rangle} \middle| \mathcal{F}_s^x \right] \\ &= \lim_{s \downarrow t} \mathbb{E}_x \left[\mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \langle u_2, X_{t_2} \rangle} \middle| \mathcal{F}_{t_1}^x \right] \middle| \mathcal{F}_s^x \right] \\ &= \Phi(t_2 - t_1, u_2) \lim_{s \downarrow t} \mathbb{E}_x \left[e^{\langle u_1 + \psi(t_2 - t_1, u_2), X_{t_1} \rangle} \middle| \mathcal{F}_s^x \right]. \end{aligned}$$

If $\Phi(t_2 - t_1, u_2) = 0$, it follows by the same step that

$$\mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \langle u_2, X_{t_2} \rangle} \middle| \mathcal{F}_t^x \right] = 0,$$

too. Otherwise, we have by Proposition 2.6 (ii), $\psi(t_2 - t_1, u_2) \in \mathcal{U}$, and by the definition of $\mathcal{U}$ also $u_1 + \psi(t_2 - t_1, u_2) \in \mathcal{U}$. Hence, again by the affine property and right-continuity of $t \mapsto X_t(\omega)$, the above becomes

$$\begin{aligned} \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \langle u_2, X_{t_2} \rangle} \middle| \mathcal{F}_{t+}^x \right] &= \Phi(t_2 - t_1, u_2) \lim_{s \downarrow t} \Phi(t_1 - s, u_1 + \psi(t_2 - t_1, u_2)) e^{\langle \psi(t_1 - s, u_1 + \psi(t_2 - t_1, u_2)), X_s \rangle} \\ &= \Phi(t_2 - t_1, u_2) \Phi(t_1 - t, u_1 + \psi(t_2 - t_1, u_2)) e^{\langle \psi(t_1 - t, u_1 + \psi(t_2 - t_1, u_2)), X_t \rangle} \\ &= \mathbb{E}_x \left[e^{\langle u_1, X_{t_1} \rangle + \langle u_2, X_{t_2} \rangle} \middle| \mathcal{F}_t^x \right]. \end{aligned}$$

This yields (4.1) and by the above arguments we conclude that $\mathcal{F}_{t+}^x = \mathcal{F}_t^x$ for all $t \geq 0$. $\square$

Remark 4.2. A consequence of Theorem 4.1 is that $(\Omega, \mathcal{F}_t, (\mathcal{F}_t^x), \mathbb{P}_x)$ satisfies the usual conditions, since

- (i) $\mathcal{F}^x$ is $\mathbb{P}_x$ -complete,
- (ii) $\mathcal{F}_0^x$ contains all $\mathbb{P}_x$ -null-sets in $\mathcal{F}^x$,
- (iii) $(\mathcal{F}_t^x)$ is right-continuous.

Let us now set

$$\mathcal{F} := \bigcap_{x \in D_\Delta} \mathcal{F}^x, \quad \mathcal{F}_t := \bigcap_{x \in D_\Delta} \mathcal{F}_t^x. \quad (4.2)$$

Then $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ does not necessarily satisfy the usual conditions, but $\mathcal{F}_t = \mathcal{F}_{t+}$ still holds true. Moreover, it follows e.g. from Revuz and Yor (1999, Proposition III.2.12, III.2.14) that, for each t , X_t is $\mathcal{F}_t$ -measurable and a Markov process with respect to $(\mathcal{F}_t)$.

Unless otherwise mentioned, we henceforth always consider affine processes on the filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t))$, where $\Omega = \mathbb{D}(D_\Delta)$, as described in Remark 3.12, and $\mathcal{F}$, $\mathcal{F}_t$ are given by (4.2). Notice that these assumptions on the probability space correspond to the standard setting considered for Feller processes (compare Rogers and Williams (1994, Definition III.7.16, III.9.2)).

Similar as in the case of Feller processes, we can now formulate and prove the strong Markov property for affine processes using the above setting and in particular the right-continuity of the sample paths.

Theorem 4.3. Let X be an affine process and let T be a $(\mathcal{F}_t)$ -stopping time. Then for each bounded Borel measurable function f and $s \geq 0$

$$\mathbb{E}_x [f(X_{T+s}) | \mathcal{F}_T] = \mathbb{E}_{X_T} [f(X_s)], \quad \mathbb{P}_x\text{-a.s.}$$

Proof. This result can be shown by the same arguments used to prove the strong Markov property of Feller processes (see, e.g., Rogers and Williams (1994, Theorem 8.3, Theorem 9.4)), namely by using a dyadic approximation of the stopping time T and applying the Markov property. Instead of using C_0 -functions and the Feller property, we here consider the family of functions $\{x \mapsto e^{\langle u, x \rangle} \mid u \in iV\}$ and the affine property, which asserts in particular that

$$x \mapsto \mathbb{E}_x [e^{\langle u, X_t \rangle}] = P_t e^{\langle u, x \rangle} = \Phi(t, u) e^{\langle \psi(t, u), x \rangle}$$

is continuous. This together with the right-continuity of paths then implies for every $\Lambda \in \mathcal{F}_T$ and $u \in iV$

$$\mathbb{E}_x [e^{\langle u, X_{T+s} \rangle} 1_\Lambda] = \mathbb{E}_x [P_s e^{\langle u, X_T \rangle} 1_\Lambda].$$

The assertion then follows by the same arguments as in Rogers and Williams (1994, Theorem 8.3) or Chung and Walsh (2005, Theorem 2.3.1). $\square$

5. SEMIMARTINGALE PROPERTY

We shall now relate affine processes to semimartingales, where, for every $x \in D$, semimartingales are understood with respect to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ defined above. By convention, we call X a semimartingale if $X 1_{[0, T_\Delta)}$ is a semimartingale, where – as a consequence of Theorem 3.10 and Corollary 3.11 – we can now define the *lifetime* T_Δ by

$$T_\Delta(\omega) = \inf\{t > 0 \mid X_t(\omega) = \Delta\}. \quad (5.1)$$

Let us start with the following definition for general Markov processes (compare Çinlar et al. (1980, Definition 7.1)):

Definition 5.1 (Extended Generator). *An operator $\mathcal{G}$ with domain $\mathcal{D}_{\mathcal{G}}$ is called extended generator for a Markov process X (relative to some filtration $(\mathcal{F}_t)$) if $\mathcal{D}_{\mathcal{G}}$ consists of those Borel measurable functions $f : D_{\Delta} \rightarrow \mathbb{C}$ for which there exists a function $\mathcal{G}f$ such that the process*

$$f(X_t) - f(x) - \int_0^t \mathcal{G}f(X_{s-}) ds$$

is a well-defined and $(\mathcal{F}_t, \mathbb{P}_x)$ -local martingale for every $x \in D_{\Delta}$.

In the following lemma we consider a particular class of functions for which it is possible to state the form of the extended generator for a Markov process in terms of its semigroup.

Lemma 5.2. *Let X be a D_{Δ} -valued Markov process relative to some filtration $(\mathcal{F}_t)$. Suppose that $u \in \mathcal{U}$ and $\eta > 0$. Consider the function*

$$g_{u,\eta} : D \rightarrow \mathbb{C}, x \mapsto g_{u,\eta}(x) := \frac{1}{\eta} \int_0^{\eta} P_s e^{\langle u, x \rangle} ds.$$

Then, for every $x \in D$,

$$M_t^u := g_{u,\eta}(X_t) - g_{u,\eta}(X_0) - \int_0^t \frac{1}{\eta} \left(P_{\eta} e^{\langle u, X_{s-} \rangle} - e^{\langle u, X_{s-} \rangle} \right) ds$$

is a (complex-valued) $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale and thus $g_{u,\eta}(X)$ is a (complex-valued) special semimartingale.

Proof. Since $g_{u,\eta}$ and $P_{\eta} e^{\langle u, \cdot \rangle} - e^{\langle u, \cdot \rangle}$ are bounded, M_t^u is integrable for each t and we have

$$\begin{aligned} \mathbb{E}_x [M_t^u | \mathcal{F}_r] &= M_r^u + \mathbb{E}_x \left[g_{u,\eta}(X_t) - g_{u,\eta}(X_r) - \int_r^t \frac{1}{\eta} \left(P_{\eta} e^{\langle u, X_{s-} \rangle} - e^{\langle u, X_{s-} \rangle} \right) ds \middle| \mathcal{F}_r \right] \\ &= M_r^u + \mathbb{E}_{X_r} \left[g_{u,\eta}(X_{t-r}) - g_{u,\eta}(X_0) - \int_0^{t-r} \frac{1}{\eta} \left(P_{\eta} e^{\langle u, X_{s-} \rangle} - e^{\langle u, X_{s-} \rangle} \right) ds \right] \\ &= M_r^u + \frac{1}{\eta} \int_{t-r}^{t-r+\eta} P_s e^{\langle u, X_r \rangle} ds - \frac{1}{\eta} \int_0^{\eta} P_s e^{\langle u, X_r \rangle} ds \\ &\quad - \frac{1}{\eta} \int_{\eta}^{t-r+\eta} P_s e^{\langle u, X_r \rangle} ds + \frac{1}{\eta} \int_0^{t-r} P_s e^{\langle u, X_r \rangle} ds \\ &= M_r^u. \end{aligned}$$

Hence M^u is $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale and thus $g_{u,\eta}(X)$ is a special semimartingale, since it is the sum of a martingale and a predictable finite variation process. $\square$

Remark 5.3. *Lemma 5.2 asserts that the extended generator applied to $g_{u,\eta}$ is given by $\mathcal{G}g_{u,\eta}(x) = \frac{1}{\eta} (P_{\eta} e^{\langle u, x \rangle} - e^{\langle u, x \rangle})$. Note that for general Markov processes and even for affine processes we do not know whether the “pointwise” infinitesimal generator applied to*

$$e^{\langle u, x \rangle} = \lim_{\eta \rightarrow 0} g_{u,\eta} = \lim_{\eta \rightarrow 0} \frac{1}{\eta} \int_0^{\eta} P_s e^{\langle u, x \rangle} ds,$$

that is,

$$\lim_{\eta \rightarrow 0} \frac{1}{\eta} \left(P_\eta e^{\langle u, x \rangle} - e^{\langle u, x \rangle} \right),$$

is well-defined or not.⁷ For this reason we consider the family of functions $\{x \mapsto g_{u,\eta}(x) \mid u \in \mathcal{U}, \eta > 0\}$, which exhibits in the case of affine processes similar properties as $\{x \mapsto e^{\langle u, x \rangle} \mid u \in \mathcal{U}\}$ (see Remark 5.6 (ii) and Lemma 5.7 below). These properties are introduced in the following definitions (compare Çinlar et al. (1980, Definition 7.7, 7.8)).

Definition 5.4 (Full Class). A class $\mathcal{C}$ of Borel measurable functions from D to $\mathbb{C}$ is said to be a full class if, for all $r \in \mathbb{N}$, there exists a finite family $\{f_1, \dots, f_N\} \in \mathcal{C}$ and a function $h \in C^2(\mathbb{C}^N, D)$ such that

$$x = h(f_1(x), \dots, f_N(x)) \quad (5.2)$$

for all $x \in D$ with $\|x\| \leq r$.

Definition 5.5 (Complete Class). Let $\beta \in V$, $\gamma \in S_+(V)$, where $S_+(V)$ denotes the positive semidefinite matrices over V , and let F be a positive measure on V , which integrates $(\|\xi\|^2 \wedge 1)$, satisfies $F(\{0\}) = 0$ and $x + \text{supp}(F) \subseteq D_\Delta$ for all $x \in D$. Moreover, let $\chi : V \rightarrow V$ denote some truncation function, that is, χ is bounded and satisfies $\chi(\xi) = \xi$ in a neighborhood of 0. A countable subset of functions $\tilde{\mathcal{C}} \subset C_b^2(D)$ is called complete if, for every $x \in D$, the countable collection of numbers

$$\begin{aligned} \kappa(f(x)) &= \langle \beta, \nabla f(x) \rangle + \frac{1}{2} \sum_{i,j} \gamma_{ij} D_{ij} f(x) \\ &\quad + \int_V (f(x + \xi) - f(x) - \langle \nabla f(x), \chi(\xi) \rangle) F(d\xi), \quad f \in \tilde{\mathcal{C}} \end{aligned} \quad (5.3)$$

completely determines β , γ and F . A class $\mathcal{C}$ of Borel measurable functions from D to $\mathbb{C}$ is said to be complete class if it contains such a countable set.

Remark 5.6. (i) Note that the integral in (5.3) is well-defined for all $f \in C_b^2(D)$. This is a consequence of the integrability assumption and the fact that $x + \text{supp}(F)$ is supposed to lie in D_Δ for all x .

(ii) The class of functions

$$\mathcal{C}^* := \left\{ D \rightarrow \mathbb{C}, x \mapsto e^{\langle u, x \rangle} \mid u \in iV \right\} \quad (5.4)$$

is a full and complete class. Indeed, for every $x \in D$ with $\|x\| \leq r$, we can find n linearly independent vectors $(u_1, \dots, u_n)$ such that

$$\text{Im} \langle u_i, x \rangle \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right].$$

This implies that x is given by

$$x = \left(\arcsin \left(\text{Im} e^{\langle u_1, x \rangle} \right), \dots, \arcsin \left(\text{Im} e^{\langle u_n, x \rangle} \right) \right) (\text{Im} u_1, \dots, \text{Im} u_n)^{-1}$$

and proves that $\mathcal{C}^*$ is a full class. Completeness follows by the same arguments as in Jacod and Shiryaev (2003, Lemma II.2.44).

⁷In the case of affine processes, this would be implied by the differentiability of Φ and ψ with respect to t , which we only prove in Section 6 using the results of this paragraph.

Lemma 5.7. *Let X be an affine process with Φ and ψ given in (2.4). Consider the class of functions*

$$\mathcal{C} := \left\{ D \rightarrow \mathbb{C}, x \mapsto g_{u,\eta}(x) := \frac{1}{\eta} \int_0^\eta \Phi(s, u) e^{\langle \psi(s, u), x \rangle} ds \mid u \in iV, \eta > 0 \right\}. \quad (5.5)$$

Then $\mathcal{C}$ is a full and complete class.

Proof. Let $(u_1, \dots, u_n) \in iV$ be n linearly independent vectors and define a function $f_\eta : D \rightarrow \mathbb{C}^n$ by $f_{\eta,i}(x) = g_{u_i,\eta}(x)$. Then the Jacobi matrix $J_{f_\eta}(x)$ is given by

$$\begin{pmatrix} \frac{1}{\eta} \int_0^\eta \Phi(s, u_1) e^{\langle \psi(s, u_1), x \rangle} \psi_1(s, u_1) ds & \dots & \frac{1}{\eta} \int_0^\eta \Phi(s, u_1) e^{\langle \psi(s, u_1), x \rangle} \psi_n(s, u_1) ds \\ \vdots & \ddots & \vdots \\ \frac{1}{\eta} \int_0^\eta \Phi(s, u_n) e^{\langle \psi(s, u_n), x \rangle} \psi_1(s, u_n) ds & \dots & \frac{1}{\eta} \int_0^\eta \Phi(s, u_n) e^{\langle \psi(s, u_n), x \rangle} \psi_n(s, u_n) ds \end{pmatrix}.$$

In particular, the imaginary part of each row tends to $(\cos(\operatorname{Im}\langle u_i, x \rangle) \operatorname{Im} u_i)^\top$ for $\eta \rightarrow 0$. Hence there exists some $\eta > 0$ such that the rows of $\operatorname{Im} J_{f_\eta}$ are linearly independent. As $\operatorname{Im} f_\eta : D \rightarrow \mathbb{R}^n$ is a $C^\infty(D)$ -function and as $J_{\operatorname{Im} f_\eta} = \operatorname{Im} J_{f_\eta}$, it follows from the inverse function theorem that, for each $x_0 \in D$, there exists some $r_0 > 0$ such that $\operatorname{Im} f_\eta : B(x_0, r_0) \rightarrow W$ has a $C^\infty(W)$ inverse, where $W = \operatorname{Im} f_\eta(B(x_0, r_0))$.

Let now $r \in \mathbb{N}$ and consider $x \in D$ with $\|x\| \leq r$. By choosing the linearly independent vectors $(u_1, \dots, u_n)$ and $\eta > 0$ appropriately, we can guarantee that $r_0 \geq \|x_0\| + r$. Indeed, for every $\delta > 0$, we can choose the linearly independent vectors $(u_1, \dots, u_n)$ such that $|\langle u_i, x \rangle| < \delta$. Assume now without loss of generality that $0 \in D$ and let $x_0 = 0$. As

$$\lim_{\eta \rightarrow 0} J_{\operatorname{Im} f_\eta}(x) = (\cos(\operatorname{Im}\langle u_1, x \rangle) \operatorname{Im} u_1, \dots, \cos(\operatorname{Im}\langle u_n, x \rangle) \operatorname{Im} u_n)^\top,$$

we can thus assure that for all $x \in \overline{B}(0, r) \cap D$

$$\begin{aligned} & \left\| \lim_{\eta \rightarrow 0} J_{\operatorname{Im} f_\eta}^{-1}(0) \lim_{\eta \rightarrow 0} J_{\operatorname{Im} f_\eta}(x) - I \right\| \\ &= \|(\operatorname{Im} u_1, \dots, \operatorname{Im} u_n)^{-\top} (\cos(\operatorname{Im}\langle u_1, x \rangle) \operatorname{Im} u_1, \dots, \cos(\operatorname{Im}\langle u_n, x \rangle) \operatorname{Im} u_n)^\top - I\| < 1 \end{aligned}$$

and by the continuity of the matrix inverse the same holds true for η small enough. The proof of the inverse function theorem (see, e.g., Howard (1997, Theorem 4.2) or Lang (1993, Lemma XIV.1.3)) thus implies that r_0 can be chosen to be r . This proves that $\mathcal{C}$ is a full class.

Concerning completeness, note that

$$\begin{aligned} \kappa(g_{u,\eta}(x)) &= \frac{1}{\eta} \int_0^\eta \Phi(s, u) e^{\langle \psi(s, u), x \rangle} \left(\langle \beta, \psi(s, u) \rangle + \frac{1}{2} \langle \psi(s, u), \gamma \psi(s, u) \rangle \right. \\ &\quad \left. + \int_V \left(e^{\langle \psi(s, u), \xi \rangle} - 1 - \langle \psi(s, u), \chi(\xi) \rangle \right) F(d\xi) \right) ds. \end{aligned} \quad (5.6)$$

By Remark 5.6 (i), the integral

$$\int_V \int_0^\eta \left| \Phi(s, u) e^{\langle \psi(s, u), x \rangle} \left(e^{\langle \psi(s, u), \xi \rangle} - 1 - \langle \psi(s, u), \chi(\xi) \rangle \right) \right| ds F(d\xi)$$

is well-defined, whence by Fubini's theorem we can interchange the integration. In particular, we have

$$\begin{aligned} \lim_{\eta \rightarrow 0} \kappa(g_{u,\eta}(x)) &= \kappa(e^{\langle u, x \rangle}) \\ &= e^{\langle u, x \rangle} \left(\langle \beta, u \rangle + \frac{1}{2} \langle u, \gamma u \rangle + \int_V \left(e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \right) F(d\xi) \right). \end{aligned} \quad (5.7)$$

Moreover, by Jacod and Shiryaev (2003, Lemma II.2.44) or simply as a consequence of the completeness of the class C^* , as defined in (5.4), the function $u \mapsto \kappa(e^{\langle u, x \rangle})$ admits a unique representation of form (5.7), that is, if $\kappa(e^{\langle \cdot, x \rangle})$ also satisfies (5.7) with $(\tilde{\beta}, \tilde{\gamma}, \tilde{F})$, then $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $F = \tilde{F}$. This property carries over to the class $\mathcal{C}$. Indeed, for every $x \in D$, there exists some $\eta > 0$ such that $\beta = \tilde{\beta}$, $\gamma = \tilde{\gamma}$ and $F = \tilde{F}$ if $u \mapsto \kappa(g_{u,\eta}(x))$ also satisfies (5.6) with $(\tilde{\beta}, \tilde{\gamma}, \tilde{F})$. This proves that $\mathcal{C}$ is a complete class. $\square$

In order to establish the semimartingale property of X and to study its characteristics, we need to handle explosions and killing. Similar to Cheridito et al. (2005), we consider again the stopping times T_Δ defined in (5.1) and T'_k given by

$$T'_k := \inf\{t \mid \|X_{t-}\| \geq k \text{ or } \|X_t\| \geq k\}, \quad k \geq 1.$$

By the convention $\|\Delta\| = \infty$, $T'_k \leq T_\Delta$ for all $k \geq 1$. As a transition to Δ occurs either by a jump or by explosion, we additionally define the stopping times:

$$\begin{aligned} T_{\text{jump}} &= \begin{cases} T_\Delta, & \text{if } T'_k = T_\Delta \text{ for some } k, \\ \infty, & \text{if } T'_k < T_\Delta \text{ for all } k, \end{cases} \\ T_{\text{expl}} &= \begin{cases} T_\Delta, & \text{if } T'_k < T_\Delta \text{ for all } k, \\ \infty, & \text{if } T'_k = T_\Delta \text{ for some } k, \end{cases} \\ T_k &= \begin{cases} T'_k, & \text{if } T'_k < T_\Delta, \\ \infty, & \text{if } T'_k = T_\Delta. \end{cases} \end{aligned} \quad (5.8)$$

Note that $\{T_{\text{jump}} < \infty\} \cap \{T_{\text{expl}} < \infty\} = \emptyset$ and $\lim_{k \rightarrow \infty} T_k = T_{\text{expl}}$ with $T_k < T_{\text{expl}}$ on $\{T_{\text{expl}} < \infty\}$. Hence T_{expl} is predictable with announcing sequence $T_k \wedge k$. In order to turn X into a semimartingale and to get explicit expressions for the characteristics, we stop X before it explodes, which is possible, since T_{expl} is predictable. Note that we cannot stop X before it is killed, as T_{jump} is totally inaccessible. For this reason we shall concentrate on the process $(X_t^\tau) := (X_{t \wedge \tau})$, where τ is a stopping time satisfying $0 < \tau < T_{\text{expl}}$, which exists by the above argument and the càdlàg property of X . Since $X = X^{T_\Delta}$, we have

$$\begin{aligned} X_t^\tau &= X_t 1_{\{t < (\tau \wedge T_\Delta)\}} + X_{\tau \wedge T_\Delta} 1_{\{t \geq (\tau \wedge T_\Delta)\}} \\ &= X_t 1_{\{t < (\tau \wedge T_{\text{jump}})\}} + X_{\tau \wedge T_{\text{jump}}} 1_{\{t \geq (\tau \wedge T_{\text{jump}})\}}, \end{aligned}$$

which implies that a transition to Δ can only occur through a jump.

Recall that Δ is assumed to be an arbitrary point which does not lie in D . We can thus identify Δ with some point in $V \setminus D$ such that every $C_b^2(D)$ -function f can be extended continuously to D_Δ with $f(\Delta) = 0$. Indeed, without loss of generality we may assume that such a point exists, because otherwise we can always embed D_Δ in $V \times \mathbb{R}$.

Theorem 5.8. *Let X be an affine process and let τ be a stopping time with $\tau < T_{\text{expl}}$, where T_{expl} is defined in (5.8). Then $X 1_{[0, T_\Delta]}$ and X^τ are semimartingales*

with state space $D \cup \{0\}$ and D_Δ , respectively. Moreover, let (B, C, ν) denote the characteristics of X^τ relative to some truncation function χ . Then there exists a version of (B, C, ν) , which is of the form

$$\begin{aligned} B_{t,i} &= \int_0^{t \wedge \tau} b_i(X_{s-}) ds, \\ C_{t,ij} &= \int_0^{t \wedge \tau} c_{ij}(X_{s-}) ds, \\ \nu(\omega; dt, d\xi) &= K(X_t, d\xi) 1_{[0, \tau]} dt, \end{aligned} \tag{5.9}$$

where $b : D_\Delta \rightarrow V$ and $c : D_\Delta \rightarrow S_+(V)$ are measurable functions and $K(x, d\xi)$ is a positive kernel from $(D_\Delta, \mathcal{D}_\Delta)$ into $(V, \mathcal{B}(V))$, which satisfies $\int_V (\|\xi\|^2 \wedge 1) K(x, d\xi) < \infty$, $K(x, \{0\}) = 0$ and $x + \text{supp}(K(x, \cdot)) \subseteq D_\Delta$ for all $x \in D$.

Proof. We adapt the proof of Çinlar et al. (1980, Theorem 7.9 (ii), (iii)) to our setting. By Lemma 5.2,

$$g_{u,\eta}(X) = \frac{1}{\eta} \int_0^\eta \Phi(s, u) e^{\langle \psi(s, u), X \rangle} ds$$

is a semimartingale for every $u \in \mathcal{U}$ and $\eta > 0$. Since Lemma 5.7 asserts that $\mathcal{C}$, as defined in (5.5), is a full class, an application of Itô's formula to the function h_i appearing in (5.2) shows that X_i coincides with a semimartingale on each stochastic interval $[0, \tau_r[$, where

$$\tau_r = \inf\{t \geq 0 \mid \|X_t\| \geq r\} \wedge T_\Delta.$$

Since we have $\mathbb{P}_x$ -a.s. $\lim_{r \rightarrow \infty} \tau_r = T_\Delta$ and since being a semimartingale is a local property (see Jacod and Shiryaev (2003, Proposition I.4.25)), we conclude that $X 1_{[0, T_\Delta]}$ is a semimartingale.

Let now τ denote a stopping time with $\tau < T_{\text{expl}}$. Then X^τ is also a semimartingale with state space D_Δ , since explosion is avoided and the transition to Δ can only occur via killing, that is, a jump to Δ , which is incorporated in the jump characteristic (see Cheridito et al. (2005, Section 3)).

By Çinlar et al. (1980, Theorem 6.25), one can find a version of the characteristics (B, C, ν) of X^τ , which is of the form

$$\begin{aligned} B_{t,i} &= \int_0^{t \wedge \tau} \tilde{b}_{s-,i} dF_s, \\ C_{t,ij} &= \int_0^{t \wedge \tau} \tilde{c}_{s-,ij} dF_s, \\ \nu(\omega; dt, d\xi) &= 1_{[0, \tau]} dF_t(\omega) \tilde{K}_{\omega,t}(d\xi), \end{aligned} \tag{5.10}$$

where F is an additive process of finite variation, which is $\mathbb{P}_x$ -indistinguishable from an $(\mathcal{F}_t)$ -predictable process, $\tilde{b}$ and $\tilde{c}$ are $(\mathcal{F}_t)$ -optional processes with values in V and $S_+(V)$, respectively, and $\tilde{K}_{\omega,t}(d\xi)$ is a positive kernel from $(\Omega \times \mathbb{R}_+, \mathcal{O}(\mathcal{F}_t))$ ⁸ into $(V, \mathcal{B}(V))$, which satisfies $\int_V (\|\xi\|^2 \wedge 1) \tilde{K}_{\omega,t}(d\xi) < \infty$, $\tilde{K}_{\omega,t}(\{0\}) = 0$ and $X_t(\omega) + \text{supp}(\tilde{K}_{\omega,t}) \subseteq D_\Delta$ for all $t \in [0, \tau]$ and $\mathbb{P}_x$ -almost all ω . Moreover,

⁸Here, $\mathcal{O}(\mathcal{F}_t)$ denotes the $(\mathcal{F}_t)$ -optional σ -algebra.

by Jacod and Shiryaev (2003, Theorem II.2.42), for every $f \in C_b^2(D_\Delta)$, the process

$$\begin{aligned} f(X_t^\tau) - f(x) - \int_0^{t \wedge \tau} \langle \tilde{b}_{s-}, \nabla f(X_{s-}) \rangle dF_s - \frac{1}{2} \int_0^{t \wedge \tau} \sum_{i,j} \tilde{c}_{s-,ij} D_{ij} f(X_{s-}) dF_s \\ - \int_0^{t \wedge \tau} \int_V (f(X_{s-} + \xi) - f(X_{s-}) - \langle \nabla f(X_{s-}), \chi(\xi) \rangle) \tilde{K}_{\omega, s-}(d\xi) dF_s \end{aligned} \quad (5.11)$$

is a $(\mathcal{F}_t, \mathbb{P}_x)$ -local martingale and the last three terms are of finite variation. Note here that Δ is assumed to be an arbitrary point in $V \setminus D$ such that we can extend $f \in C_b^2(D)$ continuously with $f(\Delta) = 0$. Let us denote

$$\begin{aligned} \tilde{\mathcal{L}}f(X_{t-}(\omega)) &:= \langle \tilde{b}_{t-}, \nabla f(X_{t-}(\omega)) \rangle - \frac{1}{2} \sum_{i,j} \tilde{c}_{t-,ij} D_{ij} f(X_{t-}(\omega)) \\ &- \int_V (f(X_{t-}(\omega) + \xi) - f(X_{t-}(\omega)) - \langle \nabla f(X_{t-}(\omega)), \chi(\xi) \rangle) \tilde{K}_{\omega, t-}(d\xi). \end{aligned} \quad (5.12)$$

As proved in Lemma 5.7, the class of functions $\mathcal{C}$ defined in (5.5) is complete. Let now $\tilde{\mathcal{C}} \subset \mathcal{C}$ be the countable set satisfying the property stated in Definition 5.5 and let $g_{\eta,u} \in \tilde{\mathcal{C}}$ for some $u \in iV$ and $\eta > 0$. Then Lemma 5.2 and Remark 5.1 imply that

$$\begin{aligned} g_{\eta,u}(X_t^\tau) - g_{\eta,u}(x) - \int_0^{t \wedge \tau} \mathcal{G}g_{\eta,u}(X_{s-}) ds \\ = g_{\eta,u}(X_t^\tau) - g_{\eta,u}(x) - \int_0^{t \wedge \tau} \frac{1}{\eta} \left(P_\eta e^{\langle u, X_{s-} \rangle} - e^{\langle u, X_{s-} \rangle} \right) ds \end{aligned} \quad (5.13)$$

is a $(\mathcal{F}_t, \mathbb{P}_x)$ -martingale, while $(\int_0^{t \wedge \tau} \mathcal{G}g_{\eta,u}(X_{s-}) ds)$ is a predictable finite variation process. Due to (5.11), (5.12) and uniqueness of the canonical decomposition of the special semimartingale $g_{\eta,u}(X^\tau)$ (see Jacod and Shiryaev (2003, Definition I.4.22, Corollary I.3.16)), we thus have

$$\int_0^{t \wedge \tau} \tilde{\mathcal{L}}g_{\eta,u}(X_{s-}) dF_s = \int_0^{t \wedge \tau} \mathcal{G}g_{\eta,u}(X_{s-}) ds \quad \text{up to an evanescent set.} \quad (5.14)$$

Set now

$$\Lambda = \left\{ (\omega, t) : \tilde{\mathcal{L}}g_{\eta,u}(X_{(t \wedge \tau \wedge T_\Delta)-}(\omega)) = 0 \text{ for every } g_{\eta,u} \in \tilde{\mathcal{C}} \right\}.$$

Then the characteristic property (5.3) of $\tilde{\mathcal{C}}$ implies that Λ is exactly the set where $\tilde{b} = 0$, $\tilde{c} = 0$ and $\tilde{K} = 0$. Hence we may replace F by $1_\Lambda F$ without altering (5.10), that is, we can suppose that $1_\Lambda F = 0$. This property together with (5.14) implies that $dF_t \ll dt$ $\mathbb{P}_x$ -a.s. Hence we know that there exists a triplet (b', c', K') such that F replaced by t and $(\tilde{b}, \tilde{c}, \tilde{K})$ replaced by (b', c', K') satisfy all the conditions of (5.10). In particular, we have by Jacod and Shiryaev (2003, Proposition II.2.9 (i)) that X^τ is quasi-left continuous. Due to Çinlar et al. (1980, Theorem 6.27), it thus follows that

$$\begin{aligned} b'_t &= b(X_t)1_{[0,\tau]}, \\ c'_t &= c(X_t)1_{[0,\tau]}, \\ K'_{\omega,t}(d\xi) &= K(X_t, d\xi)1_{[0,\tau]}, \end{aligned}$$

where the functions b , c and the kernel K have the properties stated in (5.9). This proves the assertion. $\square$

6. REGULARITY

By means of the above derived semimartingale property, in particular the fact that the characteristics are absolutely continuous with respect to the Lebesgue measure, we can prove that every affine process is regular in the following sense:

Definition 6.1 (Regularity). *An affine process X is called regular if for every $u \in \mathcal{U}$ the derivatives*

$$F(u) = \left. \frac{\partial \Phi(t, u)}{\partial t} \right|_{t=0}, \quad R(u) = \left. \frac{\partial \psi(t, u)}{\partial t} \right|_{t=0} \quad (6.1)$$

exist and are continuous on $\mathcal{U}^m$ for every $m \geq 1$.

Remark 6.2. *In the case of the canonical state space $D = \mathbb{R}_+^m \times \mathbb{R}^{n-m}$, the derivative of $\phi(t, u)$ at $t = 0$ is also denoted by $F(u)$ (see Duffie et al. (2003, Equation (3.10)) and Remark 2.5). Since $\Phi(t, u) = e^{\phi(t, u)}$, we have*

$$\partial_t \Phi(t, u)|_{t=0} = e^{\phi(0, u)} \partial_t \phi(t, u)|_{t=0} = \partial_t \phi(t, u)|_{t=0}.$$

Hence our definition of F coincides with the one in Duffie et al. (2003).

Lemma 6.3. *Let X be an affine process. Then the functions $t \mapsto \Phi(t, u)$ and $t \mapsto \psi_i(t, u)$, $i \in \{1, \dots, n\}$, defined in (2.4) are of finite variation for all $u \in \mathcal{U}$.*

Proof. Due to Assumption 2.3, there exist $n + 1$ vectors such that $(x_1, \dots, x_n)$ are linearly independent and $x_{n+1} = \sum_{i=1}^n \lambda_i x_i$ for some $\lambda \in V$ with $\sum_{i=1}^n \lambda_i \neq 1$.

Let us now take $n + 1$ affine processes $X^1, \dots, X^{n+1}$ such that

$$\mathbb{P}_{x_i}[X_0^i = x_i] = 1$$

for all $i \in \{1, \dots, n + 1\}$. It then follows from Theorem 5.8 that, for every $i \in \{1, \dots, n + 1\}$, X^i is a semimartingale with respect to the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_{x_i})$. We can then construct a filtered probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t), \mathbb{P}')$, with respect to which $X_1, \dots, X_{n+1}$ are independent semimartingales such that $\mathbb{P}' \circ (X^i)^{-1} = \mathbb{P}_{x_i}$. One possible construction is the product probability space $(\Omega^{n+1}, \otimes_{i=1}^{n+1} \mathcal{F}, (\otimes_{i=1}^{n+1} \mathcal{F}_t), (\otimes_{i=1}^{n+1} \mathbb{P}_{x_i}))$.

We write $y_i = (1, x_i)^\top$ and $Y^i = (1, X^i)^\top$ for $i \in \{1, \dots, n + 1\}$. Then the definition of x_i implies that $(y_1, \dots, y_{n+1})$ are linearly independent. Moreover, as X^i exhibits càdlàg paths for all $i \in \{1, \dots, n + 1\}$, there exists some stopping time $\delta > 0$ such that, for all $\omega \in \Omega'$ and $t \in [0, \delta(\omega))$, the vectors $(Y_t^1(\omega), \dots, Y_t^{n+1}(\omega))$ are also linearly independent. Let now $T > 0$ and $u \in \mathcal{U}$ be fixed and choose some $0 < \varepsilon(\omega) \leq \delta(\omega)$ such that, for all $t \in [0, \varepsilon(\omega))$, $\Phi(T - t, u) \neq 0$.

Denoting the $(\mathcal{F}'_t, \mathbb{P}')$ -martingales $\Phi(T - t, u)e^{\langle \psi(T-t, u), X_t^i \rangle}$ by $M_t^{T, u, i}$ and choosing the right branch of the complex logarithm, we thus have for all $t \in [0, \varepsilon(\omega))$

$$\begin{pmatrix} 1 & X_{t,1}^1(\omega) & \dots & X_{t,n}^1(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{t,1}^{n+1}(\omega) & \dots & X_{t,n}^{n+1}(\omega) \end{pmatrix}^{-1} \begin{pmatrix} \ln M_t^{T, u, 1}(\omega) \\ \vdots \\ \ln M_t^{T, u, n+1}(\omega) \end{pmatrix} = \begin{pmatrix} \ln \Phi(T - t, u) \\ \psi_1(T - t, u) \\ \vdots \\ \psi_n(T - t, u) \end{pmatrix}.$$

This implies that $(\Phi(s, u))_s$ and $(\psi(s, u))_s$ coincide on the stochastic interval $(T - \varepsilon(\omega), T]$ with deterministic semimartingales and are thus of finite variation. As this

holds true for all $T > 0$, we conclude that $t \mapsto \Phi(t, u)$ and $t \mapsto \psi_i(t, u)$ are of finite variation. $\square$

Using Lemma 6.3 and Theorem 5.8, we are now prepared to prove regularity of affine processes. Additionally, our proof reveals that the functions F and R have parameterizations of Lévy-Khintchine type and that the (differential) semimartingale characteristics introduced in (5.9) depend in an affine way on X .

Theorem 6.4. *Every affine process is regular. Moreover, the functions F and R , as defined in (6.1), are of the form*

$$\begin{aligned} F(u) &= \langle u, b \rangle + \frac{1}{2} \langle u, au \rangle - c \\ &\quad + \int_V \left(e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \right) m(d\xi), \quad u \in \mathcal{U}, \\ \langle R(u), x \rangle &= \langle u, B(x) \rangle + \frac{1}{2} \langle u, A(x)u \rangle - \langle \gamma, x \rangle \\ &\quad + \int_V \left(e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \right) M(x, d\xi), \quad u \in \mathcal{U}, \end{aligned}$$

where $\chi : V \rightarrow V$ denotes some truncation function such that $\chi(\Delta - x) = 0$ for all $x \in D$, $b \in V$, $a \in S(V)$, m is a (signed) measure, $c \in \mathbb{R}$, $\gamma \in V$ and $x \mapsto B(x)$, $x \mapsto A(x)$, $x \mapsto M(x, d\xi)$ are restrictions of $\mathbb{R}$ -linear maps on V such that

$$\begin{aligned} b(x) &= b + B(x), \\ c(x) &= c + A(x), \\ K(x, d\xi) &= m(d\xi) + M(x, d\xi) + (c + \langle \gamma, x \rangle) \delta_{(\Delta - x)}(d\xi). \end{aligned}$$

Here, the left hand side corresponds to the (differential) semimartingale characteristics introduced in (5.9).

Furthermore, on the set $\mathcal{Q} = \{(t, u) \in \mathbb{R}_+ \times \mathcal{U} \mid \Phi(s, u) \neq 0, \text{ for all } s \in [0, t]\}$, the functions Φ and ψ satisfy the ordinary differential equations

$$\partial_t \Phi(t, u) = \Phi(t, u) F(\psi(t, u)), \quad \Phi(0, u) = 1, \quad (6.2)$$

$$\partial_t \psi(t, u) = R(\psi(t, u)), \quad \psi(0, u) = u \in \mathcal{U}. \quad (6.3)$$

Remark 6.5. Recall that without loss of generality we identify Δ with some point in $V \setminus D$ such that every $f \in C_b^2(D)$ can be extended continuously to D_Δ with $f(\Delta) = 0$.

Proof. Let $m \geq 1$ and $u \in \mathcal{U}^m$ be fixed and choose $T_u > 0$ such that $\Phi(T_u - t, u) \neq 0$ for all $t \in [0, T_u]$. As $t \mapsto \Phi(t, u)$ and $t \mapsto \psi(t, u)$ are of finite variation by Lemma 6.3, their derivatives with respect to t exist almost everywhere and we can write

$$\begin{aligned} \Phi(T_u - t, u) - \Phi(T_u, u) &= \int_0^t -d\Phi(T_u - s, u), \\ \psi_i(T_u - t, u) - \psi_i(T_u, u) &= \int_0^t -d\psi_i(T_u - s, u), \end{aligned}$$

for $i \in \{1, \dots, n\}$. Moreover, by the semiflow property of Φ and ψ (see Proposition 2.6 (iv)), differentiability of $\Phi(t, u)$ and $\psi(t, u)$ with respect to t at some $T_u \geq \varepsilon > 0$ implies that the derivatives $\partial_t|_{t=0} \psi(t, \psi(\varepsilon, u))$ and $\partial_t|_{t=0} \Phi(t, \psi(\varepsilon, u))$ exist as well. Let now $(\varepsilon_k)_{k \in \mathbb{N}}$ denote a sequence of points where $\Phi(t, u)$ and $\psi(t, u)$

are differentiable such that $\lim_{k \rightarrow \infty} \varepsilon_k = 0$. Then there exists a sequence $(u_k)_{k \in \mathbb{N}}$ given by

$$u_k = \psi(\varepsilon_k, u) \in \mathcal{U} \text{ with } \lim_{k \rightarrow \infty} u_k = u \quad (6.4)$$

such that the derivatives

$$\partial_t|_{t=0} \psi(t, u_k), \quad \partial_t|_{t=0} \Phi(t, u_k) \quad (6.5)$$

exist for every $k \in \mathbb{N}$. Moreover, since $|\mathbb{E}_x[\exp(\langle u, X_{\varepsilon_k} \rangle)]| < m$, there exists some constant M such that $u_k \in \mathcal{U}^M$ for all $k \in \mathbb{N}$.

Furthermore, due to Theorem 5.8, the canonical semimartingale representation of X^τ (see Jacod and Shiryaev (2003, Theorem II.2.34)), where τ is a stopping time with $\tau < T_{\text{expl}}$, is given by

$$X_t^\tau = x + \int_0^{t \wedge \tau} b(X_{s-}) ds + N_t^\tau + \int_0^{t \wedge \tau} \int_V (\xi - \chi(\xi)) \mu^{X^\tau}(\omega; ds, d\xi),$$

where μ^{X^τ} is the random measure associated with the jumps of X^τ and N^τ is a local martingale, namely the sum of the continuous martingale part and the purely discontinuous one, that is,

$$\int_0^{t \wedge \tau} \int_V \chi(\xi) (\mu^{X^\tau}(\omega; ds, d\xi) - K(X_{s-}, d\xi) ds).$$

Applying Itô's formula (relative to the measure $\mathbb{P}_x$) to the martingale $M_{t \wedge \tau}^{T_u, u} = \Phi(T_u - (t \wedge \tau), u) e^{\langle \psi(T_u - (t \wedge \tau), u), X_{t \wedge \tau} \rangle}$, we obtain

$$\begin{aligned} M_{t \wedge \tau}^{T_u, u} &= M_0^{T_u, u} + \int_0^{t \wedge \tau} M_{s-}^{T_u, u} \left(\frac{-d\Phi(T_u - s, u)}{\Phi(T_u - s, u)} + \langle -d\psi(T_u - s, u), X_{s-} \rangle \right) \\ &\quad + \int_0^{t \wedge \tau} M_{s-}^{T_u, u} \left(\langle \psi(T_u - s, u), b(X_{s-}) \rangle \right. \\ &\quad + \frac{1}{2} \langle \psi(T_u - s, u), c(X_{s-}) \psi(T_u - s, u) \rangle \\ &\quad + \int_V \left(e^{\langle \psi(T_u - s, u), \xi \rangle} - 1 - \langle \psi(T_u - s, u), \chi(\xi) \rangle \right) K(X_{s-}, d\xi) \Big) ds \\ &\quad + \int_0^{t \wedge \tau} M_{s-}^{T_u, u} \langle \psi(T_u - s, u), dN_s^\tau \rangle \\ &\quad + \int_0^{t \wedge \tau} \int_V M_{s-}^{T_u, u} \left(e^{\langle \psi(T_u - s, u), \xi \rangle} - 1 - \langle \psi(T_u - s, u), \chi(\xi) \rangle \right) \\ &\quad \times \left(\mu^{X^\tau}(\omega; ds, d\xi) - K(X_{s-}, d\xi) ds \right). \end{aligned}$$

As the last two terms are local martingales and as the rest is of finite variation, we thus have, for almost all $t \in [0, T_u \wedge \tau]$, $\mathbb{P}_x$ -a.s. for every $x \in D$,

$$\begin{aligned} &\frac{d\Phi(T_u - t, u)}{\Phi(T_u - t, u)} + \langle d\psi(T_u - t, u), X_{t-} \rangle \\ &= \langle \psi(T_u - t, u), b(X_{t-}) \rangle dt + \frac{1}{2} \langle \psi(T_u - t, u), c(X_{t-}) \psi(T_u - t, u) \rangle dt \\ &\quad + \int_V \left(e^{\langle \psi(T_u - t, u), \xi \rangle} - 1 - \langle \psi(T_u - t, u), \chi(\xi) \rangle \right) K(X_{t-}, d\xi) dt. \end{aligned} \quad (6.6)$$

Note in particular that due to $x + \text{supp}(K(x, \cdot)) \subseteq D_\Delta$ for every $x \in D$, the above integral is well-defined. By setting $t = T_u$ on a set of positive measure with $\mathbb{P}_x[\tau \geq T_u]$ and letting $T_u \rightarrow 0$, we obtain due to (6.5) for all $k \in \mathbb{N}$ and $x \in D$

$$\begin{aligned} & \partial_t|_{t=0}\Phi(t, u_k) + \langle \partial_t|_{t=0}\psi(t, u_k), x \rangle \\ &= \langle u_k, b(x) \rangle dt + \frac{1}{2} \langle u_k, c(x)u_k \rangle dt + \int_V \left(e^{\langle u_k, \xi \rangle} - 1 - \langle u_k, \chi(\xi) \rangle \right) K(x, d\xi) dt, \end{aligned} \quad (6.7)$$

where (u_k) is given by (6.4). Since the right hand side is continuous in u_k , which is again a consequence of the support properties of $K(x, \cdot)$ and the fact that $u_k \in \mathcal{U}^M$ for all $k \in \mathbb{N}$, the limit for $u_k \rightarrow u$ of the left hand side exists as well. By the affine independence of the $n+1$ elements in D , the coefficients $\partial_t|_{t=0}\Phi(t, u_k)$ and $\partial_t|_{t=0}\psi(t, u_k)$ converge for $u_k \rightarrow u$, whence the limit is affine, too. Since $m \geq 1$ and u was arbitrary, it follows that

$$\langle u, b(x) \rangle dt + \frac{1}{2} \langle u, c(x)u \rangle dt + \int_V \left(e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \right) K(x, d\xi) dt$$

is an affine function in x for all $u \in \mathcal{U}$.

By uniqueness of the Lévy-Khintchine representation and the assumption that D contains $n+1$ affinely independent elements, this implies that $x \mapsto b(x)$, $x \mapsto c(x)$ and $x \mapsto K(x, d\xi)$ are affine functions in the following sense:

$$\begin{aligned} b(x) &= b + B(x), \\ c(x) &= a + A(x), \\ K(x, d\xi) &= m(d\xi) + M(x, d\xi) + (c + \langle \gamma, x \rangle) \delta_{(\Delta-x)}(d\xi), \end{aligned}$$

where $b \in V$, $a \in S(V)$, m a (signed) measure, $c \in \mathbb{R}$, $\gamma \in V$ and $x \mapsto B(x)$, $x \mapsto A(x)$, $x \mapsto M(x, d\xi)$ are restriction of $\mathbb{R}$ -linear maps on V . Indeed, $c + \langle \gamma, x \rangle$ corresponds to the killing rate of the process, which is incorporated in the jump measure. Here, we explicitly use the convention $e^{\langle u, \Delta \rangle} = 0$ and the fact that $\chi(\Delta - x) = 0$ for all $x \in D$.

Moreover, for t small enough, we have for all $u \in \mathcal{U}$

$$\begin{aligned} \Phi(t, u) - \Phi(0, u) &= \int_0^t \Phi(s, u) \left(\langle \psi(s, u), b \rangle + \frac{1}{2} \langle \psi(s, u), a\psi(s, u) \rangle - c \right. \\ &\quad \left. + \int_V \left(e^{\langle \psi(s, u), \xi \rangle} - 1 - \langle \psi(s, u), \chi(\xi) \rangle \right) m(d\xi) \right) ds, \\ \langle \psi(t, u) - \psi(0, u), x \rangle &= \int_0^t \left(\langle \psi(s, u), B(x) \rangle + \frac{1}{2} \langle \psi(s, u), A(x)\psi(s, u) \rangle - \langle \gamma, x \rangle \right. \\ &\quad \left. + \int_V \left(e^{\langle \psi(s, u), \xi \rangle} - 1 - \langle \psi(s, u), \chi(\xi) \rangle \right) M(x, d\xi) \right) ds. \end{aligned}$$

Note again that the properties of the support of $K(x, \cdot)$ carry over to the measures $M(x, \cdot)$ and $m(\cdot)$ implying that the above integrals are well-defined. Due to the continuity of $t \mapsto \Phi(t, u)$ and $t \mapsto \psi(t, u)$, we can conclude that the derivatives of Φ and ψ exist at 0 and are continuous on $\mathcal{U}^m$ for every $m \geq 1$, since they are given

by

$$\begin{aligned}
F(u) &= \left. \frac{\partial \Phi(t, u)}{\partial t} \right|_{t=0} = \langle u, b \rangle + \frac{1}{2} \langle u, au \rangle - c \\
&\quad + \int_V \left(e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \right) m(d\xi), \\
\langle R(u), x \rangle &= \left\langle \left. \frac{\partial \psi(t, u)}{\partial t} \right|_{t=0}, x \right\rangle = \langle u, B(x) \rangle + \frac{1}{2} \langle u, A(x)u \rangle - \langle \gamma, x \rangle \\
&\quad + \int_V \left(e^{\langle u, \xi \rangle} - 1 - \langle u, \chi(\xi) \rangle \right) M(x, d\xi).
\end{aligned}$$

This proves the first part of the theorem.

By the regularity of X , we are now allowed to differentiate the semiflow equations (2.5) on the set $\mathcal{Q} = \{(t, u) \in \mathbb{R}_+ \times \mathcal{U} \mid \Phi(s, u) \neq 0, \text{ for all } s \in [0, t]\}$ with respect to s and evaluate them at $s = 0$. As a consequence, Φ and ψ satisfy (6.2) and (6.3). $\square$

Remark 6.6. *The differential equations (6.2) and (6.3) are called generalized Riccati equations, which is due to the particular form of F and R .*

REFERENCES

- H. Bauer. *Probability theory*, volume 23 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1996. Translated from the fourth (1991) German edition by Robert B. Burckel and revised by the author.
- R. M. Blumenthal and R. K. Gettoor. *Markov processes and potential theory*. Pure and Applied Mathematics, Vol. 29. Academic Press, New York, 1968.
- M.-F. Bru. Wishart processes. *J. Theoret. Probab.*, 4(4):725–751, 1991.
- W. Bruns and J. Gubeladze. *Polytopes, rings, and K-theory*. Springer Monographs in Mathematics. Springer, Dordrecht, 2009.
- E. Çinlar, J. Jacod, P. Protter, and M. J. Sharpe. Semimartingales and Markov processes. *Z. Wahrsch. Verw. Gebiete*, 54(2):161–219, 1980.
- P. Cheridito, D. Filipović, and M. Yor. Equivalent and absolutely continuous measure changes for jump-diffusion processes. *Ann. Appl. Probab.*, 15(3):1713–1732, 2005.
- K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, New York, second edition, 2005.
- J. C. Cox, J. E. J. Ingersoll, and S. A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.
- C. Cuchiero, D. Filipović, E. Mayerhofer, and J. Teichmann. Affine processes on positive semidefinite matrices. *Ann. Appl. Probab.*, 2011.
- J. Da Fonseca, M. Grasselli, and C. Tebaldi. A multifactor volatility Heston model. *Quant. Finance*, 8(6):591–604, 2008.
- Q. Dai and K. J. Singleton. Specification analysis of affine term structure models. *J. Finance*, 55(5):1943–1978, 2000.
- D. Duffie and R. Kan. A yield-factor model of interest rates. *Math. Finance*, 6(4):379–406, 1996.

- D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68(6):1343–1376, 2000.
- D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Ann. Appl. Probab.*, 13(3):984–1053, 2003.
- S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986.
- I. I. Gihman and A. V. Skorohod. *The theory of stochastic processes II*. Grundlehren der Mathematischen Wissenschaften, Bd. 218. Berlin etc.: Springer-Verlag, 1983.
- C. Gourieroux and R. Sufana. Wishart quadratic term structure models. SSRN eLibrary, 2003.
- S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.*, 6:327–343, 1993.
- R. Howard. The inverse function theorem for Lipschitz maps. Lecture Notes, 1997.
- J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Fundamental Principles of Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 2003.
- K. Kawazu and S. Watanabe. Branching processes with immigration and related limit theorems. *Teor. Veroyatnost. i Primenen.*, 16:34–51, 1971.
- M. Keller-Ressel, W. Schachermayer, and J. Teichmann. Affine processes are regular. *Probab. Theory Related Fields, Forthcoming*, 2010.
- M. Keller-Ressel, W. Schachermayer, and J. Teichmann. Regularity of affine processes on general state spaces. Preprint, 2011.
- S. Lang. *Real and functional analysis*, volume 142 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1993.
- M. Leippold and F. Trojani. Asset pricing with matrix jump diffusions. SSRN eLibrary, 2008.
- G. Letac and H. Massam. The noncentral Wishart as an exponential family, and its moments. *J. Multivariate Anal.*, 99(7):1393–1417, 2008.
- E. Mayerhofer. On the existence of noncentral Wishart distributions. Preprint, 2011.
- P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.
- D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 1*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Ltd., Chichester, second edition, 1994. Foundations.