

# Change point analysis of an exponential model based on Phi-divergence test-statistics: simulated critical points case

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## Abstract

Recently Batsidis *et al.* (2011) have presented a new procedure based on divergence measures for testing the hypothesis of the existence of a change point in exponential populations. A simulation study was carried out, in this paper, using the asymptotic critical points obtained from the asymptotic distribution of the new test statistics introduced there. The main purpose of this paper is to study the behavior of the test statistics introduced in the cited paper of Batsidis *et al.* (2011), using simulated critical points.

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## 1 Introduction

In a previous paper Batsidis *et al.* (2011) introduced new test statistics for a change point in a sequence of independent exponentially distributed random variables and studied their asymptotic distribution. Based on the asymptotic critical point they presented a simulation study in order to analyze the behavior of the new test statistics in relation with the Likelihood Ratio Test (LRT) introduced and studied in Worsley (1986) and Haccou *et al.* (1985, 1988). The simulation study is based in obtaining the error of type I as well as the power using the asymptotic critical points.

In this paper we shall study the behavior of the test statistics introduced in the cited paper of Batsidis *et al.* (2011) but using simulated critical points instead of asymptotic critical points. In Section 2 we shall outline the different test statistics used for this purpose, while in Section 3 a simulation study is carried out.

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## 2 Families of test statistics

Let  $X_1, \dots, X_K$  be a sequence of  $K$  independent exponential random variables, with density function  $f(x_i, \theta_i) = \theta_i \exp(-\theta_i x_i)$ ,  $i = 1, \dots, K$ , respectively, where  $\theta_i$  and  $x_i$  are positive real numbers. The change point problem is to test hypothesis about the equality of means  $1/\theta_i$ ,  $i = 1, \dots, K$ , or what is equivalent about the equality of the parameters  $\theta_i$ ,  $i = 1, \dots, K$ ,

$$H_0 : \theta_1 = \theta_2 = \dots = \theta_K \quad (= \theta_0, \theta_0 \text{ unknown}), \quad (1)$$

versus the alternative

$$H_1 : \theta_1 = \dots = \theta_{k_1} \neq \theta_{k_1+1} = \dots = \theta_{k_2} \neq \dots \neq \theta_{k_{q-1}+1} = \dots = \theta_{k_q} = \theta_K, \quad (2)$$

where  $q$ ,  $1 \leq q \leq K$ , is the unknown number of changes and  $k_1, k_2, \dots, k_q$  are the unknown positions of the change points.

Similar to the classical literature of change point analysis we just need to test the single change point problem by means of the binary segmentation procedure. This procedure was proposed by Vostrikova (1981). Based on this procedure we just need to test the single change point hypothesis and then to repeat the procedure for each subsequence. Therefore initially, we test for no change point versus one change point, that is, we test the null hypothesis

$$H_0 : X_i \text{ are described by } f_{\theta_0}(x) = \theta_0 \exp(-\theta_0 x), \quad x > 0, \theta_0 > 0; \quad i = 1, \dots, K \text{ and } \theta_0 \text{ unknown}, \quad (3)$$

versus the alternative

$$\begin{aligned} H_1 : X_i \text{ are described by } f_{\theta_0}(x) &= \theta_0 \exp(-\theta_0 x), \quad x > 0, \theta_0 > 0; \quad i = 1, \dots, k \text{ and} \\ X_i \text{ are described by } f_{\theta_1}(x) &= \theta_1 \exp(-\theta_1 x), \quad x > 0, \theta_1 > 0; \quad i = k + 1, \dots, K. \end{aligned} \quad (4)$$

Here,  $k$  is the unknown location of the single change point. If  $H_0$  is not rejected, then the procedure is finished and there is no change point. If  $H_0$  is rejected, then there is a change point and we continue with the step 2. In the second step we test separately the two subsequences before and after the change point found in the first step. In the sequel, we repeat these two steps until no further subsequences have change points. At the end of the procedure, the collection of change point locations found by the previous steps constitute the set of the change points. Therefore, we will concentrate in the sequel on the case of a single change point in the random sequence.

It is well known that the maximum likelihood estimator (MLE),  $\hat{\theta}_{0,k}^{(K)}$ , of the parameter  $\theta_0$  in the exponential model, which is based on the random sample  $X_1, \dots, X_k$  from  $f_{\theta_0}(x) = \theta_0 \exp(-\theta_0 x)$ , is given by

$$\hat{\theta}_{0,k}^{(K)} = \frac{1}{\bar{X}_{k,0}}$$

with  $\bar{X}_{k,0} = \frac{1}{k} \sum_{i=1}^k X_i$  and the MLE of  $\theta_1$  in the exponential model, which is based on the random sample  $X_{k+1}, \dots, X_K$  from  $f_{\theta_1}(x) = \theta_1 \exp(-\theta_1 x)$ , is given by

$$\hat{\theta}_{1,k}^{(K)} = \frac{1}{\bar{X}_{K-k,1}}$$

with  $\bar{X}_{K-k,1} = \frac{1}{K-k} \sum_{i=k+1}^K X_i$ . We are using the subscript "0" to declare that we refer to the first  $k$  observations and the subscript "1" to declare that we refer to the last  $K - k$  last observations.

We denote,

$$N(\epsilon) = \left\{ k/k \in \{1, \dots, K-1\} \text{ and } \frac{k}{K} \in [\epsilon, 1 - \epsilon], \text{ with } \epsilon > 0 \text{ and small enough} \right\}. \quad (5)$$

Batsidis *et al.* (2011b) considered the following family of test statistics

$$\epsilon T_{\phi_\lambda}^{(K)} = \begin{cases} \max_{k \in N(\epsilon)} \frac{2k(K-k)}{K\lambda(\lambda+1)} \left\{ \frac{\bar{X}_{k,0}^{-\lambda} \bar{X}_{K-k,1}^{\lambda+1}}{(\lambda+1)\bar{X}_{K-k,1} - \lambda\bar{X}_{k,0}} - 1 \right\}, & (\lambda+1)\bar{X}_{K-k,1} - \lambda\bar{X}_{k,0} > 0, \lambda \neq 0, -1 \\ \max_{k \in N(\epsilon)} \frac{2k(K-k)}{K} \left\{ \ln \frac{\bar{X}_{K-k,1}}{\bar{X}_{k,0}} + \frac{\bar{X}_{k,0}}{\bar{X}_{K-k,1}} - 1 \right\}, & \lambda = 0 \\ \max_{k \in N(\epsilon)} \frac{2k(K-k)}{K} \left\{ \ln \frac{\bar{X}_{k,0}}{\bar{X}_{K-k,1}} + \frac{\bar{X}_{K-k,1}}{\bar{X}_{k,0}} - 1 \right\}, & \lambda = -1 \end{cases}, \quad (6)$$

for testing (3) versus (4). We have to note that when  $\lambda \neq 0$ ,  $\lambda \neq -1$  the condition  $(\lambda+1)\bar{X}_{K-k,1} - \lambda\bar{X}_{k,0} > 0$  should be satisfied in order to ensure the existence of the divergence. Taking into account that  $\bar{X}_{K-k,1} > 0$  and  $\bar{X}_{k,0} > 0$  we will restrict to values of  $\lambda$  in the interval  $(-1, 0)$ , so as the previous mentioned condition to be always satisfied.

Worsley (1986) and Haccou *et al.* (1985, 1988) used maximum likelihood methods in order to test for a change in a sequence of independent exponential family random variables, with particular emphasis on the exponential distribution. It was proved that if  $LRT^{(K)}$  is minus twice the log likelihood ratio, that is

$$LRT^{(K)} = \max_{k \in \{1, \dots, K-1\}} -2 \log \frac{\prod_{h=1}^K f_{\hat{\theta}_{0,K}^{(K)}}(X_h)}{\prod_{h=1}^k f_{\hat{\theta}_{0,k}^{(K)}}(X_h) \prod_{h=k+1}^K f_{\hat{\theta}_{1,k}^{(K)}}(X_h)}, \quad (7)$$

with  $f_{\hat{\theta}_{0,k}^{(K)}}(X_h) = \hat{\theta}_{0,k}^{(K)} \exp(-\hat{\theta}_{0,k}^{(K)} X_h)$ , and  $f_{\hat{\theta}_{1,k}^{(K)}}(X_h) = \hat{\theta}_{1,k}^{(K)} \exp(-\hat{\theta}_{1,k}^{(K)} X_h)$ ,  $k = 1, \dots, K-1$ , then its explicit expression is given by

$$LRT^{(K)} = 2 \max_{k \in \{1, \dots, K-1\}} \left( k \log \frac{\hat{\theta}_{0,k}^{(K)}}{\hat{\theta}_{0,K}^{(K)}} + (K-k) \log \frac{\hat{\theta}_{1,k}^{(K)}}{\hat{\theta}_{0,K}^{(K)}} \right). \quad (8)$$

We have to note that  $LRT^{(K)}$  can be written in terms of the Kullback-Leibler divergence measure, as follows

$$LRT^{(K)} = 2K \max_{k \in \{1, \dots, K-1\}} \left( \frac{k}{K} D_{Kull} \left( f_{\hat{\theta}_{0,k}^{(K)}}, f_{\hat{\theta}_{0,K}^{(K)}} \right) + \frac{K-k}{K} D_{Kull} \left( f_{\hat{\theta}_{1,k}^{(K)}}, f_{\hat{\theta}_{0,K}^{(K)}} \right) \right),$$

since in the special case of two exponential distributions with parameters  $\theta$  and  $\theta'$  it is easily obtained that

$$D_{Kull}(f_\theta, f_{\theta'}) = \log \frac{\theta}{\theta'} + \frac{\theta'}{\theta} - 1.$$

It is very interesting to observe that in many statistical problems LRT is obtained directly from the Kullback-Leibler divergence but we can see that in the case of the change point detection in the

exponential model it is not possible. We can only express it as a linear combination of two Kullback-Leibler divergences. This point is very important because, based on it, the LRT is not a member of the family of test statistics considered in (6). Batsidis *et al.* (2011), inspired on the third test-statistic proposed in Horváth and Serbinowska (1995, p. 373), which tends usually to behave much more better than the likelihood ratio test statistic, considered a modified likelihood ratio test as a weighted sum of two Kullback-Leibler divergences. The test statistic is given by the following formula:

$$\begin{aligned}
S^{(K)} &= 2 \max_{k \in \{1, \dots, K-1\}} \frac{k(K-k)}{K} \left( \frac{k}{K} D_{Kull} \left( f_{\widehat{\theta}_{0,k}^{(K)}}, f_{\widehat{\theta}_{0,K}^{(K)}} \right) + \frac{K-k}{K} D_{Kull} \left( f_{\widehat{\theta}_{1,k}^{(K)}}, f_{\widehat{\theta}_{0,K}^{(K)}} \right) \right) \\
&= 2 \max_{k \in \{1, \dots, K-1\}} \frac{k(K-k)}{K} \left( \frac{k}{K} \log \frac{\widehat{\theta}_{0,k}^{(K)}}{\widehat{\theta}_{0,K}^{(K)}} + \frac{K-k}{K} \log \frac{\widehat{\theta}_{1,k}^{(K)}}{\widehat{\theta}_{0,K}^{(K)}} \right). \tag{9}
\end{aligned}$$

In the next section we study the behavior of  ${}^\epsilon T_{\phi_\lambda}^{(K)}$ ,  $LRT^{(K)}$  and  $S^{(K)}$  when simulated critical values are used, instead of the asymptotic critical points considered in Batsidis *et al.* (2011).

### 3 Simulation Study

The aim of this section is the evaluation of the simulated critical values of the different test statistics presented in Section 2, subject to the assumption that there is no change point. In order to obtain the critical values for the significance levels 0.1, 0.05 and 0.01, we simulate  $B = 5000$  data sets of sample size  $K = 40, 50, 60, 64, 100, 200, 300, 400, 500$  from the standard exponential distribution. For each sample-data set, we calculate the test statistics  ${}^{0.05}T_{\phi_\lambda}^{(K)}$ , for eleven values of  $\lambda$ ,  $\lambda = -0.1(0.1)1$ , as well as the test statistics  $\widetilde{LRT}^{(K)} = a(K)\sqrt{LRT^{(K)}} - b(K)$  and  $S^{(K)}$ . Therefore, 5000 values of these statistics are obtained and the estimated 0.1, 0.05 and 0.01 critical values are given in Table 1.

From Table 1 it can be seen that the estimated critical values for the various significance levels are very different from the asymptotic critical values (see Haccou *et al.* (1985, p.10) for a similar conclusion). The disadvantage of not using the limiting distribution is that specialized extensive tables of simulated critical values for a simulation study when the sample size does not exactly match the tabled values given are needed (see Srivastava and Hui (1987)). However as for instance Romeu and Ozturk (1993) pointed out using the limiting distribution implies loss of power.

The type I error rate is an essential characteristic of the performance of a test statistic. In the sequel we present the results of a Monte Carlo study on the type I error rates, by considering data sampled from a standard exponential distribution with no change point. In the study 5000 data sets with different sample size  $K = 40, 50, 60, 64, 100, 200, 300, 500$  were generated. The simulated results on the type I error rates of  ${}^{0.05}T_{\phi_\lambda}^{(K)}$ ,  $\lambda = -0.1(0.1)1$ , as well as the test statistics  $\widetilde{LRT}^{(K)}$  and  $S^{(K)}$  for testing the existence of a change point, for significance level  $\alpha = 0.1, 0.05, 0.01$ , are presented in Table 2, when the simulated critical values of Table 1 are used respectively. In this framework the test statistics were calculated for all 5000 data sets and compared to the appropriate simulated critical value given in Table 1. The type I error rate is estimated as the proportion of rejections of the hypothesis of no change point in each situation. So in Table 2 the empirical sizes, the proportion of times that the null hypothesis is rejected when all data are distributed according to the standard exponential distribution are given.

Table 1: Simulated critical values of the test statistics based on 5000 replications.

$\alpha$	$K$	$0.05 T_{\phi_\lambda}^{(K)}$										$\widetilde{LRT}^{(K)}$	$S^{(K)}$	
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1			0
0.1	40	11.5810	9.5676	8.5411	7.9334	7.6490	7.4920	7.5585	7.8853	8.4269	9.5196	11.3937	2.3190	1.3025
	50	9.6585	8.5539	7.9585	7.5286	7.2882	7.2120	7.3280	7.5125	7.8517	8.5151	9.5691	2.2360	1.3055
	60	9.8676	8.8737	8.2602	7.8641	7.6617	7.6264	7.6725	7.8283	8.1709	8.8069	9.8090	2.2703	1.3030
	64	8.9705	8.2860	7.8496	7.5334	7.3362	7.3376	7.3600	7.4888	7.7334	8.1398	8.7467	2.2961	1.2984
	100	8.7360	8.2728	7.9100	7.7443	7.6026	7.5428	7.5825	7.7012	7.9344	8.2489	8.7130	2.2971	1.3324
	200	8.5279	8.3250	8.2117	8.0575	7.9950	7.9180	7.9329	8.0188	8.1164	8.2670	8.6134	2.3153	1.4177
	300	8.3041	8.1771	8.0462	7.9215	7.9077	7.9109	7.9861	7.9804	8.0459	8.1457	8.3344	2.3649	1.3813
	400	8.2505	8.1697	8.0972	8.0184	7.9514	7.9289	7.9042	7.9476	7.9841	8.0687	8.2334	2.3606	1.4142
500	8.2393	8.1955	8.1057	8.0541	8.0201	7.9773	7.9877	7.9788	8.0268	8.1105	8.2279	2.3646	1.4229	
$\infty$	8.31	8.31	8.31	8.31	8.31	8.31	8.31	8.31	8.31	8.31	8.31	2.9435	1.4978	
0.05	40	17.6334	13.1196	11.0430	9.8748	9.3129	9.1339	9.1892	9.7084	10.7864	12.7154	17.6140	2.7241	1.6569
	50	13.5357	11.3328	9.9970	9.2835	8.9350	8.8046	8.8077	9.1188	9.7586	11.0303	13.1107	2.6826	1.6648
	60	13.9701	11.5162	10.3217	9.6691	9.2967	9.2192	9.3095	9.7703	10.5004	11.7857	13.9239	2.6850	1.6546
	64	12.2527	10.7728	9.9618	9.4196	9.0943	9.0235	9.0149	9.2659	9.8645	10.6607	12.1021	2.7308	1.6309
	100	11.4735	10.5057	9.8783	9.3770	9.1050	9.0232	9.1798	9.4268	9.7724	10.4078	11.1852	2.7164	1.6676
	200	10.6694	10.1903	9.9174	9.6909	9.6413	9.5670	9.7049	9.8352	10.0539	10.3680	10.8759	2.8402	1.7269
	300	10.3158	10.0808	9.8837	9.7218	9.5948	9.5428	9.5377	9.6841	9.8029	9.9631	10.2826	2.8237	1.7393
	400	10.1254	9.9236	9.8319	9.6970	9.5952	9.5810	9.5850	9.7662	9.9785	10.1668	10.1668	2.8461	1.7573
500	10.1375	9.9245	9.7938	9.7101	9.6954	9.6010	9.5865	9.6099	9.6936	9.8310	10.0460	2.8415	1.7784	
$\infty$	9.90	9.90	9.90	9.90	9.90	9.90	9.90	9.90	9.90	9.90	9.90	3.6633	1.8444	
0.01	40	50.5550	24.4875	17.3032	14.1739	12.7427	12.1567	12.5086	13.6526	16.3834	22.9197	44.9917	3.5780	2.4148
	50	27.0400	19.4354	15.5167	13.6776	12.6433	12.3613	12.4914	13.2924	15.4007	19.0322	28.8758	3.5656	2.4720
	60	29.6808	19.8103	15.9025	13.7726	12.8346	12.6437	12.7626	13.8295	15.7556	19.7567	28.2007	3.5103	2.4652
	64	21.1749	16.8784	14.3429	13.1193	12.5352	12.5130	12.7935	13.5557	15.1971	17.5568	22.5916	3.6179	2.4349
	100	21.0281	17.3015	14.9784	13.6942	12.8163	12.7267	12.8364	13.3041	14.1556	16.1152	19.3112	3.4667	2.3155
	200	16.3509	15.0467	14.1423	13.9136	13.7878	13.4483	13.6832	14.3627	14.8642	15.8700	17.5407	3.7306	2.5576
	300	15.8743	15.2039	14.5089	13.9226	13.9182	13.7411	13.5544	13.5335	13.9115	14.7973	15.6554	3.7907	2.5829
	400	14.3163	14.1375	13.7048	13.3444	13.1254	13.2488	13.4639	13.7315	14.1345	14.6297	15.1618	3.8562	2.5797
500	14.8829	14.4170	14.0629	13.6848	13.3272	13.2033	13.2050	13.2802	13.6211	13.9468	14.3068	3.8130	2.5964	
$\infty$	13.45	13.45	13.45	13.45	13.45	13.45	13.45	13.45	13.45	13.45	13.45	5.2933	2.6491	

Since these type I error rates were estimated using Monte Carlo simulations, they are not free of error. So in order to decide if a test is accurate or not, in a similar manner with that of Cardoso de Oliveira and Ferreira (2010), we apply the exact binomial test for the null hypothesis  $H_0 : \alpha = 0.1(0.05, 0.01, \text{ respectively})$  versus the alternative  $H_1 : \alpha \neq 0.1(0.05, 0.01, \text{ respectively})$ , with a significance level of 0.01. In this frame, if the simulated (observed) type I error rates were not significantly different from  $\alpha$  the test would be considered accurate. Otherwise, the test would be considered conservative or liberal. In Table 2 we indicate with \* the significantly different simulated type I error rates from the nominal level  $\alpha$ . Based on Table 2 we conclude that in almost all cases presented here the test statistics considered are accurate when the simulated critical values of the Table 1 were used.

Our aim in the sequel is to estimate the power of the test statistics for detecting the existence of a change point by means of Monte Carlo methods. In this context, motivated by Haccou *et al.* (1985, 1988), we apply the test statistics to  $B = 5000$  samples, with different significance levels  $\alpha$ , as well as with several combinations of  $K$  and  $k$ , with  $K$  the sample size and  $k$  the point where change is taken place. Point  $k$  is selected such as  $[k] = \tau \times K$ , where  $0 < \tau < 1$  and  $[\cdot]$  denotes the integer part of a real number.

For each combination of sample size  $K$  and  $k$ , 5000 data sets were generated such as  $X_1, \dots, X_k$  to be distributed according to an exponential distribution with  $\theta_0 = 1$  and  $X_{k+1}, \dots, X_K$  to be described by other exponential distribution with parameter  $\theta_1 \neq 1$ . In this framework we consider the following scenarios  $\theta_1 \in \{5, 4, 3, 2, 1/2, 1/3, 1/4, 1/5\}$  and we denote by  $\rho$  the ratio  $\frac{\theta_1}{\theta_0}$ . For each sample we calculate the value of the test statistics and decide to reject or not the (false) null hypothesis, with

Table 2: Empirical sizes based on 5000 replications with simulated critical values.

$\alpha$	$K$	${}^{0.05}T_{\phi_\lambda}^{(K)}$											$\widetilde{LRT}^{(K)}$	$S^{(K)}$
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0		
0.1	40	0.0968	0.0928	0.0896	0.0932	0.0952	0.0980	0.0962	0.0960	0.0930	0.0920	0.0960	0.0928	0.1022
	50	0.0986	0.1042	0.1018	0.1032	0.1040	0.1054	0.1026	0.1040	0.1072	0.1038	0.1062	0.1106	0.1050
	60	0.0968	0.0964	0.0940	0.0940	0.0970	0.0942	0.0990	0.1008	0.1032	0.0996	0.1004	0.0992	0.1018
	64	0.1038	0.1006	0.1010	0.1032	0.1034	0.0986	0.1004	0.1012	0.1012	0.1026	0.1026	0.0978	0.1056
	100	0.1048	0.1034	0.1028	0.0976	0.0998	0.0992	0.0988	0.1004	0.0970	0.0980	0.0990	0.0960	0.1030
	200	0.0998	0.0970	0.0946	0.0958	0.0970	0.1014	0.1030	0.1002	0.1022	0.1022	0.0982	0.1066	0.0958
	300	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
500	0.1002	0.0972	0.0982	0.0988	0.0988	0.0986	0.0986	0.1012	0.1006	0.0990	0.1000	0.1016	0.1030	
0.05	40	0.0458	0.0472	0.0454	0.0424	0.0430	0.0450	0.0480	0.0484	0.0494	0.0490	0.0426	0.0460	0.0456
	50	0.0478	0.0494	0.0506	0.0524	0.0528	0.0530	0.0556	0.0568	0.0580	0.0546	0.0544	0.0516	0.0486
	60	0.0468	0.0490	0.0490	0.0480	0.0472	0.0482	0.0512	0.0478	0.0472	0.0454	0.0494	0.0536	0.0484
	64	0.0470	0.0476	0.0478	0.0484	0.0482	0.0506	0.0540	0.0536	0.0522	0.0536	0.0518	0.0470	0.0514
	100	0.0508	0.0496	0.0512	0.0542	0.0546	0.0526	0.0500	0.0492	0.0516	0.0510	0.0532	0.0512	0.0448
	200	0.0532	0.0532	0.0540	0.0534	0.0498	0.0506	0.0474	0.0466	0.0458	0.0458	0.0442	0.0484	0.0532
	300	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0502	0.0500
500	0.0476	0.0482	0.0496	0.0504	0.0496	0.0504	0.0502	0.0488	0.0490	0.0486	0.0474	0.0488	0.0526	
0.01	40	0.0076	0.0072	0.0076	0.0088	0.0092	0.0098	0.0092	0.0070	0.0070	0.0088	0.0094	0.0104	0.0116
	50	0.0114	0.0120	0.0116	0.0114	0.0114	0.0118	0.0134	0.0144*	0.0128	0.0136	0.0106	0.0116	0.0076
	60	0.0092	0.0106	0.0100	0.0110	0.0108	0.0090	0.0092	0.0096	0.0098	0.0100	0.0114	0.0100	0.0098
	64	0.0104	0.0096	0.0110	0.0114	0.0120	0.0116	0.0120	0.0140*	0.0134	0.0136	0.0154*	0.0080	0.0102
	100	0.0102	0.0104	0.0042*	0.0110	0.0124	0.0120	0.0118	0.0128	0.0154*	0.0144*	0.0126	0.0140*	0.0132
	200	0.0142*	0.0144*	0.0138*	0.0110	0.0078	0.0082	0.0078	0.0064*	0.0070	0.0074	0.0072	0.0082	0.0078
	300	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100	0.0100
500	0.0096	0.0092	0.0096	0.0092	0.0102	0.0098	0.0096	0.0094	0.0088	0.0090	0.0096	0.0094	0.0094	

\* indicates that the simulated level is significantly different from  $\alpha$ .

significance level  $\alpha$ , considering the appropriate simulated or asymptotical respectively critical value. The power is obtained calculating the proportion in times in 5000 Monte Carlo simulations that the (false) null hypothesis is rejected considering the specified significance level based on the simulated critical values of Table 1.

In the sequel, we will present in Tables 3-8 the results for  $\alpha = 0.05, 0.01$ ,  $K = 40, 50, 100, 200$  and  $\tau = 0.2, 0.3, 0.5$ .

From Tables 3-8, when the simulated critical values are used, we reach the following conclusions:

- The power increases with  $\tau$  ( $0 < \tau \leq 0.5$ ) and is optimal when  $\tau = 0.5$ .
- For  $\tau = 0.2$  the power of the  $\widetilde{LRT}^{(K)}$  is almost always greater than the power of  $S^{(K)}$ . Moreover, when  $\rho > 1$  the performance of  $\widetilde{LRT}^{(K)}$  is almost in all cases the best, while when  $\rho < 1$  then there is a value of  $\lambda$  for which the performance of  ${}^{0.05}T_{\phi_\lambda}^{(K)}$  test statistic is better. To be more specific it is suggested to use  $\lambda = -0.8, -0.9, -1$ . At this point we have to point that when  $\rho > 1$  it is suggested from Tables 3 and 6 to use  $\lambda = -0.1, -0.2, -0.3$ .
- The power is increasing rapidly as the sample size increases.
- For  $\tau = 0.3, 0.5$  the  $S^{(K)}$  has the best performance. Among the test statistics  ${}^{0.05}T_{\phi_\lambda}^{(K)}$  a similar conclusion is reached with that in b).
- The results also indicate that when  $\tau < 0.5$  the test based on  $\widetilde{LRT}^{(K)}$  as well as  $S^{(K)}$  performs less good for  $\rho < 1$  than in the opposite case of  $1/\rho$  (see Haccou et al. (1983, 1985) for a similar conclusion). However this property does not holds for  ${}^{0.05}T_{\phi_\lambda}^{(K)}$ . It is noted that initially there are values for which the opposite holds and there is a point where the behavior changes.
- When  $\tau = 0.5$  the test based on  $\widetilde{LRT}^{(K)}$  as well as  $S^{(K)}$  performs similar for  $\rho$  and  $1/\rho$ , with  $\rho < 1$ .

Table 3: Empirical powers based on 5000 replications with simulated critical values, when  $\alpha = 0.05$  and  $\tau = 0.2$ .

$K$	$\theta_1$	$0.05 T_{\phi_\lambda}^{(K)}$										$\widetilde{LRT}^{(K)}$	$S^{(K)}$	
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1			0
40	5	0.0746	0.3402	0.5550	0.6984	0.7804	0.8250	0.8528	0.8668	0.8700	0.8648	0.8246	0.9132	0.8994
	4	0.0300	0.1710	0.3492	0.5010	0.5944	0.6636	0.7072	0.7330	0.7362	0.7270	0.6630	0.8176	0.7850
	3	0.0120	0.0574	0.1434	0.2456	0.3312	0.3946	0.4492	0.4808	0.4888	0.4756	0.4018	0.5922	0.5610
	2	0.0112	0.0142	0.0286	0.0564	0.0918	0.1256	0.1562	0.1744	0.1798	0.1758	0.1440	0.2340	0.2228
	1/2	0.2396	0.2568	0.2644	0.2598	0.2376	0.2060	0.1790	0.1310	0.0848	0.0534	0.0376	0.1786	0.1844
	1/3	0.5322	0.5714	0.5776	0.5736	0.5466	0.5066	0.4498	0.3606	0.2514	0.1368	0.0452	0.4334	0.4122
	1/4	0.7536	0.7896	0.8004	0.7962	0.7780	0.7394	0.6896	0.6062	0.4606	0.2758	0.0716	0.6796	0.6432
	1/5	0.8886	0.9122	0.9190	0.9170	0.9050	0.8854	0.8546	0.7910	0.6714	0.4552	0.1436	0.8424	0.8116
50	5	0.5046	0.7160	0.8222	0.8802	0.9100	0.9302	0.9452	0.9532	0.9570	0.9556	0.9512	0.9654	0.9558
	4	0.2654	0.4786	0.6310	0.7206	0.7780	0.8154	0.8454	0.8648	0.8712	0.8700	0.8636	0.8986	0.8688
	3	0.0784	0.1984	0.3334	0.4464	0.5212	0.5798	0.6296	0.6602	0.6756	0.6708	0.6598	0.7104	0.6770
	2	0.0176	0.0394	0.0786	0.1242	0.1672	0.2040	0.2462	0.2722	0.2912	0.2882	0.2810	0.3130	0.2764
	1/2	0.3478	0.3584	0.3626	0.3482	0.3208	0.2840	0.2534	0.2102	0.1620	0.1020	0.0590	0.2280	0.2186
	1/3	0.7302	0.7442	0.7490	0.7378	0.7086	0.6758	0.6302	0.5630	0.4662	0.3254	0.1820	0.5794	0.5436
	1/4	0.9176	0.9216	0.9240	0.9170	0.9046	0.8882	0.8644	0.8230	0.7536	0.6104	0.4106	0.8304	0.7944
	1/5	0.9768	0.9800	0.9804	0.9792	0.9744	0.9664	0.9568	0.9370	0.8954	0.8050	0.6226	0.9412	0.9184
100	5	0.9924	0.9970	0.9984	0.9994	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.9498	0.9694	0.9792	0.9882	0.9912	0.9930	0.9944	0.9950	0.9956	0.9956	0.9956	0.9960	0.9944
	3	0.7272	0.8058	0.8526	0.8890	0.9118	0.9244	0.9344	0.9402	0.9448	0.9460	0.9496	0.9514	0.9432
	2	0.1744	0.2498	0.3262	0.3994	0.4524	0.4938	0.5196	0.5410	0.5584	0.5664	0.5784	0.5700	0.5528
	1/2	0.6496	0.6522	0.6446	0.6392	0.6246	0.6002	0.5552	0.5076	0.4480	0.3714	0.2976	0.4980	0.4886
	1/3	0.9738	0.9744	0.9732	0.9724	0.9686	0.9632	0.9522	0.9374	0.9168	0.8776	0.8190	0.9322	0.9270
	1/4	0.9996	0.9994	0.9996	0.9996	0.9994	0.9990	0.9982	0.9962	0.9948	0.9916	0.9842	0.9960	0.9952
	1/5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9994	0.9990	1.0000	0.9998
200	5	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	0.9944	0.9960	0.9978	0.9978	0.9980	0.9986	0.9986	0.9992	0.9994	0.9994	0.9994	0.9994	0.9984
	2	0.7046	0.7508	0.7844	0.8150	0.8328	0.8532	0.8636	0.8756	0.8828	0.8880	0.8892	0.8782	0.8738
	1/2	0.9334	0.9314	0.9258	0.9190	0.9068	0.8976	0.8820	0.8634	0.8418	0.8122	0.7682	0.8432	0.8428
	1/3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9996	0.9990	0.9984	0.9994	0.9994
	1/4	1	1	1	1	1	1	1	1	1	1	1	1	1
	1/5	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 4: Empirical powers based on 5000 replications with simulated critical values, when  $\alpha = 0.05$  and  $\tau = 0.3$ .

$K$	$\theta_1$	$0.05 T_{\phi_\lambda}^{(K)}$											$\widetilde{LRT}^{(K)}$	$S^{(K)}$
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0		
40	5	0.2422	0.6320	0.8074	0.8870	0.9266	0.9446	0.9558	0.9594	0.9590	0.9552	0.9292	0.9714	0.9802
	4	0.0998	0.3820	0.5938	0.7276	0.7992	0.8434	0.8698	0.8802	0.8798	0.8656	0.8056	0.9110	0.9316
	3	0.0304	0.1362	0.2950	0.4262	0.5212	0.5876	0.6312	0.6526	0.6494	0.6310	0.5428	0.7146	0.7678
	2	0.0154	0.0268	0.0596	0.1130	0.1640	0.2096	0.2480	0.2710	0.2698	0.2612	0.2050	0.3012	0.3548
	1/2	0.2706	0.3078	0.3226	0.3210	0.3050	0.2738	0.2344	0.1768	0.1148	0.0638	0.0318	0.2558	0.3336
	1/3	0.6232	0.6788	0.7024	0.7090	0.6916	0.6582	0.6106	0.5290	0.3984	0.2376	0.0612	0.6322	0.7250
	1/4	0.8490	0.8866	0.9010	0.9034	0.8952	0.8744	0.8436	0.7818	0.6736	0.4666	0.1452	0.8522	0.9070
	1/5	0.9554	0.9686	0.9726	0.9748	0.9706	0.9630	0.9538	0.9304	0.8640	0.7168	0.3112	0.9578	0.9790
	50	5	0.7892	0.9106	0.9534	0.9732	0.9846	0.9876	0.9900	0.9916	0.9924	0.9914	0.9906	0.9926
4		0.5456	0.7420	0.8446	0.8902	0.9214	0.9394	0.9492	0.9562	0.9586	0.9548	0.9476	0.9600	0.9716
3		0.2176	0.4054	0.5546	0.6530	0.7130	0.7602	0.7944	0.8128	0.8210	0.8092	0.7906	0.8260	0.8620
2		0.0290	0.0754	0.1434	0.2038	0.2596	0.3056	0.3486	0.3726	0.3862	0.3756	0.3548	0.3872	0.4368
1/2		0.4212	0.4408	0.4540	0.4442	0.4168	0.3792	0.3452	0.2934	0.2240	0.1364	0.0692	0.3404	0.4220
1/3		0.8252	0.8420	0.8488	0.8474	0.8338	0.8108	0.7832	0.7352	0.6480	0.5084	0.3080	0.7702	0.8388
1/4		0.9650	0.9694	0.9722	0.9710	0.9676	0.9626	0.9546	0.9338	0.8990	0.8160	0.6428	0.9474	0.9710
1/5		0.9946	0.9956	0.9956	0.9954	0.9952	0.9940	0.9918	0.9880	0.9778	0.9436	0.8558	0.9906	0.9952
100		5	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.9944	0.9978	0.9984	0.9990	0.9994	0.9994	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	1.0000
	3	0.9096	0.9414	0.9608	0.9746	0.9808	0.9842	0.9862	0.9874	0.9890	0.9892	0.9888	0.9882	0.9938
	2	0.3412	0.4398	0.5190	0.5860	0.6388	0.6734	0.6914	0.7066	0.7196	0.7208	0.7222	0.7082	0.7864
	1/2	0.7582	0.7638	0.7598	0.7564	0.7458	0.7266	0.6898	0.6508	0.5962	0.5188	0.4360	0.6646	0.7658
	1/3	0.9928	0.9932	0.9932	0.9932	0.9928	0.9924	0.9894	0.9850	0.9770	0.9638	0.9398	0.9876	0.9956
	1/4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9998	0.9996	0.9990	0.9984	0.9998	1.0000
	1/5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	1.0000	1.0000
	200	5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
4		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
3		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2		0.8920	0.9120	0.9256	0.9394	0.9464	0.9532	0.9558	0.9596	0.9618	0.9634	0.9634	0.9542	0.9748
1/2		0.9762	0.9768	0.9744	0.9716	0.9658	0.9628	0.9576	0.9492	0.9372	0.9214	0.8970	0.9442	0.9796
1/3		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1/4		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1/5		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000



Table 5: Empirical powers based on 5000 replications with simulated critical values, when  $\alpha = 0.05$  and  $\tau = 0.5$ .

$K$	$\theta_1$	$0.05 T_{\phi_\lambda}^{(K)}$										$\widetilde{LRT}^{(K)}$	$S^{(K)}$	
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1			0
40	5	0.4646	0.8180	0.9282	0.9646	0.9794	0.9856	0.9886	0.9898	0.9888	0.9870	0.9752	0.9888	0.9970
	4	0.2220	0.5834	0.7678	0.8668	0.9106	0.9346	0.9486	0.9524	0.9508	0.9420	0.9016	0.9492	0.9784
	3	0.0632	0.2624	0.4530	0.5914	0.6792	0.7304	0.7654	0.7794	0.7726	0.7496	0.6620	0.7678	0.8726
	2	0.0262	0.0512	0.1070	0.1850	0.2406	0.2910	0.3372	0.3538	0.3492	0.3302	0.2638	0.3314	0.4716
	1/2	0.2752	0.3292	0.3488	0.3566	0.3402	0.3112	0.2728	0.2056	0.1296	0.0630	0.0234	0.3522	0.4816
	1/3	0.6534	0.7274	0.7584	0.7644	0.7544	0.7248	0.6762	0.5874	0.4666	0.2816	0.0596	0.7614	0.8612
	1/4	0.8972	0.9270	0.9368	0.9408	0.9366	0.9264	0.9098	0.8688	0.7806	0.6120	0.2294	0.9372	0.9720
	1/5	0.9712	0.9834	0.9872	0.9882	0.9872	0.9846	0.9790	0.9632	0.9298	0.8374	0.4684	0.9890	0.9970
50	5	0.9212	0.9748	0.9882	0.9950	0.9970	0.9976	0.9980	0.9980	0.9982	0.9980	0.9974	0.9980	0.9998
	4	0.7448	0.8846	0.9438	0.9652	0.9760	0.9810	0.9840	0.9858	0.9866	0.9854	0.9830	0.9808	0.9928
	3	0.3800	0.5908	0.7284	0.8006	0.8438	0.8680	0.8902	0.9006	0.9036	0.8950	0.8770	0.8716	0.9408
	2	0.0620	0.1470	0.2340	0.3122	0.3676	0.4146	0.4596	0.4844	0.4928	0.4766	0.4504	0.4298	0.5818
	1/2	0.4470	0.4760	0.4904	0.4836	0.4628	0.4352	0.3970	0.3378	0.2646	0.1606	0.0706	0.4448	0.5832
	1/3	0.8734	0.8894	0.9010	0.8998	0.8882	0.8728	0.8534	0.8130	0.7436	0.6166	0.4168	0.8814	0.9432
	1/4	0.9798	0.9834	0.9850	0.9850	0.9834	0.9814	0.9762	0.9652	0.9444	0.8972	0.7766	0.9816	0.9948
	1/5	0.9978	0.9986	0.9986	0.9986	0.9986	0.9976	0.9970	0.9948	0.9888	0.9794	0.9396	0.9974	0.9994
100	5	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	0.9996	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	3	0.9730	0.9840	0.9900	0.9930	0.9950	0.9964	0.9966	0.9970	0.9972	0.9970	0.9966	0.9944	0.9988
	2	0.4946	0.5952	0.6668	0.7292	0.7714	0.7966	0.8112	0.8190	0.8260	0.8246	0.8246	0.7812	0.8872
	1/2	0.8016	0.8096	0.8104	0.8090	0.7988	0.7826	0.7532	0.7186	0.6708	0.5982	0.5200	0.7714	0.8866
	1/3	0.9978	0.9980	0.9980	0.9980	0.9976	0.9974	0.9966	0.9954	0.9930	0.9880	0.9784	0.9970	0.9994
	1/4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9996	1.0000	1.0000
	1/5	1	1	1	1	1	1	1	1	1	1	1	1	1
200	5	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	1	1	1	1	1	1	1	1	1	1	1	1	1
	3	1	1	1	1	1	1	1	1	1	1	1	1	1
	2	0.9562	0.9678	0.9736	0.9776	0.9808	0.9848	0.9860	0.9878	0.9882	0.9890	0.9884	0.9788	0.9956
	1/2	0.9858	0.9858	0.9856	0.9852	0.9832	0.9824	0.9784	0.9740	0.9668	0.9596	0.9466	0.9776	0.9952
	1/3	1	1	1	1	1	1	1	1	1	1	1	1	1
	1/4	1	1	1	1	1	1	1	1	1	1	1	1	1
	1/5	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 6: Empirical powers based on 5000 replications with simulated critical values, when  $\alpha = 0.01$  and  $\tau = 0.2$ .

$K$	$\theta_1$	$0.05 T_{\phi_\lambda}^{(K)}$										$\widetilde{LRT}^{(K)}$	$S^{(K)}$	
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1			0
40	5	0.0008	0.0052	0.1090	0.3356	0.5070	0.6232	0.6720	0.6876	0.6522	0.5440	0.3070	0.8272	0.7804
	4	0.0012	0.0012	0.0376	0.1642	0.3068	0.4186	0.4670	0.4846	0.4460	0.3378	0.1530	0.6682	0.6016
	3	0.0024	0.0016	0.0096	0.0498	0.1130	0.1808	0.2172	0.2296	0.2008	0.1314	0.0526	0.3866	0.3292
	2	0.0016	0.0012	0.0012	0.0052	0.0172	0.0332	0.0440	0.0474	0.0410	0.0252	0.0166	0.1028	0.0790
	1/2	0.0374	0.0540	0.0752	0.0864	0.0854	0.0714	0.0500	0.0292	0.0122	0.0076	0.0084	0.0520	0.0562
	1/3	0.1066	0.1980	0.2646	0.3010	0.2970	0.2672	0.1954	0.1084	0.0340	0.0062	0.0064	0.1964	0.1628
	1/4	0.2234	0.3960	0.4962	0.5392	0.5340	0.4936	0.3946	0.2544	0.0916	0.0078	0.0062	0.3790	0.3128
	1/5	0.3792	0.6034	0.7102	0.7444	0.7404	0.7098	0.6128	0.4410	0.2002	0.0200	0.0064	0.5898	0.4872
50	5	0.0074	0.1362	0.4184	0.6102	0.7354	0.7920	0.8300	0.8444	0.8328	0.8062	0.7132	0.9172	0.8758
	4	0.0012	0.0378	0.1942	0.3598	0.4992	0.5872	0.6448	0.6678	0.6464	0.6058	0.4746	0.7866	0.7118
	3	0.0004	0.0066	0.0484	0.1280	0.2168	0.2868	0.3448	0.3728	0.3454	0.3044	0.2002	0.5214	0.4384
	2	0.0022	0.0018	0.0060	0.0172	0.0360	0.0566	0.0776	0.0848	0.0788	0.0652	0.0342	0.1514	0.1132
	1/2	0.1082	0.1234	0.1442	0.1416	0.1332	0.1082	0.0796	0.0468	0.0170	0.0086	0.0064	0.0746	0.0678
	1/3	0.3668	0.4150	0.4554	0.4554	0.4396	0.3884	0.3212	0.2232	0.1006	0.0236	0.0046	0.2974	0.2274
	1/4	0.6416	0.7016	0.7418	0.7456	0.7302	0.6830	0.6080	0.4960	0.2808	0.0830	0.0078	0.5726	0.4620
	1/5	0.8276	0.8642	0.8902	0.8934	0.8834	0.8550	0.8086	0.7010	0.4772	0.1936	0.0072	0.7738	0.6572
100	5	0.8036	0.9376	0.9370	0.9886	0.9944	0.9968	0.9974	0.9976	0.9980	0.9972	0.9970	0.9994	0.9982
	4	0.4946	0.7542	0.7508	0.9292	0.9576	0.9670	0.9736	0.9762	0.9774	0.9746	0.9682	0.9880	0.9810
	3	0.1256	0.3418	0.3380	0.6714	0.7520	0.7930	0.8222	0.8354	0.8434	0.8282	0.8006	0.8926	0.8592
	2	0.0028	0.0206	0.0200	0.1312	0.1924	0.2282	0.2700	0.2958	0.3078	0.2834	0.2438	0.3992	0.3450
	1/2	0.3122	0.3494	0.2528	0.3782	0.3740	0.3360	0.2902	0.2300	0.1668	0.0898	0.0276	0.2876	0.2634
	1/3	0.8326	0.8602	0.7892	0.8864	0.8838	0.8550	0.8232	0.7646	0.6772	0.5060	0.2610	0.8132	0.7532
	1/4	0.9864	0.9898	0.9786	0.9930	0.9928	0.9906	0.9858	0.9728	0.9526	0.8746	0.6808	0.9830	0.9654
	1/5	0.9990	0.9992	0.9988	0.9994	0.9994	0.9992	0.9990	0.9986	0.9956	0.9816	0.9070	0.9990	0.9972
200	5	1	1	1	1	1	1	1	1	1	1	1	1	1
	4	0.9986	0.9996	0.9998	0.9998	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	3	0.9506	0.9726	0.9842	0.9890	0.9922	0.9930	0.9936	0.9936	0.9940	0.9940	0.9936	0.9958	0.9940
	2	0.3052	0.4120	0.4978	0.5508	0.5938	0.6486	0.6676	0.6688	0.6822	0.6784	0.6614	0.7446	0.6786
	1/2	0.7840	0.7912	0.7906	0.7706	0.7428	0.7314	0.6872	0.6104	0.5478	0.4524	0.3242	0.6528	0.5752
	1/3	0.9986	0.9988	0.9986	0.9984	0.9980	0.9978	0.9964	0.9930	0.9906	0.9818	0.9588	0.9948	0.9908
	1/4	1	1	1	1	1	1	1	1	1	1	1	1	1
	1/5	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 7: Empirical powers based on 5000 replications with simulated critical values, when  $\alpha = 0.01$  and  $\tau = 0.3$ .

$K$	$\theta_1$	$0.05 T_{\phi_\lambda}^{(K)}$										$\widetilde{LRT}^{(K)}$	$S^{(K)}$	
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1			0
40	5	0.0024	0.0250	0.3266	0.6294	0.7738	0.8456	0.8696	0.8728	0.8476	0.7584	0.4934	0.9198	0.9390
	4	0.0010	0.0066	0.1342	0.3792	0.5456	0.6534	0.6912	0.6982	0.6552	0.5286	0.2596	0.7906	0.8322
	3	0.0014	0.0016	0.0330	0.1284	0.2496	0.3452	0.3894	0.3972	0.3520	0.2400	0.0810	0.5052	0.5582
	2	0.0030	0.0018	0.0032	0.0164	0.0390	0.0706	0.0874	0.0930	0.0778	0.0426	0.0220	0.1404	0.1742
	1/2	0.0362	0.0698	0.1050	0.1244	0.1268	0.1134	0.0782	0.0390	0.0116	0.0044	0.0060	0.0928	0.1324
	1/3	0.1222	0.2812	0.3958	0.4502	0.4544	0.4240	0.3390	0.2250	0.0788	0.0068	0.0066	0.3622	0.4640
	1/4	0.2712	0.5376	0.6680	0.7186	0.7256	0.7028	0.6168	0.4604	0.2142	0.0240	0.0076	0.6408	0.7270
	1/5	0.4944	0.7690	0.8636	0.8964	0.8992	0.8834	0.8342	0.7182	0.4458	0.0680	0.0062	0.8458	0.8972
	50	5	0.0464	0.4022	0.7184	0.8592	0.9172	0.9410	0.9532	0.9564	0.9504	0.9346	0.8728	0.9726
4		0.0096	0.1638	0.4522	0.6352	0.7558	0.8114	0.8442	0.8552	0.8360	0.7942	0.6636	0.8898	0.9100
3		0.0024	0.0306	0.1540	0.2950	0.4236	0.5010	0.5522	0.5706	0.5384	0.4828	0.3334	0.6454	0.6816
2		0.0022	0.0018	0.0060	0.0172	0.0360	0.0566	0.0776	0.0848	0.0788	0.0652	0.0342	0.1514	0.1132
1/2		0.1216	0.1574	0.1938	0.1978	0.1908	0.1634	0.1244	0.0778	0.0284	0.0080	0.0028	0.1340	0.1830
1/3		0.4600	0.5436	0.6072	0.6216	0.6144	0.5704	0.5034	0.3912	0.2112	0.0578	0.0054	0.5132	0.5944
1/4		0.7796	0.8450	0.8824	0.8894	0.8856	0.8640	0.8204	0.7320	0.5192	0.2248	0.0104	0.8258	0.8724
1/5		0.9242	0.9548	0.9686	0.9720	0.9714	0.9628	0.9434	0.9034	0.7804	0.4776	0.0356	0.9452	0.9638
100		5	0.9700	0.9946	0.9942	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.8182	0.9414	0.9410	0.9894	0.9952	0.9968	0.9978	0.9980	0.9980	0.9976	0.9956	0.9990	0.9992
	3	0.3640	0.6342	0.6304	0.8772	0.9188	0.9316	0.9434	0.9470	0.9478	0.9388	0.9228	0.9662	0.9768
	2	0.0168	0.0760	0.0742	0.2708	0.3620	0.4082	0.4462	0.4660	0.4732	0.4390	0.3826	0.5322	0.5996
	1/2	0.4036	0.4582	0.3680	0.5060	0.5096	0.4748	0.4270	0.3678	0.2806	0.1554	0.0584	0.4560	0.5560
	1/3	0.9244	0.9478	0.9132	0.9624	0.9634	0.9534	0.9404	0.9134	0.8720	0.7588	0.5352	0.9472	0.9708
	1/4	0.9978	0.9984	0.9972	0.9990	0.9990	0.9986	0.9984	0.9972	0.9948	0.9774	0.9190	0.9986	
	1/5	0.9998	1.0000	0.9998	1.0000	1.0000	1.0000	0.9998	0.9998	0.9998	0.9990	0.9936	0.9998	1
	200	5	1	1	1	1	1	1	1	1	1	1	1	1
4		1	1	1	1	1	1	1	1	1	1	1	1	1
3		0.9962	0.9982	0.9992	0.9996	0.9998	0.9998	0.9998	0.9998	1.0000	1.0000	1.0000	0.9998	0.9998
2		0.5808	0.6866	0.7604	0.7912	0.8204	0.8556	0.8674	0.8650	0.8700	0.8632	0.8446	0.8886	0.9178
1/2		0.8892	0.8958	0.9004	0.8890	0.8772	0.8700	0.8424	0.7958	0.7442	0.6596	0.5364	0.8358	0.8964
1/3		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9950	1.0000	1
1/4		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9950	1.0000	1
1/5		1	1	1	1	1	1	1	1	1	1	1	1	1

Table 8: Empirical powers based on 5000 replications with simulated critical values, when  $\alpha = 0.01$  and  $\tau = 0.5$ .

$K$	$\theta_1$	$0.05 T_{\phi_\lambda}^{(K)}$											$\widetilde{LRT}^{(K)}$	$S^{(K)}$
		-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0		
40	5	0.0038	0.0902	0.5536	0.8170	0.9100	0.9458	0.9562	0.9562	0.9412	0.8818	0.6374	0.9538	0.9834
	4	0.0022	0.0242	0.2914	0.5764	0.7260	0.8120	0.8376	0.8388	0.7998	0.6846	0.3880	0.8344	0.9252
	3	0.0028	0.0046	0.0790	0.2530	0.4036	0.5014	0.5406	0.5434	0.4888	0.3646	0.1450	0.5366	0.7054
	2	0.0038	0.0028	0.0080	0.0352	0.0780	0.1204	0.1468	0.1502	0.1238	0.0758	0.0346	0.1434	0.2444
	1/2	0.0294	0.0674	0.1164	0.1470	0.1508	0.1376	0.0956	0.0486	0.0124	0.0036	0.0056	0.1586	0.2562
	1/3	0.1056	0.3064	0.4406	0.5010	0.5104	0.4900	0.4144	0.2788	0.1024	0.0074	0.0050	0.5220	0.6962
	1/4	0.3086	0.6340	0.7632	0.8088	0.8198	0.7996	0.7414	0.6150	0.3556	0.0442	0.0044	0.8286	0.9176
	1/5	0.5660	0.8496	0.9192	0.9422	0.9468	0.9388	0.9104	0.8406	0.6308	0.1468	0.0038	0.9500	0.9826
	50	5	0.1614	0.6454	0.8818	0.9542	0.9768	0.9836	0.9878	0.9884	0.9852	0.9788	0.9464	0.9868
4		0.0384	0.3450	0.6686	0.8206	0.8916	0.9240	0.9396	0.9436	0.9306	0.9026	0.8100	0.9352	0.9728
3		0.0088	0.0906	0.2970	0.4702	0.6018	0.6738	0.7164	0.7310	0.6940	0.6340	0.4612	0.7044	0.8236
2		0.0034	0.0062	0.0322	0.0860	0.1486	0.1890	0.2256	0.2374	0.2104	0.1766	0.0978	0.2142	0.3206
1/2		0.1178	0.1728	0.2200	0.2314	0.2268	0.1966	0.1526	0.0918	0.0322	0.0064	0.0030	0.2220	0.3350
1/3		0.5138	0.6234	0.6906	0.7130	0.7126	0.6722	0.6168	0.5074	0.3090	0.0996	0.0028	0.7110	0.8298
1/4		0.8424	0.8992	0.9262	0.9336	0.9338	0.9210	0.8978	0.8428	0.6838	0.3744	0.0198	0.9328	0.9698
1/5		0.9614	0.9796	0.9858	0.9868	0.9868	0.9844	0.9800	0.9632	0.8948	0.6782	0.0958	0.9868	0.9956
100		5	0.9958	0.9998	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	4	0.9502	0.9896	0.9894	0.9986	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	1
	3	0.5996	0.8232	0.8210	0.9586	0.9746	0.9790	0.9832	0.9850	0.9850	0.9800	0.9736	0.9844	0.9946
	2	0.0460	0.1612	0.1562	0.4158	0.5118	0.5564	0.5926	0.6080	0.6094	0.5716	0.5062	0.6066	0.7594
	1/2	0.4418	0.5162	0.4222	0.5822	0.5906	0.5572	0.5120	0.4420	0.3546	0.2232	0.0900	0.6010	0.7540
	1/3	0.9654	0.9780	0.9632	0.9860	0.9870	0.9830	0.9784	0.9678	0.9474	0.8830	0.7154	0.9874	0.9970
	1/4	0.9996	0.9996	0.9996	0.9996	0.9998	0.9996	0.9996	0.9994	0.9988	0.9962	0.9760	0.9998	1
	1/5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9980	1.0000	1
	200	5	1	1	1	1	1	1	1	1	1	1	1	1
4		1	1	1	1	1	1	1	1	1	1	1	1	1
3		0.9996	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1
2		0.7552	0.8358	0.8844	0.9026	0.9186	0.9336	0.9378	0.9356	0.9374	0.9350	0.9244	0.9346	0.9742
1/2		0.9368	0.9400	0.9434	0.9386	0.9312	0.9280	0.9140	0.8818	0.8508	0.7936	0.6930	0.9308	0.9744
1/3		1	1	1	1	1	1	1	1	1	1	1	1	1
1/4		1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9950	1.0000	1
1/5		1	1	1	1	1	1	1	1	1	1	1	1	1

Finally, we have to note some differences and similarities of the results obtained in this paper with the respective ones obtained in Batsidis *et al.* (2011).

Based on Table 2 of this paper, we have concluded that in almost all cases presented here the test statistics are accurate, when the simulated critical values were used. Contrary to this, when the asymptotic critical values of the test statistics were used, it was concluded in Batsidis *et al.* (2011) that for each significance level and sample size  $K \geq 40$  there are values of the parameter  $\lambda$  for which the power divergence test statistic is accurate. Moreover, the tests statistics  $\widetilde{LRT}^{(K)}$  and  $S^{(K)}$  are in almost all cases not accurate.

Based on the results of this paper and the results of Section 3 of Batsidis *et al.* (2011) related to the power of the test the following common conclusions holds: i) the power is increasing rapidly as the sample size increases, ii) the power increases with  $\tau$  ( $0 < \tau \leq 0.5$ ) and is optimal when  $\tau = 0.5$ , and iii) when  $\tau < 0.5$  the test based on  $\widetilde{LRT}^{(K)}$  as well as  $S^{(K)}$  performs less good for  $\rho < 1$  than in the opposite case of  $1/\rho$  (see Haccou et al. (1983, 1985) for a similar conclusion). However this property does not holds for  ${}^{0.05}T_{\phi_\lambda}^{(K)}$ . It is noted that initially there are values for which the opposite holds and there is a point where the behavior changes.

In Batsidis *et al.* (2011) it was concluded that there is always a value of  $\lambda$  for which  ${}^{0.05}T_{\phi_\lambda}^{(K)}$  has greater power than  $\widetilde{LRT}^{(K)}$  as well as  $S^{(K)}$ . This conclusion is not valid when the simulated critical values were used (see conclusion b) above).

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