

Dynamics of stochastic non-Newtonian fluids driven by fractional Brownian motion with Hurst parameter

$$H \in \left(\frac{1}{4}, \frac{1}{2}\right)^*$$

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Abstract

In this paper we consider the Stochastic isothermal, nonlinear, incompressible bipolar viscous fluids driven by a genuine cylindrical fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ under Dirichlet boundary condition on 2D square domain. First we prove the existence and regularity of the stochastic convolution corresponding to the stochastic non-Newtonian fluids. Then we obtain the existence and uniqueness results for the stochastic non-Newtonian fluids. Under certain condition, the random dynamical system generated by non-Newtonian fluids has a random attractor.

Keywords: fractional Brownian motion, stochastic non-Newtonian fluid, random attractor

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1 Introduction

In this paper, the stochastic non-Newtonian fluids driven by fractional Brownian motion (fBm, for short) on $[0, \pi] \times [0, \pi]$ are studied. The constitutive relations for such fluids were introduced by Bellout, Bloom and Nečas [1] to describe the isothermal, nonlinear, incompressible bipolar viscous fluid. It has the form

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0(\epsilon + |e|^2)^{-\frac{\alpha}{2}}e_{ij} - 2\mu_1\Delta e_{ij} \quad (1.1)$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k} \quad (1.2)$$

where τ_{ij} is the components of the stress tensor, τ_{ijk} is the components of the first multipolar stress tensor, and p is the pressure. e_{ij} are the components of the rate of deformation tensor, i.e.

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (1.3)$$

$\epsilon, \mu_0, \mu_1 > 1$ and $\alpha, 0 < \alpha \leq 1$, are constitutive parameters. The constitutive relation (1.1) and (1.2), and the condition of incompressibility, yield the following nonlinear partial differential equations (we call it Bellout-Bloom-Nečas fluids):

$$\rho \frac{\partial u}{\partial t} + \rho(u \cdot \nabla)u + \nabla p = \nabla \cdot (\mu(u)e - 2\mu_1\Delta e) + \rho f \quad (1.4)$$

$$\nabla \cdot u = 0, \quad (1.5)$$

where ρ is the constant density, $\mu(u) = 2\mu_0(\epsilon + |e|^2)^{-\frac{\alpha}{2}}$ is a nonlinear viscosity, and f is the external body force vector.

There are many works concerning existence and regularity of solution to the Bellout-Bloom-Nečas fluids and its dynamics (see, for instance, [2, 1, 3, 4, 5, 6]). In this paper, we consider the stochastic Bellout-Bloom-Nečas fluids driven by a genuine cylindrical fBm with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \nabla \cdot (\mu(u)e - 2\mu_1\Delta e) + \frac{dB^H(t)}{dt} \quad x \in \mathcal{O}, t > 0 \quad (1.6)$$

$$\nabla \cdot u = 0, \quad x \in \mathcal{O}, t > 0 \quad (1.7)$$

$$u = 0, \quad \tau_{ijk}\eta_j\eta_k = 0, \quad x \in \partial\mathcal{O}, t \geq 0 \quad (1.8)$$

$$u = u_0, \quad x \in \mathcal{O}, t = 0 \quad (1.9)$$

where \mathcal{O} is a 2D square, i.e., $\mathcal{O} = \{(x_1, x_2) \mid 0 < x_1 < \pi, 0 < x_2 < \pi\}$. The fractional Brownian noise enters linearly in the equation and the fBm models the noise source. The fBm is a family of Gaussian processes and some useful properties of these process were given by Mandelbrot and Van Ness [7]. For $H < \frac{1}{2}$ the fBm is not a semimartingale and they can be used in modeling phenomena with intermittency and anti-persistence such as financial turbulence.

The preprint [8] treats stochastic Bellout-Bloom-Nečas fluids that the noise term has a trace-class correlation, and moreover they treat the case $H > \frac{1}{2}$, which allows one to solve the equation using stochastic integrals understood in a pathwise way. In this paper we first provide a detailed study of the existence and regularity properties of the stochastic convolution corresponding to stochastic non-Newtonian fluids driven by fBm with $H \in (\frac{1}{4}, \frac{1}{2})$. The approach for dropping the Hilbert-Schmidt operator hypothesis follows [9]. Then we establish the existence of solution by a modified version of fixed point theorem in some specific Banach space. The a priori estimate for solution in intersection space follows [8]. Finally we construct the random dynamical system associated by non-Newtonian fluids and prove the existence of random attractor follows the framework of [10].

We emphasize four points in our paper. (i) By careful estimation, the growth speed for eigenvalues of the linear differential operator ensures the convergence of the stochastic convolution respect to fBm integral. When investigating the dynamics of Bellout-Bloom-Nečas equation perturbed by fraction Brownian noise, (ii) the ergodic property of stochastic integral with respect to fBm ensures us to construct a random dynamical system, (iii) the at most polynomial growth of sample path ensures the existence of random absorbing sets, and the regularity of stochastic convolution with respect to infinite dimensional fBm with $H \in (\frac{1}{4}, \frac{1}{2})$ ensures the existence of absorbing set with higher regularity. (iv) Under certain condition (which limits the injection parameters of fractional integral space to L^2 space), the random dynamical system has a random attractor as the case $H > \frac{1}{2}$ in [8].

The rest of the paper is organized as follow. In section 2, we formulate the mathematical setting for stochastic Bellout-Bloom-Nečas fluids with Dirichlet boundary con-

dition on 2D square domain and recall the Wiener integrals with respect to infinite dimensional fBm by the framework of [9]. Section 3 is devoted to the existence and regularity properties of the stochastic convolution. In section 4, the global existence and uniqueness of solution is obtained. In section 5, we prove the existence of a random attractor for the random dynamical system generated by non-Newtonian fluids.

2 Preliminaries

We use the standard mathematical framework of this model.

$$\mathcal{V} = \left\{ \phi = (\phi_1, \phi_2) \in (C_0^\infty(\mathcal{O}))^2 : \nabla \cdot \phi = 0 \text{ and } \phi = 0 \text{ on } \partial\mathcal{O} \right\}.$$

H = the closure of \mathcal{V} in $(L^2(\mathcal{O}))^2$ with norm $|\cdot|$.

V = the closure of \mathcal{V} in $(H^2(\mathcal{O}))^2$ with norm $|\cdot|_V$.

\dot{H}^k = the closure of \mathcal{V} in $(H^k(\mathcal{O}))^2$ with norm $|\cdot|_k$ for $k \in \mathbb{N}$.

For other Banach space Y , denote the norm $|\cdot|_Y$.

$$B_Y(M) := \{|x|_Y \leq M, x \in Y\}.$$

Let H' and V' be the dual spaces of H and V respectively. It follows that $V \subset H \equiv H' \subset V'$ and the injections are continuous.

Denote by (\cdot, \cdot) the inner product in H , $\langle \cdot, \cdot \rangle$ the dual pair between V' and V .

$\mathcal{L}(H, V) := \{ \text{all the bounded linear operator from } H \text{ to } V \}$, $\mathcal{L}(H) := \mathcal{L}(H, H)$.

$\mathcal{L}_2(H) := \{ \text{all the Hilbert-Schmidt operator from } H \text{ to } H \}$.

We use lowercase $c_i, i \in \mathbb{N}$ for global constants and capital C for local constants which may change value from line to line.

Inspired by [11], we select space $X := C([0, T]; H) \cap L^2(0, T; V)$ with norm $|\cdot|_X := |\cdot|_{C([0, T]; H)} + |\cdot|_{L^2(0, T; V)}$ for solutions. For the completeness of space X we refer to [8].

First we define a bilinear form $a(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$,

$$a(u, v) = \frac{1}{2}(\Delta u, \Delta v). \tag{2.1}$$

Proposition 2.1. *The rate of deformation tensor $e(u)$ and bilinear form $a(\cdot, \cdot)$ have the following properties:*

(i)

$$\nabla \cdot e(u) = \frac{1}{2} \Delta u \quad \forall u \in V, \quad (2.2)$$

$$\nabla \cdot (\Delta e(u)) = \frac{1}{2} \Delta^2 u \quad \forall u \in \dot{H}^4. \quad (2.3)$$

(ii) For $u, v \in V$,

$$\frac{1}{2}(\Delta u, \Delta v) = 2 \sum_{i,j,k=1}^2 \int \frac{\partial e_{ij}(u)}{\partial x_j} \cdot \frac{\partial e_{ik}(u)}{\partial x_k} dx = \sum_{i,j,k=1}^2 \int \frac{\partial e_{ij}(u)}{\partial x_k} \cdot \frac{\partial e_{ij}(u)}{\partial x_k} dx. \quad (2.4)$$

(iii) ([3] Lemma 2.3) There exist $c_1, c_2 > 0$, s.t.

$$c_1 |u|_V^2 \leq a(u, u) \leq c_2 |u|_V^2, \quad \forall u \in V. \quad (2.5)$$

Next we define the abstract differential operator A and the analytic semigroup $S(\cdot)$. According to (iii) of Proposition 2.1, we can use Lax-milgram Theorem to define $A \in \mathcal{L}(V, V')$:

$$\langle Au, v \rangle = a(u, v) \quad \forall u, v \in V. \quad (2.6)$$

And we have

Proposition 2.2. [8]

(i) Operator A is an isometric form V to V' . Furthermore, let $D(A) = \{u \in V : a(u, v) = (f, v), f \in H\}$. Then $A \in \mathcal{L}(D(A), H)$ is an isometric form $D(A)$ to H .

(ii) Operator A is self-adjoint positive with compact inverse. By Hilbert Theorem, there exist eigenvectors $\{e_i\}_{i=1}^\infty \subset D(A)$ and eigenvalues $\{\lambda_i\}_{i=1}^\infty$ s.t.

$$Ae_i = \lambda e_i, \quad e_i \in D(A), \quad i = 1, 2, \dots \quad (2.7)$$

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_i \leq \dots, \quad \lim_{i \rightarrow \infty} \lambda_i = \infty. \quad (2.8)$$

And $\{e_i\}_{i=1}^\infty$ form an orthonormal basis for H .

(iii) For $u \in D(A)$,

$$Au = \nabla \cdot (\Delta e(u)) = \frac{1}{2} \Delta^2 u, \quad (2.9)$$

i.e., $A = P\Delta^2$ where P is the Leray projection operator from $L^2(\mathcal{O})$ to H .

Noticing that A is a self-adjoint positive linear operator with discrete spectrum, we define the fractional power of A by following the framework of Chueshov [12] section 2.1:

Definition 2.3. For $\alpha > 0$,

$$D(A^\alpha) := \left\{ h = \sum_{k=1}^{\infty} c_k e_k \in H : \sum_{k=1}^{\infty} c_k^2 (\lambda_k^\alpha)^2 < \infty \right\}, \quad (2.10)$$

$$D(A^{-\alpha}) := \left\{ \text{formal serial } \sum c_k e_k \text{ such that } \sum_{k=1}^{\infty} c_k^2 (\lambda_k^\alpha)^2 < \infty \right\}, \quad (2.11)$$

$$A^\alpha h = \sum_{k=1}^{\infty} c_k \lambda_k^\alpha e_k, \quad h \in D(A^\alpha). \quad (2.12)$$

Let $\mathcal{F}_\alpha \equiv D(A^\alpha)$. Then \mathcal{F}_α is a separable Hilbert space with the inner product $(u, v)_{\mathcal{F}_\alpha} = (A^\alpha u, A^\alpha v)$ and the norm $\|u\|_{\mathcal{F}_\alpha} = |A^\alpha u|$; $\mathcal{F}_{-\alpha}$ denotes the union of bounded linear functional on \mathcal{F}_α ; Particularly, we have $\mathcal{F}_0 = H$, $\mathcal{F}_{1/2} = V$, $\mathcal{F}_{-1/2} = V'$ and the norm of V and $H^2(\mathcal{O})$ are equivalent; For $\sigma_1 > \sigma_2$, the space \mathcal{F}_{σ_1} is compactly embedded into \mathcal{F}_{σ_2} .

Since A is a densely defined self-adjoint bounded-below operator in Hilbert space H , we deduce that A is a sector operator and it generates an analytic semigroup $S \in \mathcal{L}(H)$ (see, for instance, [13] section 1.3),

$$S(t) := e^{-tA} = \int_0^\infty e^{-t\lambda} dE_\lambda. \quad (2.13)$$

For a survey of the properties of the analytic semigroup we refer to [8].

In order to write down the abstract evolution equation, we need to handle the nonlinear terms. Following the method in dealing with Navier-Stokes equation, we define the trilinear form:

$$b(u, v, w) = \sum_{i,j=1}^2 \int_{\mathcal{O}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \quad \forall u, v, w \in H_0^1(\mathcal{O}). \quad (2.14)$$

Since $V \subset H_0^1(\mathcal{O})$ is a closed subspace, $b(\cdot, \cdot, \cdot)$ is continuous in $V \times V \times V$. From [14] we have,

$$b(u, v, w) = -b(u, w, v), \quad b(u, v, v) = 0 \quad \forall u, v, w \in H_0^1(\mathcal{O}). \quad (2.15)$$

For $u, v \in V$, define the functional $B(u, v) \in V'$:

$$\langle B(u, v), w \rangle = b(u, v, w) \quad \forall w \in V \quad (2.16)$$

and denote $B(u) := B(u, u) \in V'$.

For $u \in V$, define $N(u)$ as

$$\langle N(u), v \rangle = \int_{\mathcal{O}} \mu(u) e_{ij}(u) e_{ij}(v) dx \quad \forall v \in V. \quad (2.17)$$

Then $N(\cdot)$ is a continuous from V to V' and

$$\langle N(u), v \rangle = - \int_{\mathcal{O}} (\nabla \cdot (\mu(u) e(u))) \cdot v dx. \quad (2.18)$$

Comprehensively, we have the following abstract evolution equation from problem (1.6)-(1.9):

$$\begin{cases} du + (2\mu_1 Au + B(u) + N(u)) dt = dB^H(t), \\ u(0) = u_0. \end{cases} \quad (2.19)$$

Since the derivative of fBm exists almost nowhere, we will give a mathematical interpret of above equation in the next subsection. Without loss of generality, we set $\mu_1 = 1$ in the sequel.

Next we introduce the Wiener-type stochastic integral with respect to fBm. For all $T > 0$, let $\beta^H(t)$ be the one-dimensional fBm. In this paper we only consider the case $H < \frac{1}{2}$ following the [9] and for a survey of Winer-type stochastic integral we refer to [15]. By definition β^H is a centered Gaussian process with covariance

$$R(t, s) = E(\beta^H(t)\beta^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (2.20)$$

And β^H has the following Wiener integral representation:

$$\beta^H(t) = \int_0^t K^H(t, s) dW(s), \quad (2.21)$$

where W is a Wiener process, and $K^H(t, s)$ is the kernel given by

$$K^H(t, s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + s^{\frac{1}{2}-H} F\left(\frac{t}{s}\right). \quad (2.22)$$

c_H is a constant and

$$F(z) = c_H \left(\frac{1}{2} - H\right) \int_0^{z-1} r^{H-\frac{3}{2}} \left(1 - (1+r)^{H-\frac{1}{2}}\right) dr. \quad (2.23)$$

By (2.22) we obtain

$$\frac{\partial K^H}{\partial t}(t, s) = c_H \left(H - \frac{1}{2}\right) (t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \quad (2.24)$$

Denote by \mathcal{E}_H the linear space of step function of the form

$$\varphi(t) = \sum_{i=1}^n a_i 1_{(t_i, t_{i+1}]}(t) \quad (2.25)$$

where $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and by \mathcal{H} the closure of \mathcal{E}_H with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R(t, s) \quad (2.26)$$

For $\varphi \in \mathcal{E}_H$ we define its Weiner integral with respect to the fBm as

$$\int_0^T \varphi(s) d\beta^H(s) = \sum_{i=1}^n a_i (\beta_{t_{i+1}}^H - \beta_{t_i}^H). \quad (2.27)$$

The mapping

$$\varphi = \sum_{i=1}^n a_i 1_{(t_i, t_{i+1}]} \rightarrow \int_0^T \varphi(s) d\beta^H(s) \quad (2.28)$$

is an isometry between \mathcal{E}_H and the linear space $\text{span}\{\beta^H(t), 0 \leq t \leq T\}$ viewed as a subspace of $L^2(0, T)$ and it can be extended to an isometry between \mathcal{H} and the $\overline{\text{span}}^{L^2}\{\beta^H(t), 0 \leq t \leq T\}$. The image on an element $\Psi \in \mathcal{H}$ by this isometry is called the Wiener integral of Ψ with respect to β^H .

For every $s < t$, consider the operator K^*

$$(K_t^*)\varphi(s) = K(t, s)\varphi(s) + \int_s^t (\varphi(r) - \varphi(s)) \frac{\partial K}{\partial r}(r, s) dr. \quad (2.29)$$

We refer to [16] for the proof of the fact that K^* is a isometry between \mathcal{H} and $L^2(0, T)$. For $H < \frac{1}{2}$, the reproducing kernel Hilbert space \mathcal{H} can be represented by the fractional integral space. Namely,

$$\mathcal{H} = (K_H^*)^{-1}(L^2(0, T)) = I_{T-}^{\frac{1}{2}-H}(L^2(0, T)), \quad (2.30)$$

where $I_{T-}^{\frac{1}{2}-H}(L^2(0, T))$ is the family of functions f that can be represented as a fractional $I_{T-}^{\frac{1}{2}-H}$ -integral of some function $\phi \in L^2(0, T)$. As a consequence, we have the following relationship between the Wiener integral with respect to fBm and the Wiener integral with respect to the Wiener process:

$$\int_0^t \varphi(s) d\beta^H(s) = \int_0^t (K_t^*\varphi)(s) dW(s) \quad (2.31)$$

for every $t \leq T$ and $\varphi \in \mathcal{H}$ if and only if $K_t^*\varphi \in L^2(0, T)$. Since we work only with Wiener integral over Hilbert space, we have that if $u \in L^2(0, T; H)$ is a deterministic

function, then the relation (2.31) holds and the Wiener integral on the righthand side being well defined in $L^2(\Omega; H)$ if K^*u belongs to $L^2(0, T; H)$.

In the following we concern the infinite dimensional fBm and stochastic integration (see, e.g. [17]). A standard cylindrical fractional Brownian motion is defined now.

Definition 2.4. [18] *Let (Ω, \mathcal{F}, P) be a complete probability space. A cylindrical process $\langle B^H, \cdot \rangle: \Omega \times \mathbb{R}_+ \times H \rightarrow \mathbb{R}$ on (Ω, \mathcal{F}, P) is called a standard cylindrical fractional Brownian motion with the Hurst parameter $H \in (0, 1)$ if*

1. *for each $x \in H \setminus \{0\}$, $\frac{1}{\|x\|} \langle B^H(\cdot), x \rangle$ is a standard scalar fBm with Hurst parameter H ;*

2. *for $\alpha, \beta \in \mathbb{R}$ and $x, y \in H$,*

$$\langle B^H(t), \alpha x + \beta y \rangle = \alpha \langle B^H(t), x \rangle + \beta \langle B^H(t), y \rangle \quad P\text{-a.s.} \quad (2.32)$$

For $H = \frac{1}{2}$, this definition is the usual one for a standard cylindrical Wiener process in H . For the complete orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ of H (which is generated by linear differential operator A), letting $\beta_n^H(t) = \langle B^H(t), e_n \rangle$ for $n \in \mathbb{N}$, the sequence of scalar processes $\{\beta_n^H\}_{n \in \mathbb{N}}$ is independent and B^H can be represented by the formal series

$$B^H(t) = \sum_{n=1}^{\infty} \beta_n^H(t) e_n \quad (2.33)$$

that does not converge a.s. in H . Although for any fixed t the series (2.33) is not convergent in $L^2(\Omega \times H)$, we can always consider a Hilbert space U_1 such that $H \subset U_1$ such that this inclusion is a Hilbert-Schmidt operator. In this way B^H given by (2.33) is a well-defined U_1 -valued Gaussian stochastic process.

Let $\Phi(s), 0 \leq s \leq T$ be a deterministic function with values in $\mathcal{L}_2(H)$, the space of Hilbert-Schmidt operators on H . The stochastic integral of Φ with respect to B^H is defined by

$$\int_0^t \Phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \Phi(s) e_n d\beta_n^H(s) = \sum_{n=1}^{\infty} \int_0^t (K^*(\Phi e_n))(s) d\beta_n(s) \quad (2.34)$$

where β_n is the standard Brownian motion. However, as we are about to see, the stochastic linear additive equation in its mild form can have a solution even if $\int_0^t \Phi(s) dB^H(s)$ is not properly defined as a H -valued process.

3 Linear stochastic evolution equations and stochastic convolution with fBm

In this section, we will work with a cylindrical fBm B^H on the real separable Hilbert space H . First we prove the existence of solution for linear stochastic evolution equation driven by fBm with $H \in (\frac{1}{4}, \frac{1}{2})$. Consider the equation

$$dZ = AZdt + dB^H, \quad Z(0) = u_0 \in H \quad (3.1)$$

As noted in [9] and [18], the stochastic integral $\int_0^t I_d dB^H(s)$ is not well-defined as a H -valued random variable since the identity operator $I_d \notin \mathcal{L}_2(H)$. We then consider the mild form of the equation, whose unique solution, if it exists, can be written in the evolution form

$$Z(t) = S(t)u_0 + \int_0^t S(t-s)dB^H(s). \quad (3.2)$$

Remark 3.1. In [18] they consider the noise term as ΦdB^H and assume that $\Phi \in \mathcal{L}_2$ or $S(t)\Phi \in \mathcal{L}_2$. In such case the infinite dimensional fBm noise can be viewed as finite dimensional since the Hilbert-Schmidt operator is compact. By contrast the noise we consider (i.e., $\Phi = I_d$) is rougher and there is no reason to assume that $\Phi \in \mathcal{L}_2$.

In order to obtain the existence of the stochastic convolution above, we need the following estimate about the spectrum of operator A .

Lemma 3.2. *The eigenvalues of operator A satisfy*

$$\lambda_{mn} \geq (m^2 + n^2)^2, \quad m, n \in \mathbb{N}. \quad (3.3)$$

Proof. It is well known that, the eigenfunctions and eigenvalues of Dirichlet-Laplacian on a 2D square (e.g. see [19]) are

$$\begin{cases} \phi(x_1, x_2) = C \sin mx_1 \sin nx_2 \\ \gamma_{mn} = m^2 + n^2, \quad \text{with } m, n \in \mathbb{N}. \end{cases} \quad (3.4)$$

Rewrite the index of γ and we get the spectrum

$$\sigma_D = \{\gamma_j\}_{j=1}^\infty, \quad 0 < \gamma_1 < \gamma_2 < \dots \quad (3.5)$$

Let σ_S be the spectrum of the Stokes operator (i.e. $-P\Delta$ where P is the Leray projector) with homogenous Dirichlet boundary conditions (which we refer to classical Stokes operator), and we have

$$\sigma_S = \{\tilde{\lambda}_j\}_{j=1}^{\infty}, \quad 0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots \quad (3.6)$$

According to Theorem 1.1 in [20], we have $\gamma_k < \tilde{\lambda}_k$ for all positive integers.

Back to the operator A of non-Newtonian fluids, we have

$$\begin{aligned} \lambda_j e_j &= A e_j \\ &= P\Delta^2 e_j \\ &= -P\Delta(-P\Delta) e_j \\ &= \tilde{\lambda}_j^2 e_j \end{aligned} \quad (3.7)$$

Thus we can reorder the index and estimate the spectrum of A

$$\lambda_{mn} = \tilde{\lambda}_{mn}^2 \geq \gamma_{mn}^2 = (m^2 + n^2)^2. \quad (3.8)$$

□

For technical reason we need the lemma:

Lemma 3.3. *The following inequality holds for all $\lambda > 0$.*

$$\int_0^\lambda e^{-2x} \left(\int_0^x (e^y - 1) y^{H-\frac{3}{2}} dy \right)^2 dx = C(H) < \infty. \quad (3.9)$$

Proof. Calculate the integration and we have

$$\begin{aligned} &\int_0^\lambda e^{-2x} \left(\int_0^x (e^y - 1) y^{H-\frac{3}{2}} dy \right)^2 dx \\ &\leq \int_0^\lambda e^{-2x} \left(\int_0^1 (e^y - 1) y^{H-\frac{3}{2}} dy + \int_1^{\max\{1,x\}} (e^y - 1) y^{H-\frac{3}{2}} dy \right)^2 dx. \end{aligned} \quad (3.10)$$

Firstly we estimate the first term in the integral by parts of integration.

$$\begin{aligned}
& \int_0^1 (e^y - 1)y^{H-\frac{3}{2}} dy \\
&= \frac{1}{H-\frac{1}{2}} \left((e^y - 1)y^{H-\frac{1}{2}} \Big|_0^1 - \int_0^1 e^y y^{H-\frac{1}{2}} dy \right) \\
&= \frac{1}{H-\frac{1}{2}} \left((e-1) - \lim_{y \rightarrow 0} \frac{e^y - 1}{y^{\frac{1}{2}-H}} \right) - \frac{1}{H-\frac{1}{2}} \cdot \frac{1}{H+\frac{1}{2}} \int_0^1 e^y dy^{H+\frac{1}{2}} \\
&= \frac{1}{H-\frac{1}{2}} (e-1) - \frac{1}{H-\frac{1}{2}} \cdot \frac{1}{H+\frac{1}{2}} \left(e^y y^{H+\frac{1}{2}} \Big|_0^1 - \int_0^1 e^y y^{H+\frac{1}{2}} dy \right) \\
&= C(H) < \infty.
\end{aligned} \tag{3.11}$$

Secondly we estimate the other term

$$\begin{aligned}
& \int_0^\lambda e^{-2x} \left(\int_1^{\max\{1,x\}} (e^y - 1)y^{H-\frac{3}{2}} dy \right)^2 dx \\
&= \int_1^\lambda e^{-2x} \left(\int_1^x (e^y - 1) \cdot y^{H-\frac{3}{2}} dy \right)^2 dx \\
&\leq \int_1^\lambda \int_1^x (e^y - 1)^2 \cdot y^{2H-3} e^{-2x} (x-1) dy dx \\
&\leq \int_1^\lambda \int_y^\lambda (e^y - 1)^2 \cdot y^{2H-3} e^{-2x} (x-1) dx dy \\
&\leq \int_1^\lambda (e^y - 1)^2 \cdot y^{2H-3} \int_y^\infty e^{-2x} (x-1) dx dy \\
&\leq \int_1^\lambda (e^y - 1)^2 \cdot y^{2H-3} \left(\frac{1}{2}(y-1)e^{-2y} + \frac{1}{4}e^{-2y} \right) dy \\
&\leq C \int_1^\lambda (1 - e^{-y})^2 \cdot y^{2H-2} + (1 - e^{-y})^2 \cdot y^{2H-3} dy \\
&\leq C \int_1^\infty y^{2H-2} + y^{2H-3} dy \\
&= C \left(\frac{1}{2H-1} + \frac{1}{2H-2} \right) = C(H) < \infty.
\end{aligned} \tag{3.12}$$

Finally we have

$$\begin{aligned}
& \leq \int_0^\lambda e^{-2x} \left(\int_0^1 (e^y - 1)y^{H-\frac{3}{2}} dy + \int_1^{\max\{1,x\}} (e^y - 1)y^{H-\frac{3}{2}} dy \right)^2 dx \\
& \leq \int_0^\lambda e^{-2x} \left(C(H) + \int_1^{\max\{1,x\}} (e^y - 1)y^{H-\frac{3}{2}} dy \right)^2 dx \\
& \leq 2 \int_0^\lambda e^{-2x} C(H)^2 dx + 2 \int_0^\lambda e^{-2x} \left(\int_1^{\max\{1,x\}} (e^y - 1)y^{H-\frac{3}{2}} dy \right)^2 dx \\
& \leq C(H) \int_0^\infty e^{-2x} dx + C(H) < \infty.
\end{aligned} \tag{3.13}$$

□

Remark 3.4. There is a mistake in the proof of Lemma 2 [9]. That is (in Page203), the quantity

$$K_A = \int_0^\infty \left(\int_0^x (e^y - 1)y^{A-1} dy \right)^2 dx \quad (3.14)$$

is infinite. Therefore the theorems which use this lemma (such as Theorem 1.1 in [9] and Theorem 2.1 in [21]) may not hold. However, the argument there can be modified by using our lemma and the expected result can be obtained.

The statement of existence theorem for stochastic convolution follows.

Theorem 3.5. *If the Hurst parameter $H > \frac{1}{4}$, the stochastic convolution $\int_0^t S(t-s)dB^H(s)$ is well defined.*

Proof. The proof is base on Theorem 1 of [9]. It is sufficient to estimate the mean square of the Wiener integral of (3.2).

$$E \left| \int_0^t S(t-s)dB^H(s) \right|_H^2 = E \left| \sum_{n=1}^\infty \int_0^t S(t-s)e_n d\beta_n^H(s) \right|_H^2 \quad (3.15)$$

Using (2.29) and the representation (2.31), we have

$$\begin{aligned} & E |z(t) - S(t)u_0|_H^2 \\ & \leq 2 \sum_n \int_0^t |S(t-s)e_n|^2 K^2(t,s) ds + 2 \sum_n \int_0^t \left| \int_s^t (S(t-r)e_n - S(t-s)e_n) \frac{\partial K}{\partial r}(r,s) dr \right|^2 ds \\ & \triangleq I_1 + I_2 \end{aligned} \quad (3.16)$$

By [22] Th3.2, we have

$$K(t,s) \leq C(H)(t-s)^{H-\frac{1}{2}}s^{H-\frac{1}{2}}. \quad (3.17)$$

Then,

$$\begin{aligned} I_1 & \leq C(H) \sum_n \int_0^t |e^{-(t-s)\lambda_n} e_n|^2 (t-s)^{2H-1} s^{2H-1} ds \\ & = C(H) \sum_n (2\lambda_n)^{-2H} \int_0^{2\lambda_n t} e^{-v} v^{2H-1} \left(t - \frac{v}{2\lambda_n}\right)^{2H-1} dv \end{aligned} \quad (3.18)$$

where $C(H)$ depends only on H and we use the change of variable $t - s = \frac{v}{2\lambda_n}$. Since

$$\begin{aligned}
& \int_0^{2\lambda_n t} e^{-v} v^{2H-1} \left(t - \frac{v}{2\lambda_n}\right)^{2H-1} dv \\
& \leq \int_0^{\lambda_n t} e^{-v} v^{2H-1} \left(\frac{t}{2}\right)^{2H-1} dv + \int_{\lambda_n t}^{2\lambda_n t} e^{-v} 2\lambda_n t^{2H-1} \left(t - \frac{v}{2\lambda_n}\right)^{2H-1} dv \\
& \leq C(t, H) \int_0^\infty e^{-v} v^{2H-1} dv + (2\lambda_n t)^{2H-1} \int_0^{\lambda_n t} e^{-(2\lambda_n t - v')} \left(\frac{v'}{2\lambda_n}\right)^{2H-1} dv' \\
& \leq C(t, H) \Gamma(2H) + t^{2H-1} \int_0^{\lambda_n t} e^{-2\lambda_n t} e^{v'} v'^{2H-1} dv' \\
& \leq C(t, H) + C(t, H) e^{-\lambda_n t} \int_0^{\lambda_n t} v'^{2H-1} dv' \\
& \leq C(t, H) \cdot e^{-\lambda_n t} (\lambda_n t)^{2H} \\
& \leq C(t, H),
\end{aligned} \tag{3.19}$$

we have

$$\begin{aligned}
I_1 & \leq C(H) \cdot \sum_n \lambda_n^{-2H} \cdot C(t, H) \\
& \leq C(t, H) \cdot \sum_{i,j=1}^\infty \frac{1}{(i^2 + j^2)^{4H}} \\
& \leq C(t, H) \cdot 2\beta_D(4H) \cdot \zeta(4H) < \infty
\end{aligned} \tag{3.20}$$

where $\beta_D(s)$ is the Dirichlet beta function and $\zeta(s)$ is the Riemann zeta function (for definition see [23]).

For the second sum, we have

$$\begin{aligned}
I_2 & = 2 \sum_n \int_0^t \left| \int_s^t (S(t-r)e_n - S(t-s)e_n) \frac{\partial K}{\partial r}(r, s) dr \right|_H^2 ds \\
& \leq 2 \sum_n \int_0^t \left| \int_s^t (e^{-(t-r)\lambda_n} - e^{-(t-s)\lambda_n}) \frac{\partial K}{\partial r}(r, s) dr \right|_H^2 ds
\end{aligned} \tag{3.21}$$

By the fact that $\frac{\partial K}{\partial r}(r, s) \leq 0$ for every $r, s \in [0, T]$ and

$$\left| \frac{\partial K}{\partial r}(r, s) \right| \leq C(H) \cdot (r - s)^{H - \frac{3}{2}}, \tag{3.22}$$

we have

$$\begin{aligned}
I_2 & \leq C(H) \sum_n \int_0^t \left(\int_s^t (e^{-(t-r)\lambda_n} - e^{-(t-s)\lambda_n}) (r - s)^{H - \frac{3}{2}} dr \right)^2 ds \\
& \leq C(H) \sum_n \int_0^t \left(\int_0^{t-s} (e^{-(t-s-v)\lambda_n} - e^{-(t-s)\lambda_n}) v^{H - \frac{3}{2}} dv \right)^2 ds \\
& \leq C(H) \sum_n \int_0^t \left(\int_0^u (e^{-(u-v)\lambda_n} - e^{-u\lambda_n}) v^{H - \frac{3}{2}} dv \right)^2 du
\end{aligned} \tag{3.23}$$

By the change of variables $v = \frac{y}{\lambda_n}$, $u = \frac{x}{\lambda_n}$, we have

$$\begin{aligned}
I_2 &\leq C(H) \sum_n \int_0^{\lambda_n t} e^{-2x} \left(\int_0^x (e^y - 1) \left(\frac{y}{\lambda_n} \right)^{H-\frac{3}{2}} \lambda_n^{-1} dy \right)^2 \lambda_n^{-1} dx \\
&\leq C(H) \sum_n \lambda_n^{-2H} \int_0^{\lambda_n t} e^{-2x} \left(\int_0^x (e^y - 1) y^{H-\frac{3}{2}} dy \right)^2 dx \\
&\leq C(H) \sum_n \lambda_n^{-2H} C(H) \quad (\text{by lemma 3.3}) \\
&\leq C(H) \cdot 2\beta_D(4H) \cdot \zeta(4H) < \infty \quad \text{as } 4H > 1.
\end{aligned} \tag{3.24}$$

□

In the following we investigate the regularity of solution. First we show that $B^H \in C([0, \infty); V')$.

Since

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| A^{-\frac{1}{2}} e_n \right| &= \sum_{n=1}^{\infty} \lambda_n^{-1} \\
&= 2\beta_D(2) \cdot \zeta(2) < \infty,
\end{aligned} \tag{3.25}$$

$A^{-\frac{1}{2}} : H \rightarrow V'$ is a Hilbert-Schmidt operator from H to V' . Thus the genuine cylindrical fBm B^H can be viewed as a V' -valued fBm with incremental covariance operator $A^{-\frac{1}{2}}$. And we have that B^H has continuous trajectories on V' .

Let

$$z(t) := \int_0^t S(t-s) dB^H(s), \tag{3.26}$$

$$Y(t) := \int_0^t S(t-s) B^H(s) ds, \tag{3.27}$$

and we have

Lemma 3.6. *For $H > \frac{1}{4}$, $Y(\cdot)$ belongs to $C^1([0, \infty); V)$ P-a.s. and*

$$\begin{aligned}
\frac{d}{dt} Y(t) &= \frac{d}{dt} \int_0^t S(t-s) B^H(s) ds \\
&= B^H(t) + A \int_0^t S(t-s) B^H(s) ds \\
&= z(t).
\end{aligned} \tag{3.28}$$

Since the proof is a modification (consider the base space as V') of Lemma 5.13 in [24] and Proposition 3.1 in [25], we only sketch the main ideas.

Step I:

When $A_0 : V' \rightarrow V'$ is a bounded linear operator, we have

$$z(t) = \int_0^t A_0 z(s) ds + B^H(t), \quad t \geq 0, \text{ P-a.s.} \quad (3.29)$$

Step II:

Let $A_n = n(nI - A)^{-1}A$, $n \in \mathbb{N}$ (n large enough) denote the Yosida approximations of A . Denote by $S_n(\cdot)$ the semigroup generated by A_n in V' and set $z_n(t) = \int_0^t S_n(t-s) dB^H(s)$. By using the so-called factorization method, we have for $q > 2$

$$E \sup_{t \in [0, T]} |z_n(t) - z(t)|_{V'}^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.30)$$

Step III:

Since $z_n(\cdot)$ is a strong solution to the stochastic differential equation in V'

$$dz_n(t) = A_n z_n(t) dt + dB^H(t), \quad z_n(0) = 0. \quad (3.31)$$

We have

$$z_n(t) = \int_0^t 0^t A_n z_n(s) ds + B^H(t). \quad (3.32)$$

Setting $Y_n(t) = \int_0^t z_n(s) ds$, we have $Y_n(\cdot)$ is the solution to the initial value problem

$$\frac{d}{dt} Y_n(t) = A_n Y_n(t) + B^H(t), \quad Y_n(0) = 0. \quad (3.33)$$

Thus we have

$$Y_n(t) = \int_0^t S_n(t-s) B^H(s) ds, \quad (3.34)$$

and so P-a.s.

$$z_N(t) = A_n Y_n(t) + B^H(t). \quad (3.35)$$

clearly

$$\lim_{n \rightarrow \infty} Y_n(t) = Y(t) = \int_0^t S(t-s) B^H(s) ds. \quad (3.36)$$

By step I and step II, we have

$$\lim_{n \rightarrow \infty} A_n Y_n(t) = \lim_{n \rightarrow \infty} A \left(I - \frac{1}{n} A \right)^{-1} Y_n(t) = z(t) - B^H(t). \quad (3.37)$$

Since the operator A is closed, we conclude that $Y(t) \in V$ and $AY(t) = z(t) - B^H(t)$ P-a.s..

The following proposition is an immediate consequence of above lemma.

Proposition 3.7. *If $H \in (\frac{1}{4}, \frac{1}{2})$. Then for all $u_0 \in V$ the process $Z(t, x) = S(t)u_0 + z(t)$ has an V -continuous modification. Moreover,*

$$z(t) = A \int_0^t S(t-s)B^H(s)ds + B^H(t), \quad t \geq 0 \quad (3.38)$$

holds with probability one.

4 Solution of stochastic non-Newtonian fluids

We interpret equation (2.19) as an integral equation and define the solution as follow.

Definition 4.1. *The solution to equation (2.19) is defined as a function $u \in C([0, T]; H) \cap L^2(0, T; V)$ s.t. the following integral equation holds P-a.s..*

$$u(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds - \int_0^t S(t-s)N(u(s))ds + \int_0^t S(t-s)dB^H(s) \quad (4.1)$$

where the first and second integral are Bochner integral, the last integral is a stochastic integral defined in previous section.

We seek the solution of 2D stochastic Bellout-Bloom-Nečas fluids by the fixed point in space $X = C([0, T]; H) \cap L^2(0, T; V)$. Firstly we will prove the local existence and uniqueness results.

For $u \in X$, let

$$J_1(u) := - \int_0^\cdot S(\cdot - s)B(u(s))ds, \quad (4.2)$$

$$J_2(u) := - \int_0^\cdot S(\cdot - s)N(u(s))ds. \quad (4.3)$$

We have following estimates.

Lemma 4.2. [8] $J_1 : X \rightarrow X$ and for all $u, v \in X$, we have

$$|J_1(u)|_X^2 \leq c_1 |u|_X^4, \quad (4.4)$$

$$|J_1(u) - J_1(v)|_X^2 \leq c_2 \left(|u|_{C([0,T];H)}^2 \cdot |u|_{L^2(0,T;V)}^2 + |v|_{C([0,T];H)}^2 \cdot |v|_{L^2(0,T;V)}^2 \right)^{\frac{1}{2}} \cdot |u - v|_X^2. \quad (4.5)$$

Lemma 4.3. [8] $J_2 : X \rightarrow X$ and for all $u, v \in X$, we have

$$|J_2(u)|_X^2 \leq c_3 |u|_{L^2(0,T;V)}^2, \quad (4.6)$$

$$|J_2(u) - J_2(v)|_X^2 \leq c_4 T |u - v|_X^2. \quad (4.7)$$

We apply the following version of the contraction mapping theorem.

Lemma 4.4. ([24] Lemma 15.2.6) Let F be a transformation from a Banach space E into E , $\phi \in E$ and $M > 0$ a positive number. If $F(0) = 0$, $|\phi|_E \leq \frac{1}{2}M$ and

$$|F(u) - F(v)|_E \leq \frac{1}{2} |u - v|_E \quad \forall u, v \in B_E(M), \quad (4.8)$$

then the equation

$$u = \phi + F(u) \quad (4.9)$$

has a unique solution $u \in E$ satisfying $u \in B_E(M)$.

We now prove the main result of this paper.

Theorem 4.5. (local existence and uniqueness of solution) For all $u_0 \in H$, there exists $T_0 > 0$ s.t. equation (2.19) P -a.s. admits a unique solution $u \in C([0, T_0]; H) \cap L^2(0, T_0; V)$ in the sense of (4.1).

Proof. Fix $\omega \in \Omega$. Let

$$\phi(t) = S(t)u_0 + z(t). \quad (4.10)$$

By the properties of semigroup $S(\cdot)$ and Lemma 3.7, $S(\cdot)u_0, z \in C([0, T]; V) \subset X$. Then we have

$$|\phi|_X \leq |S(\cdot)u_0|_X + |z|_X \leq 2|u_0| + |z|_X. \quad (4.11)$$

Let $M(\omega) = 2(2|u_0| + |z(\omega)|_X)$.

Construct the mapping $\mathcal{F} = J_1 + J_2$, then for all $u, v \in X$, we have

$$\begin{aligned}
& |\mathcal{F}(u) - \mathcal{F}(v)|_X \\
& \leq |J_1(u) - J_1(v)|_X + |J_2(u) - J_2(v)|_X \\
& \leq c_2^{\frac{1}{2}} \left(|u|_{C([0,T];H)}^2 \cdot |u|_{L^2(0,T;V)}^2 + |v|_{C([0,T];H)}^2 \cdot |v|_{L^2(0,T;V)}^2 \right)^{\frac{1}{4}} \cdot |u - v|_X \quad (4.12) \\
& \quad + (c_4 T)^{\frac{1}{2}} |u - v|_X \quad (\text{by Lemma 4.2 and 4.3}) \\
& \leq (c_2 M)^{\frac{1}{2}} \left(|u|_{L^2(0,T;V)}^2 + |v|_{L^2(0,T;V)}^2 \right)^{\frac{1}{4}} |u - v|_X + (c_4 T)^{\frac{1}{2}} |u - v|_X.
\end{aligned}$$

Due to the absolute continuity property of Bochner integral, we can choose $\tau \in (0, 1]$ s.t.

$$\left(|u|_{L^2(0,\tau;V)}^2 + |v|_{L^2(0,\tau;V)}^2 \right)^{\frac{1}{4}} \leq (2Mc_2)^{-\frac{1}{2}}. \quad (4.13)$$

Let $T_0 = \min\{\tau, 1, \frac{1}{16c_4}\}$ and $X_{T_0} := C([0, T_0]; H) \cap L^2(0, T_0; V)$. We have

$$|\mathcal{F}(u) - \mathcal{F}(v)|_{X_{T_0}} \leq \left(\frac{1}{4} + \frac{1}{4}\right) |u - v|_{X_{T_0}} = \frac{1}{2} |u - v|_{X_{T_0}}. \quad (4.14)$$

Applying the modified fixed point lemma 4.4, equation

$$u = \phi + \mathcal{F}(u) \equiv S(\cdot)u_0 + z + J_1(u) + J_2(u) \quad (4.15)$$

has a unique solution u in $C([0, T_0]; H) \cap L^2(0, T_0; V)$ and the solution satisfies $|u|_{X_{T_0}} \leq M$. \square

Secondly we give a priori estimates and obtain the global existence. Denote by u , the local solution of (4.1) over $[0, T_0]$. Let $v(t) = u(t) - z(t)$. Then $v(t)$ is the mild solution of equation

$$v(t) = S(t)u_0 - \int_0^t S(t-s)B(v(s) + z(s))ds - \int_0^t S(t-s)N(v(s) + z(s))ds. \quad (4.16)$$

According to [12] 2.1.20, $v(t)$ is the weak solution of the following differential equation with random parameters:

$$\begin{cases} \frac{v(t)}{dt} + Av(t) + B(v(t) + z(t)) + N(v(t) + z(t)) = 0, \\ v(0) = u_0 \end{cases} \quad (4.17)$$

Inspired by [24] Chapter 15.3, we take advantage of the relation between weak and mild solution, give a priori estimate which ensures the global existence of solution.

Proposition 4.6. [8] *Assume that v is the solution of (4.16) on the interval. Then we have*

$$\sup_{t \in [0, T]} |v(t)|^2 \leq e^{c_5 \int_0^T |z(s)|_{H_0^1}^2 ds} |u_0|^2 + \int_0^T e^{c_5 \int_s^T |z(r)|_{H_0^1}^2 dr} g_1(s) ds, \quad (4.18)$$

$$\int_0^T |v(t)|_V^2 dt \leq c_6 |u_0|^2 + c_5 c_6 \sup_{t \in [0, T]} |v(t)|^2 \int_0^T |z(s)|_{H_0^1}^2 ds + c_6 \int_0^T g_1(s) ds \quad (4.19)$$

where c_5 and c_6 are positive constants depending on λ_1 and \mathcal{O} , g_1 is an integrable function depending on z .

Since $z \in C([0, T]; V)$, the following theorem is an immediate consequence of theorem 4.5 and proposition 4.6.

Theorem 4.7. *For all $T > 0$ and $u_0 \in H$, the equation (2.19) P -a.s. has a unique solution $u \in C([0, T]; H) \cap L^2(0, T; V)$ in the sense of (4.1).*

5 Random attractor

First we introduce the fractional Ornstein-Uhlenback process (fractional O-U process) to construct the random dynamical system.

Consider the linear stochastic evolution equation

$$dz(t) = Az(t) + dB^H(t), \quad t \in \mathbb{R}. \quad (5.1)$$

By the similar argument in [25] chapter 3, we can construct a unique stationary solution $Z(t)$ (the so-called fractional O-U process) such that

$$Z(t) = Z(\theta_t \omega) = \int_{-\infty}^t S(t-r) dB^H(r) \quad (5.2)$$

$$Z(\omega) = \lim_{n \rightarrow \infty} A \int_{-n}^0 S(-r) dB^H(r) \quad (5.3)$$

Next we prove

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \int_0^n \|Z(\theta_t \omega)\|_{H_0^1}^2 dt \leq 2c_0 \beta_D(4) \cdot \zeta(4). \quad (5.4)$$

where c_0 is a constant depending on the space injection. In order to convert this integration with respect to time variable into a integration with respect to sample space,

we may use ergodic theory. Consider the real-valued continuous function $|Z(\theta.\omega)|_{H_0^1}^2$, we have

$$\begin{aligned}
\mathbb{E} |Z(\omega)|_{H_0^1}^2 &= \mathbb{E} \left| \int_{-\infty}^0 S(-r) dB^H(r) \right|_{H_0^1}^2 \\
&= \mathbb{E} \left| \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} \int_{-t}^0 S(-r) e_n d\beta_n^H(r) \right|_{H_0^1}^2 \\
&= \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} \mathbb{E} \left| \int_{-t}^0 S(-r) e_n d\beta_n^H(r) \right|_{H_0^1}^2 \\
&= \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} |K_H^*(S(\cdot) e_n)|_{L^2(-t, 0; H_0^1)}^2 \\
&= \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} |S(\cdot) e_n|_{I_T^{\frac{1}{2}-H}(L^2(-t, 0; H_0^1))}^2 \quad (\text{Since } \mathcal{H} = I_T^{\frac{1}{2}-H}(L^2(-t, 0; H_0^1))) \\
&\leq \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} c_0 |S(\cdot) e_n|_{L^2(-t, 0; H_0^1)}^2 \quad (\text{by Hardy-Littlewood theorem}) \\
&= \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} c_0 \int_0^t |A^{\frac{1}{4}} e^{-tA} e_n|^2 dt \\
&= \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} c_0 \int_0^t |\lambda_n^{\frac{1}{4}} e^{-t\lambda_n} e_n|^2 dt \\
&= c_0 \sum_{n=1}^{\infty} \lim_{t \rightarrow \infty} \lambda_n^{-\frac{1}{2}} (1 - e^{-t\lambda_n}) \\
&= 2c_0 \beta_D(4) \cdot \zeta(4)
\end{aligned} \tag{5.5}$$

Hence, we have $|Z(\theta.\omega)|_{H_0^1}^2 \in L^1(\Omega, P)$. Since $(\Omega, \mathcal{F}, \{\theta(t)\}_{t \in \mathbb{R}})$ is the metric dynamical system, we can use the Birkhoff-Chintchin Ergodic Theorem to obtain

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \int_0^n \|Z(\theta_t \omega)\|_{H_0^1}^2 dt = \mathbb{E} \|Z(\omega)\|_{H_0^1}^2 \leq 2c_0 \beta_D(4) \cdot \zeta(4). \tag{5.6}$$

According to Theorem 4.5, $\forall t_0 \in \mathbb{R}$, $u(t, \omega; t_0, u_0)$ is the unique solution of the equation

$$u(t; t_0) = S(t-t_0)u_0 - \int_{t_0}^t S(t-s)B(u(s))ds - \int_{t_0}^t S(t-s)N(u(s))ds + \int_{t_0}^t S(t-s)dB^H(s). \tag{5.7}$$

In this section, let $u(t, \omega; t_0) = v(t, \omega; t_0) + Z(t, \omega)$, we have

$$\begin{aligned} & v_2(t) + \int_{-\infty}^t S(t-s)dB^H(s) \\ = & S(t)u_0 - \int_{t_0}^t S(t-s)B(v_2(s) + Z(s))ds - \int_{t_0}^t S(t-s)N(v_2(s) + Z(s))ds + \int_{t_0}^t S(t-s)dB^H(s). \end{aligned} \quad (5.8)$$

Since

$$\int_{-\infty}^{t_0} S(t-s)dB^H(s) = S(t-t_0)Z(\theta_{t_0}\omega), \quad (5.9)$$

$v(t, \omega; t_0, u_0 - Z(\theta_{t_0}\omega))$ is the unique solution of the integral equation

$$v(t) = S(t)(u_0 - Z(\theta_{t_0}\omega)) - \int_{t_0}^t S(t-s)B(v(s) + Z(s))ds - \int_{t_0}^t S(t-s)N(v(s) + Z(s))ds. \quad (5.10)$$

According to [12] 2.1.20, v is the weak solution of the following differential equation

$$\frac{dv}{dt} + A(v + Z) + B(v + Z) = 0 \quad (5.11)$$

$$v(t_0) = u_0 - Z(\theta_{t_0}\omega) \quad (5.12)$$

We can now define an continuous mapping by setting

$$\phi(t, \omega, u_0) = v(t, \omega; 0, u_0 - Z(\omega)) + Z(\theta_t\omega), \quad \forall (t, \omega, u_0) \in \mathbb{R} \times \Omega \times H. \quad (5.13)$$

The measurability follows from the continuity dependence of solution with respect to initial value. the cocycle property follows from the uniqueness of solution. Thus, ϕ is a RDS associated with (4.1).

In the rest of this section, we will compute some estimates in spaces H and V . Then we use these estimates and compactness of the embedding $V \hookrightarrow H$ to obtain the existence of a compact random attractor. Assume that C_1 is the constant which satisfies the inequality of trilinear form b

$$b(u, v, w) \leq C_1|u|^{1/2} \cdot |u|_{H_0^1}^{1/2} \cdot |v|_{H_0^1} \cdot |w|^{1/2} \cdot |w|_{H_0^1}^{1/2} \quad (5.14)$$

as in [14].

Lemma 5.1. *If $c_0C_1^2 < \frac{1}{\beta_D(4)\zeta(4)}$, then there exist random radii $\rho_H(\omega) > 0$ and $\rho_1(\omega)$ such that for all $M > 0$ there exists $t_2(\omega) < -1$, such that whenever $t_0 < t_2$ and*

$|u_0| < M$, we have

$$|v(t, \omega; t_0, u_0 - Z(\theta_{t_0}\omega))|^2 \leq \rho_H(\omega), \quad \forall t \in [-1, 0] \quad (5.15)$$

$$|u(t, \omega; t_0, u_0)|^2 \leq \rho_H(\omega), \quad \forall t \in [-1, 0] \quad (5.16)$$

$$\int_{-1}^0 |v(t)|_V^2 dt \leq \rho_1(\omega) \quad (5.17)$$

$$\int_{-1}^0 |v(t) + Z(t)|_V^2 dt \leq \rho_1(\omega) \quad (5.18)$$

Proof. The proof is similar to [8] Proposition 3.7. Multiple (4.17) by $v(t)$ and then integrate over \mathcal{O} . We have

$$\begin{aligned} \frac{1}{2} \frac{d|v(t)|^2}{dt} + |v(t)|_V^2 &= -b(v(t) + Z(t), v(t) + Z(t), v(t)) - \langle N(v(t) + Z(t)), v(t) \rangle \\ &\leq |b(v(t) + Z(t), Z(t), v(t) + Z(t))| - \langle N(Z(t)), v(t) \rangle. \end{aligned} \quad (5.19)$$

The above inequality take advantage of $\langle N(v), v \rangle \geq 0$ (see [3]) and (2.15).

In the sequel we omit the time variable t . Firstly we estimate trilinear form b .

$$\begin{aligned} &b(v + Z, Z, v + Z) \\ &\leq C_1 |v + Z| \cdot |Z|_{H_0^1} \cdot |v + Z|_{H_0^1} \\ &\leq \frac{C_1}{2C_2} |Z|_{H_0^1}^2 \cdot |v + Z|^2 + \frac{C_1 C_2}{2} |v + Z|_{H_0^1}^2 \\ &\leq \frac{C_1}{C_2} |Z|_{H_0^1}^2 |v|^2 + C_1 C_2 |v|_{H_0^1}^2 + \frac{C_1}{C_2} |Z|^2 |Z|_{H_0^1}^2 + C_1 C_2 |Z|_{H_0^1}^2, \end{aligned} \quad (5.20)$$

where C_2 is a positive constant which will be specified later. Secondly we estimate nonlinear term N . For all $r_1 > 0$, we have

$$\begin{aligned} -\langle N(Z), v \rangle &\leq \mu_0 \epsilon^{-\alpha/2} |Z|_{H_0^1} |v|_{H_0^1} \\ &\leq r_1 |v|_{H_0^1}^2 + \frac{\mu_0^2}{4r_1 \epsilon^\alpha} |Z|_{H_0^1}^2. \end{aligned} \quad (5.21)$$

Comprehensively,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} |v|^2 + \frac{\lambda_1}{2} |v|^2 + \frac{1}{2} |v|_V^2 \\ &\leq \frac{C_1}{C_2} |Z|_{H_0^1}^2 |v|^2 + (C_1 C_2 + r_1) |v|_{H_0^1}^2 + \frac{C_1}{C_2} |Z|^2 |Z|_{H_0^1}^2 + C_1 C_2 |Z|_{H_0^1}^2 + \frac{\mu_0^2}{4r_1 \epsilon^\alpha} |Z|_{H_0^1}^2, \end{aligned} \quad (5.22)$$

where $\lambda_1 > 4$ is the first eigenvalue of operator A . Let $g_2 = \frac{C_1}{C_2} |Z|^2 |Z|_{H_0^1}^2 + C_1 C_2 |Z|_{H_0^1}^2 + \frac{\mu_0^2}{4r_1 \epsilon^\alpha} |Z|_{H_0^1}^2$. Then we have

$$\frac{d}{dt} |v|^2 + \left(\frac{1}{2} - \frac{C_1 C_2 + r_1}{\lambda_1^{\frac{1}{2}}} \right) |v|_V^2 + \left(\frac{\lambda_1}{2} - \frac{C_1 |Z|_{H_0^1}^2}{C_2} \right) |v|^2 \leq g_2. \quad (5.23)$$

By assumption we can choose $C_2 \in \left(c_0 C_1 \beta_D(4) \zeta(4), \frac{1}{C_1}\right)$ and r_1 small enough and we have

$$\frac{d}{dt}|v|^2 + \left(\frac{\lambda_1}{2} - \frac{C_1|Z|_{H_0^1}^2}{C_2}\right)|v|^2 \leq g_2. \quad (5.24)$$

By Gronwall inequality, when $t \in [-1, 0]$ and $t_0 < -1$, we have

$$\begin{aligned} & |v(t)|^2 \\ & \leq |v(t_0)|^2 e^{-\int_{t_0}^t \left(\frac{\lambda_1}{2} - \frac{C_1|Z(s)|_{H_0^1}^2}{C_2}\right) ds} + \int_{t_0}^t g_2(s_1) e^{-\int_{s_1}^t \left(\frac{\lambda_1}{2} - \frac{C_1|Z(s_2)|_{H_0^1}^2}{C_2}\right) ds_2} ds_1 \\ & \leq |v(t_0)|^2 e^{-\int_{t_0}^0 \left(\frac{\lambda_1}{2} - \frac{C_1|Z(s)|_{H_0^1}^2}{C_2}\right) ds} + \int_{t_0}^0 g_2(s_1) e^{-\int_{s_1}^0 \left(\frac{\lambda_1}{2} - \frac{C_1|Z(s_2)|_{H_0^1}^2}{C_2}\right) ds_2} ds_1. \end{aligned} \quad (5.25)$$

Due to the ergodic property of fractional O-U process (5.6), we have

$$\lim_{t_0 \rightarrow -\infty} \frac{1}{-t_0} \int_{t_0}^0 |Z(s)|_{H_0^1}^2 ds = \mathbb{E}|Z(\omega)|_{H_0^1}^2. \quad (5.26)$$

Choose r_2 small enough such that

$$\frac{C_1}{C_2} \mathbb{E}|Z(\omega)|_{H_0^1}^2 \leq \frac{C_1}{C_2} 2c_0 \beta_D(4) \cdot \zeta(4) < 2 - r_2 \leq \frac{\lambda_1}{2} - r_2. \quad (5.27)$$

Then there exists $t_1(\omega) < -1$, such that when $t_0 < t_1$ we have

$$|v(t)|^2 \leq e^{(1+t_0)r_2} |u_0|^2 + \int_{t_0}^0 e^{(1+t_0)r_2} g_2(s) ds, \quad \forall t \in [-1, 0]. \quad (5.28)$$

By Lemma 2.6 of [25], g_2 has at most polynomial growth as $t_0 \rightarrow -\infty$ for P-a.s. $\omega \in \Omega$. Thus, we have

$$\int_{t_0}^0 g_2(s) e^{(1+s)r_2} ds \leq \int_{-\infty}^0 g_2(s) e^{(1+s)r_2} ds \leq \infty, \quad \text{P-a.s.} \quad (5.29)$$

Let $\rho_H = 4 \int_{-\infty}^0 g_2(s) e^{(1+s)r_2} ds + 2 \sup_{t \in [-1, 0]} |Z(t)|^2$ and there exists $t_2(\omega) < t_1(\omega) < -1$ such that for all $|u_0| \leq M$

$$|v(-1, \omega; t_0, u_0 - Z(\theta_{t_0}\omega))|^2 \leq 2 \int_{-\infty}^0 g_2(s) e^{(1+s)r_2} ds \quad (5.30)$$

$$|u(-1, \omega; t_0, u_0)|^2 \leq 2|v(-1, \omega; t_0, u_0 - Z(\theta_{t_0}\omega))|^2 + 2 \sup_{t \in [-1, 0]} |Z(t)|^2 \quad (5.31)$$

$$\leq \rho_H(\omega), \quad \forall t_0 < t_2, t \in [-1, 0].$$

In the following we consider a bound of $\int_{-1}^0 |v(t)|_V^2 dt$. Integrating (5.23) over $[-1, 0]$ we have

$$|v(0)|^2 - |v(-1)|^2 + c_6^{-1} \int_{-1}^0 |v(t)|_V^2 dt \leq \int_{-1}^0 g_2(t) dt + \int_{-1}^0 \left(\frac{C_1}{C_2} |Z(t)|_1^2\right) |v(t)|^2 dt. \quad (5.32)$$

When $t_0 < t_2$ we have

$$\int_{-1}^0 |v(t)|_V^2 dt \leq c_6 \left(\int_{-1}^0 g_2(t) dt + \frac{C_1 \rho_H}{C_2} \int_{-1}^0 |Z(t)|_1^2 dt + |v(-1)|^2 \right) \triangleq C(\omega). \quad (5.33)$$

Similarly,

$$\int_{-1}^0 |v(t) + Z(t)|_V^2 dt \leq 2c_6 \left(\int_{-1}^0 g_2(t) dt + \frac{C_1 \rho_H}{C_2} \int_{-1}^0 |Z(t)|_1^2 dt + 2 \int_{-1}^0 |Z(t)|_V^2 dt \right) \triangleq \tilde{C}(\omega). \quad (5.34)$$

Let $\rho_1(\omega) = \max\{C(\omega), \tilde{C}(\omega)\}$ and the proof is complete. \square

By the same argument as in [8] Lemma 4.3, we have the following lemma.

Lemma 5.2. *Under the assumption of Lemma 5.1, there exists a random radius $\rho_V(\omega)$ such that for all $M > 0$ and $|u_0| < M$, there exists $t_2(\omega) < -1$ such that P-a.s.*

$$|v(t, \omega; t_0, u_0 - Z(\theta_{t_0}\omega))|_1^2 \leq \rho_V(\omega), \quad (5.35)$$

$$|u(t, \omega; t_0, u_0)|_1^2 \leq \rho_V(\omega), \quad \forall t_0 < t_2, t \in [-\frac{1}{2}, 0]. \quad (5.36)$$

Lemma 5.2 shows that there exists a bounded random ball in \dot{H}^1 which absorbs any bounded non-random subset of H . Since \dot{H}^1 is compactly embedded in H , we have established the existence of a compact random absorbing set in H . We now state the final theorem and the proof is similar to that of Theorem 5.6 in [10].

Theorem 5.3. *If $c_0 C_1^2 < \frac{1}{\beta_D(4)\zeta(4)}$, then the random dynamical system associated with (4.1) has a random attractor.*

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