

A New Class of Backward Stochastic Partial Differential Equations with Jumps and Applications

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Abstract

We formulate a new class of stochastic partial differential equations (SPDEs), named *high-order vector backward SPDEs (B-SPDEs) with jumps*, which allow the high-order integral-partial differential operators into both drift and diffusion coefficients. Under certain type of Lipschitz and linear growth conditions, we develop a method to prove the existence and uniqueness of adapted solution to these B-SPDEs with jumps. Comparing with the existing discussions on conventional backward stochastic (ordinary) differential equations (BSDEs), we need to handle the differentiability of adapted triplet solution to the B-SPDEs with jumps, which is a subtle part in justifying our main results due to the inconsistency of differential orders on two sides of the B-SPDEs and the partial differential operator appeared in the diffusion coefficient. In addition, we also address the issue about the B-SPDEs under certain Markovian random environment and employ a B-SPDE with strongly nonlinear partial differential operator in the drift coefficient to illustrate the usage of our main results in finance.

Key words and phrases: Backward Stochastic Partial Differential Equations with Jumps, High-Order Partial Differential Operator, Vector Partial Differential Equation, Existence and Uniqueness, Random Environment

1 Introduction

Motivated from mean-variance hedging (see, e.g., Dai [10]) and utility based optimal portfolio choice (see, e.g., Becherer [3], Musiela and Zariphopoulou [22]) in finance, and multi-channel (or multi-valued) image regularization such as color images in computer vision and network application (see, e.g., Caselles *et al.* [6], Tschumperlé and Deriche [33, 34, 35], and references therein), we formulate a new class of SPDEs, named *high-order vector B-SPDEs with jumps*, which allow high-order integral-partial differential operators \mathcal{L} and \mathcal{J} into both drift and diffusion coefficients as shown in the following equation (1.1),

$$(1.1) \quad V(t, x) = H(x) + \int_t^T \mathcal{L}(s^-, x, V, \cdot) ds + \int_t^T (\mathcal{J}(s^-, x, V, \cdot) - \bar{V}(s^-, x)) dW(s) \\ - \int_t^T \int_{z>0} \tilde{V}(s^-, x, z, \cdot) \tilde{N}(\lambda ds, x, dz).$$

where the operator \mathcal{L} depends not only on V, \bar{V}, \tilde{V} but also on their associated partial derivatives, i.e., for each integer $k \geq 2$ and $m \geq 0$, \mathcal{L} and \mathcal{J} are defined by

$$\begin{aligned}\mathcal{L}(s, x, V, \cdot) &\equiv \mathcal{L}(s, x, V(s, x), V^{(k)}(s, x), \bar{V}(s, x), \bar{V}^{(m)}(s, x), \tilde{V}(s, x), \cdot), \\ \mathcal{J}(s, x, V, \cdot) &= (\mathcal{J}_1(s, x, V, \cdot), \dots, \mathcal{J}_d(s, x, V, \cdot)), \\ \mathcal{J}_i(s, x, V, \cdot) &\equiv \mathcal{J}_i(s, x, V(s, x), V^{(k)}(s, x), \cdot), \quad i \in \{1, \dots, d\}.\end{aligned}$$

Under certain type of Lipschitz and linear growth conditions, we prove the existence and uniqueness of adapted triplet solution (V, \bar{V}, \tilde{V}) to these B-SPDEs. When the partial differential operator \mathcal{L} depends only on x, V, \bar{V} , and \tilde{V} but not on their associated derivatives and $\mathcal{L} = 0$, our B-SPDEs with jumps reduce to conventional BSDEs with jumps (see, e.g., Becherer [3], Dai [10], Tang and Li [32]).

BSDEs were first introduced by Bismut [5] and the first result for the existence of an adapted solution to a continuous nonlinear BSDE was obtained by Pardoux and Peng [26]. Since then, numerous extensions along the line have been conducted, such as, Tang and Li [32] get the first adapted solution to a BSDE with Poisson jumps for a fixed terminal time and Situ [31] extended the result to the case where the BSDE is with bounded random stopping time as its terminal time and non-Lipschitz coefficient. Currently, BSDEs are still an active area of research in both theory and applications, see, e.g., Becherer [3], Cohen and Elliott [8], Crépey and Matoussi [9], Dai [10], Lepeltier *et al.* [18], Yin and Mao [36], and references therein.

The study on SPDEs receives a great attention recently (see, e.g., Pardoux [25] and Hairer [14]). Particularly, Pardoux and Peng [27] introduces a system of semi-linear parabolic SPDE in a backward manner and establish the existence and uniqueness of adapted solution to the SPDE under smoothness assumptions on the coefficients, and moreover, the authors in [27] also employ backward doubly SDEs (BDSDE) to provide a probabilistic representation for the parabolic SDE. Since then, numerous researches have been conducted in terms of weak solution and stationary solution to the semi-linear SPDE (see, e.g., Bally and Matoussi [2], Zhang and Zhao [38], and references therein). However, our B-SPDEs exhibited in (1.1) are fundamentally different from the SPDEs as introduced in Pardoux and Peng [27] and as studied in most of the existing researches in the following aspects: First, our system formulation is a direct generalization of the conventional BSDEs, i.e., both the drift and diffusion coefficients of our B-SPDEs depend on the triplet (V, \bar{V}, \tilde{V}) and its associated partial derivatives not just on V and its associated partial derivatives; Second, our B-SPDEs are based on high-order partial derivatives and are subject to jumps. One special case of our B-SPDEs available in the literature is the one derived in Musiela and Zariphopoulou [22] for the purpose of optimal-utility based portfolio choice, which is strongly nonlinear in the sense that is addressed in Lions and Souganidis [19].

Note that the B-SPDEs presented in (1.1) are vector B-SPDEs with jumps, which are motivated from various aspects such as multi-channel image regularization in computer vision and network application through vector PDEs (see, e.g., Caselles *et al.* [6], Tschumperlé and Deriche [33, 34, 35], and references therein), coupling and synchronization in random

dynamic systems through vector SPDEs (see, e.g., Mueller [21], Chueshov and Schmalfuß [7], and references therein).

To show our formulated system well-posed, we develop a method based on a scheme used for conventional BSDEs (see, e.g., Yong and Zhou [37]) to prove the existence and uniqueness of adapted solution to our B-SPDEs with jumps in (1.1) under certain Lipschitz and linear growth conditions. One fundamental issue we need to handle in the method is the differentiability of the triplet solution to our B-SPDEs with jumps, which is a subtle part in the analysis due to the inconsistency of differential orders on two sides of the B-SPDEs and the partial differential operators appeared in the diffusion coefficient. So more involved functional spaces and techniques are required. In addition, although there is no perfect theory in dealing with the strongly nonlinear SPDEs (see, e.g., Pardoux [25]), our discussions about the adapted solution to (1.1) can provide some reasonable interpretation concerning the unique existence of adapted solution before a random bankruptcy time to the strongly nonlinear B-SPDE derived in Musiela and Zariphopoulou [22].

In the paper, we also provide some discussion concerning our B-SPDEs under random environment, e.g., the variable x in (1.1) is replaced by a continuous Markovian process $X(\cdot)$. To be convenient for readers, we present a rough graph in Figure 1 with respect to sample surfaces for a solution to a B-SPDE and in terms of sample curves for a solution to the B-SPDE under random environment.

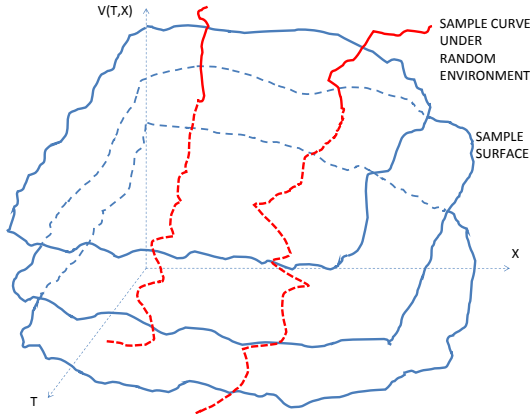


Figure 1: Sample surfaces and sample paths for a B-SPDE and a B-SPDE under random environment

The rest of the paper is organized as follows. In Section 2, we first introduce a class of B-SPDEs with jumps in finite space domain, then we state and prove our main theorem. In Section 3, we extend our discussions in the previous section to the case corresponding to infinite space domain and under random environment. In Section 4, we use an example to illustrate the usage of our main results in finance.

2 A Class of B-SPDEs with Jumps in Finite Space Domain

2.1 Required Probability and Functional Spaces

First of all, we introduce some notations to be used in the paper. Let (Ω, \mathcal{F}, P) be a fixed complete probability space on which are defined a standard d -dimensional Brownian motion $W \equiv \{W(t), t \in [0, T]\}$ with $W(t) = (W_1(t), \dots, W_d(t))'$ and h -dimensional subordinator $L \equiv \{L(t), t \in [0, T]\}$ with $L(t) \equiv (L_1(t), \dots, L_h(t))'$ and càdlàg sample paths for some fixed $T \in [0, \infty)$ (see, e.g., Applebaum [1], Bertoin [4], and Sato [30] for more details about subordinators and Lévy processes), where the prime denotes the corresponding transpose of a matrix or a vector. Moreover, W , L and their components are assumed to be independent of each other. In addition, each subordinator L_i with $i \in \{1, \dots, h\}$ can be represented by (see, e.g., Theorem 13.4 and Corollary 13.7 in Kallenberg [17])

$$(2.1) \quad L_i(t) = a_i t + \int_{(0,t]} \int_{z_i > 0} z_i N_i(ds, dz_i), \quad t \geq 0$$

where $N_i((0, t] \times A) \equiv \sum_{0 < s \leq t} I_A(L(s) - L_i(s^-))$ denotes a Poisson random measure with a deterministic, time-homogeneous intensity measure $ds\nu_i(dz_i)$, where $I_A(\cdot)$ is the index function over the set A , the constant a_i is taken to be zero, and ν_i is the Lévy measure. Related to the probability space (Ω, \mathcal{F}, P) , we suppose that there is a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ with $\mathcal{F}_t \equiv \sigma\{W(s), L(\lambda s) : 0 \leq s \leq t\}$ for each $t \in [0, T]$, $\lambda = (\lambda_1, \dots, \lambda_h)' > 0$, and $L(\lambda s) = (L_1(\lambda_1 s), \dots, L_h(\lambda_h s))'$.

Secondly, let $\mathcal{N} = \{1, 2, \dots\}$ and D be a close connected domain in R^p for a given $p \in \mathcal{N}$. Then we can use $C^k(D, R^q)$ for each $k, p, q \in \mathcal{N}$ to denote the Banach space of all functions f having continuous derivatives up to the order k with the following uniform norm,

$$(2.2) \quad \|f\|_{C^k(D, q)} = \max_{c \in \{1, \dots, k\}} \max_{j \in \{1, \dots, r(c)\}} \sup_{x \in D} |f_j^{(c)}(x)|$$

for each $f \in C^k(D, R^q)$, where $r(c)$ for each $c \in \{0, 1, \dots, k\}$ is the total number of the following partial derivatives of the order c

$$(2.3) \quad f_{r, (i_1 \dots i_p)}^{(c)}(x) = \frac{\partial^c f_r(x)}{\partial x_1^{i_1} \dots \partial x_p^{i_p}}$$

with $i_l \in \{0, 1, \dots, c\}$, $l \in \{1, \dots, p\}$, $r \in \{1, \dots, q\}$, and $i_1 + \dots + i_p = c$. Moreover, for the late purpose, we let

$$(2.4) \quad f_{(i_1, \dots, i_p)}^{(c)} \equiv (f_{1, (i_1, \dots, i_p)}^{(c)}, \dots, f_{q, (i_1, \dots, i_p)}^{(c)}),$$

$$(2.5) \quad f^{(c)}(x) \equiv (f_1^{(c)}(x), \dots, f_{r(c)}^{(c)}(x)),$$

where each $j \in \{1, \dots, r(c)\}$ corresponds to a p -tuple (i_1, \dots, i_p) and a $r \in \{1, \dots, q\}$. In addition, let $C^\infty(D, R^q)$ denote the following Banach space, i.e.,

$$(2.6) \quad C^\infty(D, R^q) \equiv \left\{ f \in \bigcap_{k=1}^{\infty} C^k(D, R^q), \|f\|_{C^\infty(D, q)} < \infty \right\}$$

where

$$(2.7) \quad \|f\|_{C^\infty(D,q)}^2 = \sum_{k=1}^{\infty} \xi(k) \|f\|_{C^k(D,q)}^2$$

for some discrete function with respect to $k \in \{0, 1, 2, \dots\}$, which is fast decaying in k . For convenience, we take $\xi(k) = e^{-k}$.

Thirdly, we introduce some measurable spaces to be used in the sequel. Let $L_{\mathcal{F}}^2([0, T]; R^q)$ denote the set of all R^q -valued measurable stochastic processes $Z(t, x)$ adapted to $\{\mathcal{F}_t, t \in [0, T]\}$ for each $x \in D$, which are in $C^\infty(D, R^q)$ for each fixed $t \in [0, T]$, such that

$$(2.8) \quad E \left[\int_0^T \|Z(t)\|_{C^\infty(D,q)}^2 dt \right] < \infty$$

and let $L_{\mathcal{F},p}^2([0, T], R^q)$ denote the corresponding set of predictable processes (see, e.g., Definition 5.2 and Definition 1.1 respectively in pages 21 and 45 of Ikeda and Watanabe [15]). Moreover, let $L_{\mathcal{F}_T}^2(\Omega; R^q)$ denote the set of all R^q -valued, \mathcal{F}_T -measurable random variables $\xi(x)$ for each $x \in D$, where $\xi(x) \in C^\infty(D, R^q)$ satisfies

$$(2.9) \quad E \left[\|\xi\|_{C^\infty(D,q)}^2 \right] < \infty.$$

In addition, let $L_p^2([0, T], R^h)$ be the set of all R^h -valued predictable processes $\tilde{V}(t, x, z) = (\tilde{V}_1(t, x, z), \dots, \tilde{V}_h(t, x, z))'$ for each $x \in D$ and $z \in R_+^h$, satisfying

$$(2.10) \quad E \left[\sum_{i=1}^h \int_0^T \int_{z_i > 0} \|\tilde{V}_i(t^-, z)\|_{C^\infty(D,q)}^2 \nu_i(dz_i) dt \right] < \infty$$

and let

$$(2.11) \quad L_{\nu,c}^2(D \times R_+^h, R^{q \times h}) \equiv \left\{ \tilde{v} : D \times R_+^h \rightarrow R^{q \times h}, \sum_{i=1}^h \int_{z_i > 0} \|\tilde{v}_i(z_i)\|_{C^c(D,q)}^2 \nu_i(dz_i) < \infty \right\}$$

with the associated norm for any $\tilde{v} \in L_{\nu,c}^2(D \times R_+^h, R^{q \times h})$ and $c \in \{0, 1, \dots, \infty\}$ as follows,

$$(2.12) \quad \|\tilde{v}\|_{\nu,c} \equiv \left(\sum_{i=1}^h \int_{z_i > 0} \|\tilde{v}_i(z_i)\|_{C^c(D,q)}^2 \lambda_i \nu_i(dz_i) \right)^{\frac{1}{2}}.$$

In the end, we define

$$(2.13) \quad \mathcal{Q}_{\mathcal{F}}^2([0, T]) \equiv L_{\mathcal{F}}^2([0, T], R^q) \times L_{\mathcal{F},p}^2([0, T], R^{q^d}) \times L_p^2([0, T], R^{q \times h}).$$

2.2 The B-SPDEs

First of all, we introduce a class of q -dimensional B-SPDEs with jumps and terminal random variable $H(x) \in L_{\mathcal{F}_T}^2(\Omega; R^q)$ for each $x \in D$ as presented in (1.1), where for each $s \in [0, T]$

and $z = (z_1, \dots, z_h) \in R_+^h$,

$$\begin{aligned}\bar{V}(s, \cdot) &= (\bar{V}_1(s, \cdot), \dots, \bar{V}_d(s, \cdot)) \in C^\infty(D, R^{q \times d}), \\ \tilde{V}(s, \cdot, z) &= (\tilde{V}_1(s, \cdot, z_1), \dots, \tilde{V}_h(s, \cdot, z_h)) \in C^\infty(D, R^{q \times h}), \\ \tilde{N}(\lambda ds, x, dz) &= (\tilde{N}_1(\lambda_1 ds, x, dz_1), \dots, \tilde{N}_h(\lambda_h ds, x, dz_h))'.\end{aligned}$$

Moreover, in (1.1), \mathcal{L} is a q -dimensional integral-partial differential operator satisfying, a.s.,

$$(2.14) \quad \left\| \Delta \mathcal{L}^{(c)}(s, x, u, v) \right\| \leq K_D \left(\|u - v\|_{C^{k+c}(D, q)} + \|\bar{u} - \bar{v}\|_{C^{m+c}(D, qd)} + \|\tilde{u} - \tilde{v}\|_{\nu, c} \right)$$

for any $(u, \bar{u}, \tilde{u}), (v, \bar{v}, \tilde{v}) \in C^k(D, R^q) \times C^m(D, R^{q \times d}) \times L_{\nu, c}^2(R_+^h, R^{q \times h})$ with $c \in \{0, 1, \dots, \infty\}$, where K_D depending on the domain D is a nonnegative constant, $\|A\|$ is the largest absolute value of entries (or components) of the given matrix (or vector) A , and

$$(2.15) \quad \Delta \mathcal{L}^{(c)}(s, x, u, v) \equiv \mathcal{L}^{(c)}(s, x, u, \cdot) - \mathcal{L}^{(c)}(s, x, v, \cdot).$$

Similarly, $\mathcal{J} = (\mathcal{J}_1, \dots, \mathcal{J}_d)$ is a $q \times d$ -dimensional partial differential operator satisfying, a.s.,

$$(2.16) \quad \|\Delta \mathcal{J}^{(c)}(s, x, u, v)\| \leq K_D \left(\|u - v\|_{C^{m+c}(D, q)} \right).$$

Moreover, we suppose that

$$(2.17) \quad \left\| \mathcal{L}^{(c)}(s, x, u, \cdot) \right\| \leq K_D \left(\|u\|_{C^{k+c}(D, q)} + \|\bar{v}\|_{C^{m+c}(D, qd)} + \|\tilde{v}\|_{\nu, c} \right),$$

$$(2.18) \quad \left\| \mathcal{J}^{(c)}(s, x, u, \cdot) \right\| \leq K_D \|u\|_{C^{m+c}(D, q)}.$$

Example 2.1 *The following conventional linear partial differential operators satisfy the conditions as stated in (2.14)-(2.18),*

$$\begin{aligned}(\mathcal{L}u)(t, x) &= \sum_{i,j=1}^p a_{ij}(x) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial u(t, x)}{\partial x_j} + c(x)u(t, x) \\ &\quad + \sum_{i,j=1}^d \bar{a}_{ij}(x) \frac{\partial^2 \bar{u}(t, x)}{\partial x_i \partial x_j} + \sum_{j=1}^d \bar{b}_j(x) \frac{\partial \bar{u}(t, x)}{\partial x_j} + \bar{c}(x)\bar{u}(t, x) \\ (\mathcal{J}u)(t, x) &= \sum_{i,j=1}^p a_{ij}(x) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x) \frac{\partial u(t, x)}{\partial x_j} + c(x)u(t, x),\end{aligned}$$

where $a_{ij}^{(c)}(x)$, $b_j^{(c)}(x)$, $c^{(c)}(x)$, $\bar{a}_{ij}^{(c)}(x)$, $\bar{b}_j^{(c)}(x)$, and $\bar{c}^{(c)}(x)$ are uniformly bounded over all $x \in D$ and $i, j \in \{1, \dots, d\}$ and $c \in \{0, 1, 2, \dots\}$.

Theorem 2.1 *Under conditions of (2.14)-(2.18), if $\mathcal{L}(t, x, v, \cdot)$ and $\mathcal{J}(t, x, v, \cdot)$ are $\{\mathcal{F}_t\}$ -adapted for each fixed $x \in D$ and any given $(v, \bar{v}, \tilde{v}) \in C^\infty(D, R^q) \times C^\infty(D, R^{q \times d}) \times L_{\nu, \infty}^2(D \times R_+^h, R^{q \times h})$ with*

$$(2.19) \quad \mathcal{L}(\cdot, x, 0, \cdot), \mathcal{J}(\cdot, x, 0, \cdot) \in L_{\mathcal{F}}^2([0, T], R^q),$$

then the B-SPDE (1.1) has a unique adapted solution satisfying, for each $x \in D$ and $z \in R_+^h$,

$$(2.20) \quad (V(\cdot, x), \bar{V}(\cdot, x), \tilde{V}(\cdot, x, z)) \in \mathcal{Q}_{\mathcal{F}}^2([0, T])$$

where V is a càdlàg process and the uniqueness is in the sense: if there exists another solution $(U(t, x), \bar{U}(t, x), \tilde{U}(t, x, z))$ as required, we have

$$E \left[\int_0^T \left(\|U(t) - V(t)\|_{C^\infty(D, q)}^2 + \|\bar{U}(t) - \bar{V}(t)\|_{C^\infty(D, qd)}^2 + \|\tilde{U}(t) - \tilde{V}(t)\|_{\nu, \infty}^2 \right) dt \right] = 0.$$

We divide the proof of Theorem 2.1 into the following three lemmas.

Lemma 2.1 *Under the conditions of Theorem 2.1, for each fixed $x \in D$, $z \in R_+^h$, and a triplet*

$$(2.21) \quad (U(\cdot, x), \bar{U}(\cdot, x), \tilde{U}(\cdot, x, z)) \in \mathcal{Q}_{\mathcal{F}}^2([0, T]),$$

there exists another triplet $(V(\cdot, x), \bar{V}(\cdot, x), \tilde{V}(\cdot, x, z))$ such that

$$(2.22) \quad \begin{aligned} V(t, x) &= H(x) + \int_t^T \mathcal{L}(s^-, x, U, \cdot) ds + \int_t^T (\mathcal{J}(s^-, x, U, \cdot) - \bar{V}(s^-, x)) dW(s) \\ &\quad - \int_t^T \int_{z>0} \tilde{V}(s^-, x, z) \tilde{N}(\lambda ds, x, dz), \end{aligned}$$

where V is a $\{\mathcal{F}_t\}$ -adapted càdlàg process, \bar{V} and \tilde{V} are the corresponding predictable processes, and for each $x \in D$,

$$(2.23) \quad E \left[\int_0^T \|V(t, x)\|^2 dt \right] < \infty,$$

$$(2.24) \quad E \left[\int_0^T \|\bar{V}(t, x)\|^2 dt \right] < \infty,$$

$$(2.25) \quad E \left[\sum_{i=1}^h \int_0^T \int_{z_i>0} \|\tilde{V}_i(t^-, x, z)\|^2 \nu_i(dz_i) dt \right] < \infty.$$

PROOF. First of all, for each fixed $x \in D$, $z \in R_+^h$, and a triplet $(U(\cdot, x), \bar{U}(\cdot, x), \tilde{U}(\cdot, x, z))$ as stated in (2.21), it follows from conditions (2.14)-(2.19) that

$$(2.26) \quad \mathcal{L}(\cdot, x, U, \cdot) \in L_{\mathcal{F}}^2([0, T], R^q), \quad \mathcal{J}(\cdot, x, U, \cdot) \in L_{\mathcal{F}}^2([0, T], R^{q \times d}).$$

Now consider \mathcal{L} and \mathcal{J} in (2.26) as two new starting $\mathcal{L}(\cdot, x, 0, \cdot)$ and $\mathcal{J}(\cdot, x, 0, \cdot)$, then it follows from the Martingale representation theorem (see, e.g., Lemma 2.3 in Tang and Li [32]) that there exists a unique pair of predictable processes $(\bar{V}(\cdot, x), \tilde{V}(\cdot, x, z))$ which are square-integrable for each $x \in D$ in the senses of (2.24)-(2.25) such that

$$(2.27) \quad \begin{aligned} \hat{V}(t, x) &\equiv E \left[H(x) + \int_0^T \mathcal{L}(s^-, x, U, \cdot) ds + \int_0^T \mathcal{J}(s^-, x, U, \cdot) dW(s) \middle| \mathcal{F}_t \right] \\ &= \hat{V}(0, x) + \int_0^t \bar{V}(s^-, x) dW(s) + \int_0^t \int_{z>0} \tilde{V}(s^-, x, z) \tilde{N}(\lambda ds, x, dz) \end{aligned}$$

which implies that

$$(2.28) \quad \begin{aligned} \hat{V}(0, x) &= H(x) + \int_0^T \mathcal{L}(s^-, x, U, \cdot) ds + \int_0^T \mathcal{J}(s^-, x, U, \cdot) dW(s) \\ &\quad - \int_0^T \bar{V}(s^-, x) dW(s) - \int_0^T \int_{z>0} \tilde{V}(s^-, x, z) \tilde{N}(\lambda ds, x, dz). \end{aligned}$$

Moreover, due to the Corollary in page 8 of Protter [29], $\hat{V}(\cdot, x)$ can be taken as a càdlàg process. Now we define a process V as follows,

$$(2.29) \quad V(t, x) \equiv E \left[H(x) + \int_t^T \mathcal{L}(s^-, x, U, \cdot) ds + \int_t^T \mathcal{J}(s^-, x, U, \cdot) dW(s) \middle| \mathcal{F}_t \right]$$

Then by simple calculation, we know that $V(\cdot, x)$ is square-integrable in the sense of (2.23), and moreover, it follows from (2.27)-(2.29) that

$$(2.30) \quad V(t, x) = \hat{V}(t, x) - \int_0^t \mathcal{L}(s^-, x, U, \cdot) ds - \int_0^t \mathcal{J}(s^-, x, U, \cdot) dW(s)$$

which indicates that $V(\cdot, x)$ is a càdlàg process. Furthermore, for a given triplet $(U(\cdot, x), \dot{U}(\cdot, x), \bar{U}(\cdot, x), \tilde{U}(\cdot, x, z))$, it follows from (2.27)-(2.28) and (2.30) that the corresponding triplet $(V(\cdot, x), \bar{V}(\cdot, x), \tilde{V}(\cdot, x, z))$ satisfies the equation (2.22) as stated in the lemma, which also implies that

$$(2.31) \quad \begin{aligned} V(t, x) &\equiv V(0, x) - \int_0^t \mathcal{L}(s^-, x, U, \cdot) ds - \int_0^t (\mathcal{J}(s^-, x, U, \cdot) - \bar{V}(s^-, x)) dW(s) \\ &\quad + \int_0^t \int_{z>0} \tilde{V}(s^-, x, z) \tilde{N}(\lambda ds, x, dz). \end{aligned}$$

Hence we complete the proof of Lemma 2.1. \square

Lemma 2.2 *Under the conditions of Theorem 2.1, for each fixed $x \in D$, $z \in R_+^h$, and a triplet as in (2.21), we define $V(t, x)$, $\bar{V}(t, x)$, $\tilde{V}(t, x, z)$ through (2.22). Then $(V^{(c)}(\cdot, x), \bar{V}^{(c)}(\cdot, x), \tilde{V}^{(c)}(\cdot, x, z))$ for each $c \in \{0, 1, \dots, c\}$ exists a.s. and satisfies a.s.*

$$(2.32) \quad \begin{aligned} V_{(i_1 \dots i_p)}^{(c)}(t, x) &= H_{(i_1 \dots i_p)}^{(c)}(x) + \int_t^T \mathcal{L}_{(i_1 \dots i_p)}^{(c)}(s^-, x, U, \cdot) ds \\ &\quad + \int_t^T \left(\mathcal{J}_{(i_1 \dots i_p)}^{(c)}(s^-, x, U, \cdot) - \bar{V}_{(i_1 \dots i_p)}^{(c)}(s^-, x) \right) dW_i(s) \\ &\quad - \int_t^T \int_{z>0} \tilde{V}_{(i_1 \dots i_p)}^{(c)}(s^-, x, z) \tilde{N}(\lambda ds, x, dz), \end{aligned}$$

where $i_1 + \dots + i_p = c$ and $i_l \in \{0, 1, \dots, c\}$ with $l \in \{1, \dots, p\}$. Moreover, $V_{(i_1 \dots i_p)}^{(c)}$ for each $c \in \{0, 1, \dots, c\}$ is a $\{\mathcal{F}_t\}$ -adapted càdlàg process, $\bar{V}_{(i_1 \dots i_p)}^{(c)}$ and $\tilde{V}_{(i_1 \dots i_p)}^{(c)}$ are the corresponding predictable processes, which are square-integrable in the senses of (2.23)-(2.25).

PROOF. First of all, we show that the claim in the lemma is true for $c = 1$. To do so, for each given $t \in [0, T]$, $x \in D$, $z \in R_+^h$ and $(U(t, x), \bar{U}(t, x), \tilde{U}(t, x, z))$ as in the lemma, let

$$(2.33) \quad (V_{(l)}^{(1)}(t, x), \bar{V}_{(l)}^{(1)}(t, x), \tilde{V}_{(l)}^{(1)}(t, x, z))$$

be defined through (2.22) where \mathcal{L} and \mathcal{J} are replaced by their first-order partial derivatives $\mathcal{L}_{(l)}^{(1)}$ and $\mathcal{J}_{(l)}^{(1)}$ in terms of x_l with $l \in \{1, \dots, p\}$. Then we can show that the triplet defined in (2.33) for each l is indeed the required first-order partial derivative of (V, \bar{V}, \tilde{V}) that is defined through (2.22) for the given (U, \bar{U}, \tilde{U}) .

As a matter of fact, for each $f \in \{U, \bar{U}, \tilde{U}, V, \bar{V}, \tilde{V}, \tilde{N}\}$, small enough positive constant δ , and $l \in \{1, \dots, p\}$, define

$$(2.34) \quad f_{(l),\delta}(t, x) \equiv f(t, x + \delta e_l),$$

where e_l is the unit vector whose l th component is one and others are zero. Moreover, let

$$(2.35) \quad \Delta f_{(l),\delta}^{(1)}(t, x) = \frac{f_{(l),\delta}(t, x) - f(t, x)}{\delta} - f_{(l)}^{(1)}(t, x)$$

for each $f \in \{U, \bar{U}, \tilde{U}, V, \bar{V}, \tilde{V}\}$. In addition, let

$$(2.36) \quad \begin{aligned} \Delta \mathcal{I}_{(l),\delta}^{(1)}(s, x, U) &= \frac{1}{\delta} (\mathcal{I}(s, x + \delta e_l, U(s, x + \delta e_l), \cdot) - \mathcal{I}(s, x, U(s, x), \cdot)) \\ &\quad - \mathcal{I}_{(l)}^{(1)}(s, x, U(s, x), \cdot) \end{aligned}$$

for each $\mathcal{I} \in \{\mathcal{L}, \mathcal{J}\}$. Then, by applying the Ito's formula (see, e.g., Theorem 1.14 and Theorem 1.16 in pages 6-9 of Øksendal and Sulem [24]) to the function

$$\zeta(\Delta V_{(l),\delta}^{(1)}(t, x)) \equiv \text{Tr} \left(\Delta V_{(l),\delta}^{(1)}(t, x) \right) e^{2\gamma t}$$

for some $\gamma > 0$, where $\text{Tr}(A)$ denotes the trace of the matrix $A'A$ for a given matrix A , we have

$$(2.37) \quad \begin{aligned} &\zeta(\Delta V_{(l),\delta}^{(1)}(t, x)) + \int_t^T \text{Tr} \left(\Delta \mathcal{J}_{(l),\delta}^{(1)}(s, x, U) - \Delta \bar{V}_{(l),\delta}^{(1)}(s, x) \right) e^{2\gamma s} ds \\ &\quad + \int_t^T \int_{z_j > 0} \text{Tr} \left(\Delta \tilde{V}_{(l),\delta}^{(1)}(s^-, x, z) \right) e^{2\gamma s} \tilde{N}(\lambda ds, x, dz) \\ &= 2 \int_t^T \left(-\gamma \text{Tr} \left(\Delta V_{(l),\delta}^{(1)}(s, x) \right) + \left(\Delta V_{(l),\delta}^{(1)}(s, x) \right)' \left(\Delta \mathcal{L}_{(l),\delta}^{(1)}(s, x, U) \right) \right) e^{2\gamma s} ds - M(t) \\ &\leq \left(-2\gamma + \frac{3K_D^2}{\hat{\gamma}} \right) \int_t^T \text{Tr} \left(\Delta V_{(l),\delta}^{(1)}(s, x) \right) e^{2\gamma s} ds + \hat{\gamma} \int_t^T \left\| \Delta \mathcal{L}_{(l),\delta}^{(1)}(s, x, U) \right\|^2 e^{2\gamma s} ds - M(t) \\ &= \hat{\gamma} \int_t^T \left\| \Delta \mathcal{L}_{(l),\delta}^{(1)}(s, x, U) \right\|^2 e^{2\gamma s} ds - M_\delta(t) \end{aligned}$$

if, in the last equality, we take

$$(2.38) \quad \hat{\gamma} = \frac{3K_D^2}{2\gamma} > 0,$$

where $M_\delta(t)$ is a martingale of the following form,

$$\begin{aligned}
& 2 \sum_{j=1}^d \int_t^T \left(\Delta V_{(l),\delta}^{(1)}(s^-, x) \right)' \left(\Delta(\mathcal{J}_j)_{(l),\delta}^{(1)}(s^-, x, U) - \Delta(\bar{V}_j)_{(l),\delta}^{(1)}(s^-, x) \right) e^{2\gamma s} dW_j(s) \\
& - 2 \sum_{j=1}^h \int_t^T \int_{z_j > 0} \left(\Delta V_{(l),\delta}^{(1)}(s^-, x) \right)' \left((1/\delta) \tilde{V}_j(s^-, x, z_j) + \tilde{V}_j^{(1)}(s^-, x, z_j) \right) e^{2\gamma s} \tilde{N}(\lambda_j ds, x, dz_j) \\
& + 2 \sum_{j=1}^h \int_t^T \int_{z_j > 0} \left(\Delta V_{(l),\delta}^{(1)}(s^-, x) \right)' \left((1/\delta) (\tilde{V}_j)_{(l),\delta}(s^-, x, z_j) \right) e^{2\gamma s} (\tilde{N}_j)_{(l),\delta}(\lambda_j ds, x, dz_j).
\end{aligned}$$

Now, it follows from Lemma 1.3 in pages 6-7 of Peskir and Shiryaev [28] that, for each $t \in [0, T]$ and $\sigma > 0$, there is a sequence of $\{\delta_n, n = 1, 2, \dots\} \subset [0, \sigma]$ such that

$$\begin{aligned}
(2.39) \quad & E \left[\text{ess sup}_{0 \leq \delta \leq \sigma} \zeta(\Delta V_{(l),\delta}^{(1)}(t, x)) \right] \\
& = E \left[\text{ess sup}_{\{\delta_n: 0 \leq \delta_n \leq \sigma, n=1,2,\dots\}} \zeta(\Delta V_{(l),\delta_n}^{(1)}(t, x)) \right] \\
& = \lim_{n \rightarrow \infty} E \left[\zeta(\Delta V_{(l),\delta_n}^{(1)}(t, x)) \right] \\
& \leq \hat{\gamma} \lim_{n \rightarrow \infty} E \left[\int_t^T \left\| \Delta \mathcal{L}_{(l),\delta_n}^{(1)}(s, x, U) \right\|^2 e^{2\gamma s} ds \right] - \lim_{n \rightarrow \infty} E [M_{\delta_n}(t)] \\
& \leq \hat{\gamma} E \left[\int_t^T \text{ess sup}_{0 \leq \delta \leq \sigma} \left\| \Delta \mathcal{L}_{(l),\delta}^{(1)}(s, x, U) \right\|^2 e^{2\gamma s} ds \right],
\end{aligned}$$

where “esssup” denotes the essential supremum and the first inequality in (2.39) follows from (2.37). So, by the Lebesgue’s dominated convergence theorem, we have

$$\begin{aligned}
(2.40) \quad & \lim_{\sigma \rightarrow 0} E \left[\text{ess sup}_{0 \leq \delta \leq \sigma} \zeta(\Delta V_{(l),\delta}^{(1)}(t, x)) \right] \\
& \leq \hat{\gamma} E \left[\int_t^T \lim_{\sigma \rightarrow 0} \text{ess sup}_{0 \leq \delta \leq \sigma} \left\| \Delta \mathcal{L}_{(l),\delta}^{(1)}(s, x, U) \right\|^2 e^{2\gamma s} ds \right],
\end{aligned}$$

since, due to the mean-value theorem and the conditions stated in (2.17), we have

$$\left\| \Delta \mathcal{L}_{(l),\delta}^{(1)}(t, x, U) \right\| \leq 2K_D \left(\|U\|_{C^{k+1}(D,q)} + \|\bar{U}\|_{C^{m+1}(D,qd)} + \|\tilde{U}\|_{\nu,1} \right).$$

Then, by (2.40) and the Fatou’s lemma, we know that, for any sequence σ_n satisfying $\sigma_n \rightarrow 0$ along $n \in \mathcal{N}$, there is a subsequence $\mathcal{N}' \subset \mathcal{N}$ such that

$$(2.41) \quad \text{ess sup}_{0 \leq \delta \leq \sigma_n} \zeta(\Delta V_{(l),\delta}^{(1)}(t, x)) \rightarrow 0 \text{ along } n \in \mathcal{N}' \text{ a.s.},$$

which implies that the first-order derivative of V in terms of x_l for each $l \in \{1, \dots, p\}$ exists and equals $V_{(l)}^{(1)}(t, x)$ a.s. for each $t \in [0, T]$ and $x \in D$, and moreover, it is $\{\mathcal{F}_t\}$ -adapted.

Next, it follows from the similar proof as used in (2.39) that

$$(2.42) \quad \begin{aligned} & \lim_{\sigma \rightarrow 0} E \left[\int_t^T \text{ess sup}_{0 \leq \delta \leq \sigma} \text{Tr} \left(\Delta \mathcal{J}_{(l),\delta}^{(1)}(s, x, U) - \Delta \bar{V}_{(l),\delta}^{(1)}(s, x) \right) e^{2\gamma s} ds \right] \\ & \leq \hat{\gamma} E \left[\int_t^T \lim_{\sigma \rightarrow 0} \text{ess sup}_{0 \leq \delta \leq \sigma} \left\| \Delta \mathcal{L}_{(l),\delta}^{(1)}(s, x, U) \right\|^2 e^{2\gamma s} ds \right]. \end{aligned}$$

Thus, by (2.41) and (2.42), we know that

$$\lim_{\delta \rightarrow 0} \Delta \bar{V}_{(l),\delta}^{(1)}(t, x) = \lim_{\delta \rightarrow 0} \Delta \mathcal{J}_{(l),\delta}^{(1)}(t, x, U) = 0 \quad \text{a.s.}$$

which implies that the first-order derivative of \bar{V} with respect to x_l for each $l \in \{1, \dots, p\}$ exists and equals $\bar{V}_{(l)}^{(1)}(t, x)$ a.s. for every $t \in [0, T]$ and $x \in D$, and moreover, it is a $\{\mathcal{F}_t\}$ -predictable process. Similarly, we can get the conclusion for $\tilde{V}_{(l)}^{(1)}(t, x, z)$ associated with each l, t, x, z .

Secondly, assuming that $(V^{(c-1)}(t, x), \bar{V}^{(c-1)}(t, x), \tilde{V}^{(c-1)}(t, x, z))$ corresponding to a given $(U(t, x), \bar{U}(t, x), \tilde{U}(t, x, z)) \in \mathcal{Q}_{\mathcal{F}}^2([0, T])$ exists for any given $c \in \{1, 2, \dots\}$. Then we can show that

$$(2.43) \quad \left(V^{(c)}(t, x), \bar{V}^{(c)}(t, x), \tilde{V}^{(c)}(t, x, z) \right)$$

exists for the given $c \in \{1, 2, \dots\}$.

As a matter of fact, consider any fixed nonnegative integer numbers i_1, \dots, i_p satisfying $i_1 + \dots + i_p = c - 1$ for the given $c \in \{1, 2, \dots\}$, each $f \in \{V, \bar{V}, \tilde{V}\}$, each $l \in \{1, \dots, p\}$, and each small enough $\delta > 0$, let

$$(2.44) \quad f_{(i_1 \dots (i_l+1) \dots i_p), \delta}^{(c-1)}(t, x) \equiv f_{(i_1 \dots i_p)}^{(c-1)}(t, x + \delta e_l)$$

correspond to $\mathcal{I}_{(i_1 \dots i_p)}^{(c-1)}(s, x + \delta e_l, U(s, x + \delta e_l), \cdot)$ with $\mathcal{I} \in \{\mathcal{L}, \mathcal{J}\}$ via (2.22), where the differential operators \mathcal{L} and \mathcal{J} are replaced by their $(c-1)$ th-order partial derivatives $\mathcal{L}_{(i_1 \dots i_p)}^{(c-1)}$ and $\mathcal{J}_{(i_1 \dots i_p)}^{(c-1)}$. Similarly, let $(V_{(i_1 \dots (i_l+1) \dots i_p)}^{(c)}(t, x), \bar{V}_{(i_1 \dots (i_l+1) \dots i_p)}^{(c)}(t, x), \tilde{V}_{(i_1 \dots (i_l+1) \dots i_p)}^{(c)}(t, x, z))$ be defined through (2.22) where \mathcal{L} and \mathcal{J} are replaced by their c th-order partial derivatives $\mathcal{L}_{i_1 \dots (i_l+1) \dots i_p}^{(c)}$ and $\mathcal{J}_{i_1 \dots (i_l+1) \dots i_p}^{(c)}$ corresponding to a given $t, x, U(t, x), \bar{U}(t, x), \tilde{U}(t, x, z)$. Moreover, define

$$(2.45) \quad \Delta f_{(i_1 \dots (i_l+1) \dots i_p), \delta}^{(c)}(t, x) = \frac{f_{(i_1 \dots (i_l+1) \dots i_p), \delta}^{(c-1)}(t, x) - f_{(i_1 \dots i_p)}^{(c-1)}(t, x)}{\delta} - f_{(i_1 \dots (i_l+1) \dots i_p)}^{(c)}(t, x)$$

for each $f \in \{U, \bar{U}, \tilde{U}, V, \bar{V}, \tilde{V}\}$ and let

$$(2.46) \quad \begin{aligned} & \Delta \mathcal{I}_{(i_1 \dots (i_l+1) \dots i_p), \delta}^{(c)}(t, x, U) \\ & = \frac{1}{\delta} \left(\mathcal{I}_{(i_1 \dots i_p)}^{(c-1)}(t, x + \delta e_l, U(t, x + \delta e_l), \cdot) \right. \\ & \quad \left. - \mathcal{I}_{(i_1 \dots i_p)}^{(c-1)}(s, x, U(s, x), \cdot) \right) - \mathcal{I}_{(i_1 \dots (i_l+1) \dots i_p)}^{(c)}(s, x, U(s, x), \cdot) \end{aligned}$$

for $\mathcal{I} \in \{\mathcal{L}, \mathcal{J}\}$. Then, by the Itô's formula and repeating the procedure as used in the second step, we know that $(V_{(i_1 \dots (i_{i+1}) \dots i_p)}^{(c)}(t, x), \bar{V}_{(i_1 \dots (i_{i+1}) \dots i_p)}^{(c)}(t, x), \tilde{V}_{(i_1 \dots (i_{i+1}) \dots i_p)}^{(c)}(t, x, z))$ exist for the given $c \in \{1, 2, \dots\}$ and all $l \in \{1, \dots, p\}$, which implies that the claim in (2.43) is true.

Thirdly, it follows from the induction method with respect to $c \in \{1, 2, \dots\}$ that the claims stated in the lemma are true. Hence we finish the proof of Lemma 2.2. \square

Lemma 2.3 *Under the conditions of Theorem 2.1, all the claims in the theorem are true.*

PROOF. Let $D_{\mathcal{F}}^2([0, T], R^q)$ be the set of R^q -valued $\{\mathcal{F}_t\}$ -adapted and square integrable càdlàg processes as in (2.8). Moreover, for any given $\gamma \in R$, define $\mathcal{M}_\gamma^D[0, T]$ to be the following Banach space (see, e.g., the similar explanation as used in Yong and Zhou [37], and Situ [31])

$$(2.47) \quad \mathcal{M}_\gamma^D[0, T] = D_{\mathcal{F}}^2([0, T], R^q) \times L_{\mathcal{F}, p}^2([0, T], R^{q \times d}) \times L_p^2([0, T], R^{q \times h})$$

endowed with the norm: for any given $(U, \bar{U}, \tilde{U}) \in \mathcal{M}_\gamma^D[0, T]$,

$$(2.48) \quad \|(U, \bar{U}, \tilde{U})\|_{\mathcal{M}_\gamma^D}^2 \equiv \sum_{k=1}^{\infty} \xi(k) \|(U, \bar{U}, \tilde{U})\|_{\mathcal{M}_{\gamma, k}^D}^2,$$

where, without loss of generality, we assume that $m = k$ in (1.1) and

$$(2.49) \quad \|(U, \bar{U}, \tilde{U})\|_{\mathcal{M}_{\gamma, k}^D}^2 = E \left[\sup_{0 \leq t \leq T} \|U(t)\|_{C^k(D, q)}^2 e^{2\gamma t} \right] + E \left[\int_0^T \|\bar{U}(t)\|_{C^k(D, qd)}^2 e^{2\gamma t} dt \right] \\ + E \left[\int_0^T \|\tilde{U}(t)\|_{\nu, k}^2 e^{2\gamma t} dt \right].$$

In addition, through (2.22), we can define the following map,

$$\Xi : (U(\cdot, x), \bar{U}(\cdot, x), \tilde{U}(\cdot, x, z)) \rightarrow (V(\cdot, x), \bar{V}(\cdot, x), \tilde{V}(\cdot, x, z)).$$

Then, based on the norm defined in (2.48), we can show that Ξ forms a contraction mapping in $\mathcal{M}_\gamma^D[0, T]$. As a matter of fact, consider $(U^i(\cdot, x), \bar{U}^i(\cdot, x), \tilde{U}^i(\cdot, x, z)) \in \mathcal{M}_\gamma^D[0, T]$ and $(V^i(\cdot, x), \bar{V}^i(\cdot, x), \tilde{V}^i(\cdot, x, z)) = \Xi(U^i(\cdot, x), \bar{U}^i(\cdot, x), \tilde{U}^i(\cdot, x, z))$ with $i \in \{1, 2, \dots\}$, define

$$\Delta f^i = f^{i+1} - f^i \quad \text{with} \quad f \in \{U, \bar{U}, \tilde{U}, V, \bar{V}, \tilde{V}\}$$

and take

$$(2.50) \quad \zeta(\Delta U^i(t, x)) = \text{Tr} \left(\Delta U^i(t, x) \right) e^{2\gamma t}.$$

Then it follows from (2.14) and the similar argument as used in proving (2.37) that, for a $\gamma > 0$ and each $i \in \{2, 3, \dots\}$,

$$(2.51) \quad \zeta(\Delta U^i(t, x)) + \int_t^T \text{Tr} \left(\Delta \mathcal{J}(s, x, U^i, U^{i-1}) - \Delta \bar{U}^i(s, x) \right) e^{2\gamma s} ds \\ + \int_t^T \int_{z>0} \text{Tr} \left(\Delta \tilde{U}^i(s^-, x, z) \right) e^{2\gamma s} \tilde{N}(\lambda ds, x, dz) \\ \leq \hat{\gamma} \int_t^T \left\| \Delta \mathcal{L}(s, x, U^i, U^{i-1}) \right\|^2 e^{2\gamma s} ds - M_{(i)}(t) \\ \leq \hat{\gamma} K_a N^{i-1}(t) - M^i(t)$$

where K_a is some nonnegative constant depending only on K_D , and for the last inequality, we have taken

$$(2.52) \quad \hat{\gamma} = \frac{3K_D^2}{2\gamma} > 0.$$

Moreover, $N^{i-1}(t)$ appeared in (2.51) is given by

$$(2.53) \quad N^{i-1}(t) = \int_t^T \left(\|\Delta U^{i-1}(s)\|_{C^k(D,q)}^2 + \|\Delta \bar{U}^{i-1}(s)\|_{C^k(D,qd)}^2 + \|\Delta \tilde{U}^{i-1}(s)\|_{\nu,k}^2 \right) e^{2\gamma s} ds$$

and $M^i(t)$ is a martingale of the following form,

$$(2.54) \quad \begin{aligned} M^i(t) &= -2 \sum_{j=1}^d \int_t^T \left((\Delta U^i)(s^-, x) \right)' \left(\Delta \mathcal{J}_j(s^-, x, U^i, U^{i-1}) - (\Delta \bar{U}^i)_j(s^-, x) \right) e^{2\gamma s} dW_j(s) \\ &\quad + 2 \sum_{j=1}^h \int_t^T \int_{z_j > 0} \left((\Delta U^i)_j(s^-, x) \right)' \left((\tilde{U}^i)_j(s^-, x, z_j) \right) e^{2\gamma s} \tilde{N}_j(\lambda_j ds, x, dz_j). \end{aligned}$$

Then, by (2.51)-(2.54) and the martingale properties related to stochastic integral, we have

$$(2.55) \quad \begin{aligned} E \left[\left\| \Delta U^i(t, x) \right\|^2 e^{2\gamma t} + \int_t^T \text{Tr} \left(\Delta \mathcal{J}(s, x, U^i, U^{i-1}) - \Delta \bar{U}^i(s, x) \right) e^{2\gamma s} ds \right. \\ \left. + \int_t^T \left\| \Delta \tilde{U}^i(s) \right\|_{\nu,0}^2 e^{2\gamma s} ds \right] \\ \leq \hat{\gamma}(T+1)K_a \left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_{\gamma,k}^D}^2. \end{aligned}$$

Next it follows from (2.54), the Burkholder-Davis-Gundy's inequality (see, e.g., Theorem 48 in page 193 of Protter [29]) that

$$(2.56) \quad \begin{aligned} E \left[\sup_{0 \leq t \leq T} |M^i(t)| \right] &\leq 4 \sum_{j=1}^d E \left[\sup_{0 \leq t \leq T} \left| \int_0^t (\Delta U^i)(s^-, x) \right)' \right. \\ &\quad \left. \left(\Delta \mathcal{J}_j(s^-, x, U^i, U^{i-1}) - (\Delta \bar{U}^i)_j(s^-, x) \right) e^{2\gamma s} dW_j(s) \right] \\ &\quad + 4 \sum_{j=1}^h E \left[\sup_{0 \leq t \leq T} \left| \int_0^t \int_{z_j > 0} (\Delta U^i)(s^-, x) \right)' (\Delta \tilde{U}^i)_j(s^-, x, z_j) e^{2\gamma s} \tilde{N}_j(\lambda_j ds, x, dz_j) \right] \\ &\leq K_b \sum_{j=1}^d E \left[\left(\int_0^T \left\| \Delta U^i(s, x) \right\|^2 \left\| (\Delta \mathcal{J}^i)_j(s, x, U^i, U^{i-1}) - (\Delta \bar{U}^i)_j(s, x) \right\|^2 e^{4\gamma s} ds \right)^{\frac{1}{2}} \right] \\ &\quad + K_b \sum_{j=1}^h E \left[\left(\int_0^T \int_{z_j > 0} \left\| \Delta U^i(s, x) \right\|^2 \left\| (\Delta \tilde{U}^i)_j(s, x, z_j) \right\|^2 e^{4\gamma s} \lambda_j \nu_j(x, dz_j) ds \right)^{\frac{1}{2}} \right] \end{aligned}$$

$$\begin{aligned}
&\leq K_b E \left[\left(\sup_{0 \leq t \leq T} \|\Delta U^i(t, x)\|^2 e^{2\gamma t} \right)^{\frac{1}{2}} \right. \\
&\quad \left. \left(\sum_{j=1}^d \left(\int_0^T \|\Delta \mathcal{J}_j(s, x, U^i, U^{i-1}) - (\Delta \bar{U}^i)_j(s, x)\|^2 e^{2\gamma s} ds \right)^{\frac{1}{2}} \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^h \left(\int_0^T \int_{z_j > 0} \|(\Delta \tilde{U}^i)_j(s, x, z_j)\|^2 e^{2\gamma s} \lambda_j \nu_j(dz_j) ds \right)^{\frac{1}{2}} \right) \right] \\
&\leq \frac{1}{2} E \left[\sup_{0 \leq t \leq T} \|\Delta U^i(t, x)\|^2 e^{2\gamma t} \right] \\
&\quad + dK_b^2 E \left[\left(\int_0^T \|\Delta \mathcal{J}_j(s, x, U^i, U^{i-1}) - (\Delta \bar{U}^i)_j(s, x)\|^2 e^{2\gamma s} ds \right) \right] \\
&\quad + K_b^2 E \left[\int_0^T \|\Delta \tilde{U}^i(s)\|_{\nu, 0}^2 e^{2\gamma s} ds \right] \\
&\leq \frac{1}{2} E \left[\sup_{0 \leq t \leq T} \|\Delta U^i(t, x)\|_{C^0(q)}^2 e^{2\gamma t} \right] + \hat{\gamma}(T+1) dK_b^2 E [N^{i-1}(t)]
\end{aligned}$$

where K_b is some nonnegative constant depending only on K_D and T , and we have used (2.55) for the last inequality of (2.56). Thus it follows from (2.51)-(2.56) that

$$\begin{aligned}
(2.57) \quad &E \left[\sup_{0 \leq t \leq T} \|\Delta U^i(t)\|_{C^0(q)}^2 e^{2\gamma t} \right] \\
&\leq 2 \left(1 + dK_b^2 \right) \hat{\gamma}(T+1) \left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_{\gamma, k}^D}^2.
\end{aligned}$$

Moreover it follows from (2.51) and (2.17) that, for $i \in \{3, 4, \dots\}$,

$$\begin{aligned}
(2.58) \quad &E \left[\int_t^T \text{Tr} \left(\Delta \bar{U}^i(s, x) \right) e^{2\gamma s} ds \right] \\
&\leq 2E \left[\int_t^T \text{Tr} \left(\Delta \mathcal{J}(s, x, U^i, U^{i-1}) - \Delta \bar{U}^i(s, x) \right) e^{2\gamma s} ds \right] \\
&\quad + 2E \left[\int_t^T \text{Tr} \left(\Delta \mathcal{J}(s, x, U^i, U^{i-1}) \right) e^{2\gamma s} ds \right] \\
&\leq 2\hat{\gamma}K_c \left(\left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_{\gamma, k}^D}^2 + \left\| (\Delta U^{i-2}, \Delta \bar{U}^{i-2}, \Delta \tilde{U}^{i-2}) \right\|_{\mathcal{M}_{\gamma, k}^D}^2 \right)
\end{aligned}$$

where K_c is some nonnegative constant depending only on K_D and T . Thus it follows from (2.51) and (2.57)-(2.58) that

$$\begin{aligned}
(2.59) \quad &\left\| (\Delta U^i, \Delta \bar{U}^i, \Delta \tilde{U}^i) \right\|_{\mathcal{M}_{\gamma, 0}^D}^2 \\
&\leq \hat{\gamma}K_d \left(\left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_{\gamma, k}^D}^2 + \left\| (\Delta U^{i-2}, \Delta \bar{U}^{i-2}, \Delta \tilde{U}^{i-2}) \right\|_{\mathcal{M}_{\gamma, k}^D}^2 \right)
\end{aligned}$$

where K_d is some nonnegative constant depending only on K_D and T .

Now, by Lemma 2.2 and the similar construction as in (2.50), for each $c \in \{1, 2, \dots\}$, we can define

$$(2.60) \quad \zeta(\Delta U^{c,i}(t, x)) \equiv \text{Tr} \left(\Delta U^{c,i}(t, x) \right) e^{2\gamma t},$$

where

$$\Delta U^{c,i}(t, x) = (\Delta U^{(0),i}(t, x), \Delta U^{(1),i}(t, x), \dots, \Delta U^{(c),i}(t, x))'.$$

Then it follows from the Itô's formula and the similar discussion for (2.59) that

$$(2.61) \quad \begin{aligned} & \left\| (\Delta U^i, \Delta \bar{U}^i, \Delta \tilde{U}^i) \right\|_{\mathcal{M}_{\gamma,c}^D}^2 \\ & \leq \hat{\gamma} K_d \left(\left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_{\gamma,k+c}^D}^2 + \left\| (\Delta U^{i-2}, \Delta \bar{U}^{i-2}, \Delta \tilde{U}^{i-2}) \right\|_{\mathcal{M}_{\gamma,k+c}^D}^2 \right), \end{aligned}$$

which implies that

$$(2.62) \quad \begin{aligned} & \left\| (\Delta U^i, \Delta \bar{U}^i, \Delta \tilde{U}^i) \right\|_{\mathcal{M}_\gamma^D}^2 \\ & \leq \hat{\gamma} K_f \left(\left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_\gamma^D}^2 + \left\| (\Delta U^{i-2}, \Delta \bar{U}^{i-2}, \Delta \tilde{U}^{i-2}) \right\|_{\mathcal{M}_\gamma^D}^2 \right). \end{aligned}$$

Since $(a^2 + b^2)^{1/2} \leq a + b$ for $a, b \geq 0$, we have

$$(2.63) \quad \begin{aligned} & \left\| (\Delta U^i, \Delta \bar{U}^i, \Delta \tilde{U}^i) \right\|_{\mathcal{M}_\gamma^D} \\ & \leq \sqrt{\hat{\gamma} K_f} \left(\left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_\gamma^D} + \left\| (\Delta U^{i-2}, \Delta \bar{U}^{i-2}, \Delta \tilde{U}^{i-2}) \right\|_{\mathcal{M}_\gamma^D} \right) \end{aligned}$$

where K_f is some nonnegative constant depending only on K_D and T . Therefore, by taking $\gamma > 0$ large enough such that $2\sqrt{\hat{\gamma}K_f}$ sufficiently small and by (2.62), we know that

$$(2.64) \quad \begin{aligned} & \sum_{i=3}^{\infty} \left\| (\Delta U^i, \Delta \bar{U}^i, \Delta \tilde{U}^i) \right\|_{\mathcal{M}_\gamma^D} \\ & \leq \frac{\sqrt{\hat{\gamma}K_f}}{1 - 2\sqrt{\hat{\gamma}K_f}} \left(2 \left\| (\Delta U^2, \Delta \bar{U}^2, \Delta \tilde{U}^2) \right\|_{\mathcal{M}_\gamma^D} + \left\| (\Delta U^1, \Delta \bar{U}^1, \Delta \tilde{U}^1) \right\|_{\mathcal{M}_\gamma^D} \right) \\ & < \infty. \end{aligned}$$

Thus, from (2.64), we see that $(U^i, \bar{U}^i, \tilde{U}^i)$ with $i \in \{1, 2, \dots\}$ forms a Cauchy sequence in $\mathcal{M}_\gamma^D[0, T]$, which implies that there is some (U, \bar{U}, \tilde{U}) such that

$$(2.65) \quad (U^i, \bar{U}^i, \tilde{U}^i) \rightarrow (U, \bar{U}, \tilde{U}) \text{ as } i \rightarrow \infty \text{ in } \mathcal{M}_\gamma^D[0, T].$$

Finally, by (2.65) and the similar procedure as used for Theorem 5.2.1 in pages 68-71 of Øksendal [23], we can finish the proof of Lemma 2.3. \square

Proof of Theorem 2.1.

By combining Lemma 2.1-Lemma 2.3, we can reach a proof for Theorem 2.1. \square

3 B-SPDEs in Infinite Space Domain and under Random Environment

3.1 B-SPDEs in the Infinite Space Domain

First of all, for a given nonnegative integer b and each $n \in \{b+1, b+2, \dots\}$, define the following sequence of sets

$$(3.1) \quad D_n = \{x \in R^p : b \leq \|x\| \leq n\}$$

and let

$$(3.2) \quad R_b^p = \{x \in R^p : \|x\| \geq b\}$$

Moreover, let $C^\infty(R_b^p, q)$ be the Banach space endowed with the following norm

$$(3.3) \quad \|f\|_{C^\infty(R_b^p, q)} \equiv \sum_{n=b+1}^{\infty} \xi(n+1) \|f\|_{C^\infty(D_n, q)}$$

for each $f \in C^\infty(R_b^p, q)$, and let

$$(3.4) \quad \bar{Q}_{\mathcal{F}}^2([0, T]) \equiv \bar{L}_{\mathcal{F}}^2([0, T], R^q) \times \bar{L}_{\mathcal{F}, p}^2([0, T], R^{qd}) \times \bar{L}_p^2([0, T], R^{q \times h})$$

be the corresponding space defined in (2.13) when the norm in (2.7) is replaced by the associated one given in (3.3).

Theorem 3.1 *Assuming that there exists a nonnegative constant $K_{R_b^p}$ such that conditions (2.14)-(2.18) are satisfied when K_D is replaced by $K_{R_b^p}$. Moreover, if $\mathcal{L}(t, x, v, \cdot)$ and $\mathcal{J}(t, x, v, \cdot)$ are $\{\mathcal{F}_t\}$ -adapted for each $x \in R_b^p$, $z \in R_+^h$, and any given $(v, \bar{v}, \tilde{v}) \in C^\infty(R_b^p, R^q) \times C^\infty(R_b^p, R^{q \times d}) \times \bar{L}_{\nu, \infty}^2(R_b^p \times R_+^h, R^{q \times h})$ with*

$$(3.5) \quad \mathcal{L}(\cdot, x, 0, \cdot), \mathcal{J}(\cdot, x, 0, \cdot) \in \bar{L}_{\mathcal{F}}^2([0, T], R^q),$$

then the B-SPDE (1.1) has a unique adapted solution satisfying,

$$(3.6) \quad (V(\cdot, x), \bar{V}(\cdot, x), \tilde{V}(\cdot, x, z)) \in \bar{Q}_{\mathcal{F}}^2([0, T])$$

where V is a càdlàg process and the uniqueness is in the sense: if there exists another solution $(U(t, x), \bar{U}(t, x), \tilde{U}(t, x, z))$ as required, we have

$$(3.7) \quad E \left[\int_0^T \left(\|U(t) - V(t)\|_{C^\infty(R_b^p, q)}^2 + \|\bar{U}(t) - \bar{V}(t)\|_{C^\infty(R_b^p, qd)}^2 + \|\tilde{U}(t) - \tilde{V}(t)\|_{L_{\nu, \infty}^2}^2 \right) dt \right] = 0.$$

PROOF. It follows from (3.5) and the similar argument used for (2.62) in the proof of Lemma 2.3 that

$$(3.8) \quad (U^1(\cdot, x), \bar{U}^1(\cdot, x), \tilde{U}^1(\cdot, x, z)) \in \bar{Q}_{\mathcal{F}}^2([0, T])$$

with $(U^0, \bar{U}^0, \tilde{U}^0) = (0, 0, 0)$, where $(U^1, \bar{U}^1, \tilde{U}^1)$ is defined through (2.22) in Lemma 2.1. Then, over each $\{D_n\}$ with $n \in \{b+1, b+2, \dots\}$, it follows from (2.62) in the proof of Lemma 2.3 that

$$(3.9) \quad \begin{aligned} & \left\| (\Delta U^i, \Delta \bar{U}^i, \Delta \tilde{U}^i) \right\|_{\mathcal{M}_\gamma^{R_b^p}}^2 \\ & \leq \hat{\gamma} K_g \left(\left\| (\Delta U^{i-1}, \Delta \bar{U}^{i-1}, \Delta \tilde{U}^{i-1}) \right\|_{\mathcal{M}_\gamma^{R_b^p}}^2 + \left\| (\Delta U^{i-2}, \Delta \bar{U}^{i-2}, \Delta \tilde{U}^{i-2}) \right\|_{\mathcal{M}_\gamma^{R_b^p}}^2 \right) \end{aligned}$$

where K_g is some nonnegative constant depending only on T and R_b^p . Then it follows from (3.9) that the remaining proof for Theorem 3.1 can be conducted similarly as in the proof of Theorem 2.1. Hence we finish the proof of Theorem 3.1. \square

3.2 B-SPDEs under Random Environment

Assuming that the random environment under consideration is characterized by a R^p -valued Markov process $X(\cdot)$ with continuous sample paths and its associated stopping time τ is defined by

$$(3.10) \quad \tau \equiv \inf\{t \in [0, T], \|X(t)\| < b\}.$$

Then the q -dimensional B-SPDEs with jumps under random environment $X(\cdot)$ can be described as follows,

$$(3.11) \quad \begin{aligned} V(t, X(t)) &= H(X(\tau)) + \int_t^\tau \mathcal{L}(s^-, X(t), V, \cdot) ds \\ &+ \int_t^\tau (\mathcal{J}(s^-, X(t), V, \cdot) - \bar{V}(s^-, X(t))) dW(s) \\ &- \int_t^\tau \int_{z>0} \tilde{V}(s^-, X(t), z, \cdot) \tilde{N}(\lambda ds, X(t), dz). \end{aligned}$$

Moreover, define

$$(3.12) \quad \hat{\mathcal{Q}}_{\mathcal{F}}^2([0, \tau]) \equiv \hat{L}_{\mathcal{F}}^2([0, \tau], R^q) \times \hat{L}_{\mathcal{F}, p}^2([0, \tau], R^{qd}) \times \hat{L}_p^2([0, \tau], R^{q \times h})$$

be the corresponding space defined in (2.13) when the norm in (2.7) is replaced by the following one,

$$(3.13) \quad \|f(x)\|_\infty \equiv \sum_{i \in \mathcal{N}} \xi(i) \sup_{j \leq i} \|f^{(j)}(x)\|,$$

where $f^{(i)}(x)$ for each $x \in R_b^p$ is defined as in (2.5).

Theorem 3.2 *Under the conditions as stated in Theorem 3.1, the B-SPDE in (3.11) under random environment $X(\cdot)$ has a unique adapted solution satisfying,*

$$(3.14) \quad (V(\cdot, X(\cdot)), \bar{V}(\cdot, X(\cdot)), \tilde{V}(\cdot, X(\cdot), z)) \in \hat{\mathcal{Q}}_{\mathcal{F}}^2([0, \tau]),$$

where V is a càdlàg process and the uniqueness is in the sense: if there exists another solution $(U(t, X(t)), \bar{U}(t, X(t)), \tilde{U}(t, X(t)))$ as required, we have

$$E \left[\int_0^T \left(\|U(t, X(t)) - V(t, X(t))\|_\infty^2 + \|\bar{U}(t, X(t)) - \bar{V}(t, X(t))\|_\infty^2 + \|\tilde{U}(t, X(t)) - \tilde{V}(t, X(t))\|_{\nu, \infty}^2 \right) dt \right] = 0.$$

PROOF. First of all, it follows from the similar discussion as in Situ [31] that $\bar{Q}_{\mathcal{F}}([0, \tau])$ is a Banach space. Then we know that all of the claims stated in Theorem 3.1 are true over the space $\bar{Q}_{\mathcal{F}}([0, \tau])$, which imply that the claims in the current theorem are true. \square

Example 3.1 *The solution $V(t, x, \cdot)$ to the B-SPDE in (1.1) is described by random surfaces and the solution $V(t, X(t))$ to the B-SPDE under random environment in (3.11) is represented by random paths, which are shown in Figure 1 presented in the Introduction.*

4 An Illustrative Example in Finance

In this section, we consider a financial market consisting of two assets and an external random factor. One asset is supposed to be a risk-free account whose price $S_0(t)$ is subject to the following ordinary differential equation,

$$(4.1) \quad \begin{cases} dS_0(t) = rS_0(t)dt, \\ S_0(0) = 1, \end{cases}$$

where the interest rate r is a nonnegative constant. Another asset is stock whose price process $S(t)$ satisfies the following SDE for each $t \in [0, T]$,

$$(4.2) \quad \begin{cases} dS(t) = S(t)\beta(Y(t))dt + S(t)\sigma(Y(t))dW_1(t), \\ S(0) = s > 0, \end{cases}$$

where the random factor $Y(t)$ with $t \in [0, T]$ satisfies

$$(4.3) \quad \begin{cases} dY(t) = c(Y(t))dt + d(Y(t)) \left(\rho dW_1(t) + \sqrt{1 - \rho^2} dW_2(t) \right), \\ Y(0) = y \in R \end{cases}$$

with $\rho \in (-1, 1)$. Moreover, we suppose that the market coefficients $f = \beta, \sigma, c, d$ satisfy the standard global Lipschitz and linear growth conditions and $\sigma(y) \geq \kappa > 0$ for all $y \in R$ and some positive constant κ .

Beginning at $t = 0$ with an initial endowment $x \in R_+$, an investor invests at any time $t > 0$ in the risky and riskless asset. The present value of the amounts invested are denoted, respectively, by $\pi_0(t)$ and $\pi_1(t)$, and then the present value of the investor's aggregate investment is given by $X^\pi(t) = \pi_0(t) + \pi_1(t)$, which satisfies (see, e.g., Musiela and Zairphopoulou [22])

$$(4.4) \quad dX^\pi(t) = \sigma(t)\pi(t) \cdot (\lambda(t)dt + dW(t))$$

where “ \cdot ” denotes the inner product, $\pi(t) = (\pi_0(t), \pi_1(t))$, $dW = (dW_1, dW_2)'$, and

$$(4.5) \quad \lambda(t) = \left(\frac{\beta(Y(t)) - r}{\sigma(y(t))}, 0 \right)'.$$

Moreover, for a given constant $b \geq 1$, let the following τ be the bankruptcy time for the investor,

$$(4.6) \quad \tau = \inf\{t > 0, X^\pi(t) < b\}.$$

One objective to study the above financial system is to find the optimal portfolio choice based on maximal expected utility of terminal wealth over all admissible strategies (see, e.g., Merton [20]), i.e., to solve the following stochastic dynamic optimization problem,

$$(4.7) \quad V(t, x) = \sup_{\mathcal{A}_\tau} E_P [u_\tau(X^\pi(\tau)) | \mathcal{F}_t, X^\pi(t) = x]$$

where \mathcal{A}_τ denotes the set of all admissible strategies π : $\pi(t)$ is self-financing and $\{\mathcal{F}_t\}$ -progressively measurable, satisfying

$$E \left[\int_0^\tau (\pi(s)\sigma(s))^2 ds \right] < \infty \text{ and } X^\pi(t) \geq 0 \text{ with } t \in [0, \tau],$$

and the utility is taken to be the following constant relative risk aversion (CRRA) case:

$$u_\tau(x) = \frac{x^\gamma}{\gamma} \text{ with } 0 < \gamma < 1, \gamma \neq 0.$$

Then it follows from the discussions in Musiela and Zariphopoulou [22], Øksendal [23], Øksendal and Sulem [24] that the value function process defined in (4.7) should satisfy the following B-SPDE,

$$(4.8) \quad \begin{aligned} V(t, x) = & u_\tau(X(\tau)) + \int_t^\tau \frac{(V_x(s, x)\lambda(t) + \sigma(t)(\sigma(t))^+ \bar{V}_x(s, x))^2}{2V_{xx}(s, x)} ds \\ & + \int_t^\tau (\bar{V}(s, x))' dW(s) \end{aligned}$$

where $\sigma(t) = (\sigma(Y(t)), 0)'$, $(\sigma(t))^+ = (1/\sigma(Y(t)), 0)$, and $dW = (dW_1, dW_2)'$.

As pointed out in Musiela and Zariphopoulou [22], the B-SPDE in (4.8) is newly derived and belongs to a class of strongly nonlinear B-SPDEs (see, e.g., the related discussion in Lions and Souganidis [19]). However, based on Theorem 3.1 in the previous section of the current paper and the discussion in Musiela and Zariphopoulou [22], we can show that there exists a unique adapted solution, before a random bankruptcy time (i.e., over $[0, \tau]$), to the B-SPDE in (4.8) over the class of functions satisfying the conditions required by Theorem 3.1. In fact, based on the discussions in Musiela and Zariphopoulou [22], Glosten *et al.* [13], we have the following observation that there is a pair of V and \bar{V} satisfying (4.8), i.e.,

$$(4.9) \quad V(t, x) = \frac{1}{\gamma} x^\gamma f(t, Y(t))^\delta$$

where f is a solution of the following partial differential equation

$$f_t + \frac{1}{2}d^2(y)f_{yy} + \left(c(y) + \frac{\rho\gamma\lambda(y)d(y)}{1-\gamma}\right)f_y + \frac{\gamma\lambda^2(y)f}{2\delta(1-\gamma)} = 0$$

with $f(\tau, y) = 1$ and $\delta = (1-\gamma)/(1-\gamma+\rho^2\gamma)$, and

$$(4.10) \quad \bar{V}_1(t, x) = \frac{\rho\delta}{\gamma}d(Y(t))f_y(t, Y(t))f(t, Y(t))^{\delta-1}$$

$$(4.11) \quad \bar{V}_2(t, x) = \frac{\delta(1-\rho^2)^{1/2}}{\gamma}x^\gamma d(Y(t))f_y(t, Y(t))f(t, Y(t))^{\delta-1}$$

Thus it follows from (4.9)-(4.11) that (V, \bar{V}) is a solution to (4.8) such that the conditions in Theorem 3.1 are satisfied over $[0, \tau]$ and hence it is the unique adapted solution to (4.8) over $[0, \tau]$. Moreover, it follows from Theorem 3.2 in the previous section and the discussion in Musiela and Zariphopoulou [22] that, for each $t \leq s \leq \tau$, the optimal feedback portfolio process is given as follows,

$$\pi^*(s, X(s)) = -\frac{(\sigma(s))^+ (V_x(s, X(s))\lambda(s) + \sigma(s)(\sigma(s))^+\bar{V}_x(s, X(s)))}{V_{xx}(s, X(s))}.$$

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