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A semiparametric estimation of copula models based on the method of moments

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Abstract

Using the classical estimation method of moments, we propose a new semiparametric estimation procedure for multi-parameter copula models. Consistency and asymptotic normality of the obtained estimators are established. By considering an Archimedean copula model, an extensive simulation study, comparing these estimators with the pseudo maximum likelihood, rho-inversion and tau-inversion ones, is carried out. We show that, with regards to the other methods, the moment based estimation is quick and simple to use with reasonable bias and root mean squared error.

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1 Introduction

Recently, considerable attention has been paid to the problem of inference about copulas. The monographs of Cherubini *et al.* (2004), Nelsen (2006) and Joe (1997) summarize to some extent the activities in this area. Roughly speaking, a copula function is a multivariate distribution function with uniform margins. It is used as a linking block between the joint distribution function (df) F of a vector of random variables $\mathbf{X} = (X_1, ..., X_d)$ and its marginal df's $F_1, ..., F_d$. This probabilistic

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interpretation of copulas is justified by the famous Sklar's theorem (Sklar, 1959) which states that, under some mild conditions, there exists a unique copula function C, such that

$$F(x_1, ..., x_d) = C\left(F_1\left(x_1\right), .., F_d\left(x_d\right)\right).$$

In other words, the copula C is the joint df of the random vector $\mathbf{U} = (U_1, ..., U_d)$, with $U_j = F_j(X_j)$. That is, for $\mathbf{u} = (u_1, ..., u_d)$, we have

$$C(\mathbf{u}) = F\left(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)\right),$$

where $F_j^{-1}(s) := \inf\{x : F_j(x) \geq s\}$ denotes the generalized inverse function (or the quantile function) of F_j.

A parametric Archimedean copula model arises for $\mathbf{X}$ when the copula C belong to a class $\mathcal{C} := \{C_{\boldsymbol{\theta}},\ \boldsymbol{\theta} \in \mathcal{O}\}$, where $\mathcal{O}$ is an open subset of $\mathbb{R}^r$ for some integer $r \geq 1$. Statistical inference on the dependence parameter $\boldsymbol{\theta}$ is one of the main topics in multivariate statistical analysis. Several methods of copula parameter estimation have been developed, including the methods of concordance (Oakes, 1982; Genest, 1987), fully maximum likelihood (ML), pseudo maximum likelihood (PML) (Genest *et al.*, 1995), inference function of margins (IFM) (Joe, 1997, 2005), and minimum distance (MD) (Tsukahara, 2005). The performance of the PML procedure vis-a-vis to the other methods has been discussed by several authors. For example, the simulation study carried out by Kim *et al.* (2007) has concluded that the PML method is conceptually almost the same as the IFM one. It overcomes its non robustness against misspecification of the marginal distributions. Moreover, by using the PML method, one would not lose any important statistical insights that would be gained by applying the IFM. An advantage of the PML over the IFM is that the former does not require modeling the marginal distributions explicitly. Therefore, the PML estimator is better than those of the ML and IFM in most practical situations. However, in time-consuming point of view the PML, ML and IFM methods require intensive computations, notably when the copula dimension increases. Moreover, when using these methods the copula density has to be involved, therefore a serious inaccuracy at boundry points arises. Several numerical methods are proposed to solve this problem, but they are still inefficient when dealing with high dimensional copula models, more precisely for $d > 2$ (see, Yan, 2007, Section 5).

The aim of this paper is to propose an alternative estimation method similar to the concordance one, avoiding technical problems caused by copula density and providing estimators with reasonable time-consuming, bias and root mean squared error (RMSE). The concordance method, also called the τ-inversion and ρ-inversion, which are based, respectively, on Kendal's τ and Spearman's ρ rank correlation coefficients, used to estimate parametric copula models with at more two parameters. Indeed, the $\tau-$inversion and $\rho-$inversion methods use the functional representations of τ and ρ in terms of the underling copula C (Schmid *et al.*, 2010), given by

$$\tau = \tau(C) = \frac{1}{2^{d-1}-1}\left\{2^d \int_{[0,1]^d} C(\mathbf{u})\, dC(\mathbf{u}) - 1\right\},$$
$$\rho = \rho(C) = \frac{d+1}{2^d-(d+1)}\left\{2^d \int_{[0,1]^d} C(\mathbf{u})\, d\mathbf{u} - 1\right\}.$$

More precisely, suppose that copula C is a parametric model, i.e. $C = C_{\boldsymbol{\theta}}$, then both τ and ρ become functions in $\boldsymbol{\theta}$ as well, that is $\tau = \tau(\boldsymbol{\theta})$ and $\rho = \rho(\boldsymbol{\theta})$. Let $\widehat{\tau}$ and $\widehat{\rho}$ be, respectively, empirical versions of τ and ρ pertaining to the sample $(\mathbf{X}_1, ..., \mathbf{X}_n)$ from the random vector $\mathbf{X}$ and suppose that $C_{\boldsymbol{\theta}}$ is one-parameter copula model (i.e. $r = 1$). Then, the estimators of θ obtained by $\tau-$inversion or $\rho-$inversion methods are defined by $\widehat{\theta} := \tau^{-1}(\widehat{\tau})$ or $\widehat{\theta} := \rho^{-1}(\widehat{\rho})$, where τ^{-1} and ρ^{-1} are the inverses, if they exist, of functions $\theta \to \tau(\theta)$ and $\theta \to \tau(\theta)$ respectively. In the case when $r = 2$, that is when $\boldsymbol{\theta} = (\theta_1, \theta_2)$, we have to use jointly the two inversion methods, called $(\tau, \rho)-$inversion, to have a system of two equations

$$\tau(\theta_1, \theta_2) = \widehat{\tau},\ \ \rho(\theta_1, \theta_2) = \widehat{\rho}. \tag{1}$$

In conclusion, when the dimension of parameter $\boldsymbol{\theta}$ equals r, we have to use r measures of association, for example Blomqvist's beta β, Gini's gamma γ, ... (see, Nelsen, 2006, page, 207) which, in general, is not convenient on the choice of measures point of view. More precisely, suppose that we are dealing with a parameter $\boldsymbol{\theta} = (\theta_1, \theta_2)$ of a copula model C_θ, then one has the right to ask the following question: What couple among all measures of association have to be chosen to get a better estimation for θ?. On the other hand, it is worth mentioning that often there exist difficulties while using Spearman's rank correlation coefficient. One such difficulty is when using very large or very small samples. For example, in the case of very large samples, it is very time consuming to perform Spearman's coefficient since it requires ranking of the data of all variables. Then we have to look for an alternative more convenient class of measures providing estimators with nice properties. A solution to this problem may be given by applying the classical method of moments to random variable (rv) $C(\mathbf{U})$. Indeed, let us define the kth-moment $M_k(C)$, called *copula moment,* of rv $C(\mathbf{U})$ as the expectation of $(C(\mathbf{U}))^k$, that is

$$M_k(C) := \mathbf{E}\left[(C(\mathbf{U}))^k\right] = \int_{[0,1]^d} (C(\mathbf{u}))^k\, dC(\mathbf{u}),\ \ k = 1, 2, ... \tag{2}$$

Notice that the case $k=1$ corresponds to

$$M_1(C) = \mathbf{E}[C(\mathbf{U})] = \frac{\left(2^{d-1}-1\right)\tau+1}{2^d}.$$

In other words, $M_k(C)$ may be considered as a generalization of Kendal's rank correlation τ. To our knowledge, the method of moments is only used in one-parameter copula models, also known by the τ-inversion method (see for instance, Tsukahara, 2005). Note that, since $0 \leq C(\mathbf{u}) \leq 1$, then $M_k(C)$ are finite for every integer k. Now we are in position to present a new estimation method that we call *copula moment (CM) estimation*. Suppose that, for unknown parameter $\boldsymbol{\theta} \in \mathcal{O} \subset \mathbb{R}^r$, we have $C = C_{\boldsymbol{\theta}}$, then $M_k(C) = M_k(\theta)$, where

$$M_k(\boldsymbol{\theta}) := \int_{[0,1]^d} (C_\theta(\mathbf{u}))^k \, dC_\theta(\mathbf{u}), \ k=1,2,... \tag{3}$$

From equations 3, we may consider $M_k : \boldsymbol{\theta} \rightarrow M_k(\boldsymbol{\theta})$ as a mapping from $\mathcal{O} \subset \mathbb{R}^r$ to $\mathbb{R}$, that will be used as a means to estimate the parameter $\boldsymbol{\theta}$. More precisely, for a given sample $(\mathbf{X}_1, ..., \mathbf{X}_n)$ of the random vector $\mathbf{X}$, let us denote $\widehat{\boldsymbol{\theta}}^{CM}$ as the estimator of θ defined by $(M_k)_{1 \leq k \leq r}$. That is

$$\widehat{\boldsymbol{\theta}}^{CM} := M_k^{-1}\left(\widehat{M_k}\right), \ k=1,...r, \tag{4}$$

where $\widehat{M_k}$ is the empirical version of $M_k(C)$ and M_k^{-1} is the inverse of the mapping M_k, provided that it exists. The rest of the paper is organized as follows. In Section 2, we present the main steps of the copula moment estimation procedure and establish the consistency and asymptotic normality of the proposed estimator. In Section 3, an application to multiparameter Archimedean copula models is given. In Section 4, an extensive simulation study is carried out to evaluate and compare the CM based estimation with the PML and (τ, ρ) $-$inversion methods. Comments and conclusion are given Section 4. The proofs are relegated to the appendix.

2 Copula Moments based estimation

In this section we present a semiparametric estimation procedure for the copula models based on the CM's 3. First suppose that the underlying copula C belongs to a parametric family C_θ, with $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_r)$, and satisfies the *concordance ordering condition* of copulas (see, Nelsen, 2006, page 135), that is:

$$\text{for every } \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathcal{O}: \ \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2 \Longrightarrow C_{\boldsymbol{\theta}_1} (> \text{ or } <) \, C_{\boldsymbol{\theta}_2}. \tag{5}$$

It is clear that this condition implies the well-known *identifiability condition* of copulas:

$$\text{for every } \boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in \mathcal{O} : \ \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_2 \Longrightarrow C_{\boldsymbol{\theta}_1} \neq C_{\boldsymbol{\theta}_2}.$$

Identifiability is a natural and even a necessary condition: if the parameter is not identifiable then consistent estimator cannot exist (see, e.g., van der Vaart, 1998, page 62).

For a given sample $(\mathbf{X}_1, ..., \mathbf{X}_n)$ from random vector $\mathbf{X} = (X_1, ..., X_d)$, we define the corresponding joint empirical df by

$$F_n(\mathbf{x}) = n^{-1} \sum_{i=1}^{n} \mathbf{1}\{X_{1i} \leq x_1, ..., X_{di} \leq x_d\},$$

with $\mathbf{x} := (x_1, ..., x_d)$, and the marginal empirical df's pertaining to the sample $(X_{j1}, ..., X_{jn})$, from rv X_j, by

$$F_{jn}(x_j) = n^{-1} \sum_{i=1}^{n} \mathbf{1}\{X_{ji} \leq x_j\}, \ j = 1, ..., d.$$

According to Deheuvels (1979), the empirical copula function is defined by

$$C_n(\mathbf{u}) := F_n\left(F_{1n}^{-1}(u_1), ..., F_{dn}^{-1}(u_d)\right), \text{ for } \mathbf{u} \in [0,1]^d,$$

where $F_{jn}^{-1}(s) := \inf\{x : F_{jn}(x) \geq s\}$ denotes the empirical quantile function pertaining df F_{jn}. We are now in position to present, in three steps, the semiparametric CM-based estimation:

- *Step* 1: For each $j = 1, ..., d$, compute $\widehat{U}_{ji} := F_{jn}(X_{ji})$, then set

$$\widehat{\mathbf{U}}_i := \left(\widehat{U}_{1i}, ..., \widehat{U}_{di}\right), i = 1, ..., n.$$

- *Step* 2: For each $k = 1, ..., r$, compute

$$\widehat{M}_k := n^{-1} \sum_{i=1}^{n} \left(C_n\left(\widehat{\mathbf{U}}_i\right)\right)^k \tag{6}$$

  as the natural estimators of CM's M_k given in equation (2).

- *Step* 3: Solve the following system

$$\begin{cases} M_1(\theta_1, ..., \theta_r) = \widehat{M}_1 \\ M_2(\theta_1, ..., \theta_r) = \widehat{M}_2 \\ \vdots \\ M_r(\theta_1, ..., \theta_r) = \widehat{M}_r. \end{cases} \tag{7}$$

  The obtained solution $\hat{\boldsymbol{\theta}}^{CM} := \left(\widehat{\theta}_1, ..., \widehat{\theta}_r\right)$ is called the CM estimator for $\boldsymbol{\theta} = (\theta_1, ..., \theta_r)$.

Consistency and asymptotic normality of $\hat{\boldsymbol{\theta}}^{CM}$ are stated in Theorem 2 below whose proof is relegated to the appendix. For convenience we set

$$L_k(\mathbf{u};\boldsymbol{\theta}) := (C_{\boldsymbol{\theta}}(\mathbf{u}))^k - M_k(\boldsymbol{\theta}) \text{ and } \mathbf{L}(\mathbf{u};\boldsymbol{\theta}) = (L_1(\mathbf{u};\boldsymbol{\theta}),...,L_r(\mathbf{u};\boldsymbol{\theta})). \tag{8}$$

Let $\boldsymbol{\theta}_0$ be the true value of $\boldsymbol{\theta}$ and assume that the following assumptions $[H.1]-[H.3]$ hold.

- $[H.1]$ $\boldsymbol{\theta}_0 \in \mathcal{O} \subset \mathbb{R}^r$ is the unique zero of the mapping $\boldsymbol{\theta} \to \int_{[0,1]^d} \mathbf{L}(\mathbf{u};\boldsymbol{\theta})\, dC_{\boldsymbol{\theta}_0}(\mathbf{u})$ which is defined from $\mathcal{O}$ to $\mathbb{R}^r$.

- $[H.2]$ $\mathbf{L}(\cdot;\boldsymbol{\theta})$ is differentiable with respect to $\boldsymbol{\theta}$ with the Jacobian matrix denoted by

$$\overset{\bullet}{\mathbf{L}}(\mathbf{u};\boldsymbol{\theta}) := \left[\frac{\partial L_k(\mathbf{u};\boldsymbol{\theta})}{\partial \theta_\ell}\right]_{r\times r},$$

  $\overset{\bullet}{\mathbf{L}}(\mathbf{u};\boldsymbol{\theta})$ is continuous both in $\mathbf{u}$ and $\boldsymbol{\theta}$, and the Euclidian norm $\left|\overset{\bullet}{\mathbf{L}}(\mathbf{u};\boldsymbol{\theta})\right|$ is dominated by a $dC_{\boldsymbol{\theta}}$-integrable function $h(\mathbf{u})$.

- $[H.3]$ The $r \times r$ matrix $A_0 := \int_{[0,1]^d} \overset{\bullet}{\mathbf{L}}(\mathbf{u};\boldsymbol{\theta}_0)\, dC_{\boldsymbol{\theta}_0}(\mathbf{u})$ is nonsingular.

Theorem 1 *Assume that the condition (5) and assumptions $[H.1]-[H.3]$ hold. Then with probability tending to one as $n \to \infty$, there exists a solution $\widehat{\boldsymbol{\theta}}^{CM}$ to the system (7) which converges to $\boldsymbol{\theta}_0$. Moreover*

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}^{CM} - \boldsymbol{\theta}_0\right) \overset{\mathcal{D}}{\to} \mathcal{N}\left(\mathbf{0}, A_0^{-1} D_0 \left(A_0^{-1}\right)^T\right), \text{ as } n \to \infty,$$

where $D_0 := var\{\mathbf{L}(\boldsymbol{\xi};\boldsymbol{\theta}_0) + \mathbf{V}(\boldsymbol{\xi};\boldsymbol{\theta}_0)\}$ and $\mathbf{V}(\boldsymbol{\xi};\boldsymbol{\theta}_0) = (V_1(\boldsymbol{\xi};\boldsymbol{\theta}_0),...,V_r(\boldsymbol{\xi};\boldsymbol{\theta}_0))$ with

$$V_k(\boldsymbol{\xi};\boldsymbol{\theta}_0) := \sum_{j=1}^d \int_{[0,1]^d} \frac{\partial (C_{\boldsymbol{\theta}_0}(\mathbf{u}))^k}{\partial u_j} \left(\mathbf{1}\{\xi_j \le u_j\} - u_j\right) dC_{\boldsymbol{\theta}_0}(\mathbf{u}),\ k = 1,...,r,$$

where $\boldsymbol{\xi} := (\xi_1,...,\xi_d)$ is a $(0,1)^d$-uniform random vector with joint df C_{θ_0}.

Remark 1 *The asymptotic variance $A_0^{-1} D_0 \left(A_0^{-1}\right)^T$ may be consistently estimated by the sample variance of $\widehat{A}_i^{-1} \widehat{D}_i \left(\widehat{A}_i^{-1}\right)^T$ where*

$$\widehat{A}_i := \int_{[0,1]^d} \overset{\bullet}{\mathbf{L}}\left(\mathbf{u};\widehat{\boldsymbol{\theta}}^{CM}\right) dC_{\widehat{\boldsymbol{\theta}}^{CM}}(\mathbf{u}) \text{ and } \widehat{D}_i := \mathbf{L}\left(\widehat{\mathbf{U}}_i;\widehat{\boldsymbol{\theta}}^{CM}\right) + \mathbf{V}\left(\widehat{\mathbf{U}}_i;\widehat{\boldsymbol{\theta}}^{CM}\right),\ i = 1,...,n,$$

as is done, in Genest et al. (1995) and Tsukahara (2005) in the case of PML's estimator and Z-estimator respectively.

3 Application: Archimedean copula models

As application to the CM estimation method, we consider the Archimedean copula family defined by $C(\mathbf{u}) = \varphi^{-1}\left(\sum_{j=1}^{d}\varphi(u_j)\right)$, where $\varphi : [0,1] \to \mathbb{R}$ is a twice differentiable function called the generator, satisfying: $\varphi(1) = 0$, $\varphi'(x) < 0$, $\varphi''(x) \geq 0$ for any $x \in (0,1)$. The notation φ^{-1} stands for the inverse function of φ. Archimedean copulas are easy to construct and have nice properties. A variety of known copula families belong to this class, including the models of Gumbel, Clayton, Frank, ... (see, Table 4.1 in Nelsen, 2006, page 116). Let $\mathbf{K}_C(s) := P(C(\mathbf{U}) \leq s)$, $s \in [0,1]$, be the df of rv $C(\mathbf{U})$, then equation 2 may be rewritten into:

$$M_k(C) = \int_0^1 s^k d\mathbf{K}_C(s),\ k = 1, 2, \ldots.$$

Suppose now, for unknown $\boldsymbol{\theta} \in \mathcal{O}$, that $\varphi = \varphi_\theta$, it follows that $C = C_{\boldsymbol{\theta}}$, $\mathbf{K}_C = \mathbf{K}_{\boldsymbol{\theta}}$ and $M_k(C) = M_k(\boldsymbol{\theta})$, that is

$$M_k(\boldsymbol{\theta}) = \int_0^1 s^k d\mathbf{K}_\theta(s),\ k = 1, 2, \ldots,$$

Notice that, one of the nice properties of Archimedean copula is that the df $\mathbf{K}_C$ of $C(\mathbf{U})$ may be represented in terms of the first and second derivatives of the generator. Indeed from Theorem 4.3.4 in Nelsen, 2006, page 127, for any $s \in [0,1]$, $\mathbf{K}_{\boldsymbol{\theta}}(s) = s - \varphi_{\boldsymbol{\theta}}(s)/\varphi'_{\boldsymbol{\theta}}(s)$, it follows that the corresponding density is $\mathbf{K}'_{\boldsymbol{\theta}}(s) = \varphi''_{\boldsymbol{\theta}}(s)\varphi_{\boldsymbol{\theta}}(s)/(\varphi'_{\boldsymbol{\theta}}(s))^2$. Therefore the kth CM, defined in (2), may be rewritten into

$$M_k(\boldsymbol{\theta}) = \int_0^1 s^k \frac{\varphi''_{\boldsymbol{\theta}}(s)\varphi_{\boldsymbol{\theta}}(s)}{\left(\varphi'_{\boldsymbol{\theta}}(s)\right)^2} ds,\ k = 1, 2, \ldots \tag{9}$$

In terms of $\mathbf{K}_{\boldsymbol{\theta}}$, the assumptions $[H.1] - [H.3]$ and Theorem 1 may be rephrased, respectively, to $[H.1'] - [H.3']$ and Theorem 2 below. For convenience, we set

$$\mathcal{L}(t;\boldsymbol{\theta}) = (\mathcal{L}_1(t;\boldsymbol{\theta}), \ldots, \mathcal{L}_r(t;\boldsymbol{\theta})) \text{ with } \mathcal{L}_k(t;\boldsymbol{\theta}) := t^k - M_k(\boldsymbol{\theta}).$$

- $[H.1']$ $\boldsymbol{\theta}_0 \in \mathcal{O} \subset \mathbb{R}^r$ is the unique zero of the mapping $\boldsymbol{\theta} \to \int_0^1 \mathcal{L}(\mathbf{t};\boldsymbol{\theta})\, d\mathbf{K}_{\theta_0}(t)$ that is defined from $\mathcal{O}$ to $\mathbb{R}^r$.

- $[H.2']$ $\mathcal{L}(\cdot;\boldsymbol{\theta})$ is differentiable with respect to $\boldsymbol{\theta}$ with the Jacobian matrix denoted by

$$\overset{\bullet}{\mathcal{L}}(t;\boldsymbol{\theta}) := \left[\frac{\partial M_k(\boldsymbol{\theta})}{\partial \theta_\ell}\right]_{r\times r},$$

  $\overset{\bullet}{\mathcal{L}}(t;\boldsymbol{\theta})$ is continuous both in t and $\boldsymbol{\theta}$, and the Euclidian norm $\left|\overset{\bullet}{\mathcal{L}}(t;\boldsymbol{\theta})\right|$ is dominated by a $d\mathbf{K}_{\boldsymbol{\theta}}$-integrable function $h(t)$.

- $[H.3']$ The $r\times r$ matrix $\mathcal{A}_0 := \int_0^1 \overset{\bullet}{\mathcal{L}}(t;\boldsymbol{\theta}_0)\, d\mathbf{K}_{\boldsymbol{\theta}_0}(t)$ is nonsingular.

Theorem 2 *Assume that the condition (5) and assumptions $[H.1'] - [H.3']$ hold. Then with probability tending to one as $n\to\infty$, there exists a solution $\widehat{\boldsymbol{\theta}}^{CM}$ to the system (7) which converges to $\boldsymbol{\theta}_0$. Moreover*

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}^{CM} - \boldsymbol{\theta}_0\right) \overset{\mathcal{D}}{\to} \mathcal{N}\left(\mathbf{0}_{\mathbb{R}^r}, \mathcal{A}_0^{-1}\mathcal{D}_0\left(\mathcal{A}_0^{-1}\right)^T\right), \ \text{as } n\to\infty,$$

where

$$\mathcal{D}_0 := var\left\{\mathcal{L}(\xi;\boldsymbol{\theta}_0) + \int_0^1 \mathbf{g}(t)\left(\mathbf{1}\{\xi\leq t\} - t\right) d\mathbf{K}_{\boldsymbol{\theta}_0}(t)\right\},$$

where ξ is a $(0,1)$-uniform rv and $\mathbf{g}(t) := \left(kt^{k-1}\right)_{k=1,r}$ is $r-$dimensional vector.

3.1 Illustrative example

The Gumbel family is an Archimedean copula defined by

$$C_\beta(\mathbf{u}) = \exp\left(-\left(\sum_{j=1}^d (-\ln u_j)^\beta\right)^{1/\beta}\right), \ \beta\geq 1,$$

with generator $\varphi_\beta(t) = (-\ln t)^\beta$, $\beta\geq 1$. For the sake of flexibility in data modelling, it is better to use the multi-parameters copula models than the one-parameter ones. To have a copula with more than one parameter, we use, for instance, the transformed (or distorted) copula defined by

$$C_\Gamma(\mathbf{u}) = \Gamma^{-1}\left(C\left(\Gamma(u_1),...,\Gamma(u_d)\right)\right),$$

where $\Gamma:[0,1]\to[0,1]$ is a continuous, concave and strictly increasing function with $\Gamma(0)=0$ and $\Gamma(1)=1$. As an example, suppose that $\Gamma=\Gamma_\alpha$, with $\Gamma_\alpha(t) = \exp\left(t^{-\alpha}-1\right)$, $\alpha>0$ and consider the Gumbel copula C_β, then the transformed copula $C_{\alpha,\beta}(\mathbf{u}) = \Gamma_\alpha^{-1}\left(C_\beta\left(\Gamma_\alpha(u_1),...,\Gamma_\alpha(u_d)\right)\right)$ is given by

$$C_{\alpha,\beta}(\mathbf{u}) := \left(\left(\sum_{j=1}^d \left(u_j^{-\alpha}-1\right)^\beta\right)^{1/\beta} + 1\right)^{-1/\alpha}, \tag{10}$$

which is also a two-parameter Archimedean copula with generator $\varphi_{\alpha,\beta}(t) := \left(t^{-\alpha}-1\right)^\beta$. Note that $C_{\alpha,\beta}$ verifies the concordance ordering condition of copulas (5) (see, Nelsen, 2006, page, 145). By an elementary calculation we get the kth CM:

$$M_k(\alpha,\beta) = \frac{(k+1)\beta + \alpha\beta - k}{(k+1)^2\beta + (k+1)\alpha\beta}.$$

In particular the first two CM's are

$$M_1(\alpha,\beta) := \frac{2\beta+\alpha\beta-1}{4\beta+2\alpha\beta} \text{ and } M_2(\alpha,\beta) := \frac{3\beta+\alpha\beta-2}{9\beta+3\alpha\beta}.$$

Let $(\mathbf{X}_1,...,\mathbf{X}_n)$ be a sample of random vector $\mathbf{X}=(X_1,...,X_d)$, then the CM estimator $\left(\widehat{\alpha},\widehat{\beta}\right)$ of (α,β) is the unique solution of the system

$$\begin{cases} M_1(\alpha,\beta)=\widehat{M}_1 \\ M_2(\alpha,\beta)=\widehat{M}_2. \end{cases}$$

That is

$$\widehat{\alpha}=\frac{8\widehat{M}_1-9\widehat{M}_2-1}{1-4\widehat{M}_1+3\widehat{M}_2},\ \widehat{\beta}=\frac{1-4\widehat{M}_1+3\widehat{M}_2}{\left(1-2\widehat{M}_1\right)\left(1-3\widehat{M}_2\right)}. \tag{11}$$

4 Simulation study

First notice that all numerical computations are performed on a personal computer with a microprocessor speed of 2.4 GHz. To evaluate and compare the performance of CM's estimator with the PML and $(\tau,\rho)-$inversion estimators, a simulation study is carried out by considering the transformed bivariate Gumbel copula family $C_{\alpha,\beta}$ defined above. The evaluation of the performance is based on the bias and the RMSE defined as follows:

$$\text{Bias}=\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i-\theta\right),\ \text{RMSE}=\left(\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i-\theta\right)^2\right)^{1/2}, \tag{12}$$

where $\hat{\theta}_i$ is an estimator (from the considered method) of θ from the ith samples for N generated samples from the underlying copula. In both parts, we selected $N=1000$. The procedure outlined in Section (2) is repeated for different sample sizes n with $n=30,50,100,200$ to assess the improvement in the bias and RMSE of the estimators with increasing sample size. Furthermore, the simulation procedure is repeated for a large set of parameters of the true copula $C_{\alpha,\beta}$. For each sample, we solve system (11) to obtain the CM-estimator $\left(\widehat{\alpha}_i,\widehat{\beta}_i\right)$ of (α,β) for $i=1,...,N$, and the estimators $\widehat{\alpha}$ and $\widehat{\beta}$ are given by $\widehat{\alpha}=\frac{1}{N}\sum_{i=1}^{N}\widehat{\alpha}_i$ and $\widehat{\beta}=\frac{1}{N}\sum_{i=1}^{N}\widehat{\beta}_i$. The choice of the true values of the parameter (α,β) have to be meaningful, in the sense that each couple of parameters assigns a value of one of the dependence measure, that is weak, moderate and strong dependence. In other words, if we consider Kendall's τ as a dependence measure, then we should select values for copula parameters that correspond to specified values of τ by means of the equation $\tau(\alpha,\beta)=4\int_{[0,1]^2}C_{\alpha,\beta}(u_1,u_2)\,dC_{\alpha,\beta}(u_1,u_2)-1$. The selected values of the true parameters are summarized in the following table:

τ	α	β
0.01	0.1	1.059
0.2	0.2	1.137
0.5	0.5	1.6
0.8	0.9	3.45

Table 1: The true parameters of transformed Gumbel copula used for the simulation study.

	$\tau = 0.01$				$\tau = 0.5$				$\tau = 0.8$				
	$\alpha = 0.1$		$\beta = 1.059$		$\alpha = 0.5$		$\beta = 1.6$		$\alpha = 0.9$		$\beta = 3.45$		
n	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	CPU
30	-0.081	0.330	0.032	0.180	-0.051	0.654	0.039	0.481	-0.073	0.907	-0.372	1.130	22.013 sec
50	-0.046	0.253	0.022	0.139	-0.043	0.487	0.018	0.367	-0.032	0.723	0.261	0.916	49.563 sec
100	-0.026	0.173	0.009	0.097	-0.023	0.350	0.012	0.262	-0.027	0.548	-0.089	0.733	2.789 mins
200	-0.011	0.122	0.002	0.064	-0.009	0.243	0.006	0.180	0.003	0.386	-0.056	0.506	10.370 mins
500	-0.005	0.075	0.000	0.041	-0.007	0.155	0.003	0.117	-0.007	0.241	-0.026	0.323	1.035 hours

Table 2: Bias and RMSE of the CLM estimator of two-parameters transformed Gumbel copula.

	$\tau = 0.01$				$\tau = 0.2$				$\tau = 0.5$				$\tau = 0.8$			
	$\alpha = 0.1$		$\beta = 1.059$		$\alpha = 0.2$		$\beta = 1.137$		$\alpha = 0.5$		$\beta = 1.6$		$\alpha = 0.9$		$\beta = 3.45$	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
$n = 30$																
CM	-0.082	0.313	0.039	0.190	-0.064	0.642	0.033	0.486	-0.068	0.567	0.074	0.466	-0.072	0.955	-0.359	1.128
PML	-0.067	0.068	-0.485	0.486	-0.117	0.128	-0.561	0.568	0.078	0.394	-0.426	0.594	-0.043	0.421	0.253	1.027
ρ-τ	1.236	2.895	-0.213	1.775	1.101	2.984	-0.913	1.641	-0.556	2.312	-0.975	1.142	-0.439	0.691	0.919	1.039
$n = 50$																
CM	-0.046	0.245	-0.021	0.141	-0.052	0.468	0.029	0.357	-0.037	0.506	0.015	0.364	-0.033	0.732	0.206	0.892
PML	-0.060	0.062	-0.478	0.482	0.102	0.112	-0.516	0.526	-0.072	0.240	-0.472	0.556	0.025	0.315	0.330	0.772
ρ-τ	1.115	2.033	-0.289	1.378	1.035	2.537	-0.354	1.341	-0.478	2.110	-0.952	1.021	-0.392	0.508	0.801	0.991
$n = 100$																
CM	-0.022	0.171	0.009	0.100	-0.019	0.342	0.016	0.258	0.029	0.367	-0.005	0.257	-0.025	0.551	0.155	0.704
PML	-0.058	0.059	-0.483	0.485	-0.109	0.113	-0.524	0.528	-0.071	0.158	-0.403	0.469	-0.023	0.167	-0.017	0.237
ρ-τ	0.973	1.220	-0.176	1.273	0.923	2.335	-0.340	1.457	-0.365	1.114	-0.852	1.001	-0.255	0.397	0.708	0.686
$n = 200$																
CM	-0.016	0.122	0.002	0.064	-0.011	0.244	0.001	0.383	-0.014	0.245	0.005	0.180	-0.001	0.396	-0.060	0.527
PML	-0.041	0.062	-0.503	0.505	-0.099	0.102	-0.514	0.516	-0.050	0.116	-0.333	0.415	-0.059	0.144	0.038	0.400
ρ-τ	0.874	1.025	-0.235	1.215	-0.890	2.414	-0.330	1.041	-0.321	0.997	-0.786	0.988	-0.239	0.331	0.580	0.625

Table 3: Bias and RMSE of the CM, PML and τ-ρ estimators of two-parameters transformed Gumbel copula.

5 Comments and conclusions

From Table (4) , we conclude that by considering three dependence cases: weak $(\tau = 0.01)$, moderate $(\tau = 0.5)$ and strong $(\tau = 0.8)$, the performance, in terms of bias and RMSE, of the CM based estimation is well justified. In each case, for small and large samples, the bias and RMSE are sufficiently small. Moreover, in time-consuming point of view, we observe that for a sample size $n = 30$ and for $N = 1000$ replications, the central processing unit (CPU) time to process CM's method took 22.013 seconds, which is relatively small. For one replication $N = 1$, the CPU time (in seconds) for different sample sizes are summarized as follows: $(n, CPU) = (30, 0.437)$, $(100, 0.312)$, $(200, 0.844)$, $(500, 3.922)$. Table (4) shows that both the PML and the CM based estimation perform better than the (τ, ρ)-inversion method. However, in weak dependence case $\tau = 0.01$, the CM method provides better results than the PML one, mainly when the sample size increases. On the other hand, it is worth mentioning that our method is quick with respect to the PML one. The main advantage of our method is that it provides estimators with explicit forms, as far as Archimedean copula models are concerned. This is not the case of the other methods which require numerical procedures leading to eventual problems in execution time and inaccuracy issues. In conclusion, the CM based estimation method performs well for the chosen model. Furthermore, its usefulness in the weak dependence case particularly makes it a good candidate for statistical tests of independence.

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A Appendix

A.1 Proof of Theorem 1

By considering CM's estimator as a RAZ-estimator (van der Vaart, 1998, page 41), a straight application of Theorem 1 in Tsukahara (2005) leads to the consistency and asymptotic normality of the considered estimator. Indeed, the existence of a sequence of consistent roots $\widehat{\boldsymbol{\theta}}^{CM}$ to (4), may be verified by using similar arguments as the proof of Theorem 1 in Tsukahara (2005). More precisely, we have to check only the conditions in Theorem A.10.2 in Bickel *et al.* (1993). Indeed, first recall 8 and set

$$\Phi\left(\boldsymbol{\theta}\right) := \int_{\mathbb{I}^d} \mathbf{L}\left(\mathbf{u};\boldsymbol{\theta}\right) dC_{\boldsymbol{\theta}_0}\left(\mathbf{u}\right), \quad \text{and } \Phi_n\left(\boldsymbol{\theta}\right) := n^{-1}\sum_{i=1}^{n} \mathbf{L}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}\right),$$

where $\widehat{\mathbf{U}}_i = \left(F_{1n}\left(X_{1i}\right), ..., F_{dn}\left(X_{di}\right)\right)$, with $\left(X_{j1}, ..., X_{jn}\right)$ is a given random sample from the r.v. X_j. In view of assumption $[H.2]$ the following derivatives exist

$$\overset{\bullet}{\Phi}\left(\boldsymbol{\theta}\right) = \int_{\mathbb{I}^d} \overset{\bullet}{\mathbf{L}}\left(\mathbf{u};\boldsymbol{\theta}\right) dC_{\boldsymbol{\theta}_0}\left(\mathbf{u}\right), \ \overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}\right) = \frac{1}{n}\sum_{i=1}^{n} \overset{\bullet}{\mathbf{L}}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}\right).$$

Next, we verify that

$$\sup\left\{\left|\overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}\right) - \overset{\bullet}{\Phi}\left(\boldsymbol{\theta}\right)\right| : \left|\boldsymbol{\theta} - \boldsymbol{\theta}_0\right| < \epsilon_n\right\} \overset{\mathbf{P}}{\to} 0, \ \text{as } n \to \infty, \tag{13}$$

for any real sequence $\epsilon_n \to 0$. Indeed, since $\overset{\bullet}{\mathbf{L}}$ is continuous in $\boldsymbol{\theta}$, then

$$\sup\left\{\left|\overset{\bullet}{\mathbf{L}}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}\right) - \overset{\bullet}{\mathbf{L}}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}_0\right)\right| : \left|\boldsymbol{\theta} - \boldsymbol{\theta}_0\right| < \epsilon_n\right\} = o_{\mathbf{P}}\left(1\right), \ i = 1, ..., n,$$

and the fact that

$$\left|\overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}\right) - \overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}_0\right)\right| \leq \frac{1}{n}\sum_{i=1}^{n}\left|\overset{\bullet}{\mathbf{L}}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}\right) - \overset{\bullet}{\mathbf{L}}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}_0\right)\right|.$$

implies

$$\sup\left\{\left|\overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}\right) - \overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}_0\right)\right| : \left|\boldsymbol{\theta} - \boldsymbol{\theta}_0\right| < \epsilon_n\right\} \overset{\mathbf{P}}{\to} 0, \ \text{as } n \to \infty. \tag{14}$$

On the other hand, in view of the law of the large number, we have

$$\frac{1}{n}\sum_{i=1}^{n} \overset{\bullet}{\mathbf{L}}\left(\mathbf{U}_i;\boldsymbol{\theta}_0\right) \overset{\mathbf{P}}{\to} \overset{\bullet}{\Phi}\left(\boldsymbol{\theta}_0\right), \ \text{as } n \to \infty,$$

where $\mathbf{U}_i = \left\{F_j\left(X_{ji}\right)\right\}_{j=1,d}$. Moreover, in view of the continuity of function $\overset{\bullet}{\mathbf{L}}$ in $\mathbf{u}$ and Glivenko-Cantelli, that is

$$\sup_{x_j}\left|F_{jn}\left(x_j\right) - F_j\left(x_j\right)\right| \to 0, \ j = 1, ..., d, \ \text{almost surely, as } n \to \infty,$$

we have

$$n^{-1}\sum_{i=1}^{n}\left|\overset{\bullet}{\mathbf{L}}\left(\widehat{\mathbf{U}}_i;\boldsymbol{\theta}_0\right) - \overset{\bullet}{\mathbf{L}}\left(\mathbf{U}_i;\boldsymbol{\theta}_0\right)\right| \overset{\mathbf{P}}{\to} 0,$$

it follows that $\left|\overset{\bullet}{\Phi}_n\left(\boldsymbol{\theta}_0\right) - \overset{\bullet}{\Phi}\left(\boldsymbol{\theta}_0\right)\right| \overset{\mathbf{P}}{\to} 0$, which together with (14), implies (13). Conditions (MG0) and (MG3) in Theorem A.10.2 in Bickel *et al.* (1993) are trivially satisfied by our assumptions

$[H1]-[H3]$. In view of the general theorem for Z-estimators (see, van der Vaart and Wellner, 1996, Theorem 3.3.1), it remains to prove that $\sqrt{n}\left(\overset{\bullet}{\Phi}_n - \overset{\bullet}{\Phi}\right)(\boldsymbol{\theta}_0)$ converges in law to the appropriate limit. But this follows from Proposition 3 in Tsukahara (2005), which achieves the proof of Theorem 1. □

A.2 Proof Theorem 2

The proof of Theorem 2 is straightforward by using similar argument as the proof of Theorem 1, therefore the details are omitted. □