On Log-concavity of the Generalized Marcum Q Function

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***Abstract*—It is shown that, if $\nu \geq 1/2$ then the generalized Marcum Q function $Q_\nu(a, b)$ is log-concave in $b \in [0, \infty)$. This proves a conjecture of Sun, Baricz and Zhou (2010). We also point out relevant results in the statistics literature.**

***Index Terms*—increasing failure rate; log-concavity; modified Bessel function; noncentral chi square.**

I. Introduction

The generalized Marcum Q function [14] has important applications in radar detection and communications over fading channels and has received much attention; see, e.g., [3], [8], [10], [13]-[17] and [19]-[21]. It is defined as

$$Q_\nu(a, b) = \int_b^\infty \frac{t^\nu}{a^{\nu-1}} \exp\left(-\frac{t^2 + a^2}{2}\right) I_{\nu-1}(at)\, \mathrm{d}t \qquad (1)$$

where $\nu > 0,\ a, b \geq 0$ and I_ν denotes the modified Bessel function of the first kind of order ν defined by the series [1] (9.6.10)

$$I_\nu(t) = \sum_{k=0}^\infty \frac{(t/2)^{2k+\nu}}{k!\Gamma(\nu + k + 1)}.$$

($Q_\nu(0, b)$ is defined by taking $a \downarrow 0$.) Recently, Sun, Baricz and Zhou [21] have studied the monotonicity, log-concavity, and tight bounds of $Q_\nu(a, b)$ in great detail. We are concerned with log-concavity, which has intrinsic interest, and can help establish useful bounds; see [21] and the references therein for the large literature in information theory and communications on numerical calculations of $Q_\nu(a, b)$.

This note resolves some of the conjectures made by [21]. We also point out relevant literature in statistics on both theoretical properties and numerical computation of $Q_\nu(a, b)$. Our Theorem 1 proves Conjecture 1 of [21].

Theorem 1: The function $Q_\nu(a, b)$ is log-concave in $b \in [0, \infty)$ for all $a \geq 0$ if and only if $\nu \geq 1/2$.

A sufficient condition for log-concavity of an integral like (1) is that the integrand is log-concave in t. Proposition 1 and Theorem 2 take this approach.

Proposition 1: The integrand in (1) is log-concave in $t \in (0, \infty)$ for all $\nu \geq 1/2$ if and only if $0 \leq a \leq 1$.

Theorem 2: The integrand in (1) is log-concave in $t \in (0, \infty)$ for all $a \geq 0$ if and only if $\nu \geq \nu_0$ where $\nu_0 \approx 0.78449776$ is the unique solution of the equation

$$\frac{I_\nu(\sqrt{5 - 2\nu})}{I_{\nu-1}(\sqrt{5 - 2\nu})} = \frac{3 - 2\nu}{\sqrt{5 - 2\nu}}$$

in the interval $\nu \in (1/2, 3/2)$.

Note the difference between Proposition 1 and Theorem 2: the former gives a criterion for log-concavity in t for all $\nu \geq 1/2$ whereas the latter gives one for all $a \geq 0$. From Proposition 1 and Theorem 2 we obtain Corollary 1, which confirms part of Conjecture 2 of [21].

Corollary 1: The function $1 - Q_\nu(a, b)$ is log-concave in $b \in [0, \infty)$, if either (i) $\nu \geq 1/2$ and $0 \leq a \leq 1$, or (ii) $\nu \geq \nu_0$ as in Theorem 2.

The case of $Q_1(a, b)$ (Marcum's original Q function) is especially interesting. If $\nu = 1$ then the integrand in (1) is the probability density function (PDF) of a Rice distribution, $Q_1(a, b)$ being the corresponding tail probability, or survival function. Therefore Theorem 2 yields

Corollary 2: The probability density function, cumulative distribution function (CDF), and survival function of a Rice distribution are all log-concave.

In general, let X be a noncentral χ^2 random variable with 2ν degrees of freedom and noncentrality parameter a^2. Then

$$Q_\nu(a, b) = \Pr\left(\sqrt{X} > b\right).$$

Equivalently, $1 - Q_\nu(\sqrt{a}, \sqrt{b})$ is the CDF of a noncentral χ^2 random variable with 2ν degrees of freedom and noncentrality parameter a. The noncentral χ^2 distribution plays an important role in statistical hypothesis testing and has been extensively studied. We mention [6], [12] on numerical computation and [7], [9], [18] on theoretical properties. Its CDF, and hence $Q_\nu(a, b)$, can be routinely calculated (e.g., using `pchisq()` in the R package).

Concerning theoretical properties, Finner and Roters [7] (see also [5]) have obtained the following results using tools from total positivity [11].

Theorem 3 ([7], Theorems 3.4, 3.9; Remark 3.6): The function $1 - Q_\nu(\sqrt{a}, \sqrt{b})$ is log-concave

- in $b \in [0, \infty)$ for $\nu > 0,\ a \geq 0$;
- in $\nu > 0$ for $a,\ b \geq 0$;
- in $a \geq 0$ for $\nu > 0,\ b \geq 0$.

The function $Q_\nu(\sqrt{a}, \sqrt{b})$ is log-concave

- in $b \in [0, \infty)$ for $\nu \geq 1,\ a \geq 0$;
- in $\nu \in [1/2, \infty)$ for $a,\ b \geq 0$;
- in $a \geq 0$ for $\nu > 0,\ b \geq 0$.

Theorem 3 and Corollary 1 cover several results of [21], including part of their Conjectures 2 and 3 (see also [20]). The parts of these conjectures that remain open are

- $1 - Q_\nu(a, b)$ is log-concave in $b \in [0, \infty)$ for $\nu \in [1/2, \nu_0)$ and $a > 1$;
- $Q_\nu(a, b)$ is log-concave in $\nu \in (0, 1/2]$ for $a,\ b \geq 0$.

In Section II we prove Theorems 1, 2 and Proposition 1. The proof of Theorem 1 uses a general technique which may be helpful in related problems. The proof of Theorem 2 relies partly on numerical verification as theoretical analysis appears quite cumbersome.

II. Proof of Main Results

The following observation, which is of independent interest, is key to our proof of Theorem 1.

Lemma 1: Let $f(t)$ be a probability density function on $\mathbf{R} \equiv (-\infty, \infty)$. Assume (i) $f(t)$ is unimodal, i.e., there exists $t_0 \in \mathbf{R}$ such that $f(t)$ increases on $(-\infty, t_0]$ and decreases

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on $[t_0,\infty)$; (ii) $f(t_0-)\le f(t_0+)$; (iii) $f(t)$ is log-concave in the declining phase $t\in(t_0,\infty)$. Then the survival function $\bar F(b)\equiv\int_b^\infty f(t)\,\mathrm{d}t$ is log-concave in $b\in\mathbf{R}$.

Proof: Assumption (iii) implies that $\bar F(b)$ is log-concave in $b\in[t_0,\infty)$. Because $f(t)$ increases on $(-\infty,t_0]$, we know $\bar F(b)$ is concave and hence log-concave on $(-\infty,t_0]$. By Assumption (ii) we have

$$\bar F'(t_0-)=-f(t_0-)\ge -f(t_0+)=\bar F'(t_0+).$$

Hence $\bar F(b)$ is log-concave in $b\in\mathbf{R}$ overall. ■

Remark 1. A distribution whose survival function is log-concave is said to have an increasing failure rate (IFR) [4]. Distributions with IFR form an important class in reliability and survival analysis. Lemma 1 provides a simple sufficient condition for IFR distributions.

Henceforth let $f(t)$ be the integrand in (1) for $t>0$. Equivalently, $f(t)$ is the density function of a noncentral χ random variable with 2ν degrees of freedom. Define

$$r_\nu(t)=\frac{I_\nu(t)}{I_{\nu-1}(t)}. \tag{2}$$

We use $r'_\nu(t)$ to denote the derivative with respect to t.

Lemma 2: If $\nu\ge 1/2$ then $f'(t)/(tf(t))$ decreases in $t\in(0,\infty)$.

Proof: Let us assume $\nu>1/2$ and $a>0$. The boundary cases follow by taking limits. Direct calculation yields

$$\begin{aligned}\frac{f'(t)}{tf(t)}&=\frac{\nu}{t^2}-1+\frac{aI'_{\nu-1}(at)}{tI_{\nu-1}(at)}\\ &=\frac{2\nu-1}{t^2}-1+\frac{ar_\nu(at)}{t}\end{aligned} \tag{3}$$

where (3) uses (2) and the formula [1] (9.6.26)

$$I'_{\nu-1}(t)=I_\nu(t)+\frac{\nu-1}{t}I_{\nu-1}(t). \tag{4}$$

Since $(2\nu-1)/t^2$ decreases in t, we only need to show that $r_\nu(t)/t$ decreases in t. We may use the integral formula of [1] (9.6.18) and obtain

$$\frac{r_\nu(t)}{t}=\frac{\int_0^1(1-s^2)g(s,t)\,\mathrm{d}s}{(2\nu-1)\int_0^1 g(s,t)\,\mathrm{d}s}$$

where

$$g(s,t)=(1-s^2)^{\nu-3/2}\cosh(ts).$$

As can be easily verified, if $0<t_1<t_2$ then $g(s,t_2)/g(s,t_1)$ increases in $s\in(0,1)$. That is, $g(s,t)$ is TP_2 [11]. Since $1-s^2$ decreases in $s\in(0,1)$, by Proposition 3.1 in Chapter 1 of [11], the ratio $\int_0^1(1-s^2)g(s,t)\,\mathrm{d}s/\int_0^1 g(s,t)\,\mathrm{d}s$ decreases in $t\in(0,\infty)$, as required. ■

Proof of Theorem 1: Let us assume $\nu>1/2$ and show log-concavity. By Lemma 2, either (i) $f'(t)<0$ for all $t\in(0,\infty)$ or (ii) there exists some $t_0\in(0,\infty)$ such that $f'(t)\ge 0$ when $t<t_0$ and $f'(t)\le 0$ when $t>t_0$. (Since $\int_0^\infty f(t)\,\mathrm{d}t=Q_\nu(a,0)=1$, it cannot happen that $f'(t)>0$ for all $t\in(0,\infty)$.) In either case $f(t)$ satisfies Assumptions (i) and (ii) of Lemma 1 ($f(t)\equiv 0$ for $t\le 0$). Let us consider Case (ii); the same argument applies to Case (i). For $t\in(t_0,\infty)$ we have $f'(t)\le 0$, and hence

$$\begin{aligned}\frac{1}{t}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log f(t)&\le\frac{1}{t}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log f(t)-\frac{f'(t)}{t^2f(t)}\\ &=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\frac{\mathrm{d}}{\mathrm{d}t}\log f(t)\right)\le 0\end{aligned}$$

where the last step holds by Lemma 2. Thus $f(t)$ is log-concave in $t\in(t_0,\infty)$ and Assumption (iii) of Lemma 1 is satisfied. We conclude that $Q_\nu(a,b)=\int_b^\infty f(t)\,\mathrm{d}t$ is log-concave in $b\in[0,\infty)$.

It remains to show that, if $Q_\nu(a,b)$ is log-concave in $b\in[0,\infty)$ for all $a\ge 0$, then we must have $\nu\ge 1/2$. Let us consider $a=0$. We have

$$Q_\nu(0,b)=1-\frac{1}{2^\nu\Gamma(\nu)}\int_0^{b^2}t^{\nu-1}e^{-t/2}\,\mathrm{d}t.$$

As $b\downarrow 0$, it is easy to see that $\log Q_\nu(0,b)$ behaves like

$$\log\left(1-Cb^{2\nu}+o(b^{2\nu})\right)=-Cb^{2\nu}+o(b^{2\nu})$$

with $C=2^{-\nu}/\Gamma(\nu+1)$. Hence, if $\nu<1/2$ then $Q_\nu(0,b)$ is no longer log-concave for b near zero. It follows that the $1/2$ in Theorem 1 is the best possible. ■

Proof of Proposition 1: Using (3) we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log f(t)=-\frac{2\nu-1}{t^2}-1+a^2r'_\nu(at). \tag{5}$$

However,

$$\begin{aligned}r'_\nu(t)&=\frac{I'_\nu(t)}{I_{\nu-1}(t)}-\frac{I_\nu(t)I'_{\nu-1}(t)}{I^2_{\nu-1}(t)}\\ &=1-\frac{2\nu-1}{t}r_\nu(t)-r^2_\nu(t)\end{aligned} \tag{6}$$

where (6) holds by applying (2), (4) and the recursion [1] (9.6.26)

$$I_{\nu+1}(t)=I_{\nu-1}(t)-\frac{2\nu}{t}I_\nu(t).$$

If $\nu\ge 1/2$ and $0<a\le 1$ then $r'_\nu(at)\le 1$ by (6), and we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\log f(t)\le a^2-1\le 0.$$

Hence $f(t)$ is log-concave in $t\in(0,\infty)$.

To show the converse, suppose $f(t)$ is log-concave in t for all $\nu\ge 1/2$. Consider $\nu=1/2$. As $t\downarrow 0$ we have $r_\nu(t)\to 0$, and $\mathrm{d}^2\log f(t)/\mathrm{d}t^2\to a^2-1$. Hence we must have $a\le 1$. ■

Remark 2. For $\nu\ge 1/2$, the function $f(t)$ is log-concave in its declining phase, as shown in the proof of Theorem 1. If $a\in[0,1]$ in addition, then Proposition 1 shows that $f(t)$ is log-concave in all $t\in(0,\infty)$. For $a>1$ and $\nu\ge 1/2$, however, numerical evidence suggests that $f(t)$ may not be log-concave in its rising phase. Hence a version of Lemma 1 cannot be applied to $1-Q_\nu(a,b)$. Log-concavity of $1-Q_\nu(a,b)$ in b appears to be a difficult problem.

Let us establish two lemmas before proving Theorem 2.

Lemma 3: The function $f(t)$ is log-concave in $t\in(0,\infty)$ for all $a\ge 0$ if and only if the function

$$h_\nu(t)=1-\frac{2\nu-1}{t^2}-\frac{2\nu-1}{t}r_\nu(t)-r^2_\nu(t) \tag{7}$$

is nonpositive for $t \in (0, \infty)$.

Proof: By (6) we get

$$h_\nu(t) = r'_\nu(t) - \frac{2\nu - 1}{t^2}. \quad (8)$$

If $h_\nu(t) \le 0$ then by (5) we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log f(t) = a^2 h_\nu(at) - 1 < 0.$$

Conversely, if $f(t)$ is log-concave in $t \in (0, \infty)$ for all $a \ge 0$, then holding at constant while letting $a \to \infty$ yields $h_\nu(s) \le 0$ for each $s \in (0, \infty)$. ∎

Lemma 4: The function

$$r_\nu(\sqrt{5 - 2\nu}) - \frac{3 - 2\nu}{\sqrt{5 - 2\nu}}$$

strictly increases in $\nu \in [1/2, 3/2]$ and has a zero at $\nu_0 \approx 0.78449776$.

Proof: Although this only involves a one-variable function over a small interval, it is verified by numerical calculations, as theoretical analysis becomes complicated. The value of ν_0 is computed by a fixed point algorithm. ∎

Proof of Theorem 2: Define $h_\nu(t)$ as in (7) and ν_0 as in Lemma 4. We examine the intervals $(0, 1/2]$, $(1/2, \nu_0)$ and $[\nu_0, \infty)$ for ν in turn. If $0 < \nu \le 1/2$ then letting $t \downarrow 0$ we have $r_\nu(t) \to 0$ and $h_\nu(t) > 0$ for small t. By Lemma 3, $f(t)$ is not log-concave for all $a \ge 0$.

Let us assume $\nu > 1/2$. Differentiating (7) with respect to t and applying (8) we get

$$h'_\nu(t) = -\frac{2\nu - 1}{t^2} l_\nu(t) - \left(\frac{2\nu - 1}{t} + 2r_\nu(t)\right) h_\nu(t) \quad (9)$$

where

$$l_\nu(t) = r_\nu(t) - \frac{3 - 2\nu}{t}. \quad (10)$$

For $\nu > 1/2$ we know $r_\nu(t)$ increases from 0 to 1 as t increases from 0 to ∞ (see [2]). Hence, if $1/2 < \nu < 3/2$, then $l_\nu(t)$ strictly increases and $l_\nu(t) = 0$ has a unique solution, say at $t_1 \in (0, \infty)$. If $1/2 < \nu < \nu_0$, then by Lemma 4, $l_\nu(\sqrt{5 - 2\nu}) < 0$, and hence $t_1 > \sqrt{5 - 2\nu}$. In view of (7) and (10) we have

$$h_\nu(t_1) = 1 - \frac{2\nu - 1}{t_1^2} - \frac{2\nu - 1}{t_1}\left(\frac{3 - 2\nu}{t_1}\right) - \frac{(3 - 2\nu)^2}{t_1^2} \quad (11)$$

$$= 1 - \frac{5 - 2\nu}{t_1^2} > 0. \quad (12)$$

By Lemma 3, $f(t)$ is no longer log-concave for all $a \ge 0$.

Suppose $\nu > \nu_0$. We have $h_\nu(t) \to -\infty$ as $t \downarrow 0$ and $h_\nu(t) \to 0$ as $t \to \infty$. If $h_\nu(t)$ does become positive, then there exists a finite $t_0 > 0$ such that $h_\nu(t_0) = 0$ and $h'_\nu(t_0) \ge 0$ (at least one sign change should be from $-$ to $+$). We get $l_\nu(t_0) \le 0$ from (9). If $\nu \ge 3/2$ then (10) yields $l_\nu(t_0) \ge r_\nu(t_0) > 0$, a contradiction. Hence $h_\nu(t) \le 0$ for all $t \in (0, \infty)$ if $\nu \ge 3/2$.

Suppose $\nu_0 < \nu < 3/2$. If $l_\nu(t_0) = h'_\nu(t_0) = 0$ then we deduce $t_0 = \sqrt{5 - 2\nu}$ from (7) and (9) by a calculation similar to (11)–(12). But $l_\nu(\sqrt{5 - 2\nu}) = 0$ contradicts Lemma 4. Hence we may assume $h'_\nu(t_0) > 0$ and $l_\nu(t_0) < 0$. By Lemma 4 we have $l_\nu(\sqrt{5 - 2\nu}) > 0$. Because $l_\nu(t)$ is strictly increasing, and t_1 is the solution of $l_\nu(t) = 0$, we obtain $t_0 < t_1 < \sqrt{5 - 2\nu}$. The calculation (11)–(12) now yields $h_\nu(t_1) < 0$. Because $h_\nu(t_0) = 0,\ h'_\nu(t_0) > 0$ there exists $t_* \in (t_0, t_1)$ such that $h_\nu(t_*) = 0$ and $h'_\nu(t_*) \le 0$. By (9), we get $l_\nu(t_*) \ge 0$, which contradicts the strict monotonicity of $l_\nu(t)$ as $l_\nu(t_1) = 0$. It follows that $h_\nu(t) \le 0,\ t \in (0, \infty)$, and $f(t)$ is log-concave. Taking the limit we extend this log-concavity to $\nu = \nu_0$. ∎

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