A Simple Spatial Dependence Test Robust to Local and Distributional Misspecifications

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Abstract

It is well known that the standard Lagrange multiplier (LM) test loses its local optimality when the true non-null model is not correctly specified. In this paper, we derive a score test robust to local and distributional misspecifications for spatial error autocorrelation and spatial lag dependence. The proposed test is general enough to include several popular tests for the spatial dependence as special cases. In our framework, we find that Burridge (1980) and Anselin, Bera, Florax and Yoon (1996)'s tests are automatically robust to distributional misspecification in some special cases. The size and power performances of our proposed score tests are investigated by a Monte-Carlo simulation.

JEL code: C12; C21; R10.
Keywords: Spatial dependence; Score test; Robust test; Distribution misspecification; Local misspecification.

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1 Introduction

The spatial autoregressive model with a spatial autoregressive disturbance allows rich spatial dependence and has attracted a lot of attentions in theoretical and empirical spatial data analysis. Recently, spatial econometric models that deal with spatial dependence across economic units in cross-sectional and/or panel data have been applied to a wide range of empirical investigations, for examples, Case (1991), Kelejian and Robinson (1992), Case, Rosen and Hines (1993), Holtz-Eakin (1994), Aten (1996), Goodchild, Anselin, Appelbaum and Harthorn (2000), Kim, Phipps and Anselin (2003), Azomahou, Diebolt and Mishra (2009), and Arbia, Battisti and Vaio (2010) among others. Along with this fast growing interests many studies also developed various testing procedures for spatial dependence, for examples, Anselin (1988b), Anselin, Bera, Florax and Yoon (1996), Anselin and Kelejian (1997), Saavedra (2003) and Yang (2009) among others. Except for a few studies such as Saavedra (2003) most testing procedures are constructed based on the normality assumption of the disturbance term. However, in many situations, the normality assumption of the error term is highly likely to be violated. This deviation from the universal normality assumption could yield incorrect asymptotic inferences for the spatial dependence.

Spatial dependence can arise from many different sources. The two most frequently cited sources are spatial error autocorrelation and spatial lag dependence. Diagnostic tests for spatial dependence are needed to detect the source of spatial dependence, which has motivated a large amount of research on the spatial dependence tests. Moran (1950) proposes a seminal test, Moran's I-test, for spatial autocorrelation in the regression model. However, the test does not provide an indication of the nature of the spatial process that causes spatial autocorrelation, particularly, whether the spatial dependence is due to the autoregressive error process or omitted spatially lagged dependent variables. Burridge (1980) extends Moran's I-test based on the Lagrange multiplier (LM) principle to test the spatial error autocorrelation in the absence of spatially lagged dependent variable. Anselin (1988b) proposes a LM test for the spatial error autocorrelation in the presence of the spatially lagged dependent variable. However, the test involves a nonlinear optimization or the application of a numerical search technique. Kelejian and Prucha (1999) find that the computational complexities of Anselin's LM test could be overwhelmed if the spatial weights are not symmetric, even if the sample size is only moderate. Anselin, Bera, Florax and Yoon (1996) propose a modified score test for the spatial error autocorrelation in the presence of local misspecification to the parameter corresponding to the spatial lag dependence. Comparing to Ansenlin's LM test, the latter only requires the ordinary least squares (OLS) residuals under the null hypothesis and has little computational burden (Bera and Bilias, 2001). However, one potential problem of the above tests is that the underlying probability density may not be correctly specified, i.e., there may exist the distributional misspecification problem.

In this paper, we propose a robust score test for spatial dependence which is robust to both the local and distributional misspecifications. Local parametric misspecification arises when some nuisance parameters deviate locally from the true values. Distributional misspecification occurs when the underlying data generating process (DGP) is not correctly specified. When nuisance parameters are locally deviated from the true values, i.e., the alternative hypothesis is not correctly specified, the score statistic has a non-zero drift term in general (Davidson and Mackinnon, 1987; Saikkonen, 1989). Thus the score test statistic follows the non-central χ^2 distribution asymptotically, and therefore, it rejects the null hypothesis too often. Bera and Yoon (1993) propose a modified score test robust to local misspecification. They also show that it is asymptotically equivalent to Neyman's $C(\alpha)$ test under the local deviation from the true non-null model. On the other hand, when the underlying probability distribution is misspecified, some standard results are not valid any more. For example, the information matrix (IM) equality is invalid under distributional misspecification. Making inference without paying attention to the distributional misspecification can cause size distortion of the test statistics asymptotically. White (1982) suggests a modified

LM test by adjusting the variance of the score function. This test is based on the restricted quasi-maximum likelihood (QML) estimator, and therefore, it is robust to distributional misspecification. Bera, Bilias and Yoon (2007) propose a score test that is not only robust to local misspecification but also to distributional misspecification in the sprit of White (1982) and Bera and Yoon (1993). In the spatial econometrics literature, although the asymptotic properties of QML estimator has been extensively studied by Lee (2004), the corresponding robust score (LM) test which takes care of distributional misspecification or both local and distributional misspecifications has not been studied yet. There are some studies that suggest some other types of robust tests. However, their test statistics are based on other different estimation methods and, moreover, they do not consider local misspecification [Anselin (1990), Kelejian and Robinson (1992), Anselin and Kelejian (1997), Kelejian and Robinson (1998) and Anselin and Moreno (2003)].

We present general score tests for spatial lag dependence and spatial error autocorrelation, which are shown to be robust to both local and distributional misspecifications. These tests are constructed by adjusting the mean and variance of the usual score test statistics and have correct size asymptotically. The proposed tests can be simplified when either the error term follows the normal distribution or the nuisance parameters are estimated consistently. We discuss the property of the score test under distributional misspecification without considering nuisance parameter and show that Burridge (1980)' test is robust to distributional misspecification. As an expansion of Anselin, Bera, Florax and Yoon (1996)'s tests, we also derive robust score tests when the nuisance parameter is locally misspecified from non-zero constant. Interestingly, we show that Anselin, Bera, Florax and Yoon (1996)'s tests are robust to local and distributional misspecifications. In other words, our results support Anselin, Bera, Florax and Yoon (1996) since their tests are even robust to misspecification of the underlying distribution. These tests are easy to be implemented, and our Monte-Carlo simulation results show that they have good finite sample properties.

Recent contributions on spatial dependence tests in the presence of local deviation and/or distributional misspecification include Saavedra (2003) and Yang (2009). Instead of testing the spatial dependence, Yang (2009) proposes a modified Anselin (2001)'s LM statistic which is robust to spatial layouts and distributional misspecification in spatial error components model. Saavedra (2003) proposes the generalized method of moment (GMM) version of three conventional statistics (Wald, LM, and LR) to test spatial lag dependence in the spatial autoregressive model with autocorrelated errors. Due to the semiparametric nature of GMM estimation, these GMM based tests are free of distributional misspecification. However, the estimation procedure for GMM based tests contains the non-linear optimization so that it may have a considerable computational burden. Moreover, Saavedra (2003) shows that the finite sample performance of GMM based tests are not quite satisfactory under some circumstances.

The rest of the paper is organized as follows: Section 2 briefly summarizes the main results of score tests under misspecification. Section 3 discusses spatial dependence tests in a spatial autoregressive model with a spatial autoregressive disturbance. Section 4 develops new diagnostic score type tests robust to both local parametric and distributional misspecifications. Section 5 presents a Monte Carlo simulation to examine the size and power performance in small samples. The paper concludes in Section 6.

2 Score test under misspecification

Suppose the DGP can be fully characterized by $\theta = (\beta', \lambda, \rho)'$ and correctly specified by the probability density $f(y; \theta)$ which satisfies the regularity conditions of White (1982). For $\theta = (\beta', \lambda, \rho)'$, β is a parameter vector, and for simplicity, λ and ρ are assumed to be scalars. The null hypothesis of interest is $H_0^\lambda : \lambda = \lambda_0$. The properties of the test for H_0^λ depend on how β is estimated and whether $H_0^\rho : \rho = \rho_0$ is true or not. We consider three alternative hypotheses, $H_a^\lambda : \lambda = \lambda_0 + \delta_1/\sqrt{N}$, $H_a^\rho : \rho = \rho_0 + \delta_2/\sqrt{N}$ and $H_a^{\lambda\rho} : \lambda = \lambda_0 + \delta_1/\sqrt{N}$ and $\rho = \rho_0 + \delta_2/\sqrt{N}$, where N is the sample size.

The score vector, the negative Hessian matrix and the information matrix are defined, respectively, as

$$
\begin{aligned}
d(\theta) &= \frac{\partial \ln L(\theta)}{\partial \theta} = \begin{pmatrix} \frac{\partial \ln L(\theta)}{\partial \beta} \\ \frac{\partial \ln L(\theta)}{\partial \lambda} \\ \frac{\partial \ln L(\theta)}{\partial \rho} \end{pmatrix}, \\
J(\theta) &= -E_f\left(\frac{1}{N}\frac{\partial^2 \ln L(\theta)}{\partial\theta\partial\theta'}\right) = \begin{pmatrix} J_\beta & J_{\beta\lambda} & J_{\beta\rho} \\ J_{\lambda\beta} & J_\lambda & J_{\lambda\rho} \\ J_{\rho\beta} & J_{\rho\lambda} & J_\rho \end{pmatrix}, \\
K(\theta) &= E_f\left(\frac{1}{N}\frac{\partial \ln L(\theta)}{\partial\theta} \cdot \frac{\partial \ln L(\theta)}{\partial\theta'}\right) = \begin{pmatrix} K_\beta & K_{\beta\lambda} & K_{\beta\rho} \\ K_{\lambda\beta} & K_\lambda & K_{\lambda\rho} \\ K_{\rho\beta} & K_{\rho\lambda} & K_\rho \end{pmatrix},
\end{aligned}
$$

where $E_f(\cdot)$ denotes the expectation under $f(y;\theta)$, and $\ln L(\theta)$ is the corresponding log-likelihood function. It can be shown that Rao's score test statistic for H_0^λ under H_a^λ and H_0^ρ can be expressed by

$$RS_\lambda(\tilde{\theta}) = \frac{1}{N} d'_\lambda(\tilde{\theta}) J^{-1}_{\lambda\cdot\beta}(\tilde{\theta}) d_\lambda(\tilde{\theta}) \rightsquigarrow \chi^2_1(\zeta_1), \tag{2.1}$$

where $J_{\lambda\cdot\beta} = J_\lambda - J_{\lambda\beta}J_\beta^{-1}J_{\beta\lambda}$, $\tilde{\theta} = (\tilde{\beta}', \lambda_0, \rho_0)'$ is the ML estimator under H_0^λ and H_0^ρ, and the non-centrality parameter $\zeta_1 = \delta_1' J_{\lambda\cdot\beta}\delta_1$. Since $\delta_1 = 0$ under H_0^λ, RS_λ has the central chi-square distribution under the joint null H_0^λ and H_0^ρ. However, this approach can have misleading consequences if the untested hypothesis is false, i.e., H_a^ρ is true. Two approaches can be used to address this problem. The first obvious approach is to test both parameters jointly. However, a more general model may be inferred if only a subset of the joint hypothesis is false, which in turn lead to over-parameterization (Jaggia and Trivedi, 1994). The second approach is to test H_0^λ allowing for the dependence of the test on certain nuisance parameters β and ρ, i.e., ρ needs to be estimated. Since the test statistic depends on an estimator $\hat{\rho}$, the estimation of ρ sometimes makes the procedure quite complicate. One interesting way to avoid estimation procedure for ρ is that ρ is assumed to be locally misspecified. Davidson and Mackinnon (1987), Saikkonen (1989) and Godfrey (1996) show that the score test statistic converges to the following non-central chi-square distribution under H_0^λ and H_a^ρ

$$RS_\lambda(\tilde{\theta}) = \frac{1}{N} d_\lambda(\tilde{\theta})' J_{\lambda\cdot\beta}^{-1}(\tilde{\theta}) d_\lambda(\tilde{\theta}) \rightsquigarrow \chi_1^2(\zeta_2), \tag{2.2}$$

where $\zeta_2 = \delta_2' J_{\rho\lambda\cdot\beta} J_{\lambda\cdot\beta}^{-1} J_{\lambda\rho\cdot\beta} \delta_2$ and $J_{\lambda\rho\cdot\beta} = J_{\lambda\rho} - J_{\lambda\beta} J_\beta^{-1} J_{\beta\rho} = J'_{\rho\lambda\cdot\beta}$. Thus under locally misspecified alternative, the score test does not follow the central chi-square distribution asymptotically even under the null hypothesis. This causes incorrect asymptotic size of the test statistic. Davidson and Mackinnon (1987) and Saikkonen (1989) investigate the power properties of RS_λ for the various choices of ρ.

In order to construct score tests robust to the misspecified local alternative, Bera and Yoon (1993) suggest a modified score test by adjusting the asymptotic mean and variance of RS_λ. The resulting test has the central χ_1^2 limiting distribution and asymptotically correct size under local misspecification. Under $H_a^{\lambda\rho}$, they propose a robust score test

$$RS_\lambda^P(\tilde{\theta}) = \frac{1}{N}[d_\lambda(\tilde{\theta}) - D_\lambda(\tilde{\theta})]' V(\tilde{\theta})^{-1} [d_\lambda(\tilde{\theta}) - D_\lambda(\tilde{\theta})] \rightsquigarrow \chi_1^2(\zeta_3), \tag{2.3}$$

where $D_\lambda = J_{\lambda\rho\cdot\beta} J_{\rho\cdot\beta}^{-1} d_\rho$, $V = J_{\lambda\cdot\beta} - J_{\lambda\rho\cdot\beta} J_{\rho\cdot\beta}^{-1} J_{\rho\lambda\cdot\beta}$ and $\zeta_3 = \delta_1' V \delta_1$. The supper-script "P" denotes a test which is robust to local misspecification. The modified RS_λ^P is asymptotically valid while the classical RS_λ usually has the size distortion under local misspecification. However, since $\zeta_1 - \zeta_3 = \delta_1' J_{\lambda\rho\cdot\beta} J_{\rho\cdot\beta}^{-1} J_{\rho\lambda\cdot\beta} \delta_1 \geq 0$, the asymptotic power of RS_λ^P is lower than that of RS_λ when H_0^ρ is true.

Even though RS_λ^P is robust to local misspecification, it is invalid if the assumed density $f(y;\theta)$ deviates from the true density, say, $g(y)$. When the assumed density $f(y;\theta)$ differs from the true density $g(y)$, in other words, $g(y)$ does not contain $f(y;\theta)$ as a special case, we have the problem of distributional (model) misspecification (see White (1994)). White (1982) provides a robust form of LM test under distributional misspecification. Let $\zeta = (\beta', \rho)'$ under H_0^λ, the score test statistic can be expressed as

$$RS_\lambda^D(\tilde{\theta}) = \frac{1}{N} d_\lambda(\tilde{\theta})' B_{\lambda\cdot\zeta}^{-1}(\tilde{\theta}) d_\lambda(\tilde{\theta}) \rightsquigarrow \chi_1^2(0), \tag{2.4}$$

where $B_{\lambda\cdot\zeta} = K_\lambda + J_{\lambda\zeta}J_\zeta^{-1}K_\zeta J_\zeta^{-1}J_{\zeta\lambda} - J_{\lambda\zeta}J_\zeta^{-1}K_{\zeta\lambda} - K_{\lambda\zeta}J_\zeta^{-1}J_{\zeta\lambda}$. The supper-script "D" denotes a test robust to distributional misspecification.

It is quite natural that one can consider two types of misspecification at the same time. In the literature, Bera, Bilias and Yoon (2007) derive a general score test which is robust to the joint presence of distributional and local misspecifications. Under H_0^λ, their test can be written by

$$RS_\lambda^{PD}(\tilde{\theta}) = \frac{1}{N}[d'_\lambda(\tilde{\theta}) - D_\lambda(\tilde{\theta})]'[B_{\lambda\cdot\beta}(\tilde{\theta}) + C_\lambda(\tilde{\theta})]^{-1}[d'_\lambda(\tilde{\theta}) - D_\lambda(\tilde{\theta})] \rightsquigarrow \chi_1^2(0), \tag{2.5}$$

where $C_\lambda = J_{\lambda\rho\cdot\beta}J_{\rho\cdot\beta}^{-1}B_{\rho\cdot\beta}J_{\rho\cdot\beta}^{-1}J_{\rho\lambda\cdot\beta} - J_{\lambda\rho\cdot\beta}J_{\rho\cdot\beta}^{-1}B_{\rho\lambda\cdot\beta} - B_{\lambda\rho\cdot\beta}J_{\rho\cdot\beta}^{-1}J_{\rho\lambda\cdot\beta}$ and $B_{\lambda\rho\cdot\beta} = K_{\lambda\rho} + J_{\lambda\beta}J_\beta^{-1}K_\beta J_\beta^{-1}J_{\beta\rho} - J_{\lambda\beta}J_\beta^{-1}K_{\beta\rho} - K_{\lambda\beta}J_\beta^{-1}J_{\beta\rho}$. The supper-script "PD" denotes a test robust to both local and distributional misspecifications.

We should note that (2.5) is a general form of the score test which is easy to compute since it only requires the QML estimator of β under the null hypothesis. When ρ is replaced by the QML estimator or $J_{\lambda\rho\cdot\beta} = 0$, (2.5) is reduced to the classical score test under distributional misspecification proposed by White (1982). When $f(y,\theta) \equiv g(y)$, i.e., $J = K$, (2.5) is exactly the score test under local misspecification proposed by Bera and Yoon (1993).

3 Spatial Dependence Tests

Consider the following spatial autoregressive model with a spatial autoregressive disturbance:

$$y = \rho W_1 y + X\beta + \epsilon, \tag{3.1}$$

$$\epsilon = \lambda W_2 \epsilon + u, \tag{3.2}$$

where y is the dependent variable, X is a $N \times k$ matrix of explanatory variables, β denotes a $k \times 1$ unknown parameter vector, ρ and λ are scalar spatial parameters, W_1 and W_2 are $N \times N$ known spatial weight matrices, ϵ is the $N \times 1$ vector of regression disturbances, and u is the $N \times 1$ vector of innovations with $u_i \sim i.i.d(0, \sigma^2)$ for $i = 1, 2, \cdots, N$. This model is a generalization of the model introduced by Cliff and Ord (1972). It is fairly general in the sense that it allows for spatial spill-overs in the dependent variable and disturbances as well. For the notational convenience, we denote $\theta = (\beta', \sigma^2, \lambda, \rho)'$, $\gamma = (\beta', \sigma^2, \lambda)'$, $\eta = (\beta', \sigma^2)$, $A = I_N - \rho W_1$, $B = I_N - \lambda W_2$, $G_A = W_1 A^{-1}$ and $G_B = W_2 B^{-1}$.

Suppose that the DGP can be fully characterized by $\theta = (\beta', \sigma^2, \lambda, \rho)'$ and a correctly specified probability density $f(y; \theta)$ which satisfies the regularity conditions in Lee (2004). Under the normality assumption, the log-likelihood function of (3.1) and (3.2) can be written by (see Anselin (1988a))

$$\ln L = -\frac{N}{2} \ln 2\pi - \frac{N}{2} \ln \sigma^2 + \ln |A| + \ln |B| - \frac{1}{2\sigma^2} u'u, \tag{3.3}$$

where $u = B(Ay - X\beta)$. The null hypothesis of interest is $H_0^\lambda : \lambda = 0$. The performance of the test for H_0^λ depends on how β is estimated and whether the hypothesis for the nuisance parameter H_0^ρ : $\rho = 0$ is true or not. We consider three alternative hypotheses, $H_a^\lambda : \lambda = \delta_1 / \sqrt{N}$, $H_a^\rho : \rho = \delta_2 / \sqrt{N}$ and $H_a^{\lambda\rho} : \lambda = \delta_1 / \sqrt{N}$ and $\rho = \delta_2 / \sqrt{N}$.

Burridge (1980) proposes an one-directional score test for H_0^λ assuming $\rho = 0$ in (3.1). The score test statistic in (2.1) is expressed by

$$RS_\lambda = \frac{\left(\tilde{u}' W_2 \tilde{u} / \tilde{\sigma}^2\right)^2}{T_{22}}, \tag{3.4}$$

where $T_{22} = \mathrm{tr}\left[(W_2 + W_2')W_2\right]$, $\tilde{u} = y - X\tilde{\beta}$ and $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$. Here $\tilde{\theta} = (\tilde{\beta}', \tilde{\sigma}^2, 0, 0)'$ denotes the constraint maximum likelihood estimator (MLE) under H_0^λ. The test statistic, RS_λ, converges to $\chi_1^2(0)$ under H_0^λ. However, if the above assumption, $\rho = 0$, is contaminated by a local deviation, say, $\rho = \delta_2 / \sqrt{N}$, it can be expressed by

$$\frac{1}{\sqrt{N}} d_\lambda(\tilde{\theta}) \rightsquigarrow N(J_{\rho\lambda\cdot\eta}\delta_2, J_{\lambda\cdot\eta}), \tag{3.5}$$

where $J_{\lambda\cdot\eta} = J_\lambda - J_{\lambda\eta} J_\eta^{-1} J_{\eta\lambda}$ and $J_{\rho\lambda\cdot\eta} = J_{\rho\lambda} - J_{\rho\eta} J_\eta^{-1} J_{\lambda\eta}$. Due to the non-zero drift term, $J_{\rho\lambda\cdot\eta}\delta_2$, of the limiting distribution of $d_\lambda(\theta)$, RS_λ will generally over-reject H_0^λ even if H_0^λ is true.

Two approaches are readily available in the literature to deal with the above problem. Anselin (1988b) proposes a LM test for the spatial error autocorrelation by estimating η and ρ jointly. Specifically, he shows under H_0^{λ}

$$LM_{\lambda}^{P} = \frac{\left[\tilde{u}' W_2 \tilde{u}/\tilde{\sigma}^2\right]^2}{T_{22} - \tilde{T}_{2A}^2 var(\tilde{\rho})}, \tag{3.6}$$

where $\tilde{u} = y - \tilde{\rho} W_1 y - X\tilde{\beta}$, $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$ and $\tilde{T}_{2A} = \text{tr}\left[(W_2' + W_2)G_A\right]$ for $G_A = W_1(I_N - \rho_0 W_1)^{-1}$. LM_{λ}^{P} converges in distribution to $\chi_1^2(0)$ under H_0^{λ}. Alternatively, Anselin, Bera, Florax and Yoon (1996) construct a robust score test for the spatial error autocorrelation by eliminating the non-central term in (2.2). Using the one-step method-of-scoring estimator, a modified score test is given by

$$RS_{\lambda}^{P} = \frac{\left[\tilde{u}' W_2 \tilde{u}/\tilde{\sigma}^2 - T_{21}\left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1} \tilde{u}' W_1 y/\tilde{\sigma}^2\right]^2}{T_{22} - T_{21}^2 \left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1}}, \tag{3.7}$$

where $\tilde{u}$ are the OLS residuals, $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$, $T_{21} = \text{tr}\left[(W_2 + W_2')W_1\right]$, $\tilde{J}_{\rho\cdot\eta} = \frac{1}{N\tilde{\sigma}^2}[\tilde{T}_{11}\tilde{\sigma}^2 + (W_1 X\beta)' M_X (W_1 X\beta)]$ for $T_{11} = \text{tr}\left[(W_1 + W_1')W_1\right]$ and $M_X = I_N - X(X'X)^{-1}X$. RS_{λ}^{P} converges to $\chi_1^2(0)$ under H_0^{λ} and $\rho = \delta_2/\sqrt{N}$. Note that the above test is derived under the assumption $\rho = \delta_2/\sqrt{N}$ that is ρ deviates locally from 0.

Similarly, there are some score tests for $H_0^{\rho} : \rho = 0$ in the literature. Anselin (1988b) proposes an one-directional score test for $H_0^{\rho} : \rho = 0$ assuming $\lambda = 0$ in (3.2). The test statistic is given by

$$RS_{\rho} = \frac{\left(\tilde{u}' W_1 y/\tilde{\sigma}^2\right)^2}{N\tilde{J}_{\rho\cdot\eta}}, \tag{3.8}$$

where $\tilde{u} = y - X\tilde{\beta}$ and $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$. Anselin (1988a) proposes a LM test for the spatial error autocorrelation by estimating β and λ jointly. The test statistic is given by

$$LM_{\rho}^{P} = \frac{\left[\tilde{u}' B' B W_1 y/\tilde{\sigma}^2\right]^2}{H_{\rho} - H_{\gamma\rho} var(\tilde{\gamma}) H_{\gamma\rho}'}, \tag{3.9}$$

where $\tilde{u}$ is a vector of residuals in the ML estimation of the null model, $H_{\rho} = T_{C_1C_1} + (BW_1X\beta)'(BW_1 X\beta)/\sigma^2$ and $H_{\gamma\rho} = ((BX)'(BW_1X\beta)/\sigma^2, 0, tr(G_B^s C_1))$ for $T_{C_1C_1} = \text{tr}\left[(C_1' + C_1)C_1\right]$, $C_1 = BW_1B^{-1}$,

$G_B^s = G_B + G_B'$ and $B = I_N - \tilde{\lambda} W_2$. Anselin, Bera, Florax and Yoon (1996) propose a robust score test for spatial lag dependence

$$RS_{\rho}^{P} = \frac{\left[\tilde{u}' W_1 y/\tilde{\sigma}^2 - T_{12} T_{22}^{-1} \left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1} \tilde{u}' W_2 \tilde{u}/\tilde{\sigma}^2\right]^2}{N\tilde{J}_{\rho\cdot\eta} - T_{21}^2 T_{22}^{-1}}, \tag{3.10}$$

where $J_{\rho\cdot\eta} = \frac{1}{N}\left[T_{C_1C_1} + \frac{1}{\sigma^2}(BW_1X\beta)' M_{BX} (BW_1X\beta)\right]$ and $M_{BX} = I_N - (BX)[(BX)'(BX)]^{-1}(BX)$. RS_{ρ}^{P} converges to $\chi_1^2(0)$ under H_0^{ρ} and $\lambda = \delta_1/\sqrt{N}$. We should note that all the above test procedures are constructed under the normality of random disturbance term. This implies their models are correctly specified. However, it is well known that, in many cases, the error term does not satisfy the normality assumption. Thus a robust testing procedure is of importance to take care of deviation from the true model.

4 A Modified Score Test Robust to Local and Distributional Misspecifications

Suppose that the true DGP characterized by $g(y)$ could be different from the assumed probability distribution $f(y,\theta)$. Although the LM_{λ}^{P} and RS_{λ}^{P} are robust to local misspecification, both tests are generally invalid under distributional (model) misspecification. We assume that $g(y)$ and $f(y,\theta)$ satisfy the regularity conditions in Lee (2004). When $g(y) \neq f(y,\theta)$, the information matrix, $K(\theta)$, and the negative Hessian matrix, $J(\theta)$, are not equivalent any more. As a result, the variance-covariance matrix of the score statistic has to be modified. In general, not only mean but also variance of the score test statistic have to be adjusted accordingly to take care of the model misspecification.

Under local and distributional misspecifications, i.e., $\rho = \rho_0 + \delta_2/\sqrt{N}$ for $\delta_2 > 0$ and $g(y) \neq f(y,\theta)$, we propose a modified score test for spatial dependence $H_0^{\lambda} : \lambda = 0$ as follows [for derivation, see Appendix B]

$$RS_{\lambda}^{PD} = \frac{\left\{\tilde{u}'W_2\tilde{u}/\tilde{\sigma}^2 - \tilde{T}_{2A}\left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1}\left[-\mathrm{tr}(G_A) + \tilde{u}'W_1y/\tilde{\sigma}^2\right]\right\}^2}{T_{22} - \tilde{T}_{2A}^2\left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1} + \frac{1}{N}\tilde{T}_{2A}^2\left(\tilde{J}_{\rho\cdot\eta}^{-2}\tilde{B}_{\rho\cdot\eta}^{*}\right)}, \tag{4.1}$$

where $\tilde{u} = y - \rho_0 W_1 y - X\tilde{\beta}$ is OLS residuals, $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$, $\tilde{T}_{2A} = \mathrm{tr}\left[(W_2' + W_2)G_A\right]$ and $G_A = W_1\left(I - \rho_0 W_1\right)^{-1}$. When $\rho_0 = 0$ and $\tilde{T}_{2A} = T_{21}$, it can be easily checked that $B_{\rho\eta}^{*} = 0$, and therefore, (4.1) is simplified to (3.7), i.e., the test statistic is equivalent to that of Anselin, Bera, Florax and Yoon (1996). When $g(y) = f(y,\theta)$ so that $J(\theta) = K(\theta)$, the denominator in (4.1) is exactly the same as the denominator in RS_{λ}^{P}. Note that the terms in the denominator in (4.1) are

$$\tilde{J}_{\rho\cdot\eta} = \frac{1}{N}\left[\tilde{T}_{AA} - \frac{2}{N}tr^2(G_A) + \frac{1}{\sigma^2}\left(G_AX\tilde{\beta}\right)' M_X\left(G_AX\tilde{\beta}\right)\right], \tag{4.2}$$

$$\tilde{B}_{\rho\cdot\eta}^{*} = \frac{1}{N\sigma^4}\left\{2\tilde{\mu}_3\left(G_AX\tilde{\beta}\right)' M_X\bar{G}_A + \tilde{\kappa}_4\bar{G}_A'\bar{G}_A\right\}, \tag{4.3}$$

where $\tilde{T}_{AA} = \mathrm{tr}\left[\left(G_A' + G_A\right)G_A\right]$, $\bar{G}_A = vec_D(G_A) - \frac{1}{N}l_N\mathrm{tr}(G_A)$, $\tilde{\mu}_3 = \frac{1}{N}\sum_{i=1}^{N}\tilde{u}_i^3$, $\tilde{\mu}_4 = \frac{1}{N}\sum_{i=1}^{N}\tilde{u}_i^4$ and $\tilde{\kappa}_4 = \tilde{\mu}_4 - 3\tilde{\sigma}^4$. Here l_N is a $N \times 1$ vector of ones and $vec_D(G_A)$ is a column vector formed by the diagonal elements of G_A. If $\rho_0 = 0$, it is easy to check that $G_A = W_1$, $vec_D(G_A)$ and $\mathrm{tr}(G_A)$ are equal to 0. Thus $\rho_0 = 0$ yields $B_{\rho\cdot\eta}^{*} = 0$, and therefore, Anselin, Bera, Florax and Yoon (1996)'s test is automatically robust to distributional misspecification. Moreover, $J_{\lambda\eta} = 0_{1\times(k+1)}$, $K_\lambda = J_\lambda$ and $B_{\lambda\cdot\eta} = J_{\lambda\cdot\eta}$ under H_0^{λ} when $\rho = 0$. Thus RS_{λ}^{PD} in (4.1) is reduced to Burridge (1980)'s test in (3.4). Thus it can be also shown that Burridge's test is also robust to distributional misspecification when $\rho = 0$ and $\lambda = 0$. When the true density is given by the normal distribution, $\tilde{\mu}_3$ and $\tilde{\kappa}_4$ in (4.3) are equal to zero which yields $\tilde{B}_{\rho\cdot\eta}^{*} = 0$. However, when $\rho_0 = 0$, our results show that $\tilde{B}_{\rho\cdot\eta}^{*} = 0$ even though $g(y)$ is different from the normal distribution. Thus the non-normality does not affect the score test statistic when $\rho_0 = 0$.

We can consider the case that ρ is a parameter to be estimated under H_0^{λ}. Since $(\tilde{\beta}', \tilde{\sigma}^2, \tilde{\rho})'$ is the constrained MLE under H_0^λ, the score term $-\mathrm{tr}(G_A) + \tilde{u}' W_1 y/\tilde{\sigma}^2 = 0$, and RS_λ^{PD} can be rewritten by

$$\overline{RS}_\lambda^{PD} = \frac{\left(\tilde{u}' W_2 \tilde{u}/\tilde{\sigma}^2\right)^2}{T_{22} - \tilde{T}_{2A}^2 \left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1} + \frac{1}{N}\tilde{T}_{2A}^2 \left(\tilde{J}_{\rho\cdot\eta}^{-2}\tilde{B}_{\rho\cdot\eta}^*\right)}. \quad (4.4)$$

Note that $var(\tilde{\rho}) = \left(N\tilde{J}_{\rho\cdot\eta}\right)$ in this case. Compared to Anselin (1998b)'s test in (3.6), the only difference comes form the fact that we correct the variance of the score test in (4.4) to incorporate the distributional misspecification.

Similarly, let us consider tests for the null hypothesis $H_0^\rho : \rho = 0$. In this case, a modified score test for spatial dependence H_0^ρ can be derived as [for derivation, see Appendix C]

$$RS_\rho^{PD} = \frac{\left\{\tilde{u}' B W_1 y/\tilde{\sigma}^2 - \tilde{T}_{BC_1}(N\tilde{J}_{\lambda\cdot\eta})^{-1}\left[-\mathrm{tr}(G_B) + \tilde{u}' G_B \tilde{u}/\tilde{\sigma}^2\right]\right\}^2}{N\tilde{J}_{\rho\cdot\eta} - \frac{1}{N}\tilde{T}_{BC_1}^2(\tilde{J}_{\lambda\cdot\eta})^{-1} + N\left(\tilde{B}_{\rho\cdot\eta}^* + \tilde{C}_\rho^*\right)}, \quad (4.5)$$

where $\tilde{u} = (I - \lambda_0 W_2)(y - X\tilde{\beta})$ and $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$, $\tilde{T}_{BC_1} = \mathrm{tr}\left[(G_B' + G_B)C_1\right]$, $\tilde{J}_{\lambda\cdot\eta} = \frac{1}{N^2}\left[NT_{BB} - 2tr^2(G_B))\right]$, $\tilde{J}_{\rho\cdot\eta} = \frac{1}{N}\left[T_{C_1C_1} + \frac{1}{\tilde{\sigma}^2}\left(BW_1X\tilde{\beta}\right)' M_{BX}\left(BW_1X\tilde{\beta}\right)\right]$, $\tilde{B}_{\rho\cdot\eta}^* + \tilde{C}_\rho^* = \frac{1}{N\tilde{\sigma}^4}\left[\tilde{\kappa}_4\tilde{F}'\tilde{F} + 2\tilde{\mu}_3(BW_1X\tilde{\beta})'M_{BX}\tilde{F}\right]$ for $\tilde{F} = vec_D(C_1) - \tilde{J}_{\rho\lambda\cdot\eta}\tilde{J}_{\lambda\cdot\eta}^{-1}\tilde{\bar{G}}_B$, $\tilde{J}_{\rho\lambda\cdot\eta} = \frac{1}{N}T_{BC_1}$ and $\tilde{\bar{G}}_B = vec_D(G_B) - \frac{1}{N}l_N\mathrm{tr}(G_B)$. When λ is absent, RS_ρ^{PD} in (4.5) is reduced to the test statistic proposed by Anselin (1988b) which is not originally robust to the distributional misspecification. This implies that Anselin (1988b)'s test is robust to distributional misspecification when $\rho = 0$ and $\lambda = 0$. Note that when $\lambda_0 = 0$, $B = I$, $G_B = W_2$, $C_1 = W_2$, $vec_D(G_B) = 0$, $\mathrm{tr}(G_B) = 0$, and $vec_D(C_1) = 0$. Thus the drift term in (4.5) is reduced to the drift term in Anselin, Bera, Florax and Yoon (1996). Moreover, since $\tilde{F} = 0$ yields $\tilde{B}_{\rho\cdot\eta}^* + \tilde{C}_\rho^* = 0$ in this case, the denominator in (4.5) is exactly the same as the denominator in Anselin, Bera, Florax and Yoon (1996). It implies Anselin, Bera, Florax and Yoon (1996)'s test is robust to distributional misspecification when $\lambda_0 = 0$. Thus the non-normality does not give any impact to the test statistics whenever $\lambda_0 = 0$.

Let us consider λ is a parameter to be estimated under H_0^ρ. Since $(\tilde{\beta}', \tilde{\sigma}^2, \tilde{\lambda})'$ is the constrained MLE under the null H_0^ρ, the score term $\tilde{S}_{\lambda_1} = 0$, and therefore, RS_ρ^{PD} can be expressed by

$$\overline{RS}_\rho^{PD} = \frac{\left(\tilde{u}' B W_1 y/\tilde{\sigma}^2\right)^2}{N\tilde{J}_{\rho\cdot\eta} - \frac{1}{N}\tilde{T}_{BC_1}^2(\tilde{J}_{\lambda\cdot\eta})^{-1} + N\left(\tilde{B}_{\rho\cdot\eta}^* + \tilde{C}_\rho^*\right)}. \quad (4.6)$$

Compared to Anselin (1998a)'s test, the main difference comes from the fact that we correct the variance of the score test in (4.6) to adopt the distributional misspecification.

5 Monte Carlo simulation

We conduct some Monte Carlo simulations to examine the finite sample performance of the proposed test in the presence of local and distributional misspecifications. We consider the following model

$$y = \rho W_1 y + X_1\beta_1 + X_2\beta_2 + X_3\beta_3 + \epsilon, \quad \epsilon = \lambda W_2 \epsilon + u. \tag{5.1}$$

Following the simulation design in Liu, Lee and Bollinger (2006), we consider four different distributions of u_i. All of distributions have mean 0 and variance 2. The first distribution for u_i explored in the experiment is *i.i.d.* normal distribution, which is regarded as a benchmark model for the maximum likelihood estimation. The second distribution is Student's t distribution with the degree of freedom k (t_k) whose skewness (η_3) and kurtosis (η_4) are 0 and 9, respectively. More specifically, we consider that $u_i = \sqrt{6/5}v$, where $v \sim t_5$. The third distribution is gamma distribution with $\eta_3 = \sqrt{2}$ and $\eta_4 = 6$, where $u_i = v - 2$ and $v \sim gamma(2,1)$. The fourth one is an asymmetric bimodal mixture normal with $\eta_3 \approx 0.84$ and $\eta_4 \approx 2.79$, where $u_i = v/2$ and $v \sim .5N(-3,1) + .5N(3,13)$. For the choice of weight matrices, we follow Kelejian and Prucha (1999)'s procedure and specify a "circular" world so that u_N is directly connected to u_1 and u_{N-1}. Similarly, u_1 is related to u_2 and u_N. The sample sizes we considered are 45, 90 and 180. For each set of the generated sample observations, we select the weight matrix by "2 ahead and 2 behind" setup that is each element of u_i is directly connected to the two elements before and after it. In our simulations, all weight matrices are row-standardized, and the same weight matrices are used in both spatial lag and error autoregressive terms. The first regressor X_1 is given by a vector of 1. The remaining regressors, X_2 and X_3, are randomly drawn from the uniform distribution with range [0,10]. The values of coefficients β are set to 1. The nominal size of all tests are set to be 0.05. The number of repetition is 2000, and the actual sizes and powers are reported.

We consider three test statistics: (i) Burridge (1980)'s test statistics, RS_λ and RS_ρ; (ii) Anselin (1988b)'s LM statistics, LM_λ^P and LM_ρ^P; (iii) RS_λ^{PD} and RS_ρ^{PD} test statistics proposed in the paper. Our test statistics are identical to those of Anselin, Bera, Florax and Yoon (1996) when the nuisance parameters in the true non-null model deviate locally from zero. Recall that the other two test statistics are also robust test statistics in the sense that Burridge's test is robust to distributional misspecification, and Anselin's tests are robust to local misspecification since it incorporates the estimation procedure for the nuisance parameters.

[Table 1]

Table 1 reports the empirical sizes of all tests for different experimental designs. The first four columns with numbers report the actual rejection probability of the tests for spatial lag dependence under different distributions and four values of $\lambda = (0, 0.1, 0.2, 0.3)$, while the next four columns present the actual sizes of the tests for spatial error autocorrelation. Using the normal approximation to the binomial distribution, 95% confidence intervals for the estimated actual sizes are [0.041, 0.059] for 2000 replications. When the model is correctly specified, and there exists no parametric misspecification, i.e., either $\lambda = 0$ or $\rho = 0$, the rejection frequencies of all tests for $N = 90$ and $N = 180$ and all distributional specifications belong to the 95% confidence interval. However, when local misspecification is present, the sizes of RS_ρ and RS_λ increase rapidly as the degree of local misspecification increases even in the case of the normal distribution. This implies that RS_ρ and RS_λ are unable to capture the source of the spatial dependence. On the contrary, LM_ρ^P and LM_λ^P are quite stable under local misspecification since they take into account the estimation procedure of λ and ρ, respectively. The empirical size of RS_ρ^{PD} tends to increase as λ increases, but the magnitude of such increase is not as large as that of RS_ρ. One interesting finding is that the empirical sizes of RS_{λ}^{PD} are relatively stable comparing to those of RS_{ρ}^{PD} and even quite similar to those of LM_{λ}^{P}.

The impact of distributional misspecification on the empirical sizes turns out to be quite small. This could be owing to the robust characteristics of Burridge's and Anselin, Bera, Florax and Yoon's tests. Moreover, when we do not consider local misspecification, for almost all cases, the empirical sizes of LM_{ρ}^{P} and LM_{λ}^{P} are in the 95% confidence interval. Only exceptions are LM_{ρ}^{P} with $N = 45$ and normal and gamma distributions. This phenomenon might be due to the specification of the experimental designs. For small samples as in our simulations, implied skewness and kurtosis of considered distributions are not distinct enough to distinguish from those of the normal distribution. Saavedra (2003) also performed a Monte-Carlo simulation to analyze the empirical sizes and powers of GMM based tests and the robust tests of Anselin, Bera, Florax and Yoon (1996). He shows that GMM based tests have considerable distortions for the irregular weight matrices while the robust tests show better size performance in this case. Moreover, one can check in Table 3 in Saavedra (2003), there are no huge differences of LM_{ρ}^{P} with the normal and log-normal distributions. Even the empirical sizes are better in the log-normal situation when sample size is 51. We can say that these findings also support the robust score tests considered in our study.

[Tables 2-7]

The power performances are demonstrated from Tables 2 to 7 for different sample sizes. Tables 2-4 report the powers of the tests for spatial lag dependence when the sample sizes are 45, 90 and 180, respectively. Tables 5-7 show the power performance of the tests for spatial error autocorrelation. Both of the tests have very similar patterns in power performance. First of all, all the tests have reasonable powers under all situations even when the sample size is relative small, say, $N = 45$. When the sample size increases, the rejection probabilities of all the tests approach to one very quickly. For example, in the case of $N = 180$ and $\lambda = 0$ (Table 4), the rejection probability of RS_ρ is equal to 1 when $\rho = 0.3$, and those of LM_ρ^P and RS_ρ^{PD} are 1 when $\rho = 0.4$. The tests for spatial error autocorrelation seem to be less powerful than those for spatial lag dependence. However, the rejection probability converges to 1 reasonably quickly as the sample size increases. In all Tables, RS_ρ (RS_λ) is more powerful than LM_ρ^P and RS_ρ^{PD} (LM_λ^P and RS_λ^{PD}), however, RS_ρ and RS_λ are invalid since they have extreme size distortion under local misspecification.

RS_ρ^{PD} and RS_λ^{PD} achieve considerable powers when the model is locally and/or globally misspecified. There is little loss of powers for RS_ρ^{PD} when λ changes form 0 to 0.3, and the magnitude of the loss can be negligible as the sample size increases. RS_ρ^{PD} and RS_λ^{PD} are uniformly more powerful than LM_ρ^P and LM_λ^P, respectively, when $N = 45$. The difference of the powers between RS_ρ^{PD} (RS_λ^{PD}) and LM_ρ^P (LM_λ^P) decreases rapidly as the sample sizes increase. This could be due to the errors in the estimation procedure for LM_ρ^P and LM_λ^P. These estimation errors tend to decrease quickly as the sample size increase. An interesting finding is that the power of RS_ρ^{PD} is positively correlated with kurtosis of the distribution. Note that the Student's t and gamma distributions have significantly larger kurtosis than those of the normal and mixture normal distributions. In Tables 2-4, we can observe that the rejection probability generating from the former two distributions are uniformly larger than those in the latter under all situations.

6 Conclusion

In this paper we derive modified score tests robust to both local and distributional misspecifications in the spatial autoregressive model with a spatial autoregressive error term. We show that some popular spatial dependence tests, such as Burridge (1980), Anselin (1988b), and Anselin, Bera, Florax, and Yoon (1996), can be expressed as special cases of our tests. We also find that Burridge (1980) and Anselin, Bera, Florax, and Yoon (1996)'s tests are automatically robust to distributional misspecification under some special cases. Our findings in this paper support the usage of Anselin, Bera, Florax, and Yoon (1996)'s tests for spatial dependence since they are also robust to distributional misspecification. The Monte Carlo simulations demonstrate that the proposed tests have good sizes and powers in the finite sample.

Table 1: Empirical size of test statistics

N	$Dist.$	$Test$	$\rho = 0.0$				$Test$	$\lambda = 0.0$			
			$\lambda = 0.0$	$\lambda = 0.1$	$\lambda = 0.2$	$\lambda = 0.3$		$\rho = 0.0$	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$
45	$Normal$	RS_ρ	0.057	0.058	0.084	0.169	RS_λ	0.042	0.056	0.167	0.322
		LM_ρ^P	0.067	0.049	0.056	0.064	LM_λ^P	0.048	0.052	0.061	0.061
		RS_ρ^{PD}	0.061	0.060	0.075	0.096	RS_λ^{PD}	0.046	0.046	0.050	0.048
	$Student$	RS_ρ	0.048	0.064	0.085	0.154	RS_λ	0.031	0.059	0.180	0.394
		LM_ρ^P	0.055	0.068	0.053	0.057	LM_λ^P	0.038	0.045	0.048	0.049
		RS_ρ^{PD}	0.052	0.071	0.076	0.101	RS_λ^{PD}	0.033	0.046	0.040	0.041
	$Gamma$	RS_ρ	0.053	0.070	0.094	0.144	RS_λ	0.031	0.068	0.168	0.405
		LM_ρ^P	0.062	0.055	0.064	0.060	LM_λ^P	0.038	0.043	0.044	0.042
		RS_ρ^{PD}	0.053	0.060	0.077	0.094	RS_λ^{PD}	0.039	0.042	0.031	0.039
	Asy	RS_ρ	0.047	0.058	0.107	0.187	RS_λ	0.050	0.066	0.142	0.312
		LM_ρ^P	0.055	0.060	0.066	0.052	LM_λ^P	0.054	0.054	0.051	0.041
		RS_ρ^{PD}	0.056	0.069	0.085	0.088	RS_λ^{PD}	0.046	0.052	0.056	0.042
90	$Normal$	RS_ρ	0.050	0.089	0.145	0.290	RS_λ	0.047	0.098	0.317	0.629
		LM_ρ^P	0.058	0.059	0.049	0.042	LM_λ^P	0.055	0.046	0.049	0.054
		RS_ρ^{PD}	0.058	0.069	0.072	0.081	RS_λ^{PD}	0.049	0.044	0.047	0.045
	$Student$	RS_ρ	0.049	0.074	0.125	0.238	RS_λ	0.047	0.095	0.363	0.753
		LM_ρ^P	0.055	0.059	0.048	0.063	LM_λ^P	0.048	0.049	0.049	0.042
		RS_ρ^{PD}	0.054	0.062	0.071	0.093	RS_λ^{PD}	0.046	0.047	0.049	0.055
	$Gamma$	RS_ρ	0.047	0.065	0.126	0.220	RS_λ	0.043	0.103	0.341	0.736
		LM_ρ^P	0.054	0.054	0.051	0.051	LM_λ^P	0.053	0.041	0.047	0.051
		RS_ρ^{PD}	0.053	0.065	0.069	0.090	RS_λ^{PD}	0.050	0.036	0.044	0.053
	Asy	RS_ρ	0.056	0.078	0.179	0.346	RS_λ	0.051	0.105	0.294	0.603
		LM_ρ^P	0.055	0.062	0.059	0.055	LM_λ^P	0.053	0.057	0.045	0.047
		RS_ρ^{PD}	0.052	0.066	0.079	0.086	RS_λ^{PD}	0.053	0.059	0.051	0.047
180	$Normal$	RS_ρ	0.051	0.106	0.254	0.512	RS_λ	0.044	0.166	0.583	0.920
		LM_ρ^P	0.050	0.059	0.052	0.052	LM_λ^P	0.048	0.050	0.044	0.052
		RS_ρ^{PD}	0.055	0.066	0.070	0.080	RS_λ^{PD}	0.048	0.048	0.047	0.064
	$Student$	RS_ρ	0.054	0.090	0.194	0.368	RS_λ	0.044	0.178	0.653	0.967
		LM_ρ^P	0.059	0.056	0.052	0.056	LM_λ^P	0.050	0.049	0.038	0.051
		RS_ρ^{PD}	0.050	0.062	0.070	0.090	RS_λ^{PD}	0.048	0.049	0.047	0.082
	$Gamma$	RS_ρ	0.047	0.092	0.201	0.376	RS_λ	0.044	0.183	0.634	0.968
		LM_ρ^P	0.051	0.056	0.051	0.057	LM_λ^P	0.047	0.045	0.055	0.047
		RS_ρ^{PD}	0.046	0.066	0.071	0.087	RS_λ^{PD}	0.046	0.045	0.058	0.088
	Asy	RS_ρ	0.052	0.111	0.278	0.557	RS_λ	0.052	0.148	0.551	0.905
		LM_ρ^P	0.054	0.050	0.047	0.060	LM_λ^P	0.047	0.046	0.050	0.042
		RS_ρ^{PD}	0.057	0.063	0.065	0.081	RS_λ^{PD}	0.046	0.051	0.052	0.059

Note: A 95 % confidence interval for p=0.05 with 2000 replications is $0.0408 < p < 0.0591$

Table 2: Empirical power of test statistics for H_0^ρ: N=45

λ	ρ	*Normal*			*Student*			*Gamma*			*Asy − normal*		
		RS_ρ	LM_ρ^P	RS_ρ^{PD}	RS_ρ	LM_ρ^P	RS_ρ^{PD}	RS_ρ	LM_ρ^P	RS_ρ^{PD}	RS_ρ	LM_ρ^P	RS_ρ^{PD}
$\lambda = 0.0$	$\rho = 0.1$	0.152	0.139	0.148	0.251	0.219	0.240	0.263	0.232	0.248	0.129	0.112	0.116
	$\rho = 0.2$	0.507	0.344	0.387	0.729	0.586	0.634	0.731	0.611	0.649	0.443	0.289	0.318
	$\rho = 0.3$	0.834	0.601	0.664	0.959	0.857	0.913	0.957	0.853	0.907	0.781	0.516	0.581
	$\rho = 0.4$	0.975	0.784	0.884	0.998	0.955	0.985	0.998	0.947	0.983	0.952	0.667	0.786
	$\rho = 0.5$	0.999	0.849	0.964	1.000	0.967	0.996	1.000	0.970	0.997	0.996	0.751	0.905
	$\rho = 0.6$	1.000	0.866	0.988	1.000	0.964	0.999	1.000	0.968	1.000	1.000	0.791	0.971
	$\rho = 0.7$	1.000	0.846	0.999	1.000	0.967	1.000	1.000	0.968	1.000	1.000	0.767	0.986
	$\rho = 0.8$	1.000	0.852	1.000	1.000	0.969	1.000	1.000	0.975	1.000	1.000	0.756	0.994
$\lambda = 0.1$	$\rho = 0.1$	0.255	0.128	0.141	0.355	0.206	0.228	0.361	0.214	0.245	0.248	0.119	0.125
	$\rho = 0.2$	0.647	0.340	0.361	0.814	0.615	0.645	0.812	0.592	0.632	0.566	0.258	0.287
	$\rho = 0.3$	0.904	0.583	0.651	0.980	0.855	0.900	0.984	0.851	0.900	0.869	0.506	0.577
	$\rho = 0.4$	0.987	0.765	0.859	0.999	0.942	0.977	0.998	0.938	0.973	0.977	0.658	0.762
	$\rho = 0.5$	1.000	0.849	0.950	1.000	0.962	0.995	1.000	0.964	0.993	0.997	0.746	0.892
	$\rho = 0.6$	1.000	0.864	0.979	1.000	0.972	0.998	1.000	0.973	0.999	1.000	0.794	0.951
	$\rho = 0.7$	1.000	0.879	0.991	1.000	0.978	1.000	1.000	0.982	0.999	1.000	0.806	0.971
	$\rho = 0.8$	1.000	0.898	0.998	1.000	0.984	1.000	1.000	0.986	1.000	1.000	0.809	0.984
$\lambda = 0.2$	$\rho = 0.1$	0.409	0.125	0.138	0.513	0.230	0.250	0.461	0.236	0.250	0.374	0.113	0.123
	$\rho = 0.2$	0.754	0.329	0.368	0.869	0.563	0.609	0.862	0.577	0.598	0.730	0.269	0.313
	$\rho = 0.3$	0.945	0.567	0.627	0.983	0.836	0.870	0.986	0.832	0.857	0.927	0.464	0.530
	$\rho = 0.4$	0.995	0.755	0.821	0.999	0.933	0.963	0.996	0.934	0.960	0.987	0.651	0.710
	$\rho = 0.5$	1.000	0.852	0.916	1.000	0.957	0.988	1.000	0.959	0.989	0.999	0.725	0.855
	$\rho = 0.6$	1.000	0.878	0.965	1.000	0.977	0.995	1.000	0.975	0.997	1.000	0.812	0.921
	$\rho = 0.7$	1.000	0.903	0.981	1.000	0.984	0.998	1.000	0.983	1.000	1.000	0.837	0.959
	$\rho = 0.8$	1.000	0.937	0.989	1.000	0.985	0.999	1.000	0.987	1.000	1.000	0.862	0.976
$\lambda = 0.3$	$\rho = 0.1$	0.544	0.116	0.132	0.575	0.212	0.237	0.578	0.208	0.227	0.572	0.122	0.145
	$\rho = 0.2$	0.839	0.334	0.356	0.896	0.549	0.580	0.901	0.535	0.556	0.836	0.260	0.295
	$\rho = 0.3$	0.970	0.551	0.575	0.990	0.817	0.836	0.990	0.806	0.832	0.968	0.443	0.481
	$\rho = 0.4$	0.993	0.719	0.762	1.000	0.938	0.939	1.000	0.918	0.935	1.000	0.650	0.668
	$\rho = 0.5$	1.000	0.815	0.886	1.000	0.960	0.978	1.000	0.977	0.983	1.000	0.737	0.807
	$\rho = 0.6$	1.000	0.879	0.946	1.000	0.982	0.991	1.000	0.983	0.993	1.000	0.814	0.887
	$\rho = 0.7$	1.000	0.928	0.962	1.000	0.989	0.995	1.000	0.989	0.996	1.000	0.870	0.927
	$\rho = 0.8$	1.000	0.946	0.984	1.000	0.996	0.997	1.000	0.996	0.999	1.000	0.902	0.957

Table 3: Empirical power of test statistics for H_0^{ρ}: N=90

λ	ρ	*Normal*			*Student*			*Gamma*			*Asy − normal*		
		RS_{ρ}	LM_{ρ}^{P}	RS_{ρ}^{PD}	RS_{ρ}	LM_{ρ}^{P}	RS_{ρ}^{PD}	RS_{ρ}	LM_{ρ}^{P}	RS_{ρ}^{PD}	RS_{ρ}	LM_{ρ}^{P}	RS_{ρ}^{PD}
$\lambda = 0.0$	$\rho = 0.1$	0.319	0.227	0.237	0.491	0.413	0.429	0.475	0.401	0.419	0.250	0.171	0.181
	$\rho = 0.2$	0.834	0.648	0.673	0.964	0.907	0.916	0.961	0.900	0.920	0.760	0.508	0.543
	$\rho = 0.3$	0.991	0.894	0.926	0.999	0.994	0.996	1.000	0.995	0.999	0.981	0.838	0.884
	$\rho = 0.4$	1.000	0.984	0.990	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.935	0.980
	$\rho = 0.5$	1.000	0.990	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.947	0.996
	$\rho = 0.6$	1.000	0.985	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.955	1.000
	$\rho = 0.7$	1.000	0.983	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.961	1.000
	$\rho = 0.8$	1.000	0.985	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.966	1.000
$\lambda = 0.1$	$\rho = 0.1$	0.508	0.213	0.227	0.627	0.383	0.389	0.629	0.382	0.392	0.502	0.177	0.188
	$\rho = 0.2$	0.933	0.611	0.636	0.986	0.881	0.901	0.984	0.880	0.896	0.910	0.525	0.553
	$\rho = 0.3$	0.998	0.906	0.935	1.000	0.989	0.997	1.000	0.987	0.992	0.996	0.787	0.838
	$\rho = 0.4$	1.000	0.965	0.987	1.000	0.998	1.000	1.000	0.999	1.000	1.000	0.928	0.971
	$\rho = 0.5$	1.000	0.992	0.999	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.958	0.994
	$\rho = 0.6$	1.000	0.982	1.000	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.961	0.998
	$\rho = 0.7$	1.000	0.989	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.969	1.000
	$\rho = 0.8$	1.000	0.991	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.970	1.000
$\lambda = 0.2$	$\rho = 0.1$	0.710	0.213	0.237	0.776	0.376	0.384	0.791	0.366	0.389	0.716	0.170	0.186
	$\rho = 0.2$	0.970	0.587	0.609	0.995	0.868	0.871	0.991	0.853	0.863	0.968	0.494	0.512
	$\rho = 0.3$	1.000	0.873	0.904	1.000	0.986	0.987	1.000	0.983	0.984	0.998	0.761	0.808
	$\rho = 0.4$	1.000	0.963	0.981	1.000	0.999	1.000	1.000	0.999	0.999	1.000	0.910	0.947
	$\rho = 0.5$	1.000	0.983	0.997	1.000	0.998	0.999	1.000	1.000	1.000	1.000	0.956	0.986
	$\rho = 0.6$	1.000	0.991	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.964	0.997
	$\rho = 0.7$	1.000	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.970	1.000
	$\rho = 0.8$	1.000	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.980	1.000
$\lambda = 0.3$	$\rho = 0.1$	0.866	0.209	0.217	0.872	0.355	0.350	0.877	0.360	0.360	0.878	0.178	0.184
	$\rho = 0.2$	0.996	0.574	0.588	0.995	0.835	0.838	0.997	0.841	0.825	0.992	0.450	0.461
	$\rho = 0.3$	1.000	0.842	0.848	1.000	0.979	0.979	1.000	0.974	0.979	1.000	0.752	0.750
	$\rho = 0.4$	1.000	0.949	0.964	1.000	0.997	0.999	1.000	0.996	0.996	1.000	0.904	0.914
	$\rho = 0.5$	1.000	0.986	0.989	1.000	0.999	1.000	1.000	1.000	1.000	1.000	0.951	0.969
	$\rho = 0.6$	1.000	0.993	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.975	0.990
	$\rho = 0.7$	1.000	0.996	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.987	0.997
	$\rho = 0.8$	1.000	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.993	0.999

Table 4: Empirical power of test statistics for H_0^{ρ}: N=180

λ	ρ	*Normal*			*Student*			*Gamma*			*Asy − normal*		
		RS_ρ	LM_ρ^P	RS_ρ^{PD}	RS_ρ	LM_ρ^P	RS_ρ^{PD}	RS_ρ	LM_ρ^P	RS_ρ^{PD}	RS_ρ	LM_ρ^P	RS_ρ^{PD}
$\lambda = 0.0$	$\rho = 0.1$	0.599	0.411	0.426	0.783	0.679	0.687	0.785	0.689	0.694	0.504	0.323	0.332
	$\rho = 0.2$	0.993	0.910	0.926	1.000	0.997	0.999	1.000	0.995	0.996	0.979	0.815	0.832
	$\rho = 0.3$	1.000	0.996	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.985	0.992
	$\rho = 0.4$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000
	$\rho = 0.5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.6$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\lambda = 0.1$	$\rho = 0.1$	0.823	0.387	0.389	0.905	0.652	0.657	0.913	0.662	0.672	0.808	0.322	0.337
	$\rho = 0.2$	0.999	0.900	0.915	1.000	0.992	0.991	1.000	0.992	0.993	0.996	0.815	0.835
	$\rho = 0.3$	1.000	0.994	0.997	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.981	0.990
	$\rho = 0.4$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000
	$\rho = 0.5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000
	$\rho = 0.6$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\lambda = 0.2$	$\rho = 0.1$	0.959	0.381	0.393	0.975	0.651	0.652	0.966	0.634	0.638	0.954	0.308	0.320
	$\rho = 0.2$	1.000	0.888	0.894	1.000	0.991	0.991	1.000	0.992	0.990	1.000	0.778	0.788
	$\rho = 0.3$	1.000	0.995	0.995	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.966	0.970
	$\rho = 0.4$	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.999
	$\rho = 0.5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.6$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\lambda = 0.3$	$\rho = 0.1$	0.992	0.350	0.355	0.994	0.621	0.606	0.995	0.618	0.612	0.995	0.289	0.298
	$\rho = 0.2$	1.000	0.850	0.850	1.000	0.987	0.983	1.000	0.986	0.979	1.000	0.751	0.742
	$\rho = 0.3$	1.000	0.988	0.987	1.000	1.000	0.998	1.000	1.000	1.000	1.000	0.965	0.960
	$\rho = 0.4$	1.000	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.992	0.999
	$\rho = 0.5$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.6$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	$\rho = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Table 5: Empirical power of test statistics for H_0^{λ}: N=45

ρ	λ	*Normal*			*Student*			*Gamma*			*Asy − normal*		
		RS_λ	LM_λ^P	RS_λ^{PD}	RS_λ	LM_λ^P	RS_λ^{PD}	RS_λ	LM_λ^P	RS_λ^{PD}	RS_λ	LM_λ^P	RS_λ^{PD}
$\rho = 0.0$	$\lambda = 0.1$	0.120	0.060	0.058	0.124	0.058	0.061	0.138	0.057	0.061	0.131	0.061	0.061
	$\lambda = 0.2$	0.255	0.094	0.098	0.251	0.101	0.113	0.245	0.096	0.105	0.243	0.096	0.105
	$\lambda = 0.3$	0.434	0.180	0.186	0.432	0.201	0.216	0.439	0.203	0.224	0.419	0.167	0.184
	$\lambda = 0.4$	0.607	0.278	0.296	0.625	0.325	0.353	0.637	0.326	0.346	0.623	0.254	0.272
	$\lambda = 0.5$	0.811	0.425	0.449	0.799	0.505	0.535	0.808	0.498	0.535	0.792	0.380	0.404
	$\lambda = 0.6$	0.912	0.563	0.615	0.917	0.676	0.709	0.921	0.663	0.715	0.916	0.524	0.544
	$\lambda = 0.7$	0.975	0.685	0.737	0.976	0.815	0.855	0.980	0.806	0.841	0.980	0.620	0.634
	$\lambda = 0.8$	0.994	0.789	0.789	0.997	0.893	0.907	0.998	0.892	0.918	0.998	0.700	0.681
$\rho = 0.1$	$\lambda = 0.1$	0.263	0.064	0.068	0.285	0.060	0.055	0.304	0.057	0.068	0.267	0.072	0.074
	$\lambda = 0.2$	0.415	0.091	0.106	0.441	0.102	0.112	0.433	0.094	0.108	0.428	0.101	0.104
	$\lambda = 0.3$	0.599	0.173	0.178	0.638	0.189	0.216	0.639	0.202	0.237	0.631	0.174	0.178
	$\lambda = 0.4$	0.756	0.289	0.292	0.778	0.316	0.341	0.787	0.337	0.368	0.763	0.268	0.280
	$\lambda = 0.5$	0.880	0.413	0.425	0.898	0.509	0.522	0.909	0.484	0.517	0.883	0.378	0.389
	$\lambda = 0.6$	0.962	0.556	0.553	0.963	0.673	0.698	0.962	0.647	0.659	0.962	0.514	0.511
	$\lambda = 0.7$	0.988	0.685	0.656	0.986	0.792	0.808	0.991	0.804	0.815	0.991	0.637	0.602
	$\lambda = 0.8$	0.997	0.783	0.704	1.000	0.897	0.861	0.999	0.895	0.873	0.999	0.712	0.620
$\rho = 0.2$	$\lambda = 0.1$	0.486	0.067	0.067	0.539	0.052	0.065	0.537	0.065	0.066	0.460	0.071	0.068
	$\lambda = 0.2$	0.656	0.107	0.106	0.695	0.100	0.118	0.672	0.107	0.119	0.630	0.110	0.107
	$\lambda = 0.3$	0.763	0.172	0.159	0.800	0.209	0.225	0.834	0.205	0.220	0.757	0.173	0.171
	$\lambda = 0.4$	0.880	0.286	0.280	0.893	0.336	0.347	0.911	0.345	0.331	0.878	0.266	0.241
	$\lambda = 0.5$	0.945	0.402	0.372	0.950	0.497	0.490	0.959	0.495	0.491	0.951	0.385	0.333
	$\lambda = 0.6$	0.983	0.547	0.504	0.985	0.660	0.640	0.982	0.675	0.644	0.979	0.504	0.446
	$\lambda = 0.7$	0.997	0.667	0.593	0.994	0.797	0.740	0.997	0.800	0.757	0.997	0.614	0.511
	$\lambda = 0.8$	1.000	0.770	0.641	1.000	0.890	0.804	0.999	0.891	0.791	1.000	0.725	0.528
$\rho = 0.3$	$\lambda = 0.1$	0.703	0.064	0.048	0.785	0.064	0.064	0.792	0.059	0.062	0.667	0.053	0.048
	$\lambda = 0.2$	0.815	0.099	0.096	0.861	0.112	0.112	0.849	0.108	0.111	0.798	0.098	0.086
	$\lambda = 0.3$	0.895	0.183	0.160	0.921	0.189	0.185	0.926	0.207	0.202	0.890	0.169	0.158
	$\lambda = 0.4$	0.946	0.293	0.249	0.965	0.348	0.306	0.958	0.312	0.279	0.953	0.270	0.235
	$\lambda = 0.5$	0.982	0.423	0.344	0.985	0.486	0.424	0.989	0.501	0.416	0.985	0.413	0.334
	$\lambda = 0.6$	0.991	0.566	0.445	0.995	0.651	0.543	0.993	0.663	0.553	0.993	0.524	0.388
	$\lambda = 0.7$	0.997	0.700	0.531	1.000	0.789	0.642	0.999	0.790	0.643	0.997	0.641	0.425
	$\lambda = 0.8$	1.000	0.780	0.548	1.000	0.884	0.663	1.000	0.899	0.693	1.000	0.742	0.431

Table 6: Empirical power of test statistics for H_0^{λ}: N=90

ρ	λ	*Normal*			*Student*			*Gamma*			*Asy − normal*		
		RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}	RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}	RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}	RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}
$\rho = 0.0$	$\lambda = 0.1$	0.253	0.072	0.082	0.268	0.076	0.083	0.257	0.090	0.095	0.257	0.073	0.080
	$\lambda = 0.2$	0.509	0.182	0.200	0.503	0.190	0.227	0.508	0.192	0.224	0.494	0.157	0.171
	$\lambda = 0.3$	0.749	0.346	0.387	0.782	0.408	0.474	0.777	0.410	0.458	0.755	0.313	0.373
	$\lambda = 0.4$	0.925	0.565	0.621	0.924	0.642	0.706	0.935	0.645	0.726	0.927	0.506	0.561
	$\lambda = 0.5$	0.986	0.756	0.821	0.983	0.846	0.889	0.992	0.860	0.906	0.983	0.683	0.739
	$\lambda = 0.6$	0.999	0.897	0.936	1.000	0.941	0.973	0.999	0.953	0.975	0.997	0.824	0.877
	$\lambda = 0.7$	1.000	0.948	0.972	1.000	0.986	0.994	1.000	0.988	0.998	1.000	0.920	0.948
	$\lambda = 0.8$	1.000	0.981	0.983	1.000	0.998	0.999	1.000	0.996	0.999	1.000	0.952	0.963
$\rho = 0.1$	$\lambda = 0.1$	0.544	0.083	0.097	0.594	0.082	0.115	0.570	0.081	0.106	0.522	0.081	0.103
	$\lambda = 0.2$	0.762	0.169	0.206	0.802	0.198	0.267	0.796	0.195	0.267	0.763	0.173	0.192
	$\lambda = 0.3$	0.922	0.334	0.404	0.926	0.400	0.501	0.927	0.419	0.500	0.920	0.319	0.373
	$\lambda = 0.4$	0.981	0.540	0.621	0.986	0.647	0.741	0.985	0.657	0.751	0.976	0.510	0.554
	$\lambda = 0.5$	0.997	0.739	0.799	0.996	0.829	0.896	0.999	0.847	0.904	0.997	0.705	0.752
	$\lambda = 0.6$	1.000	0.885	0.912	1.000	0.940	0.968	1.000	0.949	0.977	1.000	0.810	0.859
	$\lambda = 0.7$	1.000	0.942	0.960	1.000	0.988	0.994	1.000	0.982	0.995	1.000	0.913	0.937
	$\lambda = 0.8$	1.000	0.976	0.975	1.000	0.995	0.997	1.000	0.996	0.996	1.000	0.948	0.958
$\rho = 0.2$	$\lambda = 0.1$	0.815	0.082	0.108	0.882	0.085	0.148	0.881	0.077	0.131	0.793	0.084	0.107
	$\lambda = 0.2$	0.932	0.176	0.232	0.955	0.186	0.290	0.958	0.191	0.305	0.928	0.168	0.197
	$\lambda = 0.3$	0.982	0.355	0.421	0.990	0.405	0.535	0.989	0.418	0.527	0.980	0.302	0.356
	$\lambda = 0.4$	0.997	0.562	0.619	0.998	0.655	0.759	0.999	0.645	0.748	0.995	0.496	0.537
	$\lambda = 0.5$	1.000	0.756	0.789	1.000	0.835	0.888	1.000	0.833	0.893	1.000	0.692	0.714
	$\lambda = 0.6$	1.000	0.873	0.885	1.000	0.949	0.962	1.000	0.950	0.970	1.000	0.832	0.835
	$\lambda = 0.7$	1.000	0.946	0.927	1.000	0.989	0.986	1.000	0.984	0.986	1.000	0.921	0.882
	$\lambda = 0.8$	1.000	0.970	0.936	1.000	0.997	0.987	1.000	0.996	0.987	1.000	0.954	0.895
$\rho = 0.3$	$\lambda = 0.1$	0.969	0.080	0.106	0.986	0.088	0.176	0.985	0.074	0.154	0.958	0.077	0.100
	$\lambda = 0.2$	0.989	0.167	0.212	0.995	0.198	0.323	0.997	0.194	0.331	0.984	0.151	0.179
	$\lambda = 0.3$	0.998	0.334	0.379	1.000	0.381	0.508	1.000	0.392	0.513	0.997	0.325	0.342
	$\lambda = 0.4$	0.999	0.560	0.573	1.000	0.653	0.705	1.000	0.658	0.729	1.000	0.527	0.549
	$\lambda = 0.5$	1.000	0.755	0.735	1.000	0.841	0.853	1.000	0.847	0.856	1.000	0.707	0.667
	$\lambda = 0.6$	1.000	0.885	0.831	1.000	0.945	0.939	1.000	0.942	0.937	1.000	0.826	0.759
	$\lambda = 0.7$	1.000	0.948	0.874	1.000	0.985	0.965	1.000	0.989	0.964	1.000	0.917	0.819
	$\lambda = 0.8$	1.000	0.982	0.897	1.000	0.997	0.962	1.000	0.995	0.968	1.000	0.961	0.829

Table 7: Empirical power of test statistics for H_0^{λ}: N=180

ρ	λ	*Normal*			*Student*			*Gamma*			*Asy − normal*		
		RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}	RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}	RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}	RS_{λ}	LM_{λ}^{P}	RS_{λ}^{PD}
$\rho = 0.0$	$\lambda = 0.1$	0.478	0.109	0.131	0.501	0.124	0.158	0.499	0.123	0.148	0.480	0.111	0.119
	$\lambda = 0.2$	0.819	0.325	0.377	0.832	0.391	0.451	0.842	0.387	0.447	0.819	0.298	0.346
	$\lambda = 0.3$	0.968	0.613	0.683	0.968	0.715	0.787	0.978	0.721	0.786	0.966	0.571	0.644
	$\lambda = 0.4$	0.999	0.858	0.916	1.000	0.932	0.955	0.999	0.933	0.965	0.999	0.821	0.895
	$\lambda = 0.5$	1.000	0.967	0.985	1.000	0.991	0.999	1.000	0.990	0.997	1.000	0.956	0.984
	$\lambda = 0.6$	1.000	0.993	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.983	0.995
	$\lambda = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	1.000
	$\lambda = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\rho = 0.1$	$\lambda = 0.1$	0.858	0.121	0.168	0.903	0.118	0.231	0.892	0.115	0.194	0.859	0.105	0.147
	$\lambda = 0.2$	0.971	0.316	0.415	0.976	0.375	0.538	0.984	0.377	0.540	0.975	0.293	0.361
	$\lambda = 0.3$	0.997	0.647	0.750	0.998	0.719	0.857	0.999	0.710	0.847	0.997	0.557	0.677
	$\lambda = 0.4$	1.000	0.854	0.920	1.000	0.929	0.977	1.000	0.925	0.974	1.000	0.818	0.884
	$\lambda = 0.5$	1.000	0.967	0.985	1.000	0.988	0.998	1.000	0.990	0.998	1.000	0.939	0.972
	$\lambda = 0.6$	1.000	0.995	0.998	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.990	0.997
	$\lambda = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	1.000
	$\lambda = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\rho = 0.2$	$\lambda = 0.1$	0.991	0.118	0.192	0.995	0.123	0.318	0.995	0.110	0.313	0.981	0.109	0.165
	$\lambda = 0.2$	0.999	0.327	0.466	1.000	0.370	0.633	0.999	0.384	0.631	0.999	0.306	0.419
	$\lambda = 0.3$	1.000	0.616	0.765	1.000	0.707	0.875	1.000	0.726	0.885	1.000	0.559	0.692
	$\lambda = 0.4$	1.000	0.847	0.929	1.000	0.925	0.980	1.000	0.926	0.980	1.000	0.811	0.887
	$\lambda = 0.5$	1.000	0.965	0.984	1.000	0.988	0.999	1.000	0.986	0.999	1.000	0.942	0.969
	$\lambda = 0.6$	1.000	0.994	0.999	1.000	1.000	1.000	1.000	0.999	1.000	1.000	0.987	0.991
	$\lambda = 0.7$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.998	0.996
	$\lambda = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
$\rho = 0.3$	$\lambda = 0.1$	1.000	0.119	0.249	1.000	0.122	0.434	1.000	0.125	0.438	1.000	0.108	0.203
	$\lambda = 0.2$	1.000	0.326	0.517	1.000	0.383	0.702	1.000	0.371	0.702	1.000	0.316	0.451
	$\lambda = 0.3$	1.000	0.638	0.775	1.000	0.724	0.905	1.000	0.712	0.898	1.000	0.568	0.679
	$\lambda = 0.4$	1.000	0.856	0.925	1.000	0.924	0.985	1.000	0.929	0.982	1.000	0.822	0.872
	$\lambda = 0.5$	1.000	0.966	0.974	1.000	0.983	0.996	1.000	0.992	0.997	1.000	0.944	0.956
	$\lambda = 0.6$	1.000	0.992	0.993	1.000	0.998	1.000	1.000	0.999	1.000	1.000	0.988	0.980
	$\lambda = 0.7$	1.000	1.000	0.994	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.985
	$\lambda = 0.8$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

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Appendix A. The derivatives of the log-likelihood function

This appendix derives the first and second order derivatives of the log-likelihood function of the spatial autoregressive model with a spatial autoregressive disturbance. The log-likelihood function is given by

$$\ln L = -\frac{N}{2}\ln 2\pi - \frac{N}{2}\ln\sigma^2 + \ln|A| + \ln|B| - \frac{1}{2\sigma^2}u'u,$$

where $A = I_N - \rho W_1$, $B = I_N - \lambda W_2$ and $u = B(Ay - X\beta)$. Denoting $G_A = W_1 A^{-1}$ and $G_B = W_2 B^{-1}$, the partial derivatives of the log-likelihood function with respect to the parameter vector $(\beta', \sigma^2, \lambda, \rho)'$ can be written as

$$\begin{aligned}
d_\beta &= \tfrac{\partial \ln L}{\partial \beta} = \tfrac{1}{\sigma^2}(BX)'u,\\
d_{\sigma^2} &= \tfrac{\partial \ln L}{\partial \sigma^2} = -\tfrac{N}{2\sigma^2} + \tfrac{u'u}{2\sigma^4},\\
d_\lambda &= \tfrac{\partial \ln L}{\partial \lambda} = -\mathrm{tr}(G_B) + \tfrac{1}{\sigma^2}u'G_B u,\\
d_\rho &= \tfrac{\partial \ln L}{\partial \rho} = -\mathrm{tr}(G_A) + \tfrac{1}{\sigma^2}u'BW_1 y.
\end{aligned}$$

We use the fact that $\frac{\partial \ln|A|}{\partial\rho} = \mathrm{tr}\left(A^{-1}\frac{\partial A}{\partial \rho}\right)$ to derive the above equations (see Anselin (1988a, p.74)). The corresponding elements of the hessian matrix are calculated by

$$\begin{aligned}
J_\beta &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\beta\partial\beta'}\right) = \tfrac{1}{N\sigma^2}BX'BX,\\
J_{\beta\sigma^2} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\beta\partial\sigma^2}\right) = 0,\\
J_{\beta\lambda} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\beta\partial\lambda}\right) = 0,\\
J_{\beta\rho} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\beta\partial\rho}\right) = \tfrac{1}{N\sigma^2}(BX)'BG_A X\beta,\\
J_{\sigma^2} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\sigma^2\partial\sigma^2}\right) = \tfrac{1}{2\sigma^4},\\
J_{\sigma^2\lambda} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\sigma^2\partial\lambda}\right) = \tfrac{1}{N\sigma^2}\mathrm{tr}(G_B),\\
J_{\sigma^2\rho} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\sigma^2\partial\rho}\right) = \tfrac{1}{N\sigma^2}\mathrm{tr}(G_A),\\
J_\lambda &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\lambda^2}\right) = \tfrac{1}{N}T_{BB},\\
J_{\lambda\rho} &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\lambda\partial\rho}\right) = \tfrac{1}{N}T_{BC_A},\\
J_\rho &= -\tfrac{1}{N}E\left(\tfrac{\partial^2 \ln L}{\partial\rho^2}\right) = \tfrac{1}{N}T_{C_A C_A} + \tfrac{1}{N\sigma^2}\left(BG_A X\beta\right)'\left(BG_A X\beta\right),
\end{aligned}$$

where $T_{BB} = \mathrm{tr}\left[(G_B' + G_B)G_B\right]$, $T_{BC_A} = \mathrm{tr}\left[(G_B' + G_B)C_A\right]$ and $T_{C_A C_A} = \mathrm{tr}\left[(C_A' + C_A)C_A\right]$ with $C_A = BG_A B^{-1}$.

The information matrix can be expressed by the following results

$$
\begin{aligned}
K_{\beta} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \beta}\cdot\frac{\partial \ln L}{\partial \beta'}\right) = \frac{1}{N\sigma^2}BX'BX,\\
K_{\beta\sigma^2} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \beta}\cdot\frac{\partial \ln L}{\partial \sigma^2}\right) = \frac{\mu_3}{2N\sigma^6}(BX)'l_N,\\
K_{\beta\lambda} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \beta}\cdot\frac{\partial \ln L}{\partial \lambda}\right) = \frac{\mu_3}{N\sigma^4}(BX)'vec_D(G_B),\\
K_{\beta\rho} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \beta}\cdot\frac{\partial \ln L}{\partial \rho}\right) = \frac{1}{N\sigma^2}(BX)'(BG_AX\beta) + \frac{\mu_3}{N\sigma^4}(BX)'vec_D(C_A),\\
K_{\sigma^2} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \sigma^2}\cdot\frac{\partial \ln L}{\partial \sigma^2}\right) = \frac{1}{2\sigma^4} + \frac{\kappa_4}{4\sigma^8},\\
K_{\sigma^2\lambda} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \sigma^2}\cdot\frac{\partial \ln L}{\partial \lambda}\right) = \frac{1}{N\sigma^2}\mathrm{tr}(G_B) + \frac{\kappa_4}{2N\sigma^6}\mathrm{tr}(G_B),\\
K_{\sigma^2\rho} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \sigma^2}\cdot\frac{\partial \ln L}{\partial \rho}\right) = \frac{1}{N\sigma^2}\mathrm{tr}(G_A)+\frac{1}{2N\sigma^6}\left[(BG_AX\beta)'\, l_N\mu_3 + \kappa_4\mathrm{tr}(G_A)\right],\\
K_{\lambda} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \lambda}\cdot\frac{\partial \ln L}{\partial \lambda}\right) = \frac{1}{N}T_{BB} + \frac{\kappa_4}{N\sigma^4}vec_D'(G_B)\,vec_D(G_B),\\
K_{\lambda\rho} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \lambda}\cdot\frac{\partial \ln L}{\partial \rho}\right) = \frac{1}{N}T_{BC_A} + \frac{1}{N\sigma^4}\left\{\left[\mu_3(BG_AX\beta)' + \kappa_4 vec_D'(C_A)\right]vec_D(G_B)\right\},\\
K_{\rho} &= \frac{1}{N}E\left(\frac{\partial \ln L}{\partial \rho}\cdot\frac{\partial \ln L}{\partial \rho}\right),\\
&= \frac{1}{N}\left\{\frac{1}{\sigma^2}(BG_AX\beta)'(BG_AX\beta) + T_{C_AC_A} + \frac{1}{\sigma^4}\left[2(BG_AX\beta)'\mu_3 + \kappa_4 vec_D'(C_A)\right]vec_D(C_A)\right\}.
\end{aligned}
$$

In order to calculate the above equations we use the fact that $E(P'\epsilon\cdot\epsilon'Q\epsilon) = P'vec_D(Q)\mu_3$ and $E(\epsilon'P\epsilon\cdot\epsilon'Q\epsilon) = \kappa_4 vec_D'(P)vec_D(Q) + \sigma^4\left[\mathrm{tr}(P)\mathrm{tr}(Q) + \mathrm{tr}(Q^sP)\right]$, where P and Q are $N\times N$ matrices, $vec_D(Q)$ is a column vector formed by the diagonal elements of Q, $\mu_3 = E(u_i^3)$, $\mu_4 = E(u_i^4)$ and $\kappa_4 = \mu_4 - 3\sigma^4$ (See Lee (2007, pp. 494-504)).

Appendix B. Score test for $H_0^\lambda : \lambda = 0$

Note that under the null, $B = I$, $G_B = W_2$, $\mathrm{tr}(G_B) = 0$ and $vec_D(G_B) = 0$. When ρ is locally misspecified from a known constant ρ_0, under H_0^λ, $d(\theta)$, $J(\theta)$ and $K(\theta)$ can be expressed, respectively, by

$$
\begin{aligned}
d(\theta) &= \begin{pmatrix} \frac{1}{\sigma^2}X'u \\ -\frac{N}{2\sigma^2} + \frac{u'u}{2\sigma^4} \\ \frac{1}{\sigma^2}u'W_2u \\ -\mathrm{tr}(G_A) + \frac{1}{\sigma^2}u'W_1y \end{pmatrix},\\
J(\theta) &= \frac{1}{N\sigma^2}\begin{pmatrix} X'X & 0_{k\times1} & 0_{k\times1} & X'G_AX\beta \\ * & \frac{N}{2\sigma^2} & 0_{1\times1} & \mathrm{tr}(G_A) \\ * & * & \sigma^2T_{22} & \sigma^2T_{2A} \\ * & * & * & \sigma^2T_{AA} + (G_AX\beta)'(G_AX\beta) \end{pmatrix},\\
K(\theta) &= J(\theta) + K^*(\theta),
\end{aligned}
$$

where $T_{22} = \mathrm{tr}\left[(W_2' + W_2)W_2\right]$, $T_{AA} = \mathrm{tr}\left[(G_A' + G_A)G_A\right]$, $T_{2A} = \mathrm{tr}\left[(W_2' + W_2)G_A\right]$ and

$$K^*(\theta) = \frac{1}{N\sigma^4}\begin{pmatrix} 0_{k\times k} & \frac{\mu_3}{2\sigma^2}X'l_N & 0_{k\times 1} & X'vec_D(G_A)\mu_3 \\ * & \frac{N}{4\sigma^4}\kappa_4 & 0_{1\times 1} & \frac{1}{2\sigma^2}\left[(G_AX\beta)'l_N\mu_3 + \kappa_4\mathrm{tr}\,(G_A)\right] \\ * & * & 0_{1\times 1} & 0_{1\times 1} \\ * & * & * & \left[2(G_AX\beta)'\mu_3 + \kappa_4 vec_D'(G_A)\right]vec_D(G_A) \end{pmatrix}.$$

From (2.5) it follows that

$$\begin{aligned} J_{\lambda\rho\cdot\eta} &= J_{\lambda\rho} - J_{\lambda\eta}J_\eta^{-1}J_{\eta\rho} = \tfrac{1}{N}T_{2A}, \\ J_{\rho\cdot\eta} &= J_\rho - J_{\rho\eta}J_\eta^{-1}J_{\eta\rho} = \tfrac{1}{N}\left[T_{AA} - \tfrac{2}{N}\mathrm{tr}^2(G_A) + \tfrac{1}{\sigma^2}(G_AX\beta)'M_X(G_AX\beta)\right], \\ B_{\lambda\cdot\eta} &= K_\lambda + J_{\lambda\eta}J_\eta^{-1}K_\eta J_\eta^{-1}J_{\eta\lambda} - J_{\lambda\eta}J_\eta^{-1}K_{\eta\lambda} - K_{\lambda\eta}J_\eta^{-1}J_{\eta\lambda} = \tfrac{1}{N}T_{22}, \\ B_{\rho\cdot\eta} &= K_\rho + J_{\rho\eta}J_\eta^{-1}K_\eta J_\eta^{-1}J_{\eta\rho} - J_{\rho\eta}J_\eta^{-1}K_{\eta\rho} - K_{\rho\eta}J_\eta^{-1}J_{\eta\rho} = J_{\rho\cdot\eta} + B^*_{\rho\cdot\eta}, \\ B_{\rho\lambda\cdot\eta} &= K_{\rho\lambda} + J_{\rho\eta}J_\eta^{-1}K_\eta J_\eta^{-1}J_{\eta\lambda} - J_{\rho\eta}J_\eta^{-1}K_{\eta\lambda} - K_{\rho\eta}J_\eta^{-1}J_{\eta\lambda} = \tfrac{1}{N}T_{2A}, \end{aligned}$$

where $M_X = I_N - X\left(X'X\right)^{-1}X'$ and $B^*_{\rho\cdot\eta} = \frac{1}{N\sigma^4}\left[2\mu_3\left(G_AX\beta\right)'M_X\bar{G}_A + \kappa_4\bar{G}_A'\bar{G}_A\right]$ for $\bar{G}_A = vec_D(G_A) - \frac{1}{N}l_N\mathrm{tr}(G_A)$. Under H_0^λ, the resulting score test is given by

$$RS_\lambda^{PD} = \frac{\left\{\tilde{u}'W_2\tilde{u}/\tilde{\sigma}^2 - \tilde{T}_{2A}\left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1}\left[-\mathrm{tr}(G_A) + \tilde{u}'W_1y/\tilde{\sigma}^2\right]\right\}^2}{T_{22} - \tilde{T}_{2A}^2\left(N\tilde{J}_{\rho\cdot\eta}\right)^{-1} + \frac{1}{N}\tilde{T}_{2A}^2\left(\tilde{J}_{\rho\cdot\eta}^{-2}\tilde{B}^*_{\rho\cdot\eta}\right)}, \tag{A-1}$$

where $\tilde{u} = y - \rho_0W_1y - X\tilde{\beta}$ and $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$.

Appendix C. Score test for $H_0^\rho : \rho = 0$

Note that under the null hypothesis, $A = I$, $G_A = W_1$, $\mathrm{tr}\,(G_A) = 0$ and $vec_D\,(G_A) = 0$. When λ is locally misspecified from a constant λ_0, under the null hypothesis, $d(\theta)$, $J(\theta)$ and $K(\theta)$ are written by

$$\begin{aligned} d(\theta) &= \begin{pmatrix} \frac{1}{\sigma^2}(BX)'u \\ -\frac{N}{2\sigma^2} + \frac{u'u}{2\sigma^4} \\ -\mathrm{tr}(G_B) + \frac{1}{\sigma^2}u'G_Bu \\ \frac{1}{\sigma^2}u'BW_1y \end{pmatrix}, \\ J(\theta) &= \frac{1}{N\sigma^2}\begin{pmatrix} (BX)'BX & 0_{k\times 1} & 0_{k\times 1} & (BX)'BW_1X\beta \\ * & \frac{N}{2\sigma^2} & \mathrm{tr}\,(G_B) & 0_{1\times 1} \\ * & * & \sigma^2T_{BB} & \sigma^2T_{BC_1} \\ * & * & * & \sigma^2T_{C_1C_1} + (BW_1X\beta)'\,(BW_1X\beta) \end{pmatrix}, \\ K(\theta) &= J(\theta) + K^*(\theta), \end{aligned}$$

where $T_{BC_1} = \text{tr}\left[(G_B' + G_B)C_1\right]$ with $C_1 = BW_1B^{-1}$ and

$$K^*(\theta) = \frac{1}{N\sigma^4}\begin{pmatrix} 0_{k\times k} & \frac{\mu_3}{2\sigma^2}(BX)'l_N & (BX)'vec_D(G_B)\mu_3 & (BX)'vec_D(C_1)\mu_3 \\ * & \frac{N}{4\sigma^4}\kappa_4 & \frac{1}{2\sigma^2}\kappa_4\text{tr}(G_B) & \frac{1}{2\sigma^2}(BW_1X\beta)'l_N\mu_3 \\ * & * & \kappa_4 vec_D'(G_B)vec_D(G_B) & \left[\mu_3\left(BW_1X\beta\right)' + \kappa_4 vec_D'\left(C_1\right)\right]vec_D\left(G_B\right) \\ * & * & * & \left[2(BW_1X\beta)'\mu_3 + \kappa_4 vec_D'(C_1)\right]vec_D(C_1) \end{pmatrix}.$$

From (2.5) it follows that

$$\begin{aligned} J_{\rho\lambda\cdot\eta} &= J_{\rho\lambda} - J_{\rho\eta}J_\eta^{-1}J_{\eta\lambda} = \tfrac{1}{N}T_{BC_1}, \\ J_{\rho\cdot\eta} &= J_\rho - J_{\rho\eta}J_\eta^{-1}J_{\eta\rho} = \tfrac{1}{N}\left[T_{C_1C_1} + \tfrac{1}{\sigma^2}\left(BW_1X\beta\right)' M_{BX}\left(BW_1X\beta\right)\right], \\ J_{\lambda\cdot\eta} &= J_\lambda - J_{\lambda\eta}J_\eta^{-1}J_{\eta\lambda} = \tfrac{1}{N^2}\left[NT_{BB} - 2\text{tr}^2(G_B))\right], \\ B_{\lambda\cdot\eta} &= K_\lambda + J_{\lambda\eta}J_\eta^{-1}K_\eta J_\eta^{-1}J_{\eta\lambda} - J_{\lambda\eta}J_\eta^{-1}K_{\eta\lambda} - K_{\lambda\eta}J_\eta^{-1}J_{\eta\lambda} = J_{\lambda\cdot\eta} + B^*_{\lambda\cdot\eta}, \\ B_{\rho\cdot\eta} &= K_\rho + J_{\rho\eta}J_\eta^{-1}K_\eta J_\eta^{-1}J_{\eta\rho} - J_{\rho\eta}J_\eta^{-1}K_{\eta\rho} - K_{\rho\eta}J_\eta^{-1}J_{\eta\rho} = J_{\rho\cdot\eta} + B^*_{\rho\cdot\eta}, \\ B_{\rho\lambda\cdot\eta} &= K_{\rho\lambda} + J_{\rho\eta}J_\eta^{-1}K_\eta J_\eta^{-1}J_{\gamma\lambda} - J_{\rho\eta}J_\eta^{-1}K_{\eta\lambda} - K_{\rho\eta}J_\eta^{-1}J_{\eta\lambda} = J_{\rho\lambda\cdot\eta} + B^*_{\rho\lambda\cdot\eta}, \end{aligned}$$

where

$$\begin{aligned} B^*_{\lambda\cdot\eta} &= K^*_\lambda + J_{\lambda\eta}J_\eta^{-1}K^*_\eta J_\eta^{-1}J_{\eta\lambda} - J_{\lambda\eta}J_\eta^{-1}K^*_{\eta\lambda} - K^*_{\lambda\eta}J_\eta^{-1}J_{\eta\lambda} \\ &= \tfrac{\kappa_4}{N\sigma^4}\left[vec_D'(G_B)vec_D(G_B) - \tfrac{\text{tr}^2(G_B)}{N}\right], \\ B^*_{\rho\cdot\eta} &= K^*_\rho + J_{\rho\eta}J_\eta^{-1}K^*_\eta J_\eta^{-1}J_{\eta\rho} - J_{\rho\eta}J_\eta^{-1}K^*_{\eta\rho} - K^*_{\rho\eta}J_\eta^{-1}J_{\eta\rho} \\ &= \tfrac{1}{N\sigma^4}\left[2(BW_1X\beta)'M_{BX}vec_D(C_1)\mu_3 + \kappa_4 vec_D'(C_1)vec_D(C_1)\right], \\ B^*_{\rho\lambda\cdot\eta} &= K^*_{\rho\lambda} + J_{\rho\eta}J_\eta^{-1}K^*_\eta J_\eta^{-1}J_{\eta\lambda} - J_{\rho\eta}J_\eta^{-1}K^*_{\eta\lambda} - K^*_{\rho\eta}J_\eta^{-1}J_{\eta\lambda} \\ &= \tfrac{1}{N\sigma^4}\left\{(BW_1X\beta)'M_{BX}\mu_3\left[vec_D(G_B) - \tfrac{1}{N}l_N\text{tr}(G_B)\right] + \kappa_4 vec_D'\left(C_1\right)vec_D\left(G_B\right)\right\}. \end{aligned}$$

Thus

$$\begin{aligned} \tilde{B}^*_{\rho\cdot\eta} + \tilde{C}^*_\rho &= J_{\rho\lambda\cdot\eta}J^{-1}_{\lambda\cdot\eta}B^*_{\lambda\cdot\eta}J^{-1}_{\lambda\cdot\eta}J_{\lambda\rho\cdot\eta} - J_{\rho\lambda\cdot\eta}J^{-1}_{\lambda\cdot\eta}B^*_{\lambda\rho\cdot\eta} - B^*_{\rho\lambda\cdot\eta}J^{-1}_{\lambda\cdot\eta}J_{\lambda\rho\cdot\eta} + B^*_{\rho\cdot\eta} \\ &= \tfrac{1}{N\sigma^4}\left[\kappa_4 F'F + 2\mu_3(BW_1X\beta)'M_{BX}F\right], \end{aligned}$$

where $M_{BX} = I_N - (BX)\left[(BX)'(BX)\right]^{-1}(BX)'$, $F = vec_D(C_1) - J_{\rho\lambda\cdot\eta}J^{-1}_{\lambda\cdot\eta}\bar{G}_B$ and $\bar{G}_B = vec_D(G_B) - \frac{1}{N}l_N\text{tr}(G_B)$. Under H_0^ρ, the score test that robust to local and distributional misspecifications can be expressed as

$$RS_\rho^{PD} = \frac{\left\{\tilde{u}'BW_1y/\tilde{\sigma}^2 - \tilde{T}_{BC_1}(N\tilde{J}_{\lambda\cdot\eta})^{-1}\left[-\text{tr}(G_B) + \tilde{u}'G_B\tilde{u}/\tilde{\sigma}^2\right]\right\}^2}{N\tilde{J}_{\rho\cdot\eta} - \frac{1}{N}\tilde{T}^2_{BC_1}(\tilde{J}_{\lambda\cdot\eta})^{-1} + N\left(\tilde{B}^*_{\rho\cdot\eta} + \tilde{C}^*_\rho\right)}, \qquad \text{(A-2)}$$

where $\tilde{u} = (I - \lambda_0 W_2)(y - X\tilde{\beta})$ and $\tilde{\sigma}^2 = \tilde{u}'\tilde{u}/N$.