A linear relationship between market prices of risks and risk aversion in complete stochastic volatility models

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Abstract

Considering a production economy with an arbitrary von-Neumann Morgenstern utility, this paper derives a general equilibrium relationship between the market prices of risks and market risk aversion under a continuous time stochastic volatility model completed by liquidly traded options. The derived relation shows that in equilibrium the risk aversion should be a linear combination of the market price of asset risk and market price of orthogonal risk. Construction of a daily market risk aversion index is proposed to help practitioners with better risk management.

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Literature Review

Assuming no arbitrage, any traded financial asset can be priced as a mathematical expectation of its future payoff weighted by a market wide pricing kernel. Typically the pricing kernel is defined as an expression of market price of risks (MPRs) which are traditionally defined to be excess return the market compensates for taking additional unit of risk. In complete markets, additional risk in options due to higher leverage over its underlying is not rewarded as it can be perfectly hedged away by a replicating portfolio composed of the underlying and riskless assets. Market only rewards the risk associated with the underlying (called market price of asset risk, MPAR) and the risk premium is already reflected in the underlying asset price. In a diffusion-based stochastic volatility model (SVM) that introduces market incompleteness, however, risk in options will be captured not only by the MPAR, but also by market price of orthogonal risk (MPOR)[2]. Intuitively then both MPRs are intrinsically linked to market investors' risk preference. A rigorous exploration of the relationship between the MPRs and market risk aversion (MRA), however, requires a general equilibrium microeconomic model in line with either Lucas [1978] or Cox et al. [1985]. Breeden [1979], Cox et al. [1985], Duffie [1992] and references therein show that in equilibrium MPAR is a product of risk aversion and the instantaneous covariance of the underlying asset return and aggregate consumption. Bates [1988], Bakshi and Kapadia [2003] and Bollerslev et al. [2010] mention the fact that volatility risk premium (VRP), which is proportional to a linear combination of MPAR and MPOR, equals the instantaneous covariance between the pricing kernel and variance processes. This relation can be further specified to generate a linear relation between MRA and MPAR if additional assumptions, namely a square root variance process and a power utility for the representative agent, are made (Bollerslev et al. [2010])[3]. Given the documented difficulties involved in using aggregate consumption data and estimation of the market pricing kernel, however, these relations are more of academic interest than any practical use. The closest paper to ours is Lewis [2000] which provides a rigorous utility-based analysis on equilibrium MPRs under different variance process assumptions. We take a further step by relaxing the power utility assumption in Lewis [2000], and derive explicitly a linear relation between MRA and MPAR and MPOR.

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[2] It is "orthogonal" because it is the risk premium associated with the second random source of stochastic volatility which is orthogonal to the randomness driving the underlying asset price process. While not obvious, the MPOR is consistent with the traditional definition of market price of risk as excess return per unit of risk of a zero-beta hedging portfolio. See Lewis [2000] for a demonstration.

[3] In fact, with these two assumptions the equilibrium MPOR equals zero, i.e. the orthogonal risk is not priced by the market (Lewis [2000]). This corresponds to the minimal equivalent martingale $\hat{\mathrm{P}}$ -measure proposed by FÖllmer and Schweizer [1991]. Pham and Touzi [1996] show that equilibrium supporting utility functions corresponding to a $\hat{\mathrm{P}}$-measure can only take log utility form. The "discrepancy" is due to the different assumptions on dividend policy in that Pham and Touzi [1996] consider a pure exchange economy while the papers mentioned above follow Cox et al.[1985] production economy.

We establish the relation by considering the equilibrium relation between MPRs and pricing kernel first. This has been explored in several previous studies, although the focus of those studies is not the relation per se, but the consistency of the state price system, or equivalently the viability of MPRs, with respect to the economic equilibrium. He and Leland [1993] consider a pure exchange economy endowed with one unit of risky asset. Dividends are ignored and the risk free rate is treated exogenous as there is no intermediate consumption[4]. They find a partial differential equation (PDE) condition for the risk premium and the equilibrium relation between risk premium and pricing kernel on the terminal date. Examples of constant volatility are given to illustrate how specifications of utility function affect the equilibrium form of excess return. Pham and Touzi [1996] consider a similar economy but introduce stochastic volatility. The market is completed by a contingent claim on the underlying under some regularity conditions. They find necessary and sufficient conditions for viable risk premium functions and relate them to the pricing kernel in the case of positive dividends[5]. On the contrary to He and Leland [1993], they illustrate with examples how assumptions on risk premium can be supported by interim and terminal utility functions. Lewis [2000] instead considers a production economy within the framework of Cox et al. [1985] and also a SVM. Assuming power utility throughout, he derives a partial differential equation for risk premium coefficient, a concept closely related to the MPRs. Examples are given to illustrate viable forms of risk premium corresponding to different assumptions on the variance process.

While we establish the model to derive the relation between MPRs and the pricing kernel, we try to make it as realistic as possible. First, we choose to consider a production economy as in Lewis [2000] to fit the real world observation that dividends are usually exogenously determined. Stock shares is not a constant, instead it varies from time to time as companies issue new shares or buy back old shares from the market. Moreover, we specify the whole underlying asset process as exogenous to allow for a direct number-plug-in in any future empirical work. Second, to deal with market completeness, Lewis [2000] assumes identical individual preferences. Pham and Touzi [1996] complete the market with traded contingent claims under very restrictive conditions and strong assumptions on the model coefficients. We ensure the existence of a representative agent more naturally by applying more recent advances in this field[6]. Romano and Touzi [1997] show that under restrictive conditions on the drift and volatility coefficients in the state variable process and the

[4] He and Leland [1993] discuss the case involving intermediate consumption and dividends as an extension.

[5] Their results cannot be derived as a case of two risky assets in He and Leland because the total supply of the underlying risky asset is constrained to be one unit in Pham and Touzi [1996].

[6] Another reason why we complete the market using put options is that it allows for heterogeneous preferences for market players who trade different put options. And market completeness is critical in transforming the representative agent's dynamic portfolio allocation problem to its static optimization equivalence. The first order condition of the static problem is a critical link to derive the equilibrium relation between the MRA and MPRs. Details will be provided in section 2.

volatility risk premium, any European contingent claim completes the market. Davis and Obloj [2007] extend conditions proposed in Davis [2004] to a necessary and sufficient condition for market completeness using vanilla or path-dependent derivatives[7]. In particular, they show that for a SVM, if the drift and volatility coefficients in a Markov state variable process are both analytic, and other loose conditions[8] hold, then the market can be completed by any bounded European contingent claim such as a put option[9][10]. We emphasize that these conditions are imposed on the coefficients of the state variable process which are treated exogenous in our paper. They are not imposed on coefficients of the contingent claim price process which are endogenous. Therefore we avoid the issue of “endogenous completeness” as recently studied in Anderson and Raimondo [2008] and Hugonnier, Malamud and Trubowitz [2009].

Our expression of the relation between MPRs and pricing kernel is similar to that derived in He and Leland [1993] and Pham and Touzi[11], despite the fact that we are considering a production economy rather than a pure exchange economy[12]. To be more specific, in a SVM the MPRs are found in equilibrium to be linear combinations of partial derivatives of pricing kernel with respect to the underlying asset and the stochastic variance. This makes intuitive sense because pricing kernel is a continuous time version of Arrow-Debreu security prices, the derivative of which with respect to a risk factor reflects the market compensation in units of utility for taking additional risk in that factor. It is also interesting to find that, if we plug our derived equilibrium relation between MPRs and pricing kernel to the popular partial differential equation for contingent claim prices we can recover the fundamental valuation formula in Garman [1976].

We then proceed and solve the representative agent’s optimization problem and define the MRA as the Arrow-Pratt relative risk aversion in terms of the indirect utility function to arrive at a relation between the MRA and pricing kernel[13]. Using

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[7] See Theorem 4.1 in the article where an invertibility condition is imposed on the gradient matrix of the pricing functions with respect to the underlying state variables.

[8] See (4.3) and (A4) in the article.

[9] See Proposition 5.1 in the article.

[10] Using a slightly different approach, Jacod and Protter [2007] show that an unstable complete market may be obtained by imposing some complex compatibility conditions between the underlying asset price and option prices. Carr and Sun [2007] suggest that under certain hypothesis the market can completed by variance swaps now liquidly traded in the market.

[11] We also did a full analysis on viable risk premium as in Pham and Touzi. Similar partial differential equations for the MPAR and MPOR are derived. But since they are not the focus of this paper, we put them in the appendix for interested readers.

[12] This is not surprising because we can show that the relation holds as long as the market under consideration is complete.

[13] A common approach adopted in the literature to relate risk aversion to the pricing kernel is to write $G(t)=\frac{U_c(c,t)}{U_c(c,0)}$, then define the risk aversion $\gamma(t)=-c(t)\frac{U_{cc}(c,t)}{U_c(c,t)}$, hence $\gamma(t)=-s(t)\frac{G_s(t)}{G(t)}$. However,

pricing kernel as a bridge, we establish a linear relation between the MPRs and MRA. This provides a new approach to extract empirical MRA which compared with many current approaches is easier to obtain from option prices. The only paper that we are aware of and is close to our approach of estimating MRA from MPRs is Bollerslev et al. [2010]. As aforementioned however, their results depend on strong assumptions made on specific forms taken by the representative agent's utility function, the volatility of the variance process and the volatility risk premium. In particular, the MPOR is ignored and assumed to be zero in their case.

Studies on implied MRA have a long history and seem to break into two approaches. The classic equity premium literature (Friend and Blume [1975], Kydland and Prescott [1982], Mehra and Prescott [1985] and many others) estimates MRA using consumption and equity return data. Efforts are also made to estimating MRA from derivatives market. Sprenkel [1967] extracted MRA using his warrant pricing formula involving risk aversion parameter. Bartunek and Chowdhury [1997] is the first to estimate MRA from option prices. Assuming a power utility function they estimate the coefficient of constant relative risk aversion (CRRA). Adding exponential utility functions Bliss and Panigirtzoglou [2004] obtain estimates of MRA using British FTSE 100 and S&P 500 index options. Kang and Kim [2006] extend the analysis by assuming wider classes including HARA, log plus power and linear plus exponential utility functions. Despite the facility of implementing the models, however, results derived using the preference-based approach may be misleading because the models employed above either fail to incorporate stochastic volatility which should be included for any reasonable option pricing model, or depend heavily on utility function forms. Other attempts in the literature to deal with these two issues include Benth, Groth and Lindberg [2009] who estimate MRA with a utility indifference approach assuming an exponential utility under a SVM. They find a smiling MRA across strikes and time to maturities, concluding that certain crash risk is not captured by the SVM. Blackburn [2008] extracts the MRA and inter-temporal substitution assuming a non-time-separable Epstein-Zin type utility function. He finds reasonable estimates of the risk aversion parameter and claims that changes to risk aversion are closely related to changes of market risk premium, which is consistent with the theoretical results we derived. In contrast, our approach takes into account stochastic volatility and accommodates any form of von-Neumann Morgenstern utility function[14].

On the other strand of research, the MRA and pricing kernel (MRA is the negative of the derivative of log pricing kernel) are implied from the distance between the option implied risk neutral density and market subjective density of the underlying prices (Jackwerth [2000], Ait-Sahalia and Lo [2000], Rosenberg and Engel [2002], Bliss and Panigirtzoglou [2002], Perignon and Villa [2002], Figlewski [2008]).That

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the derivation requires assuming an exchange economy so that in equilibrium the underlying stock equals the consumption.

[14] In fact, the author has done a similar analysis with habit formation utility and derives identical relation between the MRA and MPRs. So the result presented is actually quite general.

the empirical pricing kernel derived is locally increasing against wealth levels (or equivalently the implied MRA is locally negative) is termed the pricing kernel puzzle[15]. Using different estimation methods for pricing kernels, Aı¨t-Sahalia and Lo [2000], Jackwerth [2000], and Rosenberg and Engle [2002] report that implied MRA exhibits strong U-shape across S&P 500 index values and the current forward prices. However, Singleton (2006) doubt that these findings may not be robust due to the simplified assumptions made about the underlying economy and the dimensionality of the state vector. Our approach of estimating MRA avoids estimating the risk neutral and subjective densities.

Instead, we require the extraction of MPOR from the market observed option prices, which is relatively easy to implement nowadays. Unfortunately little study has been done on the empirical behavior of MPOR. Most papers (Chernov et al. [2000], Coval and Shumway [2001], Bakshi and Kapadia [2003], Bollerslev et al. [2010] and others) estimate a closely related concept -VRP under SVMs by considering the difference between mean reversion parameters estimated under subjective and risk neutral measures. They report that stochastic volatility is priced in index option markets and the sign of VRP is negative which is consistent with general equilibrium theory. Guo [1998] estimates VRP in the foreign currency options market using Heston’s SVM. Doran and Ronn [2008] compute the market price of volatility risk in the energy commodity markets. Pirrong and Jermakyan [2008] extract risk premium directly from the prices of power claims. All three find significant VRP in these non-equity markets and claim that it is a critical factor in pricing contingent claims in these markets. Adopting a slightly different approach, Bollerslev et al. [2009] examine the difference between model-free risk neutral expected return variations and current realized variations as a measurement of VRP, and show that it is a strong predictor of stock market returns. Although these papers are not dealing with MPOR directly, we are happy to point out that their approaches can be readily adopted or adapted to estimate the MPOR.

With all these studies and various methods proposed in the literature to extract either MRA or VRPs, however, they almost exclusively aim toward the time-series property of the interested variables. As will be discussed in details in section 2, the idea of completing markets using traded European options with different strike prices allows us to extract an MRA or MPOR surface not only across the strikes and time to maturities for each day, but also the time evolution of the surface over periods. This should provide richer information than previous empirical studies about market players’ aggregated risk preference. Another new feature of our study is that we explicitly separate the two MPRs – MPAR and MPOR to study their individual behaviors, which are broadly ignored in most empirical works. Yet as shown in the relation we derived, the sign and value of MPOR is found to be critical in determining

[15] Jackwerth [2004] provides a good survey on this topic. Since then, Brown and Jackwerth [2004], Ziegler [2007], Chabi-Yo, Garcia and Renault [2007] offer possible explanations trying to reconcile the puzzle.

whether the risk aversion is greater or less than the classic risk aversion as in the classic Black-Scholes case.

The rest of the paper is organized as follows. Section 2 describes the economy and model assumptions. Section 3 derives the equilibrium relation between MPRs and risk aversion using a dynamic programming approach. Section 4 concludes and discusses its potential usage in practice.

Model

In this section we follow Cox et al. [1985] and consider a production economy. Capital goods are invested to solely produce a risky asset S_t with a linear technology. The production process follows a stochastic differential equation[16], the drift and volatility terms of which are dependent on state variables. There are potentially many state variables that can affect the production process. However, for simplification we assume that there is only one state variable - the variance of the risky asset ς_t - that affects the asset price. The variance itself follows a stochastic differential equation.

Assumption 1: *The joint production and state variable processes* (S_t, ς_t) *form a stochastic volatility model (SVM) as follows:*

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\begin{aligned}
dS_t &= [\mu_1(t, S_t, \varsigma_t) - D(t, S_t, \varsigma_t)]dt + \sigma_1(t, S_t, \varsigma_t) dW_{1t} \\
d\varsigma_t &= \mu_2(t, S_t, \varsigma_t)dt + \sigma_2(t, S_t, \varsigma_t) dZ_t \\
dZ_t &= \rho_t dW_{1t} + \sqrt{1-\rho_t^2}\, dW_{2t}
\end{aligned}
\tag{2.0}
$$

where ρ_t is the instantaneous correlation coefficient between the two-dimensional Brownian motion (W_{1t}, W_{2t}).

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[16] The probabilistic structure of the economy is based on a complete probability space $(\Omega, \mathbb{F}, \mathrm{P})$ where Ω is the event space with a typical element ω, $\mathbb{F}$ the sigma algebra of observable events, P a probability measure assigned on $(\Omega, \mathbb{F})$. Let I_t denote the information set faced by investors at time t, then the payoff space $\Gamma_t^+ = \{\Pi_t \in I_t : E[\Pi_t^2 | I_{t-1}] < \infty\}$ where Π_t is the asset payoff at time t. That is, the payoff space is the set of all random variables with finite conditional second moments given the previous period information. For the model we consider a market consisting of risky assets (the underlying stock and European options written on it) and a riskless asset with an instantaneous interest rate r. We use Γ_t to denote the payoff space on the market at time t, so $\Gamma_t \subset \Gamma_t^+$ for all time t.

The dividend process $D_t = \delta_t S_t$ with dividend rate δ_t. As in Lewis [2000] we treat the dividend policy as an exogenous variable to allow for the variance of stock shares. This also facilitates the later empirical study where dividends are taken directly from the market observations. Note that we intentionally leave the variance process unspecified so square root, 3/2 models are included. We can write the SVM in matrix form as $d\vec{S}_t = \vec{\mu}dt + \sigma \bullet d\vec{W}_t$.

To analyze the investors' portfolio choice problem, we need to specify the investment opportunity set.

Assumption 2: *Investors can borrow and lend at an interest rate r, which is determined endogenously in equilibrium.*

There are potentially infinitely many contingent claims traded in the market. We focus on those contingent claims in a basis[17].

Assumption 3: *A set composed of the riskless asset B, the risky underlying asset S and any traded European put or call option F written on the underlying asset form a basis.*

Assumption 3 is equivalent to assume that the exogenously given model coefficients in (2.0) are such that any of these put or call options completes the market in the sense of Davis and Obloj [2007]. Consider a finite time horizon [0, T]. The market has traded European options $\{V_{ij}\}$ with time to maturities $0 \le T_1 \le ... T_i \le ... \le T_M = T$. Let N_{T_i} denote the number of strikes for the contract with time to maturity T_i so there are (N*M) traded European options. If the current time is within $[0, T_i]$ we can essentially use any put/call option with equal or longer time to maturity to complete the market. As old options expire and new options being introduced, the hedging opportunities available to investors change. On the other hand, investors' trading with different strikes also reflects their probabilistic view of the future market movements (Breeden and Litzenberger [1978]). For each of these (N*M) markets then, there corresponds a representative agent with a certain risk attitude. It is exactly this feature that allows practitioners to be able to extract MRA cross-sectionally. For illustration purposes, in the following we use an arbitrary valid put option to complete the market. Before we define the investors' allocation problem we make the following assumption:

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[17] A basis is defined in Cox et al. [1985] as a set of production processes and a set of contingent claims that span the market.

Assumption 4: *The market is efficient such that no arbitrage principle holds.*

Combined with market completeness, this indicates a unique and strictly positive pricing kernel on the payoff space Γ_t. The price function under the original probability measure P is $p_t = E^{\mathrm{P}}\left[G(t,T)\Pi_T \left| I_t \right.\right]$ for all $\Pi_T \in \Gamma_T$ where $G(t,T)$ is the pricing kernel. Defining a risk neutral Q-measure by $\left.\frac{dQ}{dP}\right| I_t = e^{\int_0^t r_s ds} G(t)$ so $p_t = E^Q[e^{-r(T-t)}\Pi_T \left| \mathrm{I}_t \right.]$.

Using the fact that the combined stock and volatility process $\{S_t, \varsigma_t\}$ is Markovian, the put option price can be written as

$$V(t, S_t, \varsigma_t) = E^Q\left[e^{-\int_t^T r_u du} (K - S_T)^+ \left| F_t \right. \right]$$

The pricing kernel process $G(t)$ is written as

$$G(t) = \exp\left\{ -\int_0^t r_u du - \frac{1}{2}\int_0^t (\lambda^2{}_{1u} + \lambda^2{}_{2u}) du - \int_0^t \lambda_{1u} dW_{1u} - \int_0^t \lambda_{2u} dW_{2u} \right\}$$

where λ_{1u} and λ_{2u} are adapted process of market price of asset risk and market price of orthogonal risk respectively. Define a new two-dimensional Brownian motion $(\tilde{W}_{1t}, \tilde{W}_{2t})$ as $\tilde{W}_{1t} = W_{1t} + \int_0^t \lambda_{1u} \mathrm{du}$, $\widetilde{W}_{2t} = W_{2t} + \int_0^t \lambda_{2u} \mathrm{du}$. The risk neutral version of the SVM is then

(2.1)

$$dS_t = [\mu_{1t} - D_t - \lambda_{1t}\sigma_{1t}]dt + \sigma_{1t} d\tilde{W}_{1t}$$
$$d\varsigma_t = [\mu_{2t} - (\rho_t \lambda_{1t} + \sqrt{1-\rho_t^2}\lambda_{2t})\sigma_{2t}]dt + \rho_t \sigma_{2t} d\tilde{W}_{1t} + \sqrt{1-\rho_t^2}\sigma_{2t} d\tilde{W}_{2t}$$

The term $(\rho_t \lambda_{1t} + \sqrt{1-\rho_t^2}\lambda_{2t})\sigma_{2t}$ is the instantaneous volatility risk premium widely studied in the literature, and $\rho_t \lambda_{1t} + \sqrt{1-\rho_t^2}\lambda_{2t}$ is the instantaneous market price of volatility risk (MPVR). Applying Ito's lemma to the put option price, we have $dV_t = \mu_3(t, V_t)dt + F(t, s, \varsigma) d\hat{W}_t$

The representative agent's investment opportunity set includes: the stock, the put option and a riskless bond, the latter two being purely financial assets with zero net supply. Let α_t denote the amount invested in the underlying stock and β_t the amount invested in the put option and e_t her total wealth at time t. The agent's consumption is described by the pair of consumption rate process $\{c_t\}$ and final wealth e_T. The wealth process is then expressed through her portfolio process $(\alpha_t, \beta_t, e_t - \alpha_t - \beta_t)$:

(2.2) $$de_t = \left[\alpha \frac{\mu_1}{S} + \beta \frac{\mu_3}{V} + (e - \alpha - \beta)r - c\right]dt + \alpha \frac{\sigma_1}{S} dW_1 + \beta \frac{F}{V} d\hat{W}$$

A consumption and portfolio strategy (c_t, α_t, β_t) is *feasible* if it satisfies (2.2) with a nonnegative wealth process. We let $\Psi(S_0)$ denote the set of all feasible consumption-portfolio strategies with initial wealth S_0. The representative agent then pursues the following investment problem:

(2.3) $$\max_{(\alpha_t, \beta_t, c_t) \in \Psi(S_0)} E[\int_0^T u(t, c_t)dt + U(e_T)]$$

Equilibrium conditions

We say that the market is in equilibrium if the representative agent optimally chose to hold only underlying stocks (total shares normalized to be unity) and zero units of options. That is, the market clears for all t: $\alpha_t^* = S_t, \beta_t^* = 0 \ \forall t \in [0,T]$. It follows that in equilibrium the total wealth $e_t^* = S_t \ \forall t \in [0,T]$. Proposition1 states the equilibrium relation between MPRs and pricing kernel[18]:

Proposition 1*: In an economy as specified above, in equilibrium*

(1) $$G(t) = \frac{J_e(t)}{J_e(0)} = \frac{u_c\left(t, c^*(t)\right)}{u_c\left(0, c^*(0)\right)}$$

$$G(T) = \frac{U_S(T, S_T)}{U_s(0, S_0)}$$

[18] For completeness, all other non relevant results including equilibrium interest rate, the partial differential equations to be satisfied by viable[18] risk premiums, and optimal consumption rate are presented in the Appendix.

where $J(.)$ *is the value function of the optimization problem (2.3) and C* is the equilibrium consumption that will be solved endogenously .This is equivalent to state that in equilibrium the pricing kernel* $G(t)$ *can be supported by some utility functions.*

(2) *If* $(\lambda_{1t},\lambda_{2t})$ *is viable, then:*

$$\lambda_1(t)=-\sigma_1(t)G_s(t)/G-\rho(t)\sigma_2(t)G_\varsigma(t)/G$$
$$\lambda_2(t)=-\sqrt{1-\rho(t)^2}\,\sigma_2(t)G_\varsigma(t)/G$$

with terminal conditions:

$$\lambda_1(T)=-\sigma_1 U_{ss}(T)/U_s(T)$$
$$\lambda_2(T)=0$$

Proof: Let $J(t,e_t,V_t)$ be the value function for problem (2.3). Then Hamilton-Jacobi-Bellman equation gives:

$$0=\max_{(\alpha,\beta)}\left[u(t,c_t)+E_t\left(\frac{dJ}{dt}\right)\right]$$

Apply Ito's lemma to $J(t,e_t,V_t)$ and plug in the stochastic differential equations for the underlying asset, the put option and the wealth, we have:

$$(2.4)\qquad 0=u(c)+J_t+aJ_e+\mu_3 J_V+\frac{1}{2}J_{ee}(b^2+c^2)+\frac{1}{2}J_{VV}F^2+(\beta\frac{F^2}{V}+\alpha\frac{\sigma_1 F}{S}\rho^{'})J_{Ve}$$

First order conditions with respect to c, α and β are:

$$0=u_c(c,t)-J_e \qquad (A.1)$$

$$(2.5)\ 0=\left(\frac{\mu_1-D}{S}-r\right)J_e+\frac{\sigma_1}{S}\left(\frac{\alpha\sigma_1}{S}+\rho^{'}\frac{\beta F}{V}\right)J_{ee}+\frac{\sigma_1 F\rho^{'}}{S}J_{Ve} \qquad (A.2)$$

$$0=\left(\frac{\mu_3}{V}-r\right)J_e+\left[\left(\frac{\alpha\sigma_1}{S}+\rho^{'}\frac{\beta F}{V}\right)\frac{\rho^{'}F}{V}+\left(1-\rho^{'2}\right)\frac{\beta F^2}{V^2}\right]J_{ee}+\frac{F^2}{V}J_{Ve} \qquad (A.3)$$

Since the market is complete, it is well known (Cox and Huang [1989]) that the stochastic control problem (2.3) can be transformed to a static optimization problem as follows:

$$\max_{\{c_t\}} E[\int_0^T u(t,c_t,x_t)dt + U(e_T)]$$

(2.6)

$$s.t.\ E\left[\int_0^T [G(t)c(t) + G(T)e(T)]dt \middle| I_t\right] \le S_0$$

Its first order condition is $u_c(t) = \varepsilon G(t)$, which combined with (A1) in (2.5) gives our desired result $\frac{J_e(t,e_t,V_t)}{J_e(0)} = G(t,S_t,V_t)$, hence there exists a (indirect) utility function that supports the pricing kernel. The terminal condition comes from the fact that at the terminal period the agent will not care about the volatility and just consume the underlying asset.

To derive condition (2) in the proposition, we define a function H such that

$$H(t,S_t,\varsigma_t) = E\left[e^{r(T-t)}U'(S_T)\middle|S_t,\varsigma_t\right] = J_e(0)E\left[e^{r(T-t)}G(S_T)\middle|S_t,\varsigma_t\right] = J_e(0)G(t,S_t,\varsigma_t)$$

The last equivalence comes from the fact that $e^{rt}G(t)$ is a martingale. Now applying Ito's lemma to $H(t,S_t,\varsigma_t)$ to get:

$$dH(t) = (H_t + (\mu_1 - D)H_s + \mu_2 H_\varsigma + \frac{1}{2}H_{ss}\sigma_1^2 + H_{s\varsigma}\rho\sigma_1\sigma_2 + \frac{1}{2}H_{\varsigma\varsigma}\sigma_2^2)dt$$

(2.7) $$+(\sigma_1 H_s + \rho\sigma_2 H_\varsigma)dW_{1t} + \sqrt{1-\rho^2}\sigma_2 H_\varsigma dW_{2t}$$

By definition of the pricing kernel, we have

(2.8) $$dG(t) = -rGdt - \lambda_1 GdW_{1t} - \lambda_2 GdW_{2t}$$

Now comparing the volatility terms in (2.7) and (2.8) gives the desired result.

Remark: the proposition can be easily generalized to multifactor pure diffusion models, for example stochastic interest rate can be added in addition to stochastic volatility. In those cases we have $\vec{\lambda}(t) = -\sigma(t)^T \bullet \nabla' G(t)/G(t)$. ∇G_t is the gradient of G with respect to the risk factors at time t, σ the volatility matrix of the state variable process, the superscript T means transpose of the matrix. Condition (1) states that the pricing kernel in equilibrium can be supported by some utility function, hence can be interpreted as the representative agent's marginal rate of substitution. Condition (2) is similar to those derived in Pham and Touzi (1996) with the only difference being that their expression is in terms of two new processes transformed from MPRs.

Now define the market relative risk aversion $\gamma(t) = -e(t)J_{ee}(t)/J_e(t)$. Proposition 2 follows immediately.

***Proposition 2**: In an economy as specified above, in equilibrium*

$$(2.9) \quad \gamma(t) = \frac{\lambda_1(t) - \rho(t)\lambda_2(t)/\sqrt{1-\rho(t)^2}}{\sigma_1(t)/S(t)}$$

Proof: the definition of MPA combined with condition (1) in proposition 1 and market clearance condition $e_t^* = S_t$ gives an expression for the instantaneous relative risk aversion $\gamma_t = -S(t)G_S(t)/G(t)$. Using condition (2) in proposition 1, we have the desired relation between MRA and MPRs.

Remark: (2.9) implies that the MRA can be written as the Black-Scholes risk aversion plus an extra term in the case of non-zero correlation:

$$(2.10) \qquad \gamma_t = \gamma_{BS} + \frac{\lambda_2}{-\rho\sigma_1/(S\sqrt{1-\rho^2})}$$

This extra term is similar to the Black-Scholes term except that the volatility in the denominator is adjusted by the correlation. Given the empirical evidence that the correlation coefficient is generally negative for market index, whether the MRA is larger or smaller than its benchmark level is determined purely by the sign of MPOR. As pointed out in Lewis [2000] the sign depends on whether the contingent claim in the zero-beta hedging portfolio is a call or put option[19]. A zero MPOR corresponds to a logrithmatic utility form as shown in Pham and Touzi [1996]. If the MPOR is positive, indicating the market rewards investors for taking the risk due to orthogonal volatility randomness, then intuitively market players as a group should exhibit higher risk aversion than without this randomness. And a negative MPOR shall decrease the risk aversion relative to the Black-Scholes case. In addition, the risk aversion is inversely related to the volatility. If the current volatility level is low market players tend to expect higher future volatilities and hence would exhibit larger risk aversion. On the contrary if the current volatility is already very high, players would not expect too much change in volatility in the future, hence they care less about risk. In the extreme case where volatility is infinite, then everybody in the market does not care about risk at all; everybody appears risk-neutral. Notice also that in (2.10) the MRA

[19] To see this, consider a self-financing portfolio $X_t = \omega_t F_t + S_t$. The portfolio is zero-beta if dxds=0. Using Ito's Lemma and solve for ω, we have $\omega = -(F_s + \rho\sigma_2 F_\varsigma/(S\sigma_1))$. Then the volatility of the hedging portfolio can be shown to be proportional to the "delta-vega" hedged portfolio value $(F - SF_s - \rho\sigma_2 F_\varsigma/\sigma_1)$. Note that the vega is multiplied by the correlation coefficient since the portfolio only aims to hedge away the risk associated with the underlying. The sign of this "delta-vega" hedged portfolio value depends on the type of contingent claims. In empirical works one can calculate these Greeks to help judge the sign of MPRO for call and put options.

cannot be written as a function of MPVR only, suggesting the importance of separating MPAR and MPOR when MPOR is in presence.

Example (GBm and ABm): In the benchmark Black-Scholes world the underlying stock price process follows a Geometric Brownian motion (GBm). The pricing kernel function $\mathrm{G(t,S}_t;T,S_T) = A(t,T)(S_T / S_t)^{g(t)}$ where $A(t,T)$ is the discount factor, $g(t) = -\lambda(t)/\sigma(t)$. This implies that the marginal utility function is $U^{'}(S) = S^g$. By (1) and the equality $G(t) = U_c(t) = J_e(t)$ as established in the proof of proposition 1, the representative agent's utility function is then recovered to be $U(S) = S^{1+g}/(1+g)$ which is a power utility with constant relative risk aversion (CRRA) $-g = \lambda/\sigma$ as given in (2.9) with $\lambda_2 = 0$. Similarly, in the arithmetic Brownian motion (ABm) case it is known that the pricing kernel G takes the form of

$$\mathrm{G(t,S}_t;T,S_T) = \exp\{-(\lambda/\sigma)(S_T - S_t)\}\exp\{-(r + \lambda\mu/\sigma + 1/2\lambda^2)(T-t)\}$$

A little algebra then shows this is an exponential utility function with constant absolute risk aversion (CARA) λ/σ, which again is consistent with our derived risk aversion (2.9).

Example (Hull and White): two big assumptions, namely zero correlation and zero volatility risk premium are made in Hull and White [1987] stochastic volatility model. We first consider the case where only volatility risk premium is assumed to be zero. Under this assumption, $\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2 = 0$, then by (2.9) we have $\gamma_{HW,\rho\neq 0} = \left(\frac{1}{1-\rho^2}\right)\frac{\lambda_1}{\sqrt{V}}$. If the correlation is zero also then the Hull-White risk aversion is exactly the same as in the Black-Scholes case. The intuition is that investors preference should not relate to stochastic volatility under Hull and White because the underlying process is independent of the volatility process.

Example (Bollerslev et al. [2010]): we use this example to demonstrate that the relation between volatility risk premium and relative risk aversion in Bollerslev et al. [2010] effectively ignores the MPOR. The VRP is $\xi\sqrt{V}\left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right)$ where ξ is the volatility of variance, dependence on time t is omitted for brevity. As in Heston [1993], Bollerslev et al. [2010] assumes a linear VRP with respect to the variance such that we can write $\xi\left(\rho\lambda_1 + \sqrt{1-\rho^2}\lambda_2\right) = \lambda\sqrt{V}$. Let the MPOR be zero and use (2.9) to derive relative risk aversion $\gamma = \lambda / \rho\xi$, which is exactly the relation used for their empirical study of the risk aversion.

Conclusion

It is worthy at this point to re-emphasize that the market under consideration is complete under any combination of a European option and the underlying stock. Therefore there will be as many completed sub-markets as the number of valid traded options. That gives us an MRA surface across strikes and time to maturities. When practitioners commonly cross-fit the model-implied volatility smile to the market observed smile, they are actually aggregating these sub-market pricing kernels and risk aversions. We call the resulting risk aversion “average MRA” (aMRA). Theory in Cvitanic et al. [2009], however, predict that in the options market the market pricing kernel should exhibit average behavior for a certain range of strikes and is dominated by individuals outside this range. We call this the “differentiated MRA” (dMRA). In view of the above, we recommend practitioners to extract the risk aversion using three approaches. First, extract the volatility risk for each strike price, which gives MRAs for each strike. Second, extract the volatility risk by minimizing the mean squared error (RMSE) between the theoretical option prices and the market observed prices for all strikes (cross-fitting). That gives us aMRA. Third, extract the volatility risk by minimizing the RMSE for strikes between (+/- 5% of ATM strike = spot), which produces MRA in the middle range; then extract the volatility risk for each strike beyond the range to get MRA in the tails. This effectively gives us the dMRA. It will be interesting to compare these MRAs.

Future research can consider the validity of the relation under jump-diffusion models. A new market price of jump risk (MPJR) will be introduced. Another line of research could be how to model the seemingly stochastic behavior of MRA and include it into the option pricing formula.

Appendix – Equilibrium Interest Rate, consumption and MPRs

By comparing the drift terms in (2.7) and (2.8) we have:

(1) $G_t + (\mu_1 - D)G_s + \mu_2 G_\varsigma + \frac{1}{2} G_{ss}\sigma_1^2 + G_{s\varsigma}\rho\sigma_1\sigma_2 + \frac{1}{2} G_{\varsigma\varsigma}\sigma_2^2 + rG = 0$

This gives the equilibrium interest rate in terms of the pricing kernel, which is determined in turn by the indirect utility function J(t) from (2.4) and (2.5). To derive the equilibrium PDE for the MPRs, define two functions f and g such that

$$f = \frac{\lambda_1 - \rho\lambda_2 / \sqrt{1-\rho^2}}{\sigma_1}, g = \frac{\lambda_2}{\sqrt{1-\rho^2}}$$

Notice that from condition (2) in proposition 1, simple algebra shows that

$f = -\frac{G_s}{G}, g = -\frac{G_\varsigma}{G}$, so there exists a function y(.) such that

(2) $\frac{G_t}{G}(t,s,k) = -y(t) - \int_{s_0}^{s} f_t(t,\xi,k)d\xi - \int_{k_0}^{k} g_t(t,s,v)d\ v$

Now, in terms of functions f and g, (2) can be rewritten as:

(3) $\frac{G_t}{G} - (\mu_1 - D) f - \mu_2 g + \frac{1}{2}\sigma_1^2\left(f^2 - f_s\right) + \frac{1}{2}\sigma_2^2\left(g^2 - g_\varsigma\right) + \rho\sigma_1\sigma_2(f_\varsigma - fg) + r = 0$

Plugging (2) to (3), using the martingale restriction condition $\mu_1 - D - rS = \lambda_1\sigma_1$ and differentiating the resulting equation with respect to s and k, we get two PDEs that must be satisfied by equilibrium viable MPRs. Since they are similar to (3.13) and (3.14) in Pham and Touzi (1996), we do not present them here. The only difference is that we do not have the dividend terms as in Pham and Touzi, because in our model the dividend policy is treated exogenous.

Equilibrium consumption can be obtained from (A1) in (2.5) by first solving the indirect utility function J(e).

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