DEFAULTABLE BONDS VIA HKA

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Abstract. *To construct a no-arbitrage defaultable bond market, we work on the state price density framework. Using the heat kernel approach (HKA for short) with the killing of a Markov process, we construct a single defaultable bond market that enables an explicit expression of a defaultable bond and credit spread under quadratic Gaussian settings. Some simulation results show that the model is not only tractable but realistic.*

Keywords: (non-)systematic risk, state price density, killed HKA, Markov functional model, quadratic Gaussian.

1. Introduction

The HKA, which is an abbreviation of "Heat Kernel Approach to interest rate modelling, was introduced by one of the authors and his collaborators in [2]. Briefly speaking, HKA is a systematic method to produce a tractable interest rate model which is "Markov functional" in the sense of Hunt-Kennedy-Pelsser [8]. In the fundamental paper [2], four different types of implementation methods are introduced. Namely, 1) Eigenfunction models 2) Weighted HKA 3) Killed HKA and 4) Trace Approach. As is pointed out in [2], the eigenfunction models are tailor-made for swaption pricing, and a deeper understanding for its mathematical structure leads to the trace approach, which is mathematically most involved. The weighted HKA is extended to time-inhomogeneous setting and applied to information-based models by J. Akahori and A. Macrina [1].

In the present paper, we will demonstrate how the Killed HKA is applied to the modelling of defaultable bonds by constructing a market where the market price of risk and the default probability are "built in the same block" (whose precise meaning will be given later). We stress that the HKA is basically a state-price density approach where everything is written under the physical= statistical measure. Since the HKA furthermore gives an analytically tractable model in nature, the framework proposed in this paper would be promising in respect of modelling defaultable markets.

The organization of the present paper is as follows. After recalling the *plain-vanilla* HKA in section 2.1 and the killed HKA in section 2.2, we shall give the main result, a framework with in the Killed HKA to model a defaultable bond market in section 3. In section 4, we will give some simulation results of an explicit example with a quadratic form of Wiener process.

Acknowledgement

The authors are deeply grateful to Professor Dr. Jiro Akahori. His insightful comments and suggestions were an enormous help to us.

2. Heat Kernel Approach

Here we briefly recall the approach.

2.1. **Plain-Vanilla HKA.** We work on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Now we consider a general Markov process $\{X_t^x\}_{t\geq 0, x\in\mathcal{S}}$ on a polish space $\mathcal{S}$.

Definition 2.1. *Let X be an $\mathcal{S}$-valued Markov process. We shall say that a function p satisfies the* propagation property *if*

$$\mathbf{E}[p(t, X_s^x)] = p(t+s, x) \tag{1}$$

holds for any $t, s \geq 0$ and $x \in \mathcal{S}$.

The following fact is initialized by [3] and developed in [2].

Proposition 2.1 (Akahori et al. [2])**.** *Let X be a $\mathcal{S}$-valued Markov process, λ be a positive function on the half line, and p be a function with the propagation property. The bonds market given by*

$$P_f(t, T) = \frac{p(\lambda_T + T - t, X_t^x)}{p(\lambda_t, X_t^x)} \tag{2}$$

is an arbitrage free market.

Example 2.1 (Generic Example)**.** *Take a measurable, bounded $h : \mathcal{S} \to \mathbb{R}_{\geq 0}$, then*

$$p(t, x) := \mathbf{E}[h(X_t^x)] \tag{3}$$

satisfies the propagation property (1). In fact, by the Markov property, we have

$$\begin{aligned}\mathbf{E}\left[p(t, X_s)\right] &= \mathbf{E}\left[\mathbf{E}\left[h(X_t^{X_s^x})\right]\right] = \mathbf{E}\left[\mathbf{E}\left[h(X_{t+s}^x)|\mathcal{F}_s^X\right]\right] \\ &= \mathbf{E}\left[h(X_{t+s}^x)\right] = p(t+s, x).\end{aligned}$$

Proof. Let $\pi_t = p(\lambda_t, X_t^x)$. By the propagation property of p and the Markov property of X, we have

$$\begin{aligned}P_f(t, T) &= \frac{\mathbf{E}\left[\pi_T|\mathcal{F}_t\right]}{\pi_t} = \frac{\mathbf{E}\left[p(\lambda_T, X_T^x)|\mathcal{F}_t\right]}{p(\lambda_t, X_t^x)} \\ &= \frac{\mathbf{E}\left[p(\lambda_T, X_{T-t}^{X_t^x})\right]}{p(\lambda_t, X_t^x)} = \frac{p(\lambda_T + T - t, X_t^x)}{p(\lambda_t, X_t^x)}.\end{aligned}$$

This means π is the state price density of the market. □

It should be noted that we do not assume π_t to be a supermartingale in [2], i.e. in economic terms *we do not assume positive short rates*. The four implementation methods mentioned in the introduction is introduced in [2] to obtain supermartingales out of a propagator, or equivalently to obtain positive rate models.

2.2. **The Killed HKA.** We then recall, and give a more detailed description to, the Killed HKA[1]. Let V be a non-negative measurable function on $\mathcal{S}$. Put $Y_t^y = y + \int_0^t V(X_s^x)\,ds$ for $y \in \mathbf{R}$. Let us define

$$q(t,x) = \mathbf{E}[\exp(-\int_0^t V(X_s^x)\,ds)]. \tag{4}$$

Then the function

$$q(t,x,y) = e^{-y} q(t,x),$$

satisfies propagation property with respect to (X^x, Y^y);

$$\mathbf{E}[q(s, X_t^x, Y_t^y)] = q(t+s, x, y). \tag{5}$$

In fact, by the Markov property of X, we have

$$\mathbf{E}[e^{-Y_{t+s}}|\mathcal{F}_t] = \mathbf{E}[e^{-Y_s(\theta_t)}|\mathcal{F}_t] \times e^{-Y_t} = \mathbf{E}[e^{-Y_s}|X_t] \times e^{-Y_t},$$

where θ is the shift operator. Thus we obtain

$$\mathbf{E}[q(s, X_t^x, Y_t^y)] = \mathbf{E}[e^{-Y_t^y} q(s, X_t^x)] = \mathbf{E}[e^{-Y_t^y} e^{-Y_s^y \circ \theta_t}] = q(t+s, x, y).$$

This fact ensures that the bond market model constructed as

$$P(t,T) = \frac{q(\lambda_T + T - t, X_t^x)}{q(\lambda_t, X_t^x)}, \tag{6}$$

where λ is an increasing function, is arbitrage-free since we can choose

$$\pi_t = q(\lambda_t, X_t) \exp(-\int_0^t V(X_s)\,ds)$$

as a state price density of the market. In fact,

$$\begin{aligned}
\mathbf{E}[\pi_T|\mathcal{F}_t] &= \mathbf{E}[\ \mathbf{E}[\exp(-\int_T^{T+\lambda_T} V(X_u^x)\,du)|\mathcal{F}_T] \exp(-\int_0^T V(X_s)\,ds)\ |\mathcal{F}_t] \\
&= \mathbf{E}[\exp(-\int_t^{T+\lambda_T} V(X_u^x)\,du)\ |X_t] \exp(-\int_0^t V(X_s)\,ds) \\
&= q(\lambda_T + T - t, X_t) \exp(-\int_0^t V(X_s)\,ds) = \pi_t \frac{q(\lambda_T + T - t, X_t)}{q(\lambda_t, X_t)}.
\end{aligned}$$

Note that the bond price P is decreasing in T since q is increasing in t, which is ensured by the positivity of V. Thus we obtain a positive rate model.

3. HKA to Defaultable Bond

This section is the main part of the present paper. Let us now consider a defaultable bond in the following situation:

(1) The bond pays a unit account at the maturity T unless it defaults.
(2) At the default time τ, nothing will be recovered.
(3) The state variable is a Markov process $\{X_t^x; t \geq 0\}$, which is observable in the market.

[1]This part is shared with [9].

(4) The default probability is completely determined through the information of X in the following manner; the hazard rate of the default time on the filtration $\mathcal{F}^X$ is given by $\mathbf{E}[1_{\{\tau>t\}}|\mathcal{F}^X_t] = \exp\left(-\int_0^t V(X^x_u)\,du\right)$ where $\mathcal{F}^X$ is the natural filtration on X and V is a non-negative measurable function.

(5) The default come as a "surprise" to the market. To be precise, the market filtration $\{\mathcal{G}_t\}$ is defined as $\mathcal{G}_t = \sigma(X_s, \{\tau \le s\}; s \le t)$ and assume that $\mathcal{F}^X_0 = \{\Omega, \emptyset\} = \mathcal{G}_0$.

(6) A state price density of the market is given by $\pi_t := q(\lambda_t, X_t) = \mathbf{E}\left[\exp\left(-\int_0^{\lambda_t} V(X^x_u)\,du\right)\right]$ where q is defined as (4) and λ is a non-decreasing function.

Note that the assumptions 1–5 may be natural (except assumption 2, which assumes zero recovery) and very generic, while the last assumption is very specific in that the function V controls both the market price of a risk as well as the default probability of a bond. Very heuristically speaking, this market is fully subject to the risk of a defaultable bond.

We stress that this is just a toy model, which exhibits how the killed HKA is applied to a defaultable market modeling. The following is established in [9]:

Theorem 3.1. *Under the above assumptions 1–6,*

(i) *the price $P_d(t,T)$ of a defaultable zero coupon bond is given by*

$$P_d(t,T) = 1_{\{\tau>t\}} \frac{q(\lambda_T + T - t, X^x_t)}{q(\lambda_t, X^x_t)}, \tag{7}$$

(ii) *the price $P_f(t,T)$ of a default-free bond is given by*

$$P_f(t,T) = \frac{\hat{q}(\lambda_T + T - t, T - t, X^x_t)}{q(\lambda_t, X^x_t)}, \tag{8}$$

where

$$\hat{q}(t,s,x) = \mathbf{E}\left[\exp\left(-\int_s^t V(X^x_u)\,du\right)\right], \tag{9}$$

(iii) *and then the "credit spread" is given by*

$$\partial_T \log \frac{\hat{q}(\lambda_T + T - t, T - t, X^x_t)}{\hat{q}(\lambda_T + T - t, 0, X^x_t)}. \tag{10}$$

Remark 3.1. *Note that the "credit spread" makes no sense when $\tau \le t$, so we can only think of the case that $\tau > t$.*

Proof. The proof is based on the following fundamental lemma due to Dellacherie (see [6]): For any $\mathcal{F}^X_T$-integrable random variable Z and $0 < t < T$, we have

$$\mathbf{E}[1_{\{\tau>T\}} Z|\mathcal{G}_t] = \frac{1_{\{\tau>t\}}}{E[1_{\{\tau>t\}}|\mathcal{F}^X_t]} \mathbf{E}[1_{\{\tau>T\}} Z|\mathcal{F}^X_t].$$

Hence, we have

$$P_d(t,T) = \frac{1}{\pi_t} \frac{1_{\{\tau>t\}}}{\mathbf{E}\left[1_{\{\tau>t\}}|\mathcal{F}^X_t\right]} \mathbf{E}\left[1_{\{\tau>T\}}\pi_T|\mathcal{F}^X_t\right].$$

Then by a Markov property and a Tower property,

$$\mathbf{E}\left[1_{\{\tau>T\}}\pi_T|\mathcal{F}^X_t\right] = \mathbf{E}\left[\ \mathbf{E}\left[1_{\{\tau>T\}}|\mathcal{F}^X_T\right] q(\lambda_T, X^x_T)\,|\,\mathcal{F}^X_t\right],$$

and by the assumption that $\mathbf{E}[\,1_{\{\tau>t\}}\,|\,\mathcal{F}_t^X] = \exp\left(-\int_0^t V(X_u^x)\,du\right)$

$$\mathbf{E}\left[\,\mathbf{E}\left[\,1_{\{\tau>T\}}\,|\,\mathcal{F}_T^X\right]q(\lambda_T, X_T^x)\,|\,\mathcal{F}_t^X\right] = \exp\left(-\int_0^t V(X_u^x)\,du\right)\mathbf{E}\left[e^{-\int_t^T V(X_s^x)\,ds}q(\lambda_T, X_T^x)\,|\,\mathcal{F}_t^X\right].$$

Here applying the fact of the equation (5), we have

$$\mathbf{E}\left[e^{-\int_t^T V(X_s^x)\,ds}q(\lambda_T, X_T^x)|\mathcal{F}_t^X\right] = q(\lambda_T + T - t, X_t^x),$$

so that

$$P_d(t,T) = 1_{\{\tau>t\}}\frac{q(\lambda_T + T - t, X_t^x)}{q(\lambda_t, X_t^x)}.$$

On the other hand, (ii) follows a Markov property and a Tower property. And it is known that the "credit spread" is given by

$$-\partial_T \log\frac{P_d(t,T)}{P_f(t,T)}.$$

Here since $P_d(t,T)$, $P_f(t,T)$ are (i), (ii) respectively,

$$-\partial_T \log\frac{P_d(t,T)}{P_f(t,T)} = \partial_T \log\frac{\mathbf{E}\left[q(\lambda_T, X_T^x)|\mathcal{F}_t^X\right]}{q(\lambda_T + T - t, X_t^x)}$$

when $\tau > t$. Then by a Markov property and a Tower property,

$$\mathbf{E}\left[q(\lambda_T, X_T^x)|\mathcal{F}_t^X\right] = \mathbf{E}\left[e^{-\int_{T-t}^{\lambda_T+T-t} V(X_s^{X_t^x})\,ds}\right] = \hat{q}(\lambda_T + T - t, T - t, X_t^x).$$

Hence, we obtain

$$-\partial_T \log\frac{P_d(t,T)}{P_f(t,T)} = \partial_T \log\frac{\hat{q}(\lambda_T + T - t, T - t, X_t^x)}{q(\lambda_T + T - t, X_t^x)}.$$

□

4. Quadratic Example

Now we give some simulation results of an explicit example, where X is a d-dimensional Wiener process and $V(x) = \frac{\beta^2|x|^2}{2}$ ($\beta > 0$). Let $q(t,x)$ and $\hat{q}(t,x)$ be as in (4), (9), then they are explicitly given by

$$q(t,x) = (\cosh\beta t)^{-d/2}\exp\left(-\frac{\beta x^2}{2}\frac{\sinh\beta t}{\cosh\beta t}\right), \tag{11}$$

and

(12)

$$\hat{q}(t,x) = (\,\cosh\beta(t-s) + \beta s\sinh\beta(t-s)\,)^{-d/2}\exp\left(-\frac{\beta x^2}{2}\frac{\tanh\beta(t-s)}{1+\beta s\tanh\beta(t-s)}\right),$$

which result from Lemma 6.1, Corollary 6.1 in the Appendix. Hence, we obtain the analytic expression of the bond prices. The following simulated yield curves (Fig.1) and (Fig.2) implied by a default-free bond as

$$-\frac{1}{T}\log P_f(t,T)$$

are obtained by using (8) and (12). Here the parameters are set to be $\beta = 0.1$, $x = 0.01, 10, 20, 30$, $\lambda_t = e^t/10$, and the present time $t = 0$ in (Fig.1), $\beta = 1.8$ and

$\lambda_t = e^t/100$ in (Fig.2). Then x-axis stands for the maturities ranging from one year to ten years and y-axis does for the price of a default-free bond.

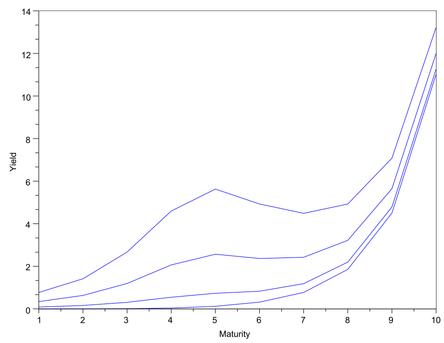

FIGURE 1. Simulated yield curves implied by a default-free bond when $\lambda_t = e^t/10$

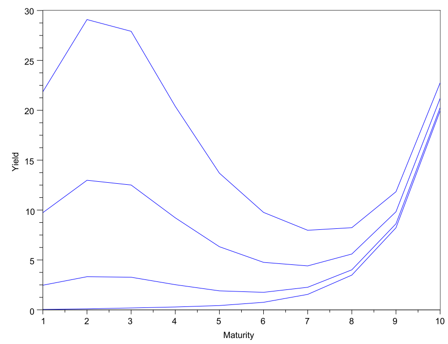

FIGURE 2. Simulated yield curves implied by a default-free bond when $\lambda_t = e^t/100$

(Fig.1) and (Fig.2) show increasing the value of x does not make the curve shift upward, but also cause a "hump" in the curve, which can not be observed in the normal affine model, with the proper choice of λ. Moreover, using the formula (10) and (12), we obtain the following simulated credit spread curves as (Fig.3). Here the parameters are set to be $\lambda_t = \sqrt{t}$, $\beta = 0.1, 0.2, ..., 1$, $x = 0$, and the present time $t = 0$.

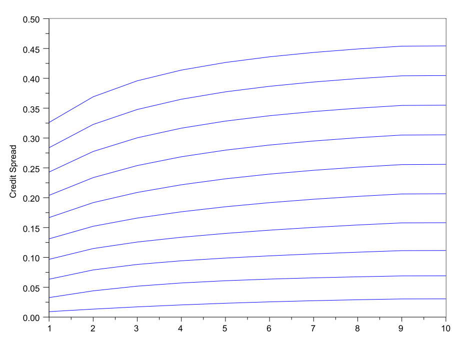

FIGURE 3. Simulated credit spread curves

As usual, the lower the credit rating of a defaultable bond is, the wider the spread is, and it is non-decreasing in the maturity time. Moreover, the spread of a defaultable bond lower rated is much wider in the maturity time than the one of a bond higher rated. It should be thought that (Fig.3) shows this fact.

5. Conclusions

We have introduced a way of constructing a single defaultable bond market model under the physical measure P by applying the killed HKA. We have also presented some simulation results in the quadratic case. Comparing the well-known Hull-White model, we can have observed a complex "hump" in the yield implied by a default-free bond, which comes from the parameter λ.

6. Appendix

Lemma 6.1. *Let X be a d-dimensional Wiener process starting at x. For $\alpha, \beta \geq 0$, it holds*

$$\mathbf{E}[e^{-\alpha|X_t^x|^2-\frac{\beta^2}{2}\int_0^t |X_s^x|^2\,ds}] = \begin{cases} \left(\cosh\beta t + \frac{2\alpha}{\beta}\sinh\beta t\right)^{-d/2} \exp\left(-\frac{\beta x^2}{2}\frac{\beta\sinh\beta t - 2\alpha\cosh\beta t}{\beta\cosh\beta t + 2\alpha\sinh\beta t}\right) & \beta > 0, \\ (2\alpha t+1)^{-d/2}\exp\left(-\frac{\alpha x^2}{2\alpha t+1}\right) & \beta = 0. \end{cases} \tag{13}$$

This is well-known formula and there are many ways to prove it. One way is presented in [9].

The following is an immediate consequence of Lemma 6.1:

Corollary 6.1. *Let X be a d-dimensional Wiener process starting at x. For $\beta > 0$, it holds*

$$\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_s^t |X_v^x|^2\,dv}\right] = (\cosh\beta(t-s) + \beta s\sinh\beta(t-s))^{-d/2} \times \exp\left(-\frac{\beta x^2}{2}\frac{\sinh\beta(t-s)}{\cosh\beta(t-s)+\beta s\sinh\beta(t-s)}\right).$$

Proof. By a Markov property, a Tower property, and Lemma 6.1,

$$\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_s^t |X_v^x|^2\,dv}\right] = \mathbf{E}\left[\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_s^t |X_v^x|^2\,dv} \mid \mathcal{F}_s^X\right]\right] = \mathbf{E}\left[\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_0^{t-s} |X_v^x|^2\,dv} \mid X_s^x\right]\right]$$
$$= \cosh\beta(t-s)^{-d/2}\,\mathbf{E}\left[\exp\left(-\frac{\beta|X_s^x|^2}{2}\frac{\sinh\beta(t-s)}{\cosh\beta(t-s)}\right)\right].$$

The proof is complete by replacing α by $\frac{\beta}{2}\frac{\sinh\beta(t-s)}{\cosh\beta(t-s)}$ in Lemma 6.1. □

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