Conservative delta hedging under transaction costs

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Abstract

Explicit robust hedging strategies for convex or concave payoffs under a continuous semimartingale model with uncertainty and small transaction costs are constructed. In an asymptotic sense, the upper and lower bounds of the cumulative volatility enable us to super-hedge convex and concave payoffs respectively. The idea is a combination of Mykland's conservative delta hedging and Leland's enlarging volatility. We use a specific sequence of stopping times as rebalancing dates, which can be superior to equidistant one even when there is no model uncertainty. A central limit theorem for the super-hedging error as the coefficient of linear transaction costs tends to zero is proved. The mean squared error is also studied.

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1 Introduction

Hedging an European option by dynamically trading the underlying asset is the very basic of mathematical finance. It is still a practical problem since liquid option markets do not exist for all kinds of payoffs written on all kinds of assets. In particular, a dynamic hedging strategy is required for option contracts with long time-to-maturity. To deal with a long period of hedging, problematic is the model uncertainty of the underlying asset price process. Super-hedging under the model uncertainty has been recently paid much attention in the literature of mathematical finance; Avellaneda, Levy and Paras [1], Lyons [17], Mykland [18, 19, 20], Denis and Martini [9], Cont [5], Peng [21, 22], Soner, Touzi and Zhang [23] among others. Another issue of dynamic hedging is that rebalancing portfolio should be done discretely and associated with small but non-zero transaction costs; see e.g., Kabanov and Safarian [15], Bouchard and Touzi [3], Leland [16], Gamys and Kabanov [14], Denis and Kabanov [8], and Fukasawa [12, 13].

The aim of this paper is to present a practical strategy which takes both the uncertainty and the transaction costs into account. We employ the familiar Black-Scholes pricing function and the delta hedging strategy with specific choices of volatility parameter and rebalancing dates. We suppose that the underlying asset price process and the zero-coupon bond price process are positive continuous semimartingale, that the option payoff is either convex or concave, and that the transaction costs follow a linear model.

To cope with the model uncertainty, we adopt a continuous-time trading strategy which is a variant of Mykland [18]'s conservative delta hedging. The same idea was also introduced by Carr and Lee [4], where emphasis was put on its application to model-free hedging of variance options. We rely only on bounds on the cumulative volatility, that is, the quadratic variation of the log price process up to the maturity of the option. It is reasonable to suppose the availability of such bounds because the volatility is typically observed to be mean-reverting, so that the cumulative volatility across a long period is naturally supposed to be stable and predictable. This is a different approach from one given by Avellaneda, Levy and Paras [1] and its extensions, which relies on bounds on not the cumulative volatility but the volatility itself. Our approach turns out to be more efficient as long as considering an European option written on one asset. Note that a bound on the volatility implies in particular a bound on the cumulative volatility. Since we are dealing with a general continuous semimartingale, it is clear that without any bounds on the cumulative volatility, we cannot have anything better than the trivial buy-and-hold super-hedging strategy.

To take the transaction costs into account, we construct a discretized version of the conservative delta hedging strategy, which is inspired by Leland [16]'s idea of enlarging volatility for the Black-Scholes model. The main result of this paper is the stable convergence of the discrete hedging strategy as the coefficient of the transaction costs converges to 0. The mean squared error is also investigated. In particular, the result implies in an asymptotic sense that by following this strategy, we can super-hedge convex and concave payoffs under the upper and lower bounds of the cumulative volatility respectively. Note that without employing such an asymptotic framework, we do not have anything better than the trivial buy-and-hold strategy, as shown in Bouchard and Touzi [3]. An alternative approach is the framework of pricing and partial hedging under loss constraints. See Bouchard and Dang [2]. The advantage of our asymptotic approach is the availability of the explicit expressions of the strategy and the corresponding loss distribution. Not only that everything is explicit but also that everything is written in terms of the Black-Scholes greeks, so it is quite easy to implement our strategy in financial practice. Our specification of rebalancing dates plays a crucial role; it is not from technical convenience but reflecting the structure of the transaction costs. Our rebalancing dates are not deterministic, so that the convergence results are new even under the Black-Scholes model that is a special case of our framework.

In Section 2, we describe the continuous-time strategy under no transaction costs. In Section 3, we construct the discrete hedging strategy and present the main result. In Section 4, we treat the mean squared error of the strategy. In Section 5, we compare the mean square errors of our strategy and Leland's strategy under the Black-Scholes model with no model uncertainty.

2 Conservative delta hedging

Here we study the super-hedging problem under no transaction costs. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$ be a filtered probability space satisfying the usual assumptions. Let S^1 be a one-dimensional positive continuous $\{\mathcal{F}_t\}$-semimartingale standing for a tradable asset price process. We consider hedging an European claim $f(S^1_T)$ by dynamically trading the underlying asset and the risk-free zero coupon bond with the same maturity T, where $T > 0$ is fixed and f is a convex or concave function whose right and left derivatives $f'_\pm$ are of polynomial growth. Here we say a function g is of polynomial growth if there exists $p > 0$ such that

$$\sup_{s\in(0,\infty)} \frac{|g(s)|}{s^p + s^{-p}} < \infty.$$

We denote by S^0 the price process of the zero coupon bond with maturity T. By definition we have $S^0_T = 1$. We extend $S^0_t = 1$ for $t > T$ and suppose that S^0 also is a positive continuous $\{\mathcal{F}_t\}$-semimartingale. Denote by $\tilde{S} = S^1/S^0$ the discounted price process. Naturally we suppose that the path of (S^1, S^0) is observable, so that in particular we know $\langle \log(\tilde{S})\rangle_t$ at time t for any $t \in [0, T]$.

Now we describe a hedging strategy under no transaction costs which is a variant of one given in Mykland [18] and essentially the same as one given in Carr and Lee [4]. Define a function $P : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ as

$$P(S, R, \Sigma) = \exp\{-R\} \int_{-\infty}^{\infty} f(S \exp\{R - \Sigma/2 + \sqrt{\Sigma} z\})\phi(z)\mathrm{d}z,$$

where ϕ is the standard normal density. Notice that $P(S^1_0, rT, \sigma^2 T)$ is the price the European option with payoff function f under the Black-Scholes model with volatility $|\sigma|$ and risk-free rate r for any constants $\sigma, r \in \mathbb{R}$. What plays an essential role is the following partial differential equations for P, which can be directly checked:

$$\frac{\partial P}{\partial \Sigma} = \frac{1}{2} S^2 \frac{\partial^2 P}{\partial S^2}, \quad \frac{\partial P}{\partial R} = S\frac{\partial P}{\partial S} - P, \quad \frac{\partial^2 P}{\partial R \partial S} = S \frac{\partial^2 P}{\partial S^2}, \quad \frac{\partial^2 P}{\partial R^2} = S^2 \frac{\partial^2 P}{\partial S^2} - \frac{\partial P}{\partial R}$$

on $(0, \infty) \times \mathbb{R} \times (0, \infty)$ with boundary condition $P(S, 0, 0) = f(S)$. Notice that if f is convex or concave, then so is P in S, which implies the monotonicity of P in Σ. For a given constant $\hat{\Sigma} > 0$, set

$$R_t = -\log(S^0_t), \quad \Sigma_t = \hat{\Sigma} - \langle \log(\tilde{S})\rangle_t, \quad \tau = \inf\{t > 0 : \langle \log(\tilde{S})\rangle_t \geq \hat{\Sigma}\}$$

and consider the portfolio strategy defined as

$$\Pi_t = \frac{\partial P}{\partial S}(S^1_t, R_t, \Sigma_t), \quad \Pi^0_t = (P(S^1_t, R_t, \Sigma_t) - \Pi_t S^1_t)/S^0_t$$

for $t \in [0, \tau)$ and $(\Pi_t, \Pi_t^0) = (a, b)$ for $t \geq \tau$, where a and b can be chosen any $\mathcal{F}_\tau$-measurable random variables satisfying $a \in [f'_-(\tilde{S}_\tau), f'_+(\tilde{S}_\tau)]$ and $b = f(\tilde{S}_\tau) - a\tilde{S}_\tau$. The following theorem is an immediate result of the above partial differential equations with the aid of Itô's formula.

Proposition 1 *The portfolio strategy* (Π, Π^0) *is self-financing up to* τ*, that is,*

$$\begin{aligned} P(S_t^1, R_t, \Sigma_t) &= \Pi_t S_t^1 + \Pi_t^0 S_t^0 \\ &= P(S_0^1, R_0, \Sigma_0) + \int_0^t \Pi_u \mathrm{d}S_u^1 + \int_0^t \Pi_u^0 \mathrm{d}S_u^0 \end{aligned}$$

for any $t \in [0, \tau]$*. Moreover,*

- *if* f *is convex, then* (Π, Π^0) *is a super-hedging strategy for the payoff* $f(S_T^1)$ *on the set* $\{\hat{\Sigma} \geq \langle \log(\tilde{S}) \rangle_T\}$*, that is, we have that*

  $$P(S_T^1, R_T, \Sigma_T) = P(S_T^1, 0, \Sigma_T) \geq f(S_T^1), \quad \tau \geq T$$

  on set $\{\hat{\Sigma} \geq \langle \log(\tilde{S}) \rangle_T\}$.

- *if* f *is concave, then* (Π, Π^0) *is a super-hedging strategy for the payoff* $f(S_T^1)$ *on the set* $\{\hat{\Sigma} \leq \langle \log(\tilde{S}) \rangle_T\}$*, that is, we have that*

  $$P(S_\tau^1, R_\tau, \Sigma_\tau) = S_\tau^0 f(\tilde{S}_\tau) = aS_\tau^1 + bS_\tau^0, \quad T \geq \tau, \quad aS_T^1 + bS_T^0 \geq f(S_T^1)$$

  on the set $\{\hat{\Sigma} \leq \langle \log(\tilde{S}) \rangle_T\}$.

This strategy requires only a suitable specification of $\hat{\Sigma}$ at time 0. We do not specify the detail of the dynamics of (S^1, S^0) other than its continuity. The strategy is efficient in the sense that it becomes the perfect hedging strategy in the case $\langle \log(\tilde{S}) \rangle_T = \hat{\Sigma}$ a.s.. The uncertainty of the whole dynamics reduces to that of the cumulative volatility of $\tilde{S}$. The cumulative volatility, that is, the quadratic variation of $\log(\tilde{S})$ is typically observed to be persistent and mean-reverting, so that it is reasonable to suppose the availability of a prediction interval of $\langle \log(\tilde{S}) \rangle_T$ at time 0 based on the past price behavior. See Mykland [19] for further discussion. The difference between our strategy and the one given in Mykland [18, 19, 20] is that ours involves dynamic trading of the zero coupon bond and we do not suppose any bounds on the cumulative interest rate. Nevertheless, our super-hedging price $P(S_0^1, R_0, \Sigma_0)$ is less than or equal to that given in Mykland [18, 20] under the same volatility bounds at least if S^0 is of bounded variation. The point of this strategy is to utilize $\langle \log(\tilde{S}) \rangle_t$ when defining the delta Π_t. A bound on the cumulative volatility does not work by itself as shown by El Karoui, Jeanblanc and Shreve [10]. The price process (S^1, S^0) are observed discretely in practice, so that the estimation error of $\langle \log(\tilde{S}) \rangle_t$ might be taken into account. Nevertheless, the observation of the prices is naturally supposed to be much more frequent than rebalancing portfolio, so that we may neglect this estimation error in this study. See e.g. Fukasawa [11] for the estimation error of $\langle \log(\tilde{S}) \rangle_t$. The approach introduced by Avellaneda, Levy and Paras [1] is based on bounds on not the cumulative volatility but the volatility itself and does not utilize the available information $\langle \log(\tilde{S}) \rangle_t$ at all. As a result, starting the same initial value of portfolio, our strategy covers a wider class of models at least when considering convex or concave European payoffs.

3 Discrete hedging under transaction costs

The use of the Black-Scholes delta would be convenient in financial practice. We believe that the above elementary result is important from the viewpoint of risk management. To make it more practical, now we consider its discretized version under transaction costs. The main source of the transaction costs is the bid-ask spread that is usually approximated by a linear model. We suppose that the loss in rebalancing portfolio from $\Pi_{t-}S^1_t + \Pi^0_{t-}S^0_t$ to $\Pi_t S^1_t + \Pi^0_t S^0_t$ is proportional to $|\Delta \Pi_t| S^1_t$. Note that the self-financing condition requires $|\Delta \Pi_t| S^1_t = |\Delta_t \Pi^0_t| S^0_t$ under no transaction costs. Further, we suppose that the coefficient of these linear transaction costs is "small". This motivates us take a positive sequence κ_n which tends to 0 as $n \to \infty$ and study the asymptotic behavior of the discrete hedging under the transaction costs $\kappa_n |\Delta \Pi_t| S^1_t$. We can expect that the limit distribution of the corresponding hedging error serves as a reasonable approximation of the error distribution corresponding to κ_n for fixed n.

Here we construct our discrete hedging strategy. The idea is a modification of Leland's strategy of enlarging volatility for the Black-Scholes model. Given a constant $\hat{\Sigma} > 0$, set

$$\Sigma^{\pm\alpha}_t = (1 \pm 2/\alpha)\Sigma_t = (1 \pm 2/\alpha)(\hat{\Sigma} - \langle \log(\tilde{S}) \rangle_t)$$

and define a portfolio strategy as

$$\Pi^{\pm\alpha}_t = \frac{\partial P}{\partial S}(S^1_t, R_t, \Sigma^{\pm\alpha}_t), \ \ \Pi^{0,\pm\alpha}_t = (P(S^1_t, R_t, \Sigma^{\alpha}_t) - \Pi^{\pm\alpha}_t S^1_t)/S^0_t$$

for $t \in [0, \tau)$ and $(\Pi^{\pm\alpha}_t, \Pi^{0,\pm\alpha}_t) = (a, b)$ for $t \geq \tau$, where P, R_t, Σ_t, a, b are the same as before. Here α is an arbitrary positive constant and we use $+\alpha$ if the payoff f is convex and use $-\alpha$ if it is concave. For the latter case we assume $\alpha > 2$. In the sequel, $\pm\alpha$ should always be understood as $+\alpha$ or $-\alpha$ if f is convex or concave respectively. We set the price of the option with payoff $f(S^1_T)$ to be

$$P(S^1_0, R_0, \Sigma^{\pm\alpha}_0) + \kappa_n |\Pi^{\pm\alpha}_0| \tilde{S}_0. \tag{1}$$

This way of enlarging or shrinking volatility is different from Leland's way. With suitable choice of rebalancing dates, this modification enables us to work beyond the Black-Scholes model. Moreover, it turns out to be more efficient than Leland's strategy even under the Black-Scholes model in some sense as we see in Section 5. While Leland used the equidistant partition of $[0, T]$ as rebalancing dates, we use the following sequence of stopping times:

$$\tau^n_0 = 0, \ \ \tau^n_{j+1} = \inf\{t > \tau^n_j ; |\Pi^{\pm\alpha}_t - \Pi^{\pm\alpha}_{\tau^n_j}| \geq \alpha \kappa_n \tilde{S}_{\tau^n_j} |\Gamma^{\pm\alpha}_{\tau^n_j}|\}, \tag{2}$$

where

$$\Gamma_u^{\pm\alpha} = S_u^0 \frac{\partial^2 P}{\partial S^2}(S_u^1, R_u, \Sigma_u^{\pm\alpha}).$$

To ensure $\tau_j^n < \tau_{j+1}^n$ a.s. for each j, we put the following condition:

Condition 1 *For all* $(S, \Sigma) \in (0, \infty) \times (0, \infty)$,

$$\left| \frac{\partial^2 P}{\partial S^2}(S, 0, \Sigma) \right| > 0.$$

Note that we are assuming that f is convex or concave, so we have already

$$\frac{\partial^2 P}{\partial S^2}(S, R, \Sigma) \geq 0 \text{ or } \frac{\partial^2 P}{\partial S^2}(S, R, \Sigma) \leq 0.$$

Therefore, Condition 1 is a fairly mild condition from practical point of view. This is violated if f'_+ is continuous singular.

We follow the delta strategy $\Pi^{\pm\alpha}$ using $\{\tau_j^n\}$ as rebalancing dates in the self-financing manner up to τ. More precisely, we define $\hat{\Pi}^{0,\pm\alpha,n}$ recursively as

$$\begin{aligned}
&\hat{\Pi}_0^{0,\pm\alpha,n} = \Pi_0^{0,\pm\alpha}, \\
&(\Pi_{\tau_{j+1}^n}^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha})S_{\tau_{j+1}^n}^1 + \kappa_n|\Pi_{\tau_{j+1}^n}^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha}|S_{\tau_{j+1}^n}^1 + (\hat{\Pi}_{j+1}^{0,\pm\alpha,n} - \hat{\Pi}_j^{0,\pm\alpha,n})S_{\tau_{j+1}^n}^0 = 0.
\end{aligned}$$

Note that the second term of (1) is to absorb the transaction cost at time 0. The value $V_t^{\pm\alpha,n}$ of our portfolio at time t ignoring the clearance cost is given by

$$V_t^{\pm\alpha,n} = \Pi_t^{\pm\alpha,n} S_t^1 + \Pi_t^{0,\pm\alpha,n} S_t^0,$$

where we put $\Pi_t^{\pm\alpha,n} = \Pi_{\tau_j^n}^{\pm\alpha}$ and $\Pi_t^{0,\pm\alpha,n} = \hat{\Pi}_j^{0,\pm\alpha,n}$ for $t \in [\tau_j^n, \tau_{j+1}^n)$. By construction we have that

$$\Delta V_j^{\pm\alpha,n} = \Pi_{\tau_j^n}^{\pm\alpha} \Delta S_j^1 + \hat{\Pi}_j^{0,\pm\alpha,n} \Delta S_j^0 - \kappa_n |\Delta \Pi_j^{\pm\alpha}| S_{\tau_{j+1}^n}^1,$$

where

$$\Delta V_j^{\pm\alpha,n} = V_{\tau_{j+1}^n}^{\pm\alpha,n} - V_{\tau_j^n}^{\pm\alpha,n}, \quad \Delta \Pi_j^{\pm\alpha} = \Pi_{\tau_{j+1}^n}^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha}, \quad \Delta S_j^i = S_{\tau_{j+1}^n}^i - S_{\tau_j^n}^i$$

for $i = 0$ and 1. Further, putting

$$\tilde{V}^{\pm\alpha,n} = V^{\pm\alpha,n}/S^0, \quad \Delta \tilde{V}_j^{\pm\alpha,n} = \tilde{V}_{\tau_{j+1}^n}^{\pm\alpha,n} - \tilde{V}_{\tau_j^n}^{\pm\alpha,n}, \quad \Delta \tilde{S}_j = \tilde{S}_{\tau_{j+1}^n} - \tilde{S}_{\tau_j^n},$$

we obtain that

$$\Delta \tilde{V}_j^{\pm\alpha,n} = \Pi_{\tau_j^n}^{\pm\alpha} \Delta \tilde{S}_j - \kappa_n |\Delta \Pi_j^{\pm\alpha}| \tilde{S}_{\tau_{j+1}^n},$$

so that

$$\tilde{V}_t^{\pm\alpha,n} = P(S_0^1, R_0, \Sigma_0^{\pm\alpha})/S_0^0 + \int_0^t \Pi_u^{\pm\alpha,n} \mathrm{d}\tilde{S}_u - \sum_{\tau_{j+1}^n \leq t} \kappa_n |\Pi_{\tau_{j+1}^n}^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha}| \tilde{S}_{\tau_{j+1}^n}. \qquad (3)$$

Our main result concerns the limit distribution of the continuous process $Z^{\pm\alpha,n}$ defined as

$$Z_t^{\pm\alpha,n} = \kappa_n^{-1}(P(S_t^1, R_t, \Sigma_t^{\pm\alpha})/S_t^0 - \tilde{V}_t^{\pm\alpha,n})$$

as $n \to \infty$, and in particular, we have a convergence

$$V_t^{\pm\alpha} \to P(S_t^1, R_t, \Sigma_t^{\pm\alpha})$$

in probability for $t \in [0, \tau)$. Note that if f is convex, then

$$P(S_T^1, R_T, \Sigma_T^{\pm\alpha}) = P(S_T^1, 0, \Sigma_T^{\pm\alpha}) \geq f(S_T^1)$$

on the set $\{\tau \geq T\}$. If f is concave,

$$P(S_\tau^1, R_\tau, \Sigma_\tau^{\pm\alpha}) = S_\tau^0 f(\tilde{S}_\tau) = aS_\tau^1 + bS_\tau^0, \quad aS_T^1 + bS_T^0 \geq f(S_T^1)$$

on the set $\{\tau \leq T\}$. Therefore, in the asymptotic sense, the self-financing strategy $(\Pi^{\pm\alpha,n}, \Pi^{0,\pm\alpha,n})$ turns out to be a super-hedging strategy for convex or concave payoffs under the cumulative volatility bounds $\langle \log(\tilde{S}) \rangle_T \leq \hat{\Sigma}$ or $\langle \log(\tilde{S}) \rangle_T \geq \hat{\Sigma}$ respectively.

Now we put a condition equivalent to *No Free Lunch with Vanishing Risk* (See Delbaen and Schachermayer [6, 7]), one of "no-arbitrage" conditions which is natural to be supposed in this financial context:

Condition 2 *There exists an equivalent local martingale measure for $\tilde{S}$, that is, an equivalent probability measure under which $\tilde{S}$ is a local martingale.*

Definition 1 *Let σ be a stopping time and $\mathcal{G} \subset \mathcal{F}$ be a sub σ-field. A sequence of continuous processes Z^n is said to converge $\mathcal{G}$-stably in $D[0, \sigma)$ to Z if there exists a sequence of stopping times σ^m which converges σ a.s. as $m \to \infty$ such that for any $m \in \mathbb{N}$ and any bounded $\mathcal{G}$-measurable random variable G, the $\mathbb{R} \times D[0, \infty)$-valued random sequence $(G, Z^n_{\cdot \wedge \sigma^m})$ converges to $(G, Z_{\cdot \wedge \sigma^m})$ in law as $n \to \infty$.*

Theorem 1 *Suppose Conditions 1 and 2 to hold. Let W be a standard Brownian motion independent of $\mathcal{F}$, possibly defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$.*

- *If f is convex, then $Z^{+\alpha,n}$ converges $\mathcal{F}$-stably in $D[0, \tau)$ to a time-changed Brownian motion W_Q, where*

$$Q = \frac{|\alpha + 2|^2}{6} \int_0^{\cdot} |\tilde{S}_u \Gamma_u^{+\alpha}|^2 \mathrm{d}\langle \tilde{S} \rangle_u.$$

- *If f is concave, then $Z^{-\alpha,n}$ converges $\mathcal{F}$-stably in $D[0, \tau)$ to a time-changed Brownian motion W_Q, where*

$$Q = \frac{|\alpha - 2|^2}{6} \int_0^{\cdot} |\tilde{S}_u \Gamma_u^{-\alpha}|^2 \mathrm{d}\langle \tilde{S} \rangle_u.$$

Proof: We take a sequence of stopping times σ^m so that $\tilde{S}$ and $1/\tilde{S}$ are bounded by m on $[0,\sigma^m]$ and $\sigma^m \leq m$, $\langle \log(\tilde{S}) \rangle_{\sigma^m} \leq \hat{\Sigma} - 1/m$ for each $m \in \mathbb{N}$ and $\sigma^m \to \tau$ a.s. as $m \to \infty$. Notice that putting

$$\Delta(S,\Sigma) = \frac{\partial P}{\partial S}(S,0,\Sigma)$$

for $(S,\Sigma) \in \mathbb{R}^2_+$, we have

$$\Pi^{\pm\alpha} = \Delta(\tilde{S},\Sigma^{\pm\alpha}), \quad \Gamma^{\pm\alpha} = \frac{\partial \Delta}{\partial S}(\tilde{S},\Sigma^{\pm\alpha}). \tag{4}$$

Hence $\Pi^{\pm\alpha}$, $\Gamma^{\pm\alpha}$ and $1/\Gamma^{\pm\alpha}$ are bounded on $[0,\sigma^m]$. By Itô's formula,

$$\begin{aligned} P(S^1_t, R_t, \Sigma^{\pm\alpha}_t) =& P(S^1_0, R_0, \Sigma^{\pm\alpha}_0) \\ &+ \int_0^t \Pi^{\pm\alpha}_u \mathrm{d}S_u + \int_0^t \Pi^{0,\pm\alpha}_u \mathrm{d}S^0_u \mp \frac{1}{\alpha}\int_0^t S^0_u \Gamma^{\pm\alpha}_u \mathrm{d}\langle \tilde{S} \rangle_u \end{aligned}$$

for $t \in [0,\tau)$. Hence, again by Itô's formula,

$$\tilde{P}^{\pm\alpha}_t = \tilde{P}^{\pm\alpha}_0 + \int_0^t \Pi^{\pm\alpha}_u \mathrm{d}\tilde{S}_u \mp \frac{1}{\alpha}\int_0^t \Gamma^{\pm\alpha}_u \mathrm{d}\langle \tilde{S} \rangle_u,$$

where $\tilde{P}^{\pm\alpha} = P(S^1, R, \Sigma^{\pm\alpha})/S^0$. Using (2) and (3), we have that

$$\begin{aligned} \tilde{P}^{\pm\alpha}_t - \tilde{V}^{\pm\alpha,n}_t =& \int_0^t (\Pi^{\pm\alpha}_u - \Pi^{\pm\alpha,n}_u)\mathrm{d}\tilde{S}_u \\ &+ \alpha\kappa_n^2 \sum_{\tau^n_{j+1}\leq t} \tilde{S}_{\tau^n_{j+1}} \tilde{S}_{\tau^n_j} |\Gamma^{\pm\alpha}_{\tau^n_j}| \mp \frac{1}{\alpha}\int_0^t \frac{\mathrm{d}\langle \Pi^{\pm\alpha}\rangle_u}{\Gamma^{\pm\alpha}_u}. \end{aligned} \tag{5}$$

Since stable convergence and in particular, convergence in probability are stable against equivalent changes of measures, we can freely choose an equivalent measure to estimate terms. First take an equivalent martingale measure for $\tilde{S}$. Then,

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq \sigma^m}\left|\sum_{\tau^n_{j+1}\leq t} \tilde{S}_{\tau^n_j}|\Gamma^{\pm\alpha}_{\tau^n_j}|(\tilde{S}_{\tau^n_{j+1}} - \tilde{S}_{\tau^n_j})\right|^2\right] \\ &\leq C_m + \mathbb{E}\left[\sup_{0\leq t\leq \sigma^m}\left|\sum_{j=0}^{\infty} \tilde{S}_{\tau^n_j}|\Gamma^{\pm\alpha}_{\tau^n_j}|(\tilde{S}_{\tau^n_{j+1}\wedge t} - \tilde{S}_{\tau^n_j\wedge t})\right|^2\right] \\ &\leq C_m + \mathbb{E}\left[\sum_{j=0}^{\infty} \tilde{S}^2_{\tau^n_j}|\Gamma^{\pm\alpha}_{\tau^n_j}|^2(\langle \tilde{S}\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle \tilde{S}\rangle_{\tau^n_j\wedge\sigma^m})\right] \end{aligned}$$

by Doob's inequality, where C_m is a constant. Therefore, using (2) again, we have that the sum of the last two terms of (5) is equal to

$$\pm\frac{1}{\alpha}\left\{\sum_{j=0}^{\infty}\frac{1}{\Gamma^{\pm\alpha}_{\tau^n_j}}|\Pi^{\pm\alpha}_{t\wedge\tau^n_{j+1}}-\Pi^{\pm\alpha}_{t\wedge\tau^n_j}|^2-\int_0^t\frac{\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u}{\Gamma^{\pm\alpha}_u}\right\}+O_p(\kappa_n^2).$$

Here we have used $|\Gamma^{\pm\alpha}|=\pm\Gamma^{\pm\alpha}$. Since

$$|\Pi^{\pm\alpha}_t-\Pi^{\pm\alpha}_s|^2=2\int_s^t(\Pi^{\pm\alpha}_u-\Pi^{\pm\alpha}_s)\mathrm{d}\Pi^{\pm\alpha}_u+\langle\Pi^{\pm\alpha}\rangle_t-\langle\Pi^{\pm\alpha}\rangle_s,$$

we obtain that

$$\begin{aligned}Z^{\pm\alpha,n}_t=&\kappa_n^{-1}\left\{1\pm\frac{2}{\alpha}\right\}\int_0^t(\Pi^{\pm\alpha}_u-\Pi^{\pm\alpha,n}_u)\mathrm{d}\tilde{S}_u\\&\pm\frac{1}{\alpha\kappa_n}\left\{\int_0^t\frac{\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u}{\Gamma^{\pm\alpha,n}_u}-\int_0^t\frac{\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u}{\Gamma^{\pm\alpha}_u}\right\}\\&\pm\frac{2}{\alpha\kappa_n}\left\{\int_0^t\frac{\Pi^{\pm\alpha}_u-\Pi^{\pm\alpha,n}_u}{\Gamma^{\pm\alpha,n}_u}\mathrm{d}\Pi^{\pm\alpha}_u-\int_0^t(\Pi^{\pm\alpha}_u-\Pi^{\pm\alpha,n}_u)\mathrm{d}\tilde{S}_u\right\}+O_p(\kappa_n),\end{aligned}\tag{6}$$

where $\Gamma^{\pm\alpha,n}_t=\Gamma^{\pm\alpha}_{\tau^n_j}$ for $t\in[\tau^n_j,\tau^n_{j+1})$. Next for each of $+\alpha$ and $-\alpha$, take an equivalent measure under which $\Pi^{\pm\alpha}_{\cdot\wedge\sigma^m}$ is a local martingale. Then, by Lemma 1 below, we have that

$$\int_0^t\frac{\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u}{\Gamma^{\pm\alpha}_u}=\int_0^t\frac{\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u}{\Gamma^{\pm\alpha,n}_u}+o_p(\kappa_n)$$

and that

$$\int_0^t\frac{\Pi^{\pm\alpha}_u-\Pi^{\pm\alpha,n}_u}{\Gamma^{\pm\alpha,n}_u}\mathrm{d}\Pi^{\pm\alpha}_u=\int_0^t(\Pi^{\pm\alpha}_u-\Pi^{\pm\alpha,n}_u)\mathrm{d}\tilde{S}_u+o_p(\kappa_n).$$

The result then follows from Theorem 2.6 of Fukasawa [12]. ////

Lemma 1 *Suppose Conditions 1 and 2 to hold. Let σ^m be a bounded stopping time such that $\tilde{S}$ and $1/\tilde{S}$ are bounded by m on $[0,\sigma^m]$ and $\langle\log(\tilde{S})\rangle_{\sigma^m}\leq\hat{\Sigma}-1/m$. Denote by $\mathbb{E}^{\pm\alpha}$ the expectation operators of equivalent measures under which $\Pi^{\pm\alpha}_{\cdot\wedge\sigma^m}$ are local martingales respectively. Let U, V be twice continuously differentiable functions on $(0,\infty)^2$. Set*

$$U^{\pm\alpha}_t=U(\tilde{S}_t,\Sigma^{\pm\alpha}_t),\quad V^{\pm\alpha}_t=V(\tilde{S}_t,\Sigma^{\pm\alpha}_t)$$

and $U^{\pm\alpha,n}_t=U^{\pm\alpha}_{\tau^n_j}$ for $t\in[\tau^n_j,\tau^n_{j+1})$. Then for any $k\in\mathbb{N}$,

$$\sup_{n\in\mathbb{N}}\kappa_n^{-2k}\mathbb{E}^{\pm\alpha}\left[\int_0^{\sigma^m}|U^{\pm\alpha}_s-U^{\pm\alpha}_s|^{2k}\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_s\right]<\infty\tag{7}$$

and

$$\lim_{n\to\infty}\kappa_n^{-2}\mathbb{E}^{\pm\alpha}\left[\sup_{t\geq0}\left|\int_0^{t\wedge\sigma^m}(U^{\pm\alpha}_s-U^{\pm\alpha,n}_s)V^{\pm\alpha}_s\mathrm{d}\langle\Pi^{\pm\alpha}\rangle_s\right|^2\right]=0.\tag{8}$$

Proof: We omit $\pm\alpha$ since everything is the same for the both cases. We use C as a generic positive constant which does not depend on n. To prove this lemma we can suppose without loss of generality that

$$\langle\Pi\rangle_t = \langle\Pi\rangle_{t\wedge\sigma^m} + (t-\sigma^m)_+, \ \ \tilde{S}_t = \tilde{S}_{t\wedge\sigma^m}, \ \ \Gamma_t = \Gamma_{t\wedge\sigma^m}$$

for all $t \geq 0$. By the Burkholder-Davis-Gundy, and Doob's inequalities, we have

$$C^{-1}\kappa_n^{2k} \leq \mathbb{E}[|\langle\Pi\rangle_{\tau_{j+1}^n} - \langle\Pi\rangle_{\tau_j^n}|^k|\mathcal{F}_{\tau_j^n}] \leq C\kappa_n^{2k}$$

uniformly in $j = 0, 1, \dots, N_t^n$ for any $t \geq 0$, where

$$N_t^n = \max\{j \geq 0; \tau_j^n \leq t\}.$$

Moreover,

$$\kappa_n^2\mathbb{E}[N_t^n] \leq \mathbb{E}\left[\sum_{j=0}^{N_t^n} \frac{|\Pi_{\tau_{j+1}^n} - \Pi_{\tau_j^n}|^2}{\alpha^2\tilde{S}^2_{\tau_j^n}\Gamma^2_{\tau_j^n}}\right] \leq Ct + C\mathbb{E}[\langle\Pi\rangle_{\sigma^m}],$$

so that

$$\begin{aligned}\mathbb{E}\left[\sum_{j=0}^{\infty} |\langle\Pi\rangle_{t\wedge\tau_{j+1}^n} - \langle\Pi\rangle_{t\wedge\tau_j^n}|^k\right] =&\mathbb{E}\left[\sum_{j=0}^{\infty} \mathbb{E}[|\langle\Pi\rangle_{t\wedge\tau_{j+1}^n} - \langle\Pi\rangle_{t\wedge\tau_j^n}|^k|\mathcal{F}_{t\wedge\tau_j^n}]\right] \\ \leq&\mathbb{E}\left[\sum_{j=0}^{\infty} \mathbb{E}[|\langle\Pi\rangle_{\tau_{j+1}^n} - \langle\Pi\rangle_{\tau_j^n}|^k|\mathcal{F}_{\tau_j^n}]1_{\{j\leq N_t^n\}}\right] \\ \leq& C\kappa_n^{2k}(1+\mathbb{E}[N_t^n]) = O(\kappa_n^{2(k-1)})\end{aligned} \tag{9}$$

for any $k \in \mathbb{N}$. Hence the same argument as in the proof of Lemma 2.9 of Fukasawa [12] is applicable to obtain (7) in cases where there exists a continuous bounded process Y such that

$$\mathrm{d}U_t = Y_t\mathrm{d}\Pi_t$$

on $[0, \sigma^m]$. In general, there exist continuous bounded processes Y^i such that

$$\mathrm{d}U_t = Y_t^1\mathrm{d}\Pi_t + Y_t^2\mathrm{d}\langle\Pi\rangle_t, \ \ \mathrm{d}V_t = Y_t^3\mathrm{d}\Pi_t + Y_t^4\mathrm{d}\langle\Pi\rangle_t$$

on $[0, \sigma^m]$. To obtain (7) in the general case, observe that

$$\lim_{n\to\infty} \kappa_n^{-2k}\mathbb{E}\left[\sum_{j=0}^{\infty}\int_{\tau_j^n\wedge\sigma^m}^{\tau_{j+1}^n\wedge\sigma^m} \left|\int_{\tau_j^n\wedge\sigma^m}^{s} Y_u^2\mathrm{d}\langle\Pi\rangle_u\right|^{2k} \mathrm{d}\langle\Pi^{\pm\alpha}\rangle_s\right] = 0.$$

This is because Y^i are bounded and we already have (9). By the same reason, we can suppose $Y^4 = 0$ to prove (8). Further, we can replace $V = V^{\pm\alpha}$ with

V^n by (7) with the aid of the Cauchy-Schwarz inequality, where $V^n_t = V_{\tau^n_j}$ for $t \in [\tau^n_j, \tau^n_{j+1})$. It suffices then to show that

$$\lim_{n\to\infty} \kappa_n^{-2} \mathbb{E}\left[\sup_{t\geq 0}\left|\sum_{j=0}^{\infty}\int_{\tau^n_j\wedge\sigma^m}^{\tau^n_{j+1}\wedge\sigma^m}\int_{\tau^n_j\wedge\sigma^m}^{t}(Y^3_s - Y^3_{\tau^n_j})\mathrm{d}\Pi_s V_{\tau^n_j}\mathrm{d}\langle\Pi\rangle_t\right|^2\right] = 0$$

and that

$$\lim_{n\to\infty} \kappa_n^{-2} \mathbb{E}\left[\sup_{t\geq 0}\left|\sum_{j=0}^{N^n_{t\wedge\sigma^m}}\int_{\tau^n_j}^{\tau^n_{j+1}}(\Pi_t - \Pi_{\tau^n_j})Y^3_{\tau^n_j} V_{\tau^n_j}\mathrm{d}\langle\Pi\rangle_t\right|^2\right] = 0.$$

The first one follows again from the same argument as in the proof of Lemma 2.9 of Fukasawa [12]. To show the second, notice that by the definition (2) and the martingale property of Π,

$$\mathbb{E}\left[\int_{\tau^n_j}^{\tau^n_{j+1}}(\Pi_t - \Pi_{\tau^n_j})\mathrm{d}\langle\Pi\rangle_t\middle|\mathcal{F}_{\tau^n_j}\right] = \frac{1}{3}\mathbb{E}[(\Pi_{\tau^n_{j+1}} - \Pi_{\tau^n_j})^3|\mathcal{F}_{\tau^n_j}] = 0.$$

Therefore, we obtain the result by observing that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=0}^{\infty}\left|\int_{\tau^n_j\wedge\sigma^m}^{\tau^n_{j+1}\wedge\sigma^m}(\Pi_t - \Pi_{\tau^n_j})Y^3_{\tau^n_j} V_{\tau^n_j}\mathrm{d}\langle\Pi\rangle_t\right|^2\right]\\
&\leq C\mathbb{E}\left[\sum_{j=0}^{\infty}|\langle\Pi\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle\Pi\rangle_{\tau^n_j\wedge\sigma^m}|^{3/2}\left\{\int_{\tau^n_j\wedge\sigma^m}^{\tau^n_{j+1}\wedge\sigma^m}|\Pi_t - \Pi_{\tau^n_j}|^4\mathrm{d}\langle\Pi\rangle_t\right\}^{1/2}\right]\\
&\leq C\left\{\mathbb{E}\left[\sum_{j=0}^{\infty}|\langle\Pi\rangle_{\tau^n_{j+1}\wedge\sigma^m} - \langle\Pi\rangle_{\tau^n_j\wedge\sigma^m}|^3\right]\right\}^{1/2}\left\{\mathbb{E}\left[\int_0^{\sigma^m}|\Pi_t - \Pi^n_t|^4\mathrm{d}\langle\Pi\rangle_t\right]\right\}^{1/2} = o(\kappa_n^2)
\end{aligned}$$

because of (9), where $\Pi^n = \Pi^{\pm\alpha,n}$. ////

4 Mean squared error

Here we study the mean squared error of the discrete hedging strategy under linear transaction costs described in the last section. We prove the following convergence, that is formally indicated by the results of the last section.

Theorem 2 *Suppose Conditions 1 and 2 to hold. Denote by $\mathbb{E}$ the equivalent local martingale measure for $\tilde{S}$. Then, there exists a sequence of stopping times σ^m such that $\sigma^m \to \tau$ a.s. as $m \to \infty$ and*

$$\lim_{n\to\infty}\mathbb{E}[|Z^{\pm\alpha,n}_{t\wedge\sigma^m}|^2] = \frac{|\alpha\pm 2|^2}{6}\mathbb{E}\left[\int_0^{t\wedge\sigma^m}|\tilde{S}_u\Gamma^{\pm\alpha}_u|^2\mathrm{d}\langle\tilde{S}\rangle_u\right]$$

for any $t \geq 0$, where as before, $\pm\alpha$ should be understood as $+\alpha$ or $-\alpha$ if f is convex or concave respectively.

Proof: For given a continuous semimartingale X with decomposition

$$X_t = X_0 + M_t + \int_0^t V_t \mathrm{d}\langle M \rangle_t,$$

where M is a continuous local martingale and V is a locally bounded adapted process, define a continuous process $e(X)$ as

$$e(X)_t = \exp\left\{ \int_0^t V_t \mathrm{d}M_t - \frac{1}{2}\int_0^t V_t^2 \mathrm{d}\langle M \rangle_t \right\}.$$

Take a sequence of stopping times σ^m so that $\tilde{S}$, $1/\tilde{S}$ and $1/e(\Pi)$ are bounded by m on $[0, \sigma^m]$ and $\sigma^m \leq m$, $\langle \log(\tilde{S}) \rangle_{\sigma^m} \leq \hat{\Sigma} - 1/m$ for each m, and $\sigma^m \to \tau$ a.s. as $m \to \infty$. We may start from (6), where the $O_p(\kappa_n)$ term is negligible in L^2. Denote by R^n the sum of the second and third terms of (6). By Lemma 1 and the Girsanov-Maruyama theorem, we have that

$$\mathbb{E}[|R^n_{t\wedge\sigma^m}|^2] \leq m\mathbb{E}[e(\Pi)_{t\wedge\sigma^m}|R^n_{t\wedge\sigma^m}|^2] \to 0$$

as $n \to \infty$. Now it remains to show

$$\lim_{n\to\infty} \kappa_n^{-2}\mathbb{E}\left[\int_0^{t\wedge\sigma^m} (\Pi_u^{\pm\alpha} - \Pi_u^{\pm\alpha,n})^2 \mathrm{d}\langle \tilde{S} \rangle_u\right] = \frac{\alpha^2}{6}\mathbb{E}\left[\int_0^{t\wedge\sigma^m} |\tilde{S}_u\Gamma_u^{\pm\alpha}|^2 \mathrm{d}\langle \tilde{S} \rangle_u\right].$$

By Lemma 1 again, we have that

$$\begin{aligned}
&\kappa_n^{-2}\mathbb{E}\left[\int_0^{t\wedge\sigma^m} (\Pi_u^{\pm\alpha} - \Pi_u^{\pm\alpha,n})^2 \left|\frac{1}{|\Gamma_u^{\pm\alpha}|^2} - \frac{1}{|\Gamma_u^{\pm\alpha,n}|^2}\right| \mathrm{d}\langle \tilde{\Pi}^{\pm\alpha} \rangle_u\right] \\
&\leq m\kappa_n^{-2}\left|\mathbb{E}\left[e(\Pi)_{t\wedge\sigma^m}\int_0^{t\wedge\sigma^m} (\Pi_u^{\pm\alpha} - \Pi_u^{\pm\alpha,n})^4 \mathrm{d}\langle \tilde{\Pi}^{\pm\alpha} \rangle_u\right]\right|^{1/2} \cdot \\
&\qquad \left|\mathbb{E}\left[e(\Pi)_{t\wedge\sigma^m}\int_0^{t\wedge\sigma^m} \left|\frac{1}{|\Gamma_u^{\pm\alpha}|^2} - \frac{1}{|\Gamma_u^{\pm\alpha,n}|^2}\right|^2 \mathrm{d}\langle \tilde{\Pi}^{\pm\alpha} \rangle_u\right]\right|^{1/2} = O(\kappa_n).
\end{aligned}$$

By Itô's formula, we have that

$$\left(\Pi^{\pm\alpha}_{\tau^n_{j+1}} - \Pi^{\pm\alpha}_{\tau^n_j}\right)^4 = 4\int_{\tau^n_j}^{\tau^n_{j+1}} \left(\Pi^{\pm\alpha}_s - \Pi^{\pm\alpha}_{\tau^n_j}\right)^3 \mathrm{d}\Pi^{\pm\alpha}_s + 6\int_{\tau^n_j}^{\tau^n_{j+1}} \left(\Pi^{\pm\alpha}_s - \Pi^{\pm\alpha}_{\tau^n_j}\right)^2 \mathrm{d}\langle \Pi^{\pm\alpha} \rangle_s,$$

so that putting $N_s^n = \max\{j \geq 0; \tau_j^n \leq s\}$,

$$
\begin{aligned}
&\lim_{n\to\infty} \kappa_n^{-2}\mathbb{E}\left[\int_0^{t\wedge\sigma^m} (\Pi_u^{\pm\alpha} - \Pi_u^{\pm\alpha,n})^2 \mathrm{d}\langle\tilde{S}\rangle_u\right] \\
&= \lim_{n\to\infty} \kappa_n^{-2}\mathbb{E}\left[\sum_{j=0}^{N^n_{t\wedge\sigma^m}} \int_{\tau_j^n}^{\tau_{j+1}^n} \frac{(\Pi_u^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha})^2}{|\Gamma_{\tau_j^n}^{\pm\alpha}|^2} \mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u\right] \\
&= \frac{1}{6}\lim_{n\to\infty} \kappa_n^{-2}\mathbb{E}\left[\sum_{j=0}^{N^n_{t\wedge\sigma^m}} \frac{\left|\Pi_{\tau_{j+1}^n}^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha}\right|^4}{|\Gamma_{\tau_j^n}^{\pm\alpha}|^2}\right] \\
&= \frac{\alpha^2}{6}\lim_{n\to\infty} \mathbb{E}\left[\sum_{j=0}^{N^n_{t\wedge\sigma^m}} |\tilde{S}_{\tau_j^n}|^2 \left|\Pi_{\tau_{j+1}^n}^{\pm\alpha} - \Pi_{\tau_j^n}^{\pm\alpha}\right|^2\right].
\end{aligned}
$$

By Lemma 1 again, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\left|\sum_{j=0}^{\infty} |\tilde{S}_{\tau_j^n}|^2 \left|\Pi_{\tau_{j+1}^n\wedge t\wedge\sigma^m}^{\pm\alpha} - \Pi_{\tau_j^n\wedge t\wedge\sigma^m}^{\pm\alpha}\right|^2 - \int_0^{t\wedge\sigma^m} |\tilde{S}_u^n|^2 \mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u\right|^2\right] \\
&\leq 2m\mathbb{E}\left[e(\Pi)_{t\wedge\sigma^m}\int_0^{t\wedge\sigma^m} \left|\Pi_u^{\pm\alpha} - \Pi_u^{\pm\alpha,n}\right|^2 |\tilde{S}_u^n|^2 \mathrm{d}\langle\Pi^{\pm\alpha}\rangle_u\right] = O(\kappa_n^2),
\end{aligned}
$$

where $\tilde{S}_t^n = \tilde{S}_{\tau_j^n}$ for $t \in [\tau_j^n, \tau_{j+1}^n)$. The rest is obvious. ////

Unfortunately, we cannot claim in general that

$$\lim_{n\to\infty} \mathbb{E}[|Z_\tau^{\pm\alpha,n}|^2] = \frac{|\alpha \pm 2|^2}{6}\mathbb{E}\left[\int_0^\tau |\tilde{S}_u\Gamma_u^{\pm\alpha}|^2 \mathrm{d}\langle\tilde{S}\rangle_u\right].$$

In particular, even under the Black-Scholes model with volatility $\sigma = \sqrt{\hat{\Sigma}/T}$, we do not have in general that

$$\lim_{n\to\infty} \kappa_n^{-2}\mathbb{E}[|f(S_T^1) - V_T^{\pm\alpha}|^2] = \frac{|\alpha \pm 2|^2}{6}\mathbb{E}\left[\int_0^T |\tilde{S}_u\Gamma_u^{\pm\alpha}|^2 \mathrm{d}\langle\tilde{S}\rangle_u\right].$$

This is mainly because we do not have enough tools to estimate the integrability of random variables depending on the stopping times τ_j^n under reasonable assumptions on the regularity of f. Nevertheless, it is practically irrelevant to dynamically rebalance portfolios up to the last moment T. It is common to avoid frequent rebalancing near the maturity due to the high sensitivity of strategies with respect to price movement. We are therefore satisfied with such a localized result as Theorem 2.

5 Leland's strategy and the choice of α

Our rebalancing dates (2) are not deterministic, while only deterministic dates have been treated in the preceding studies of Leland's strategy. Therefore, our convergence results are new even under the Black-Scholes model that is a special case of our framework. Here we compare the asymptotic variances of hedging error associated with our strategy and Leland's original one and make a remark on the choice of α in (2). Consider the Black-Scholes model with volatility σ for the risk-neutral dynamics of $\tilde{S}$ and let f be a convex payoff function. Let $T = 1$ for brevity. Leland's original strategy uses the equidistant partition j/n, $j = 0, 1, 2, \dots, n$ as rebalancing date and employs the Black-Scholes pricing and hedging strategy with enlarged volatility $\check{\sigma}$ defined as

$$\check{\sigma}^2 = \sigma^2 + \sigma n^{1/2}\kappa_n \sqrt{\frac{8}{\pi}}.$$

In the case $\kappa_n = \kappa_0 n^{-1/2}$ with constant $\kappa_0 > 0$, it is known that

$$\lim_{n\to\infty} \kappa_n^{-2}\mathbb{E}[|f(S_1^1) - V_1^n|^2] = \beta\left(\frac{\sigma}{\kappa_0}\right)\mathbb{E}\left[\int_0^1 |\tilde{S}_u\check{\Gamma}_u|^2 \mathrm{d}\langle\tilde{S}\rangle_u\right],$$

where V_t^n is the portfolio value at time 1 of the strategy,

$$\beta(x) = \frac{x^2}{2} + \sqrt{\frac{2}{\pi}}x + 1 - \frac{2}{\pi}$$

and $\check{\Gamma}$ is defined as

$$\check{\Gamma}_t = \frac{\partial\Delta}{\partial S}(\tilde{S}_t, \check{\sigma}^2(1-t)).$$

See Gamys and Kabanov [14]. Further, denoting by $\check{V}_t = P(\tilde{S}_t, 0, \check{\sigma}^2(1-t))$, it is known that $\kappa_n^{-1}(\check{V} - V^n)$ converges stably in $D[0,1]$ to a time changed Brownian motion W_Q with

$$Q = \beta\left(\frac{\sigma}{\kappa_0}\right)\int_0^{\cdot} |\tilde{S}_u\check{\Gamma}_u|^2 \mathrm{d}\langle\tilde{S}\rangle_u.$$

See Denis and Kabanov [8]. Letting $\hat{\Sigma} = \sigma^2$ and

$$\alpha = \frac{\sigma}{\kappa_0}\sqrt{\frac{\pi}{2}},$$

our strategy $(\Pi^{+\alpha}, \Pi^{0,+\alpha})$ has the same initial value as Leland's and we have $\Gamma^{+\alpha} = \check{\Gamma}$. The coefficient of the asymptotic variance is

$$\frac{|\alpha+2|^2}{6} = \hat{\beta}\left(\frac{\sigma}{\kappa_0}\right), \quad \hat{\beta}(x) = \frac{\pi}{12}x^2 + 2\sqrt{2\pi}x + \frac{2}{3}.$$

Since $1/2 > \pi/12$, if σ/κ_0, or equivalently, α is sufficiently large, our strategy results in a smaller hedging error. As shown in Fukasawa [12], hitting times such as (2) always results in a smaller discretization error of stochastic integrals than deterministic partitions. Roughly speaking, this is because the asymptotic variance of the discretization error is determined by the skewness and kurtosis of each increment of the integrand. Hitting times produce the increments with unit kurtosis and zero skewness. On the other hand, we are now dealing with not only the discretization error but also the linear transaction costs. Due to the linear structure, the first absolute moment of each increment also affects the asymptotic variance, while this is not clear from our proof. This is the reason why the equidistant partition can be superior when α is small. Recall that in our framework, α can be chosen arbitrarily. As α increases, our enlarging or shrinking volatility becomes modest and the price of the option decreases. This occurs at the expense of a larger asymptotic variance of the super-hedging error, which is clearly seen in our main theorems. The optimal choice of α therefore depends on the risk preference of agents.

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