ARBITRAGE AND HEDGING IN A NON PROBABILISTIC FRAMEWORK

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Abstract. The paper studies the concepts of hedging and arbitrage in a non probabilistic framework. It provides conditions for non probabilistic arbitrage based on the topological structure of the trajectory space and makes connections with the usual notion of arbitrage. Several examples illustrate the non probabilistic arbitrage as well perfect replication of options under continuous and discontinuous trajectories, the results can then be applied in probabilistic models path by path. The approach is related to recent financial models that go beyond semimartingales, we remark on some of these connections and provide applications of our results to some of these models.

1. Introduction

Modern mathematical finance relies on the notions of arbitrage and hedging replication; generally, these ideas are exclusively cast in probabilistic frameworks. The possibility of dispensing with probabilities in mathematical finance, wherever possible, may have already occurred to several researchers. A reason for this is that hedging results do not depend on the actual probability distributions but on the support of the probability measure. Actually, hedging is clearly a pathwise notion and a simple view of arbitrage is that there is a portfolio that will no produce any loss for all possible paths and there exists at least one path that will provide a profit. This informal reasoning suggests that there is no need to use probabilities to define the concepts even though probability has been traditionally used to do so. The paper makes an attempt to study these two notions without probabilities in a direct and simple way.

From a technical point of view, we rely on a simple calculus for non differentiable functions introduced in [12] (see also [20]). This calculus is available for a fairly large class of functions. Hedging results that only depend on this pathwise calculus can be considered independently of probabilistic assumptions. In [4] the authors take this point of view and develop a discrete framework, and its associated limit, to hedge continuous paths with a prescribed 2-variation. Reference [4] is mostly devoted to payoff replication for continuous trajectories and does not address the issue of non probabilistic arbitrage. The present paper formally defines this last notion in a context that allows for trajectories with jumps, develops some of the basic consequences and presents some simple applications including novel results to probabilistic frameworks.

Results on the existence of arbitrage in a non standard framework (i.e. a non semimartingale price process) leads to interesting and challenging problems. In order to gain a perspective on this issue, recall that a consequence of the fundamental theorem of asset pricing of Delbaen and Schachermayer in [10] is that under the NFLVR condition and considering simple predictable portfolio strategies, the price process of the risky asset necessarily must be a semi-martingale. Recent literature ([7], [16], [9], [3]) describes pricing results in non semi-martingale settings; the restriction of the possible portfolio strategies has been a central issue in these works. In [7] the permissible portfolio strategies are restricted to those for which the time between consecutive transactions is bounded below by some number h. In [16] these results are extended, also considering portfolios having a minimal fixed time between successive trades. In [9] the notion of $\mathcal{A}$-martingale is introduced in order to have the no-arbitrage property for a given class $\mathcal{A}$ of admissible strategies.

We also treat this problem but with a different perspective, our main object of study is a class of trajectories $\mathcal{J} = \mathcal{J}(x_0)$ starting at x_0. To this set of deterministic trajectories we associate a class of admissible portfolios $\mathcal{A} = \mathcal{A}_{\mathcal{J}(x_0)}$ which, under some conditions, is free of arbitrage and allow for perfect replication to take place. These two notions, arbitrage and hedging, are defined without probabilities. Once no arbitrage and hedging have been established for the non probabilistic market model $(\mathcal{J}(x_0), \mathcal{A}_{\mathcal{J}(x_0)})$, these results could be used to provide a fair price for the option being hedged. These pricing results will not be stated explicitly in the paper and will be left implicit.

Some technical aspects from our approach relate to the presentation in [3], similarities with [3] are expressed mainly in the use of a *continuity* argument which is also related to a *small ball* property. Our approach eliminates the probability structure altogether and replaces it with appropriate classes of trajectories; the new framework also allows to accommodate continuous and discontinuous trajectories.

In summary, our work intends to develop a probability-free framework that allows us to price by hedging and no-arbitrage. The results, obtained under no probabilistic assumptions, will depend however, on the topological structure of the possible trajectory space and a restriction on the admissible portfolios by requiring a certain type of continuity property. We connect our non probabilistic models with stochastic models in a way that arbitrage results can be translated from the non probabilistic models to stochastic models, even if these models are not semi-martingales. The framework handles naturally general subsets of the given trajectory space, this is not the case in probabilistic frameworks that rely in incorporating subsets of measure zero in the formalism.

The paper is organized as follows. Section 2 briefly introduces some of the technical tools we need to perform integration with respect to functions of finite quadratic variation and defines the basic notions of the non probabilistic framework. Section 3 proves two theorems, they are key technical results used throughout the rest of the paper. The theorems provide a tool connecting the usual notion of arbitrage and non probabilistic arbitrage. Section 4 introduces classes of modeling trajectories, we prove a variety of non probabilistic hedging and no arbitrage results for these classes. Section 5 presents several examples in which we apply the non probabilistic results to obtain new pricing results in several non standard stochastic models. Appendix A provides some information on the analytical version of Ito formula that we rely upon. Appendix B presents some technical results needed in our developments.

2. Non Probabilistic Framework

We make use of the definition of integral with respect to functions with unbounded variation but with finite quadratic variation given in [12].

Let $T > 0$ be a fixed real number and let $\mathcal{T} \equiv \{\tau_n\}_{n=0}^{\infty}$ where

$$\tau_n = \{0 = t_{n,0} < \cdots < t_{n,K(n)} = T\}$$

are partitions of $[0, T]$ such that:

$$\text{mesh}(\tau_n) = \max_{t_{n,k} \in \tau_n} |t_{n,k} - t_{n,k-1}| \to 0.$$

Let x be a real function on $[0, T]$ which is right continuous and has left limits (RCLL for short), the space of such functions will be denoted by $\mathcal{D}[0, T]$. The following notation will be used, $\Delta x_t = x_t - x_{t-}$ and $\Delta x_t^2 = (\Delta x_t)^2$.

Financial transactions will take place at times belonging to the above discrete grid but, otherwise, time will be treated continuously, in particular, the values $x(t)$ could be observed in a continuous way.

A real valued RCLL function x is of quadratic variation along $\mathcal{T}$ if the discrete measures

$$\xi_n = \sum_{t_i \in \tau_n} (x_{t_{i+1}} - x_{t_i})^2 \epsilon_{t_i}$$

converge weakly to a Radon measure ξ on $[0, T]$ whose atomic part is given by the quadratic jumps of x:

$$[x]_t^{\mathcal{T}} = \langle x \rangle_t^{\mathcal{T}} + \sum_{s \le t} \Delta x_s^2, \tag{1}$$

where $[x]^{\mathcal{T}}$ denotes the distribution function of ξ and $\langle x \rangle^{\mathcal{T}}$ its continuous part.

Considering x as above and y to be a function on $[0, T] \times \mathcal{D}$, we will formally define the Föllmer's integral of y respect to x along τ over the interval $[0, t]$ for every $0 < t \le T$. We should note that while the integral over $[0, t]$ for $t < T$ will be defined in a proper sense, the integral over $[0, T]$ will be defined as an improper Föllmer's integral.

Definition 1. *Let $0 < t < T$ and x and y as above, the Föllmer's integral of y with respect to x along $\mathcal{T}$ is given by*

$$\int_0^t y(s, x)\ dx_s = \lim_{n \to \infty} \sum_{\tau_n \ni t_{n,i} \le t} y(t_{n,i}, x)\ (x(t_{n,i+1}) - x(t_{n,i})), \tag{2}$$

provided the limit in the right-hand side of (2) exists. The Föllmer's integral over the whole interval $[0, T]$ is defined in an improper sense:

$$\int_0^T y(s, x)\ dx_s = \lim_{t \to T} \int_0^t y(s, x)\ dx_s,$$

provided the limit exists.

Consider $\phi \in \mathcal{C}^1(\mathbb{R})$ (i.e. a function with domain $\mathbb{R}$ and first derivative continuous), take $y(s, x) \equiv \phi(x(s^-))$ then (2) exists. More generally, if $y(s, x) = \phi(s, x(s^-), g_1(t, x^-), \ldots, g_m(t, x^-))$ where the $g_i(\cdot, x)$ are functions of bounded variation (that may depend on the past values of the trajectory x up to time t) then (2) exists. Moreover, in these two instances an Ito formula also holds, we refer to Appendix A for some details. Several of our non probabilistic arbitrage arguments will depend only on assuming the *existence* of integrals of the form $\int_0^t \phi(s,x)dx_s$ for a given generic integrand $\phi(s,x)$ that (potentially) depends on all the path values $x(t)$, $0 \leq t < s$, in these instances, and for the sake of generality, we will work under this general assumption.

Next, we introduce the concepts of predictability, admissibility and self-financing in a non probabilistic setting. The NP prefix will be used throughout the paper indicating some non probabilistic concept. For a given real number x_0, the central modeling object is a set of trajectories x starting at x_0, so x:$[0,T] \to \mathbb{R}$ with $x(0) = x_0$. We will assume that these functions are RCLL and belong to a given *set of trajectories* $\mathcal{J}(x_0)$. In order to easy the notation, this last class may be written as $\mathcal{J}$ when the point x_0 is clear from the context.

Some of our results apply to rather general trajectory classes, particular trajectory classes will be needed to deal with hedging results and applications to classical models and will be introduced at due time.

We assume the existence of a non risky asset with interest rates $r \geq 0$ which, for simplicity, we will assume constant, and a risky asset whose price trajectory belongs to a function class $\mathcal{J}(x_0)$. For convenience, in several occasions, we will restrict our arguments to the case $r = 0$.

A NP-portfolio Φ is a function Φ: $[0,T] \times \mathcal{J}(x_0) \to \mathbb{R}^2$, $\Phi = (\psi, \phi)$, satisfying $\Phi(0,x) = \Phi(0,x')$ for all $x, x' \in \mathcal{J}(x_0)$. We will also consider the associated projection functions Φ_x:$[0,T] \to \mathbb{R}^2$ and Φ_t:$\mathcal{J}(x_0) \to \mathbb{R}^2$, for fixed x and t respectively.

The value of a NP-portfolio Φ is the function V_Φ:$[0,T] \times \mathcal{J}(x_0) \to \mathbb{R}$ given by:

$$V_\Phi(t,x) \equiv \psi(t,x) + \phi(t,x)\ x(t).$$

Definition 2. *Consider a class $\mathcal{J}(x_0)$ of trajectories starting at x_0:*

i) *A portfolio Φ is said to be NP-predictable if $\Phi_t(x) = \Phi_t(x')$ for all $x, x' \in \mathcal{J}(x_0)$ such that $x(s) = x'(s)$ for all $0 \leq s < t$ and $\Phi_x(\cdot)$ is a left continuous function with right limits (LCRL functions for short) for all $x \in \mathcal{J}(x_0)$.*

ii) *A portfolio Φ is said to be NP-self-financing if the integrals $\int_0^t \psi(s,x)\ ds$ and $\int_0^t \phi(s,x)\ dx_s$ exist for all $x \in \mathcal{J}(x_0)$ as a Stieljes and Föllmer integrals respectively, and*

$$V_\Phi(t,x) = V_0 + \int_0^t \psi(s,x)\ r\ ds + \int_0^t \phi(s,x)dx_s,\ \forall x \in \mathcal{J}(x_0),$$

where $V_0 = V(0,x) = \psi(0,x) + \phi(0,x)\ x(0)$ for any $x \in \mathcal{J}(x_0)$.

iii) *A portfolio Φ is said to be NP-admissible if Φ is NP-predictable, NP-self-financing and $V_\Phi(t,x) \geq -A$, for a constant $A = A(\Phi) \geq 0$, for all $t \in [0,T]$ and all $x \in \mathcal{J}(x_0)$.*

Remark 1.

(1) *Two identical trajectories up to time t will lead to identical portfolio strategies up to time t.*

(2) *Notice that the notion of NP-admissible portfolio is relative to a given class of trajectories $\mathcal{J}$, classes of NP portfolios will be denoted $\mathcal{A}_\mathcal{J}$ or $\mathcal{A}$ for simplicity .*

The following definition is central to our approach.

Definition 3. *A NP-market is a pair $(\mathcal{J}, \mathcal{A})$ where $\mathcal{J}$ represents a class of possible trajectories for a risky asset and $\mathcal{A}$ is an admissible class of portfolios .*

The following definition provides the notion of arbitrage in a non probabilistic framework.

Definition 4. *We will say that there exists NP-arbitrage in the NP-market* $(\mathcal{J}, \mathcal{A})$ *if there exists a portfolio* $\Phi \in \mathcal{A}$ *such that* $V_\Phi(0, x) = 0$ *and* $V_\Phi(T, x) \geq 0$ *for all* $x \in \mathcal{J}$*, and there exists at least one trajectory* $x^* \in \mathcal{J}$ *such that* $V_\Phi(T, x^*) > 0$*. If no NP-arbitrage exists then we will say that the NP-market* $(\mathcal{J}, \mathcal{A})$ *is NP-arbitrage-free.*

The notion of probabilistic market that we use through the paper is similar to the one in [3]. Assume a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ is given. Let Z be an adapted stochastic process modeling asset prices defined on this space.

A portfolio strategy Φ^z is a pair of stochastic processes $\Phi^z = (\psi^z, \phi^z)$. The value of a portfolio Φ^z at time t is a random variable given by:

$$V_{\Phi^z}(t) = \psi_t^z + \phi_t^z Z_t.$$

A portfolio Φ^z is self-financing if the integrals $\int_0^t \psi_s^z(\omega) ds$ and $\int_0^t \phi_s^z(\omega) dZ_s(\omega)$ exist $\mathbb{P}$ -a.s. as a Stieltjes integral and a Föllmer stochastic integral respectively and

$$V_{\Phi^z}(t) = V_{\Phi^z}(0) + r \int_0^t \psi_s^z ds + \int_0^t \phi_s^z dZ_s, \ \mathbb{P} \ - a.s.$$

A portfolio Φ^z is admissible if Φ^z is self-financing, predictable (i.e. measurable with respect to $\mathcal{F}_{t-}$) and there exists $A^z = A^z(\Phi^z) \geq 0$ such that $V_{\Phi^z}(t) \geq - A^z$ $\mathbb{P}$ -a.s. $\forall \ t \in [0, T]$.

Definition 5. *A stochastic market defined on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ *is a pair* $(Z, \mathcal{A}^Z)$ *where* Z *is an adapted stochastic process modeling asset prices and* $\mathcal{A}^Z$ *is a class of admissible portfolio strategies.*

Remark 2. *We assume* $\mathcal{F}_0$ *is the trivial sigma algebra, furthermore, without loss of generality, we will assume that the constant* $z_0 = Z(0, w)$ *is fixed, i.e. we assume the same initial value for all paths. The constant* $V_{\Phi^z}(0, w)$ *will also be denoted* $V_{\Phi^z}(0, z_0)$.

The notion of arbitrage in a probabilistic market is the classical notion of arbitrage (which in this paper will be referred simply as *arbitrage*). The market $(Z, \mathcal{A}^Z)$ defined over $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ has arbitrage opportunities if there exists $\Phi^z \in \mathcal{A}^Z$ such that $V_{\Phi^z}(0) = 0$ and $V_{\Phi^z}(T) \geq 0$ $\mathbb{P}$ -a.s., and $\mathbb{P}(V_{\Phi^z}(T) > 0) > 0$.

3. Non Probabilistic Arbitrage Results

Our technical approach to establish NP arbitrage results is to link them to classical arbitrage results. Somehow surprisingly, this connection will allow us to apply the so obtained NP results to prove non existence of arbitrage results in new probabilistic markets. This section provides two basic theorems, Theorem 1 allows to construct NP markets free of arbitrage from a given arbitrage free probabilistic market. Applications of this theorem are given in Section 4. Theorem 2 presents a dual result allowing to construct probabilistic, arbitrage free, markets from a given NP market which is arbitrage free. Applications of this theorem are given in Section 5.

In order to avoid repetition we will make the following standing assumption for the rest of the section: for all the set of trajectories $\mathcal{J}$ and price processes Z to be considered, there exists a metric space $(\mathcal{S}, d)$ satisfying $\mathcal{J} \subseteq \mathcal{S}$ and $Z(\Omega) \subseteq \mathcal{S}$ up to a set of measure zero. All topological notions considered in the paper are relative to this metric space. Examples in later sections will use the uniform distance $d(x, y) = ||x - y||_\infty$ where $||x||_\infty \equiv \sup_{s\in[0,T]} |x(s)|$ and $\mathcal{S}$ the set of continuous functions x with $x(0) = x_0 = z_0$. For trajectories with jumps, later sections will use the Skorohod distance, denoted by d_s, and $\mathcal{S}$ the set of RCLL functions x with $x(0) = x_0 = z_0$.

While the main results in this section can be formulated in terms of isomorphic and V-continuous portfolios (see Definitions 10 and 11), the presentation makes use the following weaker notion of connected portfolios; this approach provides stronger results.

Definition 6. *Let $(\mathcal{J}, \mathcal{A})$ and $(Z, \mathcal{A}^Z)$ be respectively NP and stochastic markets. $\Phi \in \mathcal{A}$ is said to be connected to $\Phi^z \in \mathcal{A}^Z$ if the following holds in a set of full measure:*

$$V_{\Phi^z}(0, z_0) = V_\Phi(0, x_0)$$

and for any fixed $x \in \mathcal{J}$ and arbitrary $\rho > 0$ there exists $\delta = \delta(x, \rho) > 0$ such that

$$\text{if } d(Z(w), x) < \delta \text{ then } V_{\Phi^z}(T, \omega) \geq V_\Phi(T, x) - \rho. \tag{3}$$

Given a class of stochastic portfolios $\mathcal{A}^Z$, Section 4 constructs NP-admissible portfolios $\Phi \in \mathcal{A}$ with the goal of obtaining NP arbitrage free markets $(\mathcal{J}, \mathcal{A})$. Each such collection of portfolios $\mathcal{A}$ is defined as the largest class of NP admissible portfolios connected to an element from $\mathcal{A}^Z$. Here is the required definition.

Definition 7. *Let $(Z, \mathcal{A}^Z)$ be a stochastic market on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, and $\mathcal{J}$ a set of trajectories. Define:*

$$[\mathcal{A}^Z] \equiv \{\Phi : \Phi \text{ is NP-admissible}, \ \exists \ \Phi^z \in \mathcal{A}^Z \text{ s.t. } \Phi \text{ is connected to } \Phi^z\}.$$

Theorem 1. *Let $(Z, \mathcal{A}^Z)$ be a stochastic market and $\mathcal{J}$ a set of trajectories. Furthermore, assume the following conditions are satisfied:*

C_0 : $Z(\omega) \subseteq \mathcal{J}$ a.s.
C_1 : Z satisfies a small ball property with respect to the metric d and the space $\mathcal{J}$, namely for all $\epsilon > 0$:

$$P\left(d(Z, x) < \epsilon\right) > 0, \ \forall x \in \mathcal{J}.$$

Then, the following statement holds.

If $(Z, \mathcal{A}^Z)$ is arbitrage free then $(\mathcal{J}, [\mathcal{A}^Z])$ is NP-arbitrage free.

Proof. We proceed to prove the statement by contradiction. Suppose then, that there exists a NP-arbitrage portfolio $\Phi \in [\mathcal{A}^z]$; therefore $V_\Phi(0, x) = 0$ and $V_\Phi(T, x) \geq 0$ for all $x \in \mathcal{J}$ and there is also $x^* \in \mathcal{J}$ satisfying $V_\Phi(T, x^*) > 0$. From the definition of $[\mathcal{A}^z]$, it follows that there exists $\Phi^z \in \mathcal{A}^z$ connected to Φ satisfying $V_{\Phi^z}(0, z_0) = V_\Phi(0, x_0) = 0$. Using C_0, consider the case when there exist $\hat{\Omega}$, a measurable set of full measure, such that $Z(\omega) \subseteq \mathcal{J}$ hods for all $w \in \hat{\Omega}$. Assume further, there exists a measurable set $\hat{\Omega}_1 \subseteq \hat{\Omega}$ with $P(\hat{\Omega}_1) > 0$ such that $V_{\Phi^z}(T, \omega) < 0$ holds for all $\omega \in \hat{\Omega}_1$. The relation "$\Phi$ is connected to Φ^z" holds in a set of full measure which is independent on any given x, then, we may assume without loss of generality that (3) holds for all $w \in \hat{\Omega}_1$. Consider an arbitrary $\hat{\omega} \in \hat{\Omega}_1$ and use the notation $x \equiv Z(\hat{\omega}) \in \mathcal{J}$; for an arbitrary $\rho > 0$ we then have: $V_{\Phi^z}(T, \hat{\omega}) \geq V_\Phi(T, x) - \rho$; ρ being arbitrary, this gives a contradiction. Therefore,

$V_{\Phi^z}(T,\omega) \geq 0$ a.s. holds. Consider now x^* fixed as above, an arbitrary $\rho > 0$ and $\delta > 0$ given by the fact that Φ is connected to Φ^z. Condition C_1 implies that the set $B_\rho = \{w : d(Z(w), x^*) < \delta\}$ satisfies $P(B_\rho) > 0$ for any $\rho > 0$, then using (3) we obtain $V_{\Phi^z}(T,\omega) \geq V_\Phi(T,x^*) - \rho$, which we may assume without loss of generality holds for all $w \in B_\rho$. Clearly, $V_\Phi(T,x^*) > 0$ being fixed, there exist a small $\rho^* > 0$ such that $V_{\Phi^z}(T,\omega) > 0$ for all $w \in B_{\rho^*}$ and $P(B_{\rho^*}) > 0$. This concludes the proof. □

In Section 4 we are faced with the following problem: given $\mathcal{A}^z$, we need to prove that a given NP admissible portfolio Φ belongs to $[\mathcal{A}^z]$. The following proposition provides a sufficient condition to check that a given Φ is connected to a certain Φ^z. The stronger setting of the proposition also allows to see the condition (3) as a weak form of lower semi continuity of the value of the NP portfolio.

Proposition 1. *Let $(Z, \mathcal{A}^Z)$ be a stochastic market and $\mathcal{J}$ a set of trajectories and assume that C_0 from Theorem 1 holds. Then, if a NP-admissible portfolio Φ is such that $V_\Phi(T,\cdot)\colon\mathcal{J} \to \mathbb{R}$ is lower semi-continuous with respect to metric d and there exist $\Phi^z \in \mathcal{A}^Z$ such that $V_{\Phi^z}(0,z_0) = V_\Phi(0,x_0)$ and $V_{\Phi^z}(T,w) = V_\Phi(T,Z(w))$ then Φ is connected to Φ^z (so $\Phi \in [\mathcal{A}^Z]$.)*

Proof. Consider Φ and Φ^z satisfying the hypothesis of the proposition. The lower semi continuity means that for a given $x \in \mathcal{J}$ and any $\rho > 0$ there exists $\delta > 0$ satisfying: if $d(x',x) < \delta$, with $x' \in \mathcal{J}$ then

$$V_\Phi(T,x') \geq V_\Phi(T,x) - \rho. \tag{4}$$

Consider now w to be in the set of full measure where $Z(\Omega) \subseteq \mathcal{J}$ holds; fix $x \in \mathcal{J}$ and $\rho > 0$ arbitrary. Consider now δ as given by the lower semi continuity assumption, then, if $d(Z(w),x) < \delta$, taking $x' \equiv Z(w)$ we obtain $V_{\Phi^z}(T,w) = V_\Phi(T,x') \geq V_\Phi(T,x) - \rho$. □

In order to construct arbitrage free probabilistic markets from NP markets free of arbitrage we will make use of the following notion.

Definition 8. *Let $(\mathcal{J}, \mathcal{A})$ and $(Z, \mathcal{A}^Z)$ be respectively NP and stochastic markets. $\Phi^z \in \mathcal{A}^Z$ is said to be connected to $\Phi \in \mathcal{A}$ if the following holds in a set of full measure:*

$$V_{\Phi^z}(0,z_0) = V_\Phi(0,x_0)$$

and for any fixed $x \in \mathcal{J}$ and arbitrary $\rho > 0$ there exists $\delta = \delta(x,\rho) > 0$ such that

$$\text{if } d(Z(w),x) < \delta \text{ then } V_\Phi(T,x) \geq V_{\Phi^z}(T,\omega) - \rho. \tag{5}$$

In order to apply results obtained for NP-markets to stochastic markets, in particular non-semimartingale processes, Section 5 makes use of the following construction: starting from a class of NP admissible portfolios $\mathcal{A}$, a class of portfolios $\mathcal{A}^Z$ is defined as the largest collection of admissible portfolios which are connected to elements from $\mathcal{A}$. This construction will give an arbitrage free stochastic market $(Z, \mathcal{A}^Z)$. Here is the required definition.

Definition 9. *Let $(\mathcal{J}, \mathcal{A})$ be a NP market and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a filtered probability space. Let Z be an adapted stochastic process defined on this space. Define:*

$$[\mathcal{A}]^Z \equiv \{\Phi^z : \Phi^z \text{ is admissible, } \exists\, \Phi \in \mathcal{A} \text{ s.t. } \Phi^z \text{ is connected to } \Phi\}.$$

The following theorem is the dual version of Theorem 1.

Theorem 2. *Let $(\mathcal{J}, \mathcal{A})$ be a NP market and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ a filtered probability space. Let Z be an adapted stochastic process defined on this space. Furthermore, assume C_0 and C_1 from Theorem 1 hold.*

Then the following statement holds:

If $(\mathcal{J}, \mathcal{A})$ is NP-arbitrage free then $(Z, [\mathcal{A}]^z)$ is arbitrage free.

Proof. We argue by contradiction, suppose there exists an arbitrage portfolio $\Phi^z \in [\mathcal{A}]^z$; therefore, $V_{\Phi^z}(0, w) = 0$ and $V_{\Phi^z}(T, w) \geq 0$ a.s. Moreover, there exists a measurable set $D \subseteq \Omega$ satisfying

$$V_{\Phi^z}(T, w) > 0 \text{ for all } w \in D \text{ and } P(D) > 0. \tag{6}$$

Because $\Phi^z \in [\mathcal{A}]^z$, we know that Φ^z is connected to some $\Phi \in \mathcal{A}$. Then, $0 = V_{\Phi^z}(0, z_0) = V_\Phi(0, x_0) = V_\Phi(0, x)$ for all $x \in \mathcal{J}$. Assume now there exists $\tilde{x} \in \mathcal{J}$ and $V_\Phi(T, \tilde{x}) < 0$, by C_1 and (5), given $\rho > 0$, we obtain

$$V_\Phi(T, \tilde{x}) \geq V_{\Phi^z}(T, \omega) - \rho \tag{7}$$

a.s. for $w \in B_\rho \equiv \{\omega : d(Z(\omega), \tilde{x}) < \delta\}$ and $\delta > 0$ is as in (5). Using the fact that $V_{\Phi^z}(T, \omega) \geq 0$ we arrive at a contradiction and conclude $V_\Phi(T, x) \geq 0$ for all $x \in \mathcal{J}$. Assume now $V_\Phi(T, x) = 0$ for all $x \in \mathcal{J}$, because C'_0 and (6), there exists $\omega^* \in D$ and $x^* \equiv Z(\omega^*) \in \mathcal{J}$. The relation "$\Phi^z$ is connected to Φ" holds in a set of full measure which is independent on any given x, then, we may assume without loss of generality that (5) holds for ω^*. Then, using C_1 we obtain: $V_\Phi(T, x^*) \geq V_{\Phi^z}(T, \omega^*) - \rho$ for all $\rho > 0$. This implies $V_\Phi(T, x^*) > 0$. □

The following proposition provides sufficient conditions to check that a certain Φ^z is connected to a NP portfolio Φ.

Proposition 2. *Let $(\mathcal{J}, \mathcal{A})$ be a NP market, Z an adapted stochastic process defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ and assume C_0 from Theorem 1 holds. Then, if Φ^z is an admissible portfolio such that there exists $\Phi \in \mathcal{A}$ satisfying: $V_\Phi(T, \cdot) : \mathcal{J} \to \mathbb{R}$ is upper semi-continuous with respect to metric d, $V_{\Phi^z}(0, z_0) = V_\Phi(0, x_0)$ and $V_{\Phi^z}(T, w) = V_\Phi(T, Z(w))$ a.s., then Φ^z is connected to Φ (so $\Phi^z \in [\mathcal{A}]^Z$.)*

Proof. Consider Φ and Φ^z satisfying the hypothesis of the proposition. The upper semi continuity means that for a given $x \in \mathcal{J}$ and any $\rho > 0$ there exists $\delta > 0$ satisfying: if $d(x', x) < \delta$, with $x' \in \mathcal{J}$ then

$$V_\Phi(T, x) \geq V_\Phi(T, x') - \rho.$$

Consider now w to be in the set of full measure where $Z(\Omega) \subseteq \mathcal{J}$ holds; fix $x \in \mathcal{J}$ and $\rho > 0$ arbitrary. Consider now δ as given by the upper semi continuity assumption, then, if $d(Z(w), x) < \delta$, taking $x' \equiv Z(w)$ we obtain $V_\Phi(T, x) \geq V_\Phi(T, x') - \rho = V_{\Phi^z}(T, w) - \rho$. □

For simplicity, in most of our further developments, we will make use of stronger notions than connected and lower and upper semi-continuous portfolios. Namely, isomorphic and V-continuous portfolios, here are the definitions.

Definition 10. *Let $(\mathcal{J}, \mathcal{A})$ and $(Z, \mathcal{A}^Z)$ be respectively NP and stochastic markets and assume the condition C_0 from Theorem 1 holds. A NP portfolio $\Phi \in \mathcal{A}$ and $\Phi^z \in \mathcal{A}^Z$ are said to be isomorphic if $\mathbb{P}$-a.s.:*

$$\Phi^z(t, \omega) = \Phi(t, Z(\omega))$$

for all $0 \leq t \leq T$.

Definition 11. *Let $(\mathcal{J}, \mathcal{A})$ be a NP market. A NP portfolio $\Phi \in \mathcal{A}$ is said to be V-continuous with respect to d if the functional $V_\Phi(T, \cdot): \mathcal{J} \to \mathbb{R}$ is continuous with respect to the topology induced on $\mathcal{J}$ by distance d.*

Whenever the distance d is understood from the context we will only refer to the portfolio as *V-continuous*. The intuitive notion of a V-continuous portfolio is that small changes in the asset price trajectory will lead to small changes to the final value of the portfolio.

Remark 3. *Clearly, if $\Phi_T: \mathcal{J} \to \mathbb{R}^2$ is continuous then Φ is V-continuous.*

Propositions 1 and 2 plus the definition of V-continuity give the following corollary.

Corollary 1. *Consider the setup of Definition 10. In particular, consider Φ and Φ^z to be isomorphic, furthermore, assume Φ to be V-continous, then:*

- *Φ is connected to Φ^z and so $\Phi \in [\mathcal{A}^Z]$.*
- *Φ^z is connected to Φ and so $\Phi^z \in [\mathcal{A}]^Z$.*

In many of our examples, we will rely on Corollary 1 to check if given portfolios belong to $[\mathcal{A}^Z]$ or $[\mathcal{A}]^Z$. In each of our examples, introduced in later sections, it will arise the question on how large are the classes of portfolios $[\mathcal{A}^Z]$ and $[\mathcal{A}]^Z$ as the Definitions 7 and 9 do not provide a direct characterization of its elements. For each of our examples we will prove that specific classes of portfolios do belong to $[\mathcal{A}^Z]$ and $[\mathcal{A}]^Z$, answering the question in general is left to future research.

Arbitrage in subsets

Consider $\mathcal{J}^* \subseteq \mathcal{J}$, it is natural to look for conditions that provide a relationship between the arbitrage opportunities of these two sets. The NP framework allows a simple result, Proposition 3 below, which provides a clear contrast with the probabilistic framework which, in particular, is not able to provide an answer when $\mathcal{J}^*$ is a subset of measure zero (see Example 1). There exist cases, for example if $\mathcal{J}^* = \{x \in \mathcal{J} : x_T > x_0 e^{rT}\}$, for which there exists an obvious NP-arbitrage portfolio by borrowing money from the bank and investing on the asset. Proposition 3 shows that the no-arbitrage property for a NP-market $(\mathcal{J}, \mathcal{A})$ is inherited by NP-markets whose trajectories $\mathcal{J}^*$ are dense on $\mathcal{J}$.

Proposition 3. *Consider the NP-market $(\mathcal{J}, \mathcal{A})$ where $\mathcal{A}$ is some class of NP-admissible, V-continuous portfolio strategies (with respect to metric d). Let $\mathcal{J}^* \subset \mathcal{J}$ be a subclass of trajectories such that $\mathcal{J}^*$ is dense in $\mathcal{J}$ with respect to the metric d and consider $\mathcal{A}_{\mathcal{J}^*}$ to be the restriction of portfolio strategies in $\mathcal{A}$ to the subclass $\mathcal{J}^*$. Then the NP-Market $(\mathcal{J}^*, \mathcal{A}_{\mathcal{J}^*})$ is NP-arbitrage-free if $(\mathcal{J}, \mathcal{A})$ is NP-arbitrage-free.*

Proof. Assuming that there exists $\Phi \in \mathcal{A}_{\mathcal{J}^*}$, an arbitrage opportunity on $(\mathcal{J}^*, \mathcal{A}_{\mathcal{J}^*})$, we will derive an arbitrage strategy in $(\mathcal{J}, \mathcal{A})$. To achieve this end, it is enough to prove that $V_\Phi(0,x) = 0$ and $V_\Phi(T,x) \geq 0$, both relations valid for all $x \in \mathcal{J}$. The first relation should hold because of the density assumption and the fact that $V_\Phi(0,\cdot)$ is a continuous function on $\mathcal{J}$. The second relationship follows similarly using continuity of $V_\Phi(T,\cdot)$ on $\mathcal{J}$. □

4. Examples: Arbitrage and Hedging in Trajectory Classes

This section provides examples of NP markets $(\mathcal{J}, \mathcal{A})$ which are free of arbitrage and in which general classes of payoffs can be hedged. Several of the results from Section 3 are applied in order to gain a more complete understanding of these examples, in particular, we provide several details about the characterizations of the portfolios $\Phi \in \mathcal{A}$.

A main example deals with continuous trajectories (this set is denoted by $\mathcal{J}^\sigma_\tau$), other examples deal with trajectories containing jumps. Several aspects of these different examples are treated in a uniform way illustrating the flexibility of using different topologies in the trajectory space. The classes of trajectories to be introduced could be considerably enlarged by allowing the parameter σ to be a function of t (obeying some regularity conditions). Our results apply to such (extended) classes as well, in the present paper we will restrict σ to be a constant for simplicity. We also restrict to hedging results to path independent derivatives but expect the results can be extended to path independent derivatives as well.

The replicating portfolio strategies that we will obtain in a NP-market are essentially the same that in the corresponding stochastic frameworks, for example, to replicate a payoff when prices lie in our example $\mathcal{J}^\sigma_\tau$, comprised of continuous trajectories, we use the well known delta-hedging as in the Black-Scholes model. In the available literature there exist several results related to the robustness of delta hedging, see for example: [4], [19], [3] and [9]. A point to emphasize is the fact that the replication results, being valid in a different sense (probability-free), are valid also when considering subclasses of trajectories $\mathcal{J}^* \subset \mathcal{J}$. Formally, this fact is not available in probabilistic frameworks due to the technical reliance on sets of measure zero or non-measurable sets.

4.1. Non Probabilistic Black-Scholes Model.

Denote by $\mathcal{Z}_\mathcal{T}([0,T])$ the collection of all continuous functions $z(t)$ such that $[z]^\mathcal{T}_t = t$ for $0 \leq t \leq T$ and $z(0) = 0$. Notice that $\mathcal{Z}_\mathcal{T}([0,T])$ includes a.s. paths of Brownian motion if $\mathcal{T}$ is a refining sequence of partitions ([17].)

For a given sequence of subdivisions $\mathcal{T}$ define,

- Given constants $\sigma > 0$ and $x_0 > 0$, let $\mathcal{J}^\sigma_\tau(x_0)$ to be the class of all real valued functions x for which there exists $z \in \mathcal{Z}_\mathcal{T}([0,T])$ such that:

$$x(t) = x_0 \; e^{\sigma z(t)}.$$

According to (35), the class $\mathcal{J}^\sigma_\tau(x_0)$ is the class of continuous functions x with $x(0) = x_0$ and quadratic variation satisfying $d\langle x\rangle^\tau_t = \sigma^2 x(t)^2 dt$. The trajectory class $\mathcal{J}^\sigma_\tau(x_0)$ will be considered as a subset of the continuous functions with the uniform topology induced by the uniform distance.

Remark 4. *The class $\mathcal{J}^\sigma_\tau(x_0)$ includes trajectories of processes different than the geometric Brownian motion, as an illustration we indicate that if $z = B + y$, with B*

a Brownian motion and y a process with zero quadratic variation, then the trajectories of z belongs to $\mathcal{Z}_\tau([0,T])$, hence the trajectories of the process $x_0\ e^{\sigma z}$ belong to $\mathcal{J}_\tau^\sigma(x_0)$.

As previously suggested, the hedging results in this class have been already obtained, more or less explicitly, in several papers (see [4] and [19] for example).

Theorem 3. *Let $\mathcal{J}^*$ be a class of possible trajectories, $\mathcal{J}^* \subset \mathcal{J}_\tau^\sigma$ and let $v(\cdot,\cdot) : [0,T] \times \mathbb{R}^+ \to \mathbb{R}$ be the solution of the PDE*

$$\frac{\partial v}{\partial t}(t,x) + r\ x\frac{\partial v}{\partial x}(t,x) + \frac{\sigma^2 x^2}{2}\frac{\partial^2 v}{\partial x^2}(t,x) - r\ v(t,x) = 0 \tag{8}$$

with terminal condition $v(T,x) = h(x)$ where $h(\cdot)$ is Lipschitz. Then, the delta hedging NP-portfolio $\Phi_t \equiv (v(t,x(t)) - \varrho(t,x(t))x(t), \varrho(t,x(t)))$ where $\varrho(t,x) \equiv \frac{\partial v}{\partial x}(t,x)$ replicates the payoff h at maturity time T for all $x \in \mathcal{J}^*$.

Proof. The existence and uniqueness of the solution of (8) is guaranteed because h is Lipschitz. If $x \in \mathcal{J}^* \subset \mathcal{J}_\tau^\sigma(x_0)$ then we know that x is of quadratic variations and $d\langle x\rangle_t^\tau = \sigma^2 x(t)^2 dt$, so applying Itô-Föllmer formula, taking $\varrho(t,x) = \frac{\partial v}{\partial x}(t,x)$, using (8) and noticing that the integral $\int_0^T r(v(s,x(s)) - \varrho(s,(s,x(s))x(s))ds$ exists, we obtain:

$$h(x(T)) = v(T,x(T)) = \lim_{u\to T} v(u,x(u)) = \tag{9}$$

$$\lim_{u\to T}\left[v(0,x(0)) + \int_0^u \frac{\partial v}{\partial t}(s,x(s))ds + \frac{1}{2}\int_0^u \frac{\partial^2 v}{\partial x^2}(s,x(s))d\langle x\rangle_s^\tau + \int_0^u \varrho(s,x(s))dx(s)\right] =$$

$$\lim_{u\to T}\left[v(0,x(0)) + \int_0^u r[v(s,x(s)) - \varrho(s,x(s))x(s)]ds + \int_0^u \varrho(s,x(s))dx(s)\right] =$$

$$v(0,x(0)) + \int_0^T r\left[v(s,x(s)) - \varrho(s,x(s))x(s)\right]ds + \lim_{u\to T}\int_0^u \varrho(s,x(s))\ dx(s) =$$

$$v(0,x(0)) + \int_0^T r\left[v(s,x(s)) - \varrho(s,x(s))x(s)\right]ds + \int_0^T \varrho(s,x(s))dx(s).$$

The analysis in (9) implies that the NP-portfolio

$$\Phi_t = (v(t,x(t)) - \varrho(t,x(t))x(t), \varrho(t,x(t))) \tag{10}$$

replicates the payoff h at maturity time T. □

Corollary 2. *The delta hedging portfolio given by (10) is NP-admissible and V-continuous relative to the uniform topology.*

Proof. The self-financing and predictable properties follow from the definition and constructions in Theorem 3 by noticing that $x(t) = x(t-)$. The portfolio is admissible, with $A = 0$, by the known property $v(t,x(t)) \geq 0$. V-continuity follows from $V_\Phi(T,x) = h(x(T))$ and the fact that h is continuous. □

4.1.1. *Arbitrage in $\mathcal{J}_\tau^\sigma$.* We analyze next the problem of arbitrage in a market where possible trajectories are in $\mathcal{J}_\tau^\sigma$. We will make use Theorem 1 applied to the Black and Scholes model (Z, A^Z); in particular, $A^Z = \mathcal{A}_{BS}^Z$, where $\mathcal{A}_{BS}^Z$ denotes the admissible portfolios in the Black-Scholes stochastic market. Corollary 1 will be used to show that a large class of portfolios belong to $[\mathcal{A}_{BS}^Z]$; towards this end, we incorporate portfolio strategies that depend on past values of the trajectory and not just on the spot value ([3]).

Definition 12. *A hindsight factor g over some class of trajectories $\mathcal{J}$ is a mapping $g : [0,T] \times \mathcal{J} \to \mathbb{R}$ satisfying:*

i) *$g(t,\eta) = g(t,\tilde{\eta})$ whenever $\eta(s) = \tilde{\eta}(s)$ for all $0 \leq s \leq t$.*
ii) *$g(\cdot,\eta)$ is of bounded variation and continuous for every $\eta \in \mathcal{J}$.*
iii) *There is a constant K such that for every continuous function f.*

$$\left| \int_0^t f(s)dg(s,\eta) - \int_0^t f(s)dg(s,\tilde{\eta}) \right| \leq K \max_{0\leq r\leq t} f(r)\, \|\eta - \tilde{\eta}\|_\infty$$

Another definition of [3] are the smooth strategies introduced next.

Definition 13. *A portfolio strategy $\Phi = (\psi_t, \phi_t)_{0\leq t\leq T}$ over the class of trajectories $\mathcal{J}$ is called smooth if:*

i) *The number of assets held at time t, ϕ_t, has the form*

$$\phi_t(x) = \phi(t,x) = G(t, x_t, g_1(t,x), \ldots, g_m(t,x)) \tag{11}$$

for all $t \in [0,T]$ and for all $x \in \mathcal{J}$ where $G \in C^1([0,T] \times \mathbb{R} \times \mathbb{R}^m)$ and the g_i's are hindsight factors

ii) *There exists $A > 0$ such that $V_\Phi(t,x) \geq -A$ $\forall t \in [0,T]$ and $\forall x \in \mathcal{J}$.*

Given the notation and assumptions from Definition 13, an application of Itô-Föllmer formula (34) proves that the integrals $\int_0^t \phi(s,x)dx(s)$ exist for all $t \in [0,T]$ if $\Phi = (\psi_t, \phi_t)$ is smooth. Propositions 11 and 12 (stated and proven in Appendix B) relate the smoothness condition in (11) with the admissibility conditions of both, stochastic and NP portfolios respectively.

Proposition 4. *If Φ is a smooth portfolio strategy over $\mathcal{J}_\tau^\sigma(x_0)$ and d is the uniform distance then Φ is V-continuous.*

Proposition 4 follows immediately from Lemma 4.5 in [3].

The following result is well known ([14]), we present a proof for completeness.

Lemma 1. *Let y be a continuous function $y : [0,T] \to \mathbb{R}$ with $y(0) = 0$. If W is a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$ then for all $\epsilon > 0$,*

$$P\left(\omega : \sup_{s\in[0,T]} |W_s(\omega) - y(s)| < \epsilon\right) > 0. \tag{12}$$

Proof. Function y is continuous on $[0,T]$, therefore is uniformly continuous so for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t_2 - t_1| < \delta \Longrightarrow |y(t_2) - y(t_1)| < \epsilon/3$$

Let M be an integer, $M > T/\delta$, and define points $s_i = iT/M$, for $i = 0, \ldots, M$. By definition $|s_{i+1} - s_i| < \delta$, so $|y(s_{i+1}) - y(s_i)| < \epsilon/3$.

Define for all $1 \leq i \leq M$

$$A_i = \left\{ \omega : \sup_{s_{i-1} \leq t \leq s_i} \left| W_t - W_{s_{i-1}} \right| < \epsilon/2 \right\}$$

$$B_i = \left\{ \omega : \left| (W_{s_i} - W_{s_{i-1}}) - (y(s_i) - y(s_{i-1})) \right| < \epsilon/6M \right\}$$

and $\Omega_i = A_i B_i$. It is immediate, from results in [15], that $P(\Omega_i) > 0$. On the other hand, it is obvious that events Ω_i and Ω_j are independent for $i \neq j$ since increments of Brownian motions on disjoint intervals are independent. So

$$P\left(\prod_{i=1}^{M} \Omega_i \right) = \prod_{i=1}^{M} P(\Omega_i) > 0 \tag{13}$$

We will prove now that

$$\prod_{i=1}^{M} \Omega_i \subset \left\{ \omega : \sup_{s \in [0,T]} |W_s(\omega) - y(s)| < \epsilon \right\} \tag{14}$$

Let $\omega \in \prod_{i=1}^{M} \Omega_i$, then $\omega \in \prod_{i=1}^{M} B_i$ so for all $k = 1, \ldots, M-1$, applying triangular inequality:

(15)

$$|W_{s_k}(\omega) - y(s_k)| \leq \sum_{i=1}^{k} \left| (W_{s_i}(\omega) - W_{s_{i-1}}(\omega)) - (y(s_i) - y(s_{i-1})) \right| < k\epsilon/6M \leq \epsilon/6$$

Also $\omega \in A_{k+1}$, so $|W_t(\omega) - W_{s_k}(\omega)| < \epsilon/2$ for all $t \in [s_k, s_{k+1}]$. We also know that $|y(s_k) - y(t)| \leq \epsilon/3$ for all $t \in [s_k, s_{k+1}]$. Using again triangular inequality:

(16)

$$|W_t(\omega) - y(t)| \leq |W_t(\omega) - W_{s_k}(\omega)| + |W_{s_k}(\omega) - y(s_k)| + |y(s_k) - y(t)| < \epsilon/2 + \epsilon/6 + \epsilon/3 = \epsilon$$

is valid for all $t \in [s_k, s_{k+1}]$ for all k so (16) is valid for all $t \in [0,T]$ and (14) is true. From (14) and (13) we obtain (12) and the Lemma is proved □

A main consequence of Theorem 1 and the previous definitions and results is the following Theorem.

Theorem 4. *Let $(Z, \mathcal{A}^Z_{BS})$ be the Black-Scholes stochastic market defined by*

$$Z_t = x_0 e^{(\mu - \sigma^2/2)t + \sigma W_t},$$

where μ and $\sigma > 0$ are constant real numbers, W is a Brownian Motion, and $\mathcal{A}^Z_{BS}$ is the class of all admissible strategies for Z. Consider the class of trajectories $\mathcal{J}^\sigma_\tau$ with the uniform topology. We have:

i) *The NP market $(\mathcal{J}^\sigma_\tau, [\mathcal{A}^Z_{BS}])$ is NP arbitrage-free.*

ii) *$[\mathcal{A}^Z_{BS}]$ contains:*

a) the smooth strategies such that the hindsight factors g_i satisfy that $g_i(t, X)$ are $(\mathcal{F}_{t-})$-measurable,

b) delta hedging strategies.

Proof. i) By the definition of Z and $\mathcal{J}^\sigma_\tau$ clearly condition C_0 in Theorem 1 is satisfied. Also, condition C_1 from Theorem 1 follows from Lemma 1. As $(Z, \mathcal{A}^Z_{BS})$ is arbitrage-free (see for example [10]) then the NP market $(\mathcal{J}^\sigma_\tau, [\mathcal{A}^Z_{BS}])$ is NP arbitrage-free according to Theorem 1.

ii) Let Φ be a smooth strategy over $\mathcal{J}_\tau^\sigma$ As the trajectories in $\mathcal{J}_\tau^\sigma$ are continuous, condition (39) in Proposition 12 holds, therefore Φ is NP-admissible; Φ is also V-continuous as consequence of Proposition 4. Define a.s. Φ^z as $\Phi^z(t,\omega) = \Phi(t, Z(\omega))$; Proposition 11 shows that the stochastic portfolio Φ^z is predictable, LCRL and self-financing. The admissibility of Φ^z results from ii) in Definition 13, hence $\Phi^z \in \mathcal{A}_{BS}^Z$. As Φ and Φ^z are isomorphic and Φ is V-continuous, Corollary 1 applies so Φ is connected to Φ^z and $\Phi \in [\mathcal{A}_{BS}^Z]$. For the hedging strategies the same arguments apply and the V-continuity and admissibility follow by an application of Corollary 2. □

Remark 5. *In the framework of Theorem 4, where trajectories in $\mathcal{J}$ are continuous it is not difficult to see that $\tilde{g}(t,x) = \min_{0\le s\le t} x(s)$, as well as the maximum and the average, are hindsight factors over $\mathcal{J}$ (see* [3]*), moreover $\tilde{g}(t,X)$ is a $(\mathcal{F}_{t-})$-measurable random variable.*

Remark 6. *It can be proved that $[\mathcal{A}_{BS}^Z]$ also contains simple (piece-wise constant) portfolio strategies satisfying*

$$\phi_t = \sum_{l=1}^{L} 1_{(s_{l-1},s_l]}(t) G(t, x(s_{l-1}))$$

where $0 = s_0 < s_1 < \cdots < s_L = T$, the s_i are deterministic and G is C^1. This is consequence of Remark 4.6 of [3].

Theorem 4 is the analogous in our framework of the known absence of arbitrage in the Black-Scholes model, a property that in fact we use in the above proof.

In a classical stochastic framework, the absence of arbitrage is equivalent to the existence of at least one risk neutral probability measure, the next example shows a possible trajectory class which has no obvious probabilistic counterpart.

Example 1. *Define the class*

$$\mathcal{J}_{\tau,\mathbb{Q}}^\sigma = \{x \in \mathcal{J}_\tau^\sigma : x(T) \in \mathbb{Q}\}$$

where $\mathbb{Q}$ is the set of rational numbers.
Consider $[\mathcal{A}_{BS}^Z]$ as defined in Theorem 4. Let $\mathcal{A}^V \subset [\mathcal{A}_{BS}^Z]$ be the class of all V-continuous portfolios in $[\mathcal{A}_{BS}^Z]$. Item ii) in Theorem 4, $\mathcal{A}^V$ is a large class of portfolios which also satisfies that the market $(\mathcal{J}_\tau^\sigma, \mathcal{A}^V)$ is NP-arbitrage free. Let $\mathcal{A}^V_{\mathcal{J}_{\tau,\mathbb{Q}}^\sigma}$ be the restriction of portfolio strategies in $\mathcal{A}^V$ to the subclass of trajectories $\mathcal{J}_{\tau,\mathbb{Q}}^\sigma$. As $\mathcal{J}_{\tau,\mathbb{Q}}^\sigma$ is dense on $\mathcal{J}_\tau^\sigma$, applying Proposition 3 we conclude that the market $(\mathcal{J}_{\tau,\mathbb{Q}}^\sigma, \mathcal{A}^V_{\mathcal{J}_{\tau,\mathbb{Q}}^\sigma})$ is NP-arbitrage-free.

The absence of arbitrage for model in Example 1 and replicating portfolio in Theorem 3 imply that it is possible to price derivatives using the Black-Scholes formula also for this model, even if there is no obvious intuitive measure over the possible set of trajectories. In fact, the set $\mathcal{J}_{\tau,\mathbb{Q}}^\sigma$ has null probability under the Black-Scholes model, therefore if a measure is defined over this set, it will not be absolutely continuous with respect to the Wiener measure. Hence, it is not clear how to price derivatives under a stochastic model following a risk neutral approach, if the trajectories of the asset price process belong to $\mathcal{J}_{\tau,\mathbb{Q}}^\sigma$.

4.2. **Non Probabilistic Geometric Poisson Model.** This section studies hedging and arbitrage in specific examples of trajectory classes with jumps. Denote by $\mathcal{N}([0,T])$ the collection of all functions $n(t)$ such that there exists a non negative integer m and positive numbers $0 < s_1 < \ldots < s_m < T$ such that $n(t) = \sum_{s_i \leq t} 1_{[0,t]}(s_i)$. The function $n(t)$ is considered as identically zero on $[0,T]$ whenever $m = 0$.

The following class of real valued functions will be another example of possible trajectories for the asset price.

- Given constants $\mu, a \in \mathbb{R}$ and $x_0 > 0$, let $\mathcal{J}^{a,\mu}(x_0)$ to be the class of all functions x for which exists $n(t) \in \mathcal{N}([0,T])$ such that:

$$x(t) = x_0 e^{\mu t}(1+a)^{n(t)}. \tag{17}$$

The function $n(t)$ counts the number of jumps present in the path x until, and including, time t. Note also that the definition of $\mathcal{J}^{a,\mu}(x_0)$ does not depend on the particular subdivision $\mathcal{T}$ used elsewhere in the paper.

The natural probabilistic counterpart for this model is the Geometric Poisson model

$$Z_t = x_0 e^{\mu t}\ (1+a)^{N_t^p},$$

where $N^P = (N_t^P)$ is a Poisson process on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Notice that $P(Z(w) \in \mathcal{J}^{a,\mu}(x_0)) = 1$. Even if this stochastic model has limited practical use in finance, it has theoretical importance because, together with the Black-Scholes model, they are the only exponential Lévy models leading to complete markets, see [8].

Remark 7. *The class $\mathcal{J}^{a,\mu}(x_0)$ includes trajectories of processes different than the Geometric Poisson model, in fact if N is a renewal process, trajectories of the process Z defined as $Z_t = x_0 e^{\mu t}\ (1+a)^{N_t}$ are also in $\mathcal{J}^{a,\mu}(x_0)$.*

A replicating portfolio for trajectories in $\mathcal{J}^{a,\mu}$ corresponds to the probabilistic-free version of the hedging strategy associated to the Geometric Poisson model, see [6].

Suppose we have an European type derivative with payoff $h(x(T))$. For simplicity we consider interest rate $r = 0$. We are looking for a NP-admissible portfolio strategy that perfectly replicates the payoff $h(x(T))$. The next Theorem provides the answer to this NP hedging question.

Theorem 5. *Let $\mathcal{J}^*$ be a class of possible trajectories for the asset price, $\mathcal{J}^* \subset \mathcal{J}^{a,\mu}(x_0)$. Consider that $a\mu < 0$ and let $\lambda = -\mu/a$. Define $\tilde{F}(s,t)$ by:*

$$\tilde{F}(t,s) = e^{-\lambda(T-t)} \sum_{k=0}^{\infty} \frac{h\left(s e^{\mu(T-t)}(1+a)^k\right)(T-t)^k}{k!}.$$

Then, the portfolio $\Phi_t = (\psi_t, \phi_t)$ where

$$\phi_t = \frac{\tilde{F}(t,(a+1)x(t^-)) - \tilde{F}(t,x(t^-))}{a\ x(t^-)},$$

and $\psi_t = \tilde{F}(t,x(t^-)) - \phi_t x(t^-)$, whose initial value is $\tilde{F}(0,x_0)$, replicates the Lipschitz payoff $h(x(T))$ at time T for every $x \in \mathcal{J}^*$.

We will not provide a proof of Theorem 5 as it can be easily extracted from [6] even though that reference obtains a probabilistic result considering $n(t)$ (as appears in equation (17)) to be a Poisson process N_t^P. The proof in [6] can be translated to our non probabilistic model in a straightforward way. It is important to remark that such a proof would use only ordinary calculus.

Theorem 5 can be generalized to the case where a and μ are considered no longer as constants but known deterministic functions $a(t)$ and $\mu(t)$ such that $\mu(t)a(t) < 0$ for all t. A proof of this result is contained in [1].

4.2.1. *Arbitrage in* $\mathcal{J}^{a,\mu}$. Next we concentrate on establishing the absence of NP arbitrage for the class of trajectories $\mathcal{J}^{a,\mu}$ to this end we will apply Theorem 1. It remains to select an appropriate metric d. Instead of using the uniform distance, for models with jumps we will use the Skorohod's distance d_s; for a definition of this distance and its associated topology we refer the reader to [5].

The next proposition gives sufficient conditions for the V-continuity of a portfolio over $\mathcal{J}^{a,\mu}$ with respect to the Skorohod's metric.

Proposition 5. *A portfolio* $\Phi_t = (\psi_t, \phi_t)$ *on* $\mathcal{J}^{a,\mu}$ *for which the amount invested in the stock* $\phi_t = \phi(t, x_{t-})$ *is such that* $\phi \in C([0,T] \times \mathbb{R})$ *is V-continuous relative to the Skorohod's topology.*

Proof. Let $x \in \mathcal{J}^{a,\mu}$ and let $\left\{x^{(n)}\right\}_{n=0,1\ldots}$ with $x^{(n)} \in \mathcal{J}^{a,\mu}$ be a sequence that converges to x in the Skorohod's distance. Suppose that x has m jumps located at $0 < \tau_1 < \ldots < \tau_m < T$. Then for every $\epsilon > 0$ there exists an integer $K > 0$ such that for $n > K$, $x^{(n)}$ has exactly m jumps located at $0 < \tau_1^{(n)} < \ldots < \tau_m^{(n)} < T$ and satisfying that $|\tau_i^{(n)} - \tau_i| < \epsilon$ for $i = 1, \ldots, m$. For convenience denote $\tau_0 = \tau_0^{(n)} = 0$ and $\tau_{m+1} = \tau_{m+1}^{(n)} = T$.

Next we evaluate $V_\Phi(T, \cdot)$, the value of portfolio Φ at maturity time T for both trajectories x and $x^{(n)}$. Because we are restricting to the case of interest rate $r = 0$, we have $\forall x \in \mathcal{J}^{a,\mu}$:

$$V_\Phi(T, x) = V_\Phi(0, x) + \int_0^T \phi^s dx_s = V_0 + \int_0^T \phi(s, x_{s-}) dx_s \tag{18}$$

Using the particular form of x, the integral on (18) can be computed as:

$$\begin{aligned} \int_0^T \phi(s, x_{s-}) dx_s &= \sum_{i=0}^{m} \int_{\tau_i}^{\tau_{i+1}} \phi\left(s, x_0 e^{\mu s}(1+a)^i\right) \mu x_0 e^{\mu s}(1+a)^i ds \\ &+ \sum_{i=1}^{m} \left[\phi(\tau_i, x_0 e^{\mu\tau_i}(1+a)^i) - \phi(\tau_i, x_0 e^{\mu\tau_i}(1+a)^{i-1})\right] a x_0 e^{\mu\tau_i}(1+a)^{i-1} \end{aligned} \tag{19}$$

A similar expression applies for $x^{(n)}$ for all n:

$$\int_0^T \phi(s, x_{s-}^{(n)}) dx_s^{(n)} = \sum_{i=0}^{m} \int_{\tau_i^{(n)}}^{\tau_{i+1}^{(n)}} \phi\left(s, x_0 e^{\mu s}(1+a)^i\right) \mu x_0 e^{\mu s}(1+a)^i ds + \tag{20}$$

$$\sum_{i=1}^{m} \left[\phi(\tau_i^{(n)}, x_0 e^{\mu\tau_i^{(n)}}(1+a)^i) - \phi(\tau_i^{(n)}, x_0 e^{\mu\tau_i^{(n)}}(1+a)^{i-1})\right] a x_0 e^{\mu\tau_i^{(n)}}(1+a)^{i-1}$$

As $\tau_i^{(n)} \to \tau_i$ as $n \to \infty$ integrals and summands in (20) converge to analogous elements in (19), thus $V_\Phi(T, x^{(n)}) \to V_\Phi(T, x)$, which proves that Φ is a V-continuous portfolio. □

The next Theorem shows that our general Theorem 1 is also useful to establish the absence of NP-arbitrage in models with jumps.

Theorem 6. *Let $(Z, \mathcal{A}_P^Z)$ be the stochastic market defined by the geometric Poisson stochastic process introduced before:*

$$Z_t = x_0 e^{\mu\ t}\ (1+a)^{N_t^p}, \tag{21}$$

and $\mathcal{A}_P^Z$ is the class of admissible strategies for Z. Consider the class of trajectories $\mathcal{J}^{a,\mu}$ endowed with the Skorohod's topology. We have:

i) *The NP market $(\mathcal{J}_\tau^{a,\mu}, [\mathcal{A}_P^Z])$ is NP arbitrage-free.*
ii) *$[\mathcal{A}_P^Z]$ contains the portfolio strategies from Proposition 5 which furthermore satisfy that there exist $A > 0$ such that $V_\Phi(t,x) > -A$ $\forall t \in [0,T]$, $\forall x \in \mathcal{J}_\tau^{a,\mu}$. $[\mathcal{A}_P^Z]$ also contains the portfolio strategies defined in Theorem 5.*

Proof. *i)* The proof is analogous to the proof of Theorem 4. As indicated, $P(Z(\omega) \in \mathcal{J}^{a,\mu}(x_0)) = 1$, so condition C_0 from Theorem 1 holds. In order to verify condition C_1 from Theorem 1, we argue directly (another possibility would be to extract the result from the proof of the more general Lemma 8). Consider $x \in \mathcal{J}^{a,\mu}(x_0)$ and suppose that x has m jumps at times $0 < s_1 \cdots < s_m < T$. Then, for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x' \in \mathcal{J}^{a,\mu}(x_0)$ has exactly m jumps $0 < s'_1 \cdots < s'_m < T$ with $|s_i - s'_i| < \delta$ then $d_s(x, x') < \epsilon$. We know that the time between two consecutive jumps of a Poisson process has exponential distribution (which is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^+$) therefore, jumps occur in a given interval with positive probability. From previous analysis and the property of independence of the time between jumps, we conclude that the set of trajectories x' of the Geometric Poisson model (21) having jumps in a δ-neighborhood of the jumps of any $x \in \mathcal{J}^{a,\mu}(x_0)$ has positive probability, thus Condition C_1 is verified and so the market $(\mathcal{J}^{a,\mu}(x_0), [\mathcal{A}_P^Z])$ is NP-arbitrage-free.
ii) Consider Φ from Proposition 5 satisfying the lower bound assumption then, the same arguments used in the proof of *ii)* of Theorem 4 apply in this case, namely, the use of Propositions 11, 12 and 5, as well as Corollary 1 prove that $\Phi \in [\mathcal{A}_P^Z]$. Similar arguments show that $\Phi \in [\mathcal{A}_P^Z]$ whenever Φ is one of the hedging strategies introduced in Theorem 5. □

One important question at this point is whether or not simple portfolio strategies are V-continuous for the Geometric Poisson model. Next proposition addresses that question.

Proposition 6. *In general, simple portfolios strategies are not V-continuous relative to the Skorohod topology in $\mathcal{J}^{a,\mu}(x_0)$.*

Proof. We provide an example of a simple strategy that is not V-continuous. Consider $T = 1$ and let Φ the NP-portfolio with initial value x_0 defined as:

- $\phi(t,x) = 1$, $\psi(t,x) = 0$, for all $x \in \mathcal{J}^{a,\mu}(x_0)$ if $0 \le t \le 1/2$
- $\phi(t,x) = 0$, $\psi(t,x) = x_{\frac{1}{2}}$ for all $x \in \mathcal{J}^{a,\mu}(x_0)$ if $1/2 < t \le 1$

We can easily check that Φ is NP-admissible according to Definition 2.

Let $y \in \mathcal{J}^{a,\mu}(x_0)$ be the function $y_t = x_0 e^{\mu t}(1+a)^{1_{[0,t]}(1/2)}$ and let $(y^{(n)})_{n=1,\dots}$ be the sequence of functions defined by $y_t^{(n)} = x_0 e^{\mu t}(1+a)^{1_{[0,t]}(1/2+1/n)}$. Clearly $y^{(n)} \to y$ in the Skorohod topology on $D[0,1]$.

From the definition of Φ we have $V_\Phi(1,x) = x_{\frac{1}{2}}$ for all $x \in \mathcal{J}^{a,\mu}(x_0)$, in particular $V_\Phi(1,y) = x_0 e^{\mu/2}(1+a)$ and $V_\Phi(1,y^{(n)}) = x_0 e^{\mu/2}$.

As $y^{(n)} \to y$ but $V_\Phi(1,y^{(n)}) \nrightarrow V_\Phi(1,y)$ we conclude that Φ is not a V-continuous portfolio in $\mathcal{J}^{a,\mu}(x_0)$. □

It can also be shown that not necessarily a portfolio must be V-continuous in order to belong to $[\mathcal{A}_P^Z]$. An example of such portfolio is given in the next proposition, proving that $[\mathcal{A}_P^Z]$ contains more portfolios than those explicitly showed in Theorem 6.

Proposition 7. *If $a < 0$ the portfolio Φ in Proposition 6 belongs to $[\mathcal{A}_P^Z]$*

Proof. Let Φ the NP-portfolio defined in Proposition 6, and consider the isomorphic portfolio Φ^Z over the price process Z in (21) defined a.s. by $\Phi^Z(t,\omega) = \Phi(t, Z(\omega))$. The portfolio strategy Φ^Z is a simple strategy, therefore Φ^Z is admissible so $\Phi^Z \in \mathcal{A}_P^Z$. Let us show that even if Φ is not V-continuous on $\mathcal{J}^{a,\mu}(x_0)$, $\Phi \in [\mathcal{A}_P^Z]$.

If an arbitrary trajectory $x \in \mathcal{J}^{a,\mu}(x_0)$ is continuous at $t = 1/2$, then it is always possible to choose $\delta > 0$ small enough, such that for all x' satisfying $d_s(x',x) < \delta$, it holds $x'(1/2) = x(1/2)$ which in turns implies that $V_\Phi(1,x') = V_{Phi}(1,x)$. If an arbitrary x is discontinuous at $t = 1/2$, then Φ is not V-continuous at x, in fact that was the statement of Proposition 6. Nevertheless, for such trajectories x we can always choose $\delta > 0$ small enough such that for all x' satisfying $d_s(x',x) < \delta$, we have one of the two following possibilities:
1) $x'(1/2) = x(1/2)$.
2) $x'(1/2) = x(1/2)(1+a)^{-1}$.
Case 1) corresponds to those trajectories x' that jump in the interval $(1/2-\delta, 1/2]$ while case 2) corresponds to those trajectories that jump in $(1/2, 1/2+\delta)$. For trajectories $Z(\omega)$ in case 1) we have $V_\Phi(1,x') = V_\Phi(1,x)$. For those in case 2) it holds $V_\Phi(1,x') = x(1/2)(1+a)^{-1} > x(1/2) = V_\Phi(T,x) = x(1/2)$ if $a < 0$. What we have shown with previous analysis is that for an arbitrary trajectory $x \in \mathcal{J}^{a,\mu}(x_0)$, whether or not x is continuous at $t = 1/2$, it is always possible to find δ small enough such that if $d_s(x',x) < \delta$ then $V_\Phi(1,x') \geq V_\Phi(1,x)$, which implies that application $V_\Phi(1,\cdot) : \mathcal{J}^{a,\mu}(x_0) \to \mathbb{R}$ is lower semicontinuous with respect to the Skorohod topology. Applying Proposition 1, as $\Phi^Z \in \mathcal{A}_P^Z$, then Φ is connected to Φ^Z hence $\Phi \in [\mathcal{A}_P^Z]$. □

Remark 8. *It is expected that the present class $[\mathcal{A}_P^Z]$ could be considerably enlarged, and in particular simple strategies would belong to this enlarged class, once the notion of stopping times is incorporated in our non probabilistic approach. This line of research represents work in progress* [1].

4.3. **Non Probabilistic Jump Diffusions.** Fix $\sigma > 0$ and C a non empty set of real numbers such that $\inf(C) > -1$. Define $\mathcal{J}_\tau^{\sigma,C}(x_0)$ as the class of real valued functions x on $[0,T]$ such that there exits $z \in \mathcal{Z}_\mathcal{T}([0,T])$, $n(t) \in \mathcal{N}([0,T])$, and real numbers $a_i \in C$, $i = 1,2,\dots,m$, verifying:

$$x(t) = x_0 e^{\sigma z(t)} \prod_{i=1}^{n(t)} (1+a_i) \tag{22}$$

Remark 9. *The class $\mathcal{J}_\tau^{\sigma,a}(x_0)$ combines the features of classes given in Sections 4.1 and 4.2. Its probabilistic counterpart is the class of exponential jump-diffusion processes.*

In the stochastic framework, the markets where prices are driven by jump-diffusion models are not complete in general, therefore hedging is not always possible. On the other hand, we do know that these models admit many risk neutral measures, indicating that they are arbitrage free. In this section we will obtain the property of absence of arbitrage in the analogous NP framework given by trajectories belonging to $\mathcal{J}_\tau^{\sigma,C}(x_0)$.

First we give a small ball property result for a jump-diffusion model and the class of price trajectories $\mathcal{J}_\tau^{\sigma,C}(x_0)$ defined in (22) and then we derive the NP arbitrage-free result using Theorem 1 by relying on the V-continuity property of certain class of portfolios with respect to the Skorohod topology.
The following proposition provides the required small ball property.

Proposition 8. *For any $x_0 > 0$ consider in the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ the exponential jump diffusion processes, starting at x_0 given by:*

$$Z_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{i=1}^{N_t} (1 + X_i) \tag{23}$$

where W is a Brownian Motion, N is a homogeneous Poisson Process with intensity $\lambda > 0$, and the X_i are independent random variables, also independent of W and N, with common probability distribution F_X. Assume that F_X verifies the condition:

A1)For any $a \in C$ and for all $\epsilon > 0$, $F_X(a+\epsilon) - F_X(a-\epsilon) > 0$.

Then the jump-diffusion process given by (23) satisfies a small ball property on $\mathcal{J}_\tau^{\sigma,C}(x_0)$ with respect to the Skorohod metric.

Proof. Consider $x(\cdot) = x(0)e^{\sigma z(\cdot)} \prod_{i=1}^{n(\cdot)}(1+a_i) \in \mathcal{J}_\tau^{\sigma,C}(x_0)$, where $n(\cdot) \in \mathcal{N}([0,T])$ has m discontinuity points in $[0,T]$ denoted by $0 < s_1 < \ldots < s_m < T$. Also denote $\tilde{n}(t) = \sum_{i=1}^{n(t)} \ln(1+a_i)$ and $\xi_t = \sum_{i=1}^{N(t)} \ln(1+X_i)$.

Fix $\epsilon > 0$ and consider $\delta > 0$. Define Ω_1^δ as the set of $w \in \Omega$ having jump times $0 < T_1(w) < T_2(w) < \ldots < T_m(w) < T$ and $T_{m+1} > T$, satisfying also that $|T_i(w) - s_i| \leq \frac{\delta}{3}$, for $i = 1, 2, \ldots, m$. Note that the T_i's are finite sum of continuous random variables, namely the times between jumps, therefore we have that $P(\Omega_1) > 0$.

Take now Ω_2^δ as the set of $w \in \Omega$ such that:
$|\ln(1+X_i(w)) - \ln(1+a_i)| < \frac{\delta}{3m}$, for $i = 1, 2, \ldots, m$, which implies that

$$\left| \sum_{i=1}^m \ln(1+X_i(w)) - \sum_{i=1}^m \ln(1+a_i) \right| < \frac{\delta}{3}.$$

Let $\lambda(t)$ be function from $[0,T]$ onto itself defined by the polygonal through the points $(0,0), (s_1, T_1(w)), \ldots, (s_m, T_m(w)), (T,T)$.

Note that, by construction, for $w \in \Omega_1^\delta \bigcap \Omega_2^\delta$ we have:

$$\sup_{t\in[0,T]} |\lambda(t) - t| < \frac{\delta}{3}$$

$$\sup_{t\in[0,T]} |\xi_{\lambda(t)}(w) - \tilde{n}(t)| < \frac{\delta}{3}.$$

We should note that $P(\Omega_2^\delta) > 0$ for every $\delta > 0$ as consequence of condition A1). For $0 \le t \le T$ we have

$$\begin{aligned} (24) \quad & \left| \left(\mu - \frac{1}{2}\sigma^2\right)\lambda(t) + \sigma W_{\lambda(t)} + \xi_{\lambda(t)} - \sigma z(t) - \tilde{n}(t) \right| \\ \le\ & \sigma \left| W_{\lambda(t)} - z(t) + \frac{1}{\sigma}\left(\mu - \frac{1}{2}\sigma^2\right)\lambda(t) \right| + |\xi_{\lambda(t)} - \tilde{n}(t)| \\ \le\ & \sigma \left| W_{\lambda(t)} - z(\lambda(t)) + \frac{1}{\sigma}\left(\mu - \frac{1}{2}\sigma^2\right)\lambda(t) \right| + \sigma|z(\lambda(t)) - z(t)| + |\xi_{\lambda(t)} - \tilde{n}(t)|. \end{aligned}$$

Define $z'(t) = z(t) - \frac{1}{\sigma}\left(\mu - \frac{1}{2}\sigma^2\right)t$ and let $\Omega_3^\delta = \{\omega \in \Omega : \sup_{t\in[0,T]} |W_t(w) - z'(t)| < \frac{\delta}{3\sigma}\}$. By Lemma 1 we have $P(\Omega_3^\delta) > 0$. Moreover by independence between W, N and the X_i's we have that:

$$P(\Omega_1^\delta \bigcap \Omega_2^\delta \bigcap \Omega_3^\delta) = P(\Omega_1^\delta)P(\Omega_2^\delta)P(\Omega_3^\delta) > 0$$

Now, as $z(t)$ is uniformly continuous on $[0, T]$, there exists $\delta' > 0$ such that $|\lambda(t) - t| < \delta'$ implies $\sup_{t\in[0,T]} |z(\lambda(t)) - z(t)| < \frac{\epsilon}{3\sigma}$. Without loss of generality take $0 < \delta < \min(\epsilon, \delta')$

According to (24), in $\Omega_1^\delta \bigcap \Omega_2^\delta \bigcap \Omega_3^\delta$ we have, for $0 \le t \le T$:

$$\begin{aligned} & \left| \left(\mu - \frac{1}{2}\sigma^2\right)\lambda(t) + \sigma W_{\lambda(t)} + \xi_{\lambda(t)} - \sigma z(t) - \tilde{n}(t) \right| \\ \le\ & \sigma \left| W_{\lambda(t)} - z'(\lambda(t)) \right| + \sigma|z(\lambda(t)) - z(t)| + |\xi_{\lambda(t)} - \tilde{n}(t)|. \\ \le\ & \frac{\delta}{3} + \frac{\epsilon}{3} + \frac{\delta}{3} < \epsilon \end{aligned}$$

As $Z_t = \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t + \xi_t\right)$ and $x(t) = \exp(\sigma z_t + \tilde{n}(t))$, the small ball property is obtained by an argument of uniform continuity of the exponential function. □

Analogously to Proposition 5 we have the following result on V-continuity relative to the Skorohod's topology on $\mathcal{J}_\tau^{\sigma,C}(x_0)$.

Proposition 9. *Let $\Phi_t = (\psi_t, \phi_t)$ be a portfolio strategy on $\mathcal{J}_\tau^{\sigma,C}(x_0)$ for which the amount invested in the stock $\phi_t = \phi(t, x(t-))$ is such that $\phi \in C^{1,1}([0, T] \times R^+)$ and ψ is defined through the self financing condition in (40). Assume also that $\inf_{c\in C} |c| > h$ for some real number h. Then Φ is V-continuous on $\mathcal{J}_\tau^{\sigma,C}(x_0)$ relative to the Skorohod's topology.*

Proof. Let $\Phi_t = (\psi_t, \phi_t)$ be such that $\phi_t = \phi(t, x(t-))$ with $\phi \in C^{1,1}([0, T] \times R^+)$. Define the function $U_\Phi : \mathbb{R}^2 \to \mathbb{R}$ as:

$$U_\Phi(t, x) = \int_{x_0}^{x} \phi(t, \xi) d\xi \qquad (25)$$

and the functional $u_\Phi : \mathcal{J}_\tau^{\sigma,C}(x_0) \to \mathbb{R}$ as:

$$
\begin{aligned}
(26)\quad u_\Phi(x) \;=\;& U_\Phi(T, x(T)) - U_\Phi(0, x(0)) \\
& - \int_0^T \frac{\partial U_\Phi}{\partial t}(s, x(s-))ds - \frac{1}{2}\int_0^T \frac{\partial^2 U_\Phi}{\partial x^2}(s, x(s-))d\langle x\rangle_s^{\mathcal{T}} \\
& - \sum_{s\leq T}\left[U_\Phi(s, x(s)) - U_\Phi(s, x(s-)) - \frac{\partial U_\Phi}{\partial x}(s, x(s-))\Delta x(s)\right]
\end{aligned}
$$

From Ito-Föllmer formula

$$
u_\Phi(x) = \int_0^T \frac{\partial U_\Phi}{\partial x}(s, x(s-))dx(s) = \int_0^T \phi(s, x(s-))dx(s) \tag{27}
$$

which implies that portfolio Φ is V-continuous on $\mathcal{J}_\tau^{\sigma,C}(x_0)$ (with respect to the Skorohod's topology) if and only if the functional u_Φ is continuous on $\mathcal{J}_\tau^{\sigma,C}(x_0)$ with respect to the Skorohod's topology.

For all $x \in \mathcal{J}_\tau^{\sigma,C}(x_0)$, $d\langle x\rangle_s^{\mathcal{T}} = \sigma^2 x^2(s-)ds$, therefore (26) transforms into:

$$
u_\Phi(x) = U_\Phi(T, x(T)) - U_\Phi(0, x(0)) - I_\Phi(x) - S_\Phi(x) \tag{28}
$$

where

$$
I_\Phi(x) = \int_0^T \frac{\partial U_\Phi}{\partial t}(s, x(s-))ds + \frac{1}{2}\int_0^T \frac{\partial^2 U_\Phi}{\partial x^2}(s, x(s-))\sigma^2 x^2(s-)ds
$$

and

$$
S_\Phi(x) = \sum_{s\leq T}\left[U_\Phi(s, x(s)) - U_\Phi(s, x(s-)) + \frac{\partial U_\Phi}{\partial x}(s, x(s-))\Delta x(s)\right].
$$

Let $x^* \in \mathcal{J}_\tau^{\sigma,C}(x_0)$ and let $\{x^{(n)}\}_{n=0,1,\ldots}$ be a sequence of functions, $x^{(n)} \in \mathcal{J}_\tau^{\sigma,C}(x_0)$, such that $\{x^{(n)}\}$ converges to x^* in the Skorohod's topology.

From Lemma 2 and the continuity of both U_Φ and $\frac{\partial U_\Phi}{\partial x}$, it is immediate to see that $S_\Phi(x^{(n)}) \to S_\Phi(x^*)$.

Next we will prove that $I_\Phi(x^{(n)}) \to I_\Phi(x^*)$. Consider

$$
g(s, x) = \frac{\partial U_\Phi}{\partial t}(s, x) + \frac{\sigma^2\ x^2}{2}\frac{\partial^2 U_\Phi}{\partial x^2}(s, x)
$$

Then

$$
\left|I_\Phi(x^*) - I_\Phi(x^{(n)})\right| = \left|\int_0^T g(s, x^*(s)-)ds - \int_0^T g(s, x^{(n)}(s)-)ds\right|. \tag{29}
$$

Let (λ_n) be a sequence of non-decreasing functions on $[0, T]$ such that $\lambda_n(0) = 0$, $\lambda_n(T) = T$ and $\lambda_n(s) \to s$. Making the variable change $s \to \lambda_n(s)$ on the second integral, expression (29) transforms into:

$$\begin{aligned}\left|I_\Phi(x^*) - I_\Phi(x^{(n)})\right| &= \left|\int_0^T g(s,x^*(s-))ds - \int_0^T g\left(\lambda_n(s), x^{(n)}(\lambda_n(s)-)\right)d\lambda_n(s)\right| \\ &\leq \left|\int_0^T g(s,x^*(s-))ds - \int_0^T g(s,x^*(s-))d\lambda_n(s)\right| \\ &\quad + \left|\int_0^T g(s,x^*(s-))d\lambda_n(s) - \int_0^T g\left(\lambda_n(s), x^{(n)}(\lambda_n(s)-)\right)d\lambda_n(s)\right| \\ &\leq \left|\int_0^T g(s,x^*(s-))ds - \int_0^T g(s,x^*(s-))d\lambda_n(s)\right| \\ &\quad + \int_0^T \left|g(s,x^*(s-)) - g\left(\lambda_n(s), x^{(n)}(\lambda_n(s)-)\right)\right| d\lambda_n(s).\end{aligned}$$

As $\left(\lambda_n(s), x^{(n)}(\lambda_n(s)-)\right)$ converges to $(s, x^*(s-))$ uniformly in s and g is continuous, hence uniformly continuous on compact sets then:

$$\int_0^T \left|g(s,x^*(s-)) - g\left(\lambda_n(s), x^{(n)}(\lambda_n(s)-)\right)\right| d\lambda_n(s) \to 0 \tag{30}$$

Notice also that

$$\int_0^T g(s,x^*(s-))d\lambda_n(s) \to \int_0^T g(s,x_s-)ds \tag{31}$$

Expression (31) is consequence of the weak convergence of $\lambda_n(s)$ to s and the fact that $y(s) = g(s, x(s-))$ is bounded on $[0,T]$ with only a finite number of discontinuities. See for example the Continuous Mapping Theorem (page 87, [11]).

Both (30) and (31) imply that $I_\Phi(x^{(n)}) \to I_\Phi(x^*)$. As already proved $S_\Phi(x^{(n)}) \to S_\Phi(x^*)$, therefore $u_\Phi(x^{(n)}) \to u_\Phi(x^*)$, and the V-continuity of Φ is proved

□

We have now all necessary ingredient to prove an arbitrage result for the present class of trajectories.

Theorem 7. *Let $(Z, \mathcal{A}^Z_{JD})$ be the stochastic market defined by the geometric jump diffusion process introduced in (23) and $\mathcal{A}^Z_{JD}$ is the class of admissible strategies for Z. Consider the class of trajectories $\mathcal{J}^{\sigma,C}_T$ introduced in (22) endowed with the Skorohod's topology. Assume the random variables X_i to be integrable and that their common probability distribution F_X satisfies the following conditions:*
1) supp(F_X) $\subset C$ where supp(F) stands for the support of the distribution function F
2) For any $a \in C$ and for all $\epsilon > 0$, $F_X(a+\epsilon) - F_X(a-\epsilon) > 0$
Then it holds:

i) *The NP market $(\mathcal{J}^{\sigma,C}_\tau, [\mathcal{A}^Z_{JD}])$ is NP arbitrage-free*

ii) *If C satisfies that* $\inf(C) > -1$ *and* $\inf_{c \in C} |c| > h$ *for some real number* $h > 0$*, then* $[\mathcal{A}^Z_{JD}]$ *contains the portfolios from Proposition 9 which furthermore satisfy that there exist* $A > 0$ *such that* $V_\Phi(t, x) > -A$ $\forall t \in [0, T]$*,* $\forall x \in \mathcal{J}^{\sigma,C}_\tau$*.*

Proof. First note that $P(w \in \Omega : Z(w) \in \mathcal{J}^{\sigma,C}_\tau) = 1$ as consequence of 1) hence condition C_0 from Theorem 1 is fulfilled. From Proposition 8 we also have that condition C_1 from Theorem 1 holds. Therefore, in order to establish conclusion $i)$ we need to argue that the stochastic market $(Z, \mathcal{A}^Z_{JD})$ is arbitrage free. Our hypothesis allow the application of Proposition 9.9 from [8], this result establishes the existence of a probability $\mathbb{Q}$ such that e^{rt} Z_t is a martingale, therefore the probabilistic market $(Z, \mathcal{A}^Z_{JD})$ is arbitrage free.
ii) Consider Φ satisfying the conditions listed in $ii)$ and define Φ^z as $\Phi^z(t, \omega) = \Phi(t, Z(\omega))$ Proposition 11 shows that the stochastic portfolio Φ^z is predictable, LCRL and self-financing. The admissibility of Φ^z then results from our hypothesis, hence $\Phi^z \in \mathcal{A}^Z_{JD}$. Proposition 12 shows that Φ is admissible; Φ is also V-continuous as consequence of Proposition 9. The NP portfolio Φ and the stochastic portfolio $\Phi^z \in \mathcal{A}^Z_{JD}$ are isomorphic, so according to Corollary 1, Φ is connected to Φ^z and $\Phi \in [\mathcal{A}^Z_{JD}]$. □

Analogously to Proposition 6, simple portfolio strategies may not be V-continuous on the trajectory space $\mathcal{J}^{\sigma,C}_\tau$.

5. Implications to Stochastic Frameworks

In previous sections we have studied some connections between stochastic and NP-markets, in particular Theorem 1 was used to establish that some NP models are NP-arbitrage free. In this section we use Theorem 2 to prove the no arbitrage property in stochastic settings using some of the results that we have obtained previously for NP-models. In this way, results obtained in a non-probabilistic framework not only constitute a different approach to the main financial problems of hedging and arbitrage but can also be used as a technical tool to obtain new arbitrage results in stochastic settings as well.

One important property of this approach for pricing derivatives in probabilistic models is that it is applicable even in cases where prices are not semimartingales. From our point of view, this is one a main advantage of the approach, allowing to price derivatives for some models where the risk neutral approach is impossible to carry out or is not very clear. Another important feature of this approach is that it encompasses models with jumps and without jumps as we illustrate in the examples in this section.

Example 2 (Black-Scholes related models)**.** *Let* X *be a price process on* $(\Omega, \mathcal{F}, \mathcal{F}_t, P)_{0 \le t \le T}$ *defined by* $X_t = x_0\ e^{\sigma Z^{Gen}_t}$ *where* Z^{Gen} *is a (general) process adapted to* $\mathcal{F}_t$ *satisfying*
1)$[Z^{Gen}]^\tau_t = t$*.*
2)Z^{Gen} *satisfies a small ball property on* $\mathcal{Z}_\mathcal{T}([0, T])$ *with respect to the uniform metric.*
As consequence of 1) Theorem 3 applies in this case indicating that hedging is possible in a path by path sense.

Consider $\mathcal{A} \equiv [\mathcal{A}_{BS}^Z]$ introduced in Theorem 4 and $\mathcal{A}^X \equiv [\mathcal{A}]^X$, where $[\mathcal{A}]^X$ is given as in Definition 9. Under these circumstances we will argue that the stochastic market $(X, \mathcal{A}^X)$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is arbitrage-free in the classical probabilistic sense. We will apply Theorem 2 to the given NP market $(\mathcal{J}_\tau^\sigma, \mathcal{A})$ with the supremum metric and $\mathcal{A}$ as defined above. Condition C_0 is an obvious consequence of 1). Condition C_1 is derived from 2) and the fact that $\exp(\cdot)$ *(used to construct X) is a continuous function, hence uniformly continuous on compacts.*
As all conditions of Theorem 2 are satisfied and $(\mathcal{J}_\tau^\sigma, \mathcal{A})$ is NP-arbitrage free by means of Theorem 4 then the stochastic market $(X, \mathcal{A}^X)$ is arbitrage free in the classical probabilistic sense.

Using similar arguments to the ones we have used in Section 4, it is possible to verify that the smooth strategies as previously defined belong to $\mathcal{A}^X$, this means that $\mathcal{A}^X$ is a large class. Moreover, also the delta hedging strategies belong to $\mathcal{A}^X$, as well as simple portfolio strategies like those in Remark 6.

Next we will show that examples of such processes Z^{Gen} are:

- $Z^F = W + B^H$ where W is a Brownian Motion and B^H is a fractional Brownian motion with Hurst index $1/2 < H < 1$ independent of W.
- $Z^R = \rho W + \sqrt{1-\rho^2} B^R$ where W is a Brownian Motion, B^R is a reflected Brownian Motion independent of W. and ρ is a real number, $0 < \rho < 1$.
- Z^w, a Weak Brownian motion.

Mixed Fractional Brownian Model: For process $Z^F = W + B^H$ we have that if $1/2 < H < 1$ then the trajectories of B^H have zero quadratic variation, which implies that $[Z^F]_t^\tau = [W]_t^\tau = t$ almost surely. The small ball property of Z^F on $\mathcal{Z}_\mathcal{T}([0,T])$ is consequence of a small ball property of W on $\mathcal{Z}_\mathcal{T}([0,T])$ which is obtained from Lemma 1, the hypothesis on independence between W and B^H, and a small ball property of the fractional Brownian motion B^H around the identically null function ([22], [24]).

The absence of arbitrage for this model implies that pricing and hedging can be done exactly as in the Black-Scholes model. We have proven this fact using essentially that the trajectories of prices are dense in $\mathcal{J}_\tau^\sigma(x_0)$. We did not use any semi-martingale property of the price process, in fact, for $H \in (1/2, 3/4]$, X is not a semi-martingale, which is a drawback form the point of view of a risk-neutral approach for pricing.

Some results for models similar to $X_t = x(0)e^{\sigma Z_t^F}$ are presented in [19] and [2] proving that pricing and hedging procedures in the Black-Scholes model are robust against perturbations with zero quadratic variation. More recently, in [3], path-dependents options are replicated under this model.

The replication result in the previous model, which is consequence of $[Z^F]_t^\tau = [W]_t^\tau = t$ remains true for models satisfying $Z_t = W_t + Y_t$ where Y is a process with zero quadratic variation, as for example any continuous process with finite variation. While the replication is always valid for a general Y , the no-arbitrage property is less obvious to verify and will depend on the particular form of Y, in the dependence structure between W and Y, etc.

Mixed Reflected Brownian Model: Consider the process $Z^R = \rho W + \sqrt{1-\rho^2} B^R$ where W is a Brownian Motion and B^R is a reflected Brownian Motion independent of W. By a reflected Brownian motion we understand a process whose trajectories are obtained by reflecting trajectories of a standard Brownian motion on one or two reflecting boundaries. A particular example of reflected Brownian motion is given by $B_t^R = |\tilde{B}_t|$ where $\tilde{B}$ is a Brownian motion. Therefore, our reflected Brownian B^R motion will be upper bounded and/or lower bounded.

In this case we also have $[Z^R]_t^\tau = t$ almost surely. The small ball property of Z^R on $\mathcal{Z}_\mathcal{T}([0,T])$ is deduced as follows: we know from Lemma 1 that $P(\sup_{s\in[0,T]} |W_s - f(s)/\rho| < \epsilon/2) > 0$ for all f continuous such that $f(0) = 0$. On the other hand $P(\sup_{s\in[0,T]} |B_s^R| < \epsilon/2) > 0$ is also true, also as consequence of Lemma 1. Then $P(\sup_{s\in[0,T]} |Z^R(s) - f(s)| < \epsilon) > 0$ for all $\epsilon > 0$ and for all f continuous, with $f(0) = 0$ in particular for all $f \in \mathcal{Z}_\mathcal{T}([0,T])$.

A financial interpretation of this example is that the asset price is influenced by an external source of randomness B^R which is limited within some bounds. This idea has some precedents, see for example [18]. Nevertheless, to our knowledge, this model has not been considered previously for pricing and hedging purposes. The risk neutral approach for pricing does not seem to be obvious for this model, and in fact, could heavily depend on the reflected boundaries. Our approach to this model (as stated in Example 2) is simple and tells us again that pricing and hedging for this model can be done as in the Black-Scholes paradigm.

Weak Brownian Motion: A weak Brownian motion Z^w of order $k \in \mathbb{N}$ is a stochastic process whose k-marginal distributions are the same as of a Brownian motion, although it is not a Brownian motion. In particular we will consider those weak Brownian motions of order at least 4 such that their law on $C[0,T]$ are equivalent to the Wiener measure on $C[0,T]$. The existence of such processes, as well as some of their properties are established in [13]. In particular we will use that if $k \geq 4$ $\langle Z^w \rangle_t^\tau = t$ almost surely. The required small-ball property of Z^w on $\mathcal{Z}_\mathcal{T}([0,T])$ is consequence of the equivalence between the law of Z^w and the Wiener measure on $C[0,T]$.

We should remark that a Weak Brownian motion may not be a semimartingale, therefore, the risk neutral approach for pricing may be impossible for models that include them. Nonetheless, models including weak Brownian motions have been recently studied in [9], through the weaker concept of $\mathcal{A}$-martingale.

Example 3 (Renewal Process)**.** *Consider N^R to be a renewal process instead of the usual classical Poisson process:*

$$N_t^R = \sum_i 1_{[0,t]}(S_i)$$

where S_i are considered random jump times such that random variables $S_{i+1} - S_i$, representing the times between jumps, are positive, independent and identically distributed with some probability distribution $G(x)$. Consider the price process given by

$$X_t = x_0 e^{\mu t}(1+a)^{N_t^R} \tag{32}$$

The trajectories of the process (32) will be in $\mathcal{J}^{a,\mu}(x_0)$, so the replicating portfolio in Proposition 5 also applies to this case for any probability distribution G. In order to have the no-arbitrage property, additional assumptions must be made on G. For example, if the support of G is a finite interval it could lead to obvious arbitrage

opportunities related to the imminent occurrence of a jump. Nevertheless, if G is absolutely continuous with respect to the Lebesgue measure on the whole positive real line, the model is arbitrage-free for a large class of portfolio strategies as the following analysis shows.
Let X be a price process on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ as defined in (32). Consider $\mathcal{A} \equiv [\mathcal{A}_P^Z]$ introduced in Theorem 6. Let also $\mathcal{A}^X \equiv [\mathcal{A}]^X$, where $[\mathcal{A}]^X$ is given as in Definition 9. Under these circumstances we will argue that the stochastic market $(X, \mathcal{A}^X)$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is arbitrage-free in the classical probabilistic sense.
We will apply Theorem 2 to the given NP market $(\mathcal{J}^{a,\mu}, \mathcal{A})$ with the Skorohod's metric and $\mathcal{A} \equiv [\mathcal{A}_P^Z]$. As we observed, condition C_0 from Theorem 2 is satisfied. Condition C_1, also from Theorem 2, is valid because of our hypothesis on the support of G: the probability of those trajectories jumping exactly in a small neighborhood of the jumps of any $x \in \mathcal{J}^{a,\mu}(x_0)$ (which guarantees that these trajectories are close to x in the Skohorod topology) is positive. Here implicitly we also used that the times between jumps are independent random variables. As all conditions of Theorem 2 are satisfied and $(\mathcal{J}_\tau^\sigma, \mathcal{A})$ is NP-arbitrage free by means of Theorem 4 then the stochastic market $(X, \mathcal{A}^X)$ is arbitrage free in the classical probabilistic sense.

Let Φ be any of the NP portfolio strategies considered either in Theorem 5 or Proposition 5, it then follows from Corollary 1 that $\Phi^X(t, w) \equiv \Phi(t, X(w))$ belongs to $\mathcal{A}^X$.

An specific example is when $G(x)$ is such that $1-G(x) \sim x^{-(1+\beta)}$ with $\beta \in (0,1)$. This particular case is used in [23] for the approximation of a Geometric Fractional Brownian motion. There, using path-by-path arguments it is shown that model is complete and arbitrage-free for some class of portfolio strategies.

We should remark that a similar result can be obtained also if the support of G is dense in $[0, \infty)$, in other words, if every interval $[a, b]$ with $0 < a < b$ has positive probability according to G: $G(b) > G(a)$, as suggests the proof of Theorem 6. It means that if G is supported on the set of positive rational numbers $\mathbb{Q}_+$ for example, the no-arbitrage property is valid, thus pricing can be done exactly as in the Geometric Poisson model. This result, which is analogous in some way to the one presented in Example 1, is surprising in the sense that the measure that G induces on $\mathcal{J}^{a,\mu}(x_0)$ is not absolutely continuous with respect to the measure induced by the Geometric Poisson model.

Example 4 (Jump-diffusion related models)**.** *Consider a stochastic process having the form*

$$X_t = e^{(\mu-\sigma^2/2)t+\sigma Z_t^G} \prod_{i=1}^{N_t^R} (1 + X_i), \tag{33}$$

where Z^G is a continuous process satisfying that $\langle Z^G \rangle_t = t$. We also assume that Z^G satisfies a small ball property on $\mathcal{Z}_\mathcal{T}([0,T])$ with respect to the uniform norm. Examples of such Z^G are the processes Z^F, Z^R and Z^w, previously defined. Process N^R is a renewal process as considered in Example 3 and random variables X_i are considered to be independent with common distribution F_X. Set $\mathcal{A} \equiv [\mathcal{A}_{JD}^Z]$ and consider the NP market $(\mathcal{J}_\tau^{\sigma,C}, \mathcal{A})$ defined in Theorem 7. As we have defined before, let $\mathcal{A}^X \equiv [\mathcal{A}]^X$, where $[\mathcal{A}]^X$ is given by (9). Also assume F_X and the set C satisfy the assumptions 1) and 2) of Theorem 7. Using the same arguments we used there, it is possible to verify that all conditions of Theorem 2 are satisfied for the stochastic *process X in (33) and the NP market $(\mathcal{J}_{\tau}^{\sigma,C}, \mathcal{A})$, therefore the stochastic market $(X, [\mathcal{A}]^X)$ is arbitrage free in the classical probabilistic sense. Consider Φ to be any of the NP portfolio strategies considered in Proposition 9, it then follows from Corollary 1 that $\Phi^X(t,w) \equiv \Phi(t, X(w))$ belongs to $\mathcal{A}^X$.*

The examples introduced above indicate that replication and arbitrage problems in a stochastic framework could be studied without requiring the semi-martingale property. Instead, topological properties of the support of the price process play a key role. Under these circumstances, the existence of a risk neutral measure constitutes a useful tool for pricing but not a necessary condition for stating and solving a pricing problem in a coherent way.

6. Conclusions and Further Work

The present work develops a non probabilistic framework for pricing using the classical arguments of hedging and no arbitrage. We obtain general no arbitrage results in our framework that no longer rely on a probabilistic structure, but in the topological structure of the space of possible price trajectories and a convenient restriction on the admissible portfolios to those satisfying certain continuity properties. Apparently, the introduced non probabilistic framework is far from being a different and isolated approach, this is strongly suggested by the fact that our results in a non probabilistic setting have also implications on stochastic frameworks because of the existing connections between both approaches. Therefore, the results can be used to price derivatives in non standard stochastic models.

There are several possible extensions of our work, in particular, there are many spaces of trajectories that could be encoded in a non probabilistic framework. In relation to this we mention that many of the results presented in the paper can be extended by introducing the analogue of stopping times in our non probabilistic framework. It seems that a major technical advance for the proposed formalism would be to supply a proof technique that allows to establish non arbitrage results without relying on the known results for the probabilistic setting.

Appendix A. Quadratic Variation and Ito Formula

From [12] we have the following.

Proposition 10. *(Itô-Föllmer Formula) Let x be of quadratic variation along τ, and let $y^1, \ldots, y^m$ be continuous functions of bounded variation. Suppose that $f \in C^{1,2,1}([0,T) \times \mathbb{R} \times \mathbb{R}^m)$, then for all $0 \le s < t < T$:*

$$f(t, x_t, y_t^1, \ldots, y_t^m) = f(s, x_s, y_s^1, \ldots, y_s^m) + \int_s^t \frac{\partial}{\partial t} f(u, x_u, y_u^1, \ldots, y_u^m) du + \tag{34}$$

$$\int_s^t \frac{\partial}{\partial x} f(u, x_u, y_u^1, \ldots, y_u^m) dx_u + \frac{1}{2} \int_s^t \frac{\partial^2}{\partial x^2} f(u, x_u, y_u^1, \ldots, y_u^m) d\langle x \rangle_u^{\tau} +$$

$$\sum_{i=1}^m \int_s^t \frac{\partial}{\partial y^i} f(u, x_u, y_u^1, \ldots, y_u^m) dy_u^i +$$

$$\sum_{u \le t} \Big[f(u, x_u, y_u^1, \ldots, y_u^m) - f(u^-, x_{u^-}, y_{u^-}^1, \ldots, y_{u^-}^m) \Big] -$$

$$\frac{\partial}{\partial x}f(u^-, x_{u-}, y^1_{u-}, \dots, y^m_{u-})\Delta x_u - \sum_{i=1}^{m}\frac{\partial}{\partial y^i}f(u^-, x_{u^-}, y^1_{u^-,}, \dots, y^m_{u^-})\Delta y^i_s.$$

Here are some results that we use in the main body of the paper.

- If $z(s),\ s \in [0,T]$ is a continuous function with zero quadratic variation along τ and x is of quadratic variation along τ, then $[x+z]^\tau_t = [x]^\tau_t$.
- If x is of quadratic variation along τ and $f \in C^1(\mathbb{R})$ then $y = f \circ x$ is of quadratic variation along τ , moreover

$$\langle y\rangle^\tau_t = \int_0^t (f'(x(s)))^2 d\langle x\rangle^\tau_s. \tag{35}$$

For more details, see [12].

Appendix B. Technical Results

Proposition 11. *Let $(X_t)_{0\le t\le T}$ be an adapted process on the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ representing the price process. Assume that a.s. the trajectories of X belong to some class of functions $\mathcal{J}$ and let g_1, g_2, g_m be hindsight factors over $\mathcal{J}$. Let the stochastic portfolio $\Phi = (\phi, \psi)$ have the form*

$$\phi_t = G(t, X_{t-}, g_1(t,X), g_2(t,X), \dots, g_m(t,X)) \tag{36}$$

and

$$\psi_t = \tilde{V}_\Phi(t-) - \phi_t X_{t-} \tag{37}$$

with $G \in C^1([0,T]\times\mathbb{R}\times\mathbb{R}^m)$,$g_1(t,X)$, $g_2(t,X)$,..., $g_m(t,X)$ are $(\mathcal{F}_{t-})$-measurable random variables, and $\tilde{V}_\Phi(t-) = \lim_{s\to t-}\tilde{V}_\Phi(s)$ with

$$\tilde{V}_\Phi(s) = V_\Phi(0) + \int_0^s G(r, X_{r-}, g_1(r,X), g_2(r,X)..., g_m(r,X))dX_r \tag{38}$$

Then the stochastic portfolio $\Phi = (\phi, \psi)$ is predictable, LCRL, and self-financing.

Proof. The held number of stock at time t, ϕ_t, given by (36), is predictable (also LCRL) because all g_i's and X_{t-} are predictable (also LCRL) and G is continuous. The integral

$$\int_0^s G(r, X_{r-}, g_1(r,X), g_2(r,X)..., g_m(r,X))dX_r$$

in (41) can be computed using Ito's formula in terms of X_{s-} as well as predictable functions (measurable with respect to $\mathcal{F}_{s-}$) of X, which easily implies that

$$\lim_{s\to t}\int_0^s G(r, X_{r-}, g_1(r,X), g_2(r,X)..., g_m(r,X))dX_r$$

is measurable with respect to $\mathcal{F}_{t-}$, therefore $\tilde{V}_\Phi(t-)$ is predictable. As $\tilde{V}_\Phi(t-)$, X_{t-}, and ϕ_t are predictable and LCRL, ψ_t is also predictable and LCRL from expression (37). Then we conclude that Φ is predictable and LCRL.

Let us prove now the self-financing condition. From the definition of $\tilde{V}_\Phi(t)$ and $\tilde{V}_\Phi(t-)$ it follows that

$$\tilde{V}_\Phi(t) - \tilde{V}_\Phi(t-) = \phi_t(X_t - X_{t-})$$

hence we have

$$\tilde{V}_\Phi(t) = \tilde{V}_\Phi(t-) + \phi_t(X_t - X_{t-}) = \phi_t X_{t-} + \psi_t + \phi_t(X_t - X_{t-}) = \phi_t X_t + \psi_t$$

Previous expression means that $\tilde{V}_\Phi(t) = V_\Phi(t)$, hence Φ is self-financing. □

We have also an analogous result in the NP framework.

Proposition 12. *Consider a class of trajectories $\mathcal{J}$ and let g_1, g_2, g_m be hindsight factors over $\mathcal{J}$ such that all g_i satisfy the additional condition:*

$$g_i(t, x) = g_i(t, \tilde{x}) \tag{39}$$

whenever $x(s) = \tilde{x}(s)$ for all $0 \leq s < t$

For all $x \in \mathcal{J}$ let the portfolio $\Phi = (\phi, \psi)$ have the form

$$\phi(t, x) = G(t, x(t-), g_1(t, x), g_2(t, x)..., g_m(t, x))$$

and

$$\psi(t, x) = \tilde{V}_\Phi(t-, x) - \phi(t, x)x(t-) \tag{40}$$

with $G \in C^1\left([0, T] \times \mathbb{R} \times \mathbb{R}^m\right)$ and $\tilde{V}_\Phi(t-, x) = \lim_{s \to t-} \tilde{V}_\Phi(s, x)$ with

$$\tilde{V}_\Phi(s, x) = V_\Phi(0, x_0) + \int_0^s G(r, x(r-), g_1(r, x), g_2(r, x)..., g_m(r, x))dx_r \tag{41}$$

Then the NP portfolio $\Phi = (\phi, \psi)$ is NP-predictable, LCRL, and NP-self-financing.

Proof. The NP-predictability of ϕ is an obvious consequence of (39). The NP-predictable representation of $\int_0^s G(r, x(r-), g_1(r, x), g_2(r, x)..., g_m(r, x))dx_r$, which is given by Ito-Föllmer formula, guarantees that also ψ is NP-predictable, therefore portfolio $\Phi = (\phi, \psi)$ is NP predictable.

The self-financing property and the LCRL property can be proved exactly as in Proposition 11. □

Lemma 2. *Let $x^* \in \mathcal{J}_\tau^{\sigma,C}(x_0)$ and let $\left\{x^{(n)}\right\}_{n=0,1...}$ with $x^{(n)} \in \mathcal{J}_\tau^{\sigma,C}(x_0)$ be a sequence of functions converging to x^* on the Skorohod's topology. Then there exists $M \in \mathbb{N}$ such that if $n > M$ then $x^{(n)}$ has the same number of jumps as x^*. Moreover, if the jump times of x^* are denoted as $s_1, s_2, ..., s_m$ and the jump times of $x^{(n)}$ (for $n > M$) are denoted as $s_1^{(n)}, s_2^{(n)}, ..., s_m^{(n)}$, then for all $i = 1, 2..., m$:*

$$s_i^{(n)} \to s_i$$

and

$$\left[x^{(n)}(s_i^{(n)}) - x^{(n)}(s_i^{(n)}-)\right] \to \left[x^*(s_i) - x^*(s_i-)\right] \tag{42}$$

Proof. As x^* is càd-làg on $[0, T]$, x^* attains its maximum and minimum on this interval. On the other hand $x^*(t) > 0$ for all $0 \leq t \leq T$, so $\inf_{t\in[0,T]} x^*(t) = \min_{t\in[0,T]} x^*(t) = x^*_{min} > 0$.

If s is a jump time for x^*, $|x^*(s) - x^*(s-)| > x^*(s-)h \geq x^*_{min}h$, which means that the absolute size of the jumps of x^* are strictly bounded below by $x^*_{min}h$.

Let $\epsilon < \frac{x^*_{min}h}{2(h+2)}$, then there exists $M(\epsilon) \in \mathbb{N}$ such that $d(x^*, x^{(n)}) < \epsilon$ for all $n \geq M(\epsilon)$. As the Skorohod's distance between x^* and $x^{(n)}$, $d(x^*, x^{(n)}) < \epsilon$, there exist an increasing function $\lambda_n : [0, T] \to [0, T]$ with $\lambda_n(0) = 0$, $\lambda_n(1) = 1$, $|\lambda_n(t) - t| < \epsilon$ for $0 \leq t \leq T$ and

$$\left|x^*(t) - x^{(n)}(\lambda_n(t))\right| < \epsilon, \text{ for } 0 \leq t \leq T. \tag{43}$$

Let s be a jump time of x^*. As expression (43) is valid for $0 \leq t \leq T$, for any increasing sequence of positive real numbers $\{s_i\}_{i=1,2,\ldots}$ converging to s we have that:

$$\left|x^*(s_i) - x^{(n)}(\lambda_n(s_i))\right| < \epsilon, \text{ for all } i. \tag{44}$$

As x^* and $x^{(n)}$ are càd-làg function on $[0, T]$ it is possible to take limits in (44) as $i \to \infty$ obtaining:

$$\left|x^*(s-) - x^{(n)}(\lambda_n(s)-)\right| < \epsilon. \tag{45}$$

From expressions (43) and (45) we have

$$\left|(x^*(s) - x^*(s-)) - \left(x^{(n)}(\lambda_n(s)) - x^{(n)}(\lambda_n(s)-)\right)\right| < 2\epsilon, \tag{46}$$

therefore

$$\begin{aligned}
\left|x^{(n)}(\lambda_n(s)) - x^{(n)}(\lambda_n(s)-)\right| &> |x^*(s) - x^*(s-)| - 2\epsilon \\
&> x^*_{min} h - \frac{2x^*_{min} h}{2(h+2)} \\
&= x^*_{min} h \frac{h+1}{h+2} > 0.
\end{aligned}$$

This means that if s is a jump time for x^*, then $\lambda_n(s)$ is also a jump time for $x^{(n)}$.

Consider now that s' is a jump time of $x^{(n)}$ and define $s'' = \lambda_n^{-1}(s')$. We have from (46) that

$$\left|(x^*(s'') - x^*(s''-)) - \left(x^{(n)}(s') - x^{(n)}(s'-)\right)\right| < 2\epsilon.$$

Thus,

$$|x^*(s'') - x^*(s''-)| > \left|x^{(n)}(s') - x^{(n)}(s'-)\right| - 2\epsilon. \tag{47}$$

As $d(x^{(n)}, x^*) < \epsilon$ then $\inf_{s\in[0,T]}(x^{(n)}(s)) > x^*_{min} - \epsilon > 0$ uniformly on n. Therefore, (47) becomes:

$$|x^*(s'') - x^*(s''-)| > (x^*_{min} - \epsilon)h - 2\epsilon = \frac{x^*_{min} h}{2} > 0,$$

which implies that s'' is a jump time for x if $s' = \lambda_n(s'')$ is a jump time for $x^{(n)}$.

Previous analysis tells that for n large enough, $x^{(n)}$ has exactly the same number of jumps as x^* has.

In order to derive (42), consider $\Delta_i x^*$ as the size of the i-th jump of x^* and $\Delta_i x^{(n)}$ as the size of the i-th jump of $x^{(n)}$. Then:

$$\begin{aligned}\Delta_i x^{(n)} &= \left(x^{(n)}(s_i^{(n)}) - x^{(n)}(s_i^{(n)}-)\right) \\ &= \left(x^{(n)}(\lambda_n(s_i)) - x^{(n)}(\lambda_n(s_i)-)\right).\end{aligned}$$

Expression (46) implies that:

$$\left(x^{(n)}(\lambda_n(s_i)) - x^{(n)}(\lambda_n(s_i)-)\right) \to (x^*(s_i) - x^*(s_i-)),$$

as $n \to \infty$, so $\Delta_i x^{(n)} \to \Delta_i x$ as $n \to \infty$. □

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