Bayesian Model Choice of Grouped t-copula

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Abstract

One of the most popular copulas for modeling dependence structures is t-copula. Recently the grouped t-copula was generalized to allow each group to have one member only, so that *a priori* grouping is not required and the dependence modeling is more flexible. This paper describes a Markov chain Monte Carlo (MCMC) method under the Bayesian inference framework for estimating and choosing t-copula models. Using historical data of foreign exchange (FX) rates as a case study, we found that Bayesian model choice criteria overwhelmingly favor the generalized t-copula. In addition, all the criteria also agree on the second most likely model and these inferences are all consistent with classical likelihood ratio tests. Finally, we demonstrate the impact of model choice on the conditional Value-at-Risk for portfolios of six major FX rates.

Key words: grouped $t-$copula, dependence modeling, Bayesian model choice, Markov chain Monte Carlo, foreign exchange.

1   Introduction

Copula functions have become popular and flexible tools in modeling multivariate dependence among financial risk factors. In practice, one of the most popular copulas in modeling multivariate financial data is perhaps the t-copula implied by the multivariate t-distribution (hereafter referred to as *standard t-copula*); see Embrechts *et al* (2001), Fang *et al* (2002), and Demarta and McNeil (2005). This is due to its simplicity in terms of simulation and calibration, combined with its ability to model tail dependence which is often observed in financial returns data. Papers by Mashal *et al* (2003) and Breymann *et al* (2003) have demonstrated that the empirical fit of the standard t-copula is superior in most cases when compared to the Gaussian copula. However, the standard t-copula is often criticized due to the restriction of having only one parameter for the degrees of freedom (dof), which may limit its ability to model tail dependence in multivariate case. To overcome this problem, Daul *et al* (2003) proposed the use of the *grouped t-copula*, where risks are grouped into classes and each class has its own standard t-copula with a specific dof. This, however, requires an *a priori* choice of classes. It is not always obvious how the risk factors should be divided into sub-groups. An adequate choice of grouping configurations requires substantial additional effort if there is no natural grouping, for example, by sector or class of asset.

Recently, the grouped t-copula was generalized to a new t-copula with multiple dof parameters (hereafter referred to as *generalized t-copula*); see Luo and Shevchenko (2010) and Venter et al (2007). This copula can be viewed as a grouped t-copula with each group having only one member. It has the advantages of a grouped t-copula with flexible modeling of multivariate dependences, yet at the same time it overcomes the difficulties with *a priori* choice of groups. For convenience, denote the new copula as $\tilde{t}_{\boldsymbol{\nu}}$-copula, where $\boldsymbol{\nu} = (\nu_1, ..., \nu_n)$ denotes the vector of dof parameters and n is the number of dimensions. Luo and Shevchenko (2010) demonstrated that some characteristics of this new copula in

the bivariate case are quite different from those of the standard t-copula. For example, the copula is not exchangeable if $\nu_1 \neq \nu_2$ and tail dependence implied by the $\tilde{t}_{\boldsymbol{\nu}}$-copula depends on both dof parameters. The difference between $\tilde{t}_{\boldsymbol{\nu}}$- and standard t-copulas, in terms of impact on Value-at-Risk (VaR) and conditional Value-at-Risk (CVaR) of the portfolio, can be significant as demonstrated by simulation experiments for the bivariate case. This difference is even much larger than the difference between Gaussian copula and the standard t-copula. In examples of maximum likelihood fitting to USD/AUD and USD/JPY daily return data, standard t-copula was statistically rejected by a formal Likelihood Ratio test in favour of the $\tilde{t}_{\boldsymbol{\nu}}$ copula (i.e. dof parameters in the $\tilde{t}_{\boldsymbol{\nu}}$-copula were statistically different).

This paper presents a Bayesian model selection study on the t-copula models in the multivariate case. We demonstrate how to perform Bayesian inference using Markov chain Monte Carlo (MCMC) simulations to estimate parameters and make decisions on model choice. From a Bayesian point of view, model parameters are random variables whose distribution can be inferred by combining the prior density with the likelihood of observed data. The complete posterior distribution of the parameters resulting from Bayesian MCMC allows further analysis such as model selection and parameter uncertainty quantification. Specifically, we solve a variable selection problem in the same vein as discussed in Cairns (2000). Increasingly, Bayesian MCMC finds new applications in quantitative financial risk modeling. Recent examples are found in Peters et $al.$ (2009, 2010) for insurance, Shevchenko (2010) for operational risk and Luo and Shevchenko (2010) for credit risk.

As a case study, we consider the application of modeling dependence among six major foreign exchange (FX) rates (AUD, CAD, CHF, EUR, GBP and JPY, against USD) using t-copulas. Following common practice (see e.g. McNeil et al 2005), we use the GARCH(1,1) model to standardize the log-returns of the exchange rates marginally. Then the GARCH filtered residuals of the six major FX rates are modeled by a t-copula. In

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this study we consider altogether 33 competing t-copula models: the standard t-copula, 31 grouped t-copulas and the generalized t-copula (i.e. $\tilde{t}_{\boldsymbol{\nu}}$-copula). The 31 grouped t-copulas are a complete set of all possible combinations of two groups from six FXs (see Table 1 for all possible 2-group configurations for the six FX majors).

We present procedures and results of MCMC simulation for t-copula models under the Bayesian framework. Also, we demonstrate using Bayesian model inference and actual data, that the generalized t-copula ($\tilde{t}_{\boldsymbol{\nu}}$-copula) is convincingly the model of choice for modeling dependence between six FX majors, among considered 33 t-copula models. Even compared with the best grouped t-copula chosen from 31 possible two-group configurations, the $\tilde{t}_{\boldsymbol{\nu}}$-copula is overwhelmingly favoured by the Bayesian factors obtained from the MCMC posterior distribution. We demonstrate that the joint calibration of grouped t-copula can be done very efficiently by applying MCMC. Using model parameters estimated from MCMC, we also demonstrate the impact of model choice on CVaR of two portfolios of six FX majors.

The organisation of this paper is as follows. Section 2 introduces the various t-copula models and notations. Then it describes the GARCH model filtering for the six FX majors, and calibration of the t-copula models using the maximum likelihood method. Section 3 discusses the Bayesian inference formulation, the MCMC simulation algorithm, the reciprocal importance sampling estimator and the deviance information criterion for model selection. Direct computing of the posterior model probability is also discussed in Section 3. Section 4 presents MCMC results and the corresponding Bayesian model selection, in comparison with the traditional maximum likelihood results and Likelihood Ratio tests. Examples of portfolio CVaR calculation using selected models and calibrated parameters are provided in Section 5, demonstrating the impact of model choice on risk quantification. Concluding remarks are given in the final section.

2 Model, data and maximum likelihood calibration

It is well known from Sklar's theorem (see Sklar 1959 and Joe 1997) that any joint distribution function F with continuous (strictly increasing) margins $F_1, F_2, \ldots, F_n$ has a unique copula

$$C(\mathbf{u}) = F(F_1^{-1}(u_1), F_2^{-1}(u_2), \ldots, F_n^{-1}(u_n)). \tag{1}$$

The t-copulas are most easily described and understood by a stochastic representation, as defined below.

2.1 t-copula models

We introduce notation and definitions as follows:

- $\mathbf{Z} = (Z_1, \ldots, Z_n)'$ is a random vector from the multivariate normal distribution $\Phi_{\boldsymbol{\Sigma}}(\mathbf{z})$ with zero mean vector, unit variances and correlation matrix $\boldsymbol{\Sigma}$.

- $\mathbf{U} = (U_1, \ldots, U_n)'$ is defined on $[0,1]^n$ domain.

- V is a random variable from the uniform (0,1) distribution independent of $\mathbf{Z}$.

- $W = G_\nu^{-1}(V)$, where $G_\nu(\cdot)$ is the distribution function of $\sqrt{\nu/S}$ with S distributed from the chi-square distribution with ν dof, i.e. W and $\mathbf{Z}$ are independent.

- $t_\nu(\cdot)$ is the standard univariate t-distribution and $t_\nu^{-1}(\cdot)$ is its inverse.

Then we have the following representations.

Standard t-copula

The random vector

$$\mathbf{X} = W \times \mathbf{Z} \tag{2}$$

is distributed from a multivariate t-distribution and random vector

$$\mathbf{U} = (t_\nu(X_1), \ldots, t_\nu(X_n))' \tag{3}$$

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is distributed from the standard t-copula.

Grouped t-copula

Partition $\{1, 2, \ldots, n\}$ into m non-overlapping sub-groups of sizes $n_1, \ldots, n_m$. Then the copula of the random vector

$$\mathbf{X} = (W_1 Z_1, \ldots, W_1 Z_{n_1}, W_2 Z_{n_1+1}, \ldots, W_2 Z_{n_1+n_2}, \ldots, W_m Z_n)', \tag{4}$$

where $W_k = G_{\nu_k}^{-1}(V)$, $k = 1, \ldots, m$, is the grouped t-copula. That is,

$$\mathbf{U} = (t_{\nu_1}(X_1), \ldots, t_{\nu_1}(X_{n_1}), t_{\nu_2}(X_{n_1+1}), \ldots, t_{\nu_2}(X_{n_1+n_2}), \ldots, t_{\nu_m}(X_n))' \tag{5}$$

is a random vector from the grouped t-copula. Here, the copula for each group is a standard t-copula with its own dof parameter (i.e. ν_k is dof parameter of the standard t-copula for the k-th group).

Generalized t-copula with multiple dof ($\tilde{t}_{\boldsymbol{\nu}}$-copula)

Consider the grouped t-copula where each group has a single member. In this case the copula of the random vector

$$\mathbf{X} = (W_1 Z_1, \ W_2 Z_2, \ldots, W_n Z_n)' \tag{6}$$

is said to have a t-copula with multiple dof parameters $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$, which we denote as $\tilde{t}_{\boldsymbol{\nu}}$-copula. That is,

$$\mathbf{U} = (t_{\nu_1}(X_1), t_{\nu_2}(X_2), \ldots, t_{\nu_n}(X_n))' \tag{7}$$

is a random vector distributed according to $\tilde{t}_{\boldsymbol{\nu}}$-copula. Note, all W_i are perfectly dependent.

Remark: Given the above stochastic representation, simulation of the $\tilde{t}_{\boldsymbol{\nu}}$ copula is straightforward. In the case of standard t-copula $\nu_1 = \cdots = \nu_n = \nu$ and in the case of grouped t-copula the corresponding subsets have the same dof parameter. Note that, the standard t-copula and grouped t-copula are special cases of $\tilde{t}_{\boldsymbol{\nu}}$-copula.

From the stochastic representation (6-7), it is easy to show that the $\tilde{t}_{\boldsymbol{\nu}}$-copula distribution has the following explicit integral expression

$$C_{\boldsymbol{\nu}}^{\boldsymbol{\Sigma}}(\mathbf{u}) = \int\limits_0^1 \Phi_{\boldsymbol{\Sigma}}(z_1(u_1, s), \ldots, z_n(u_n, s)) ds \tag{8}$$

and its density is

$$c_{\boldsymbol{\nu}}^{\Sigma}(\mathbf{u}) = \frac{\partial^n C_{\boldsymbol{\nu}}^{\boldsymbol{\Sigma}}(\mathbf{u})}{\partial u_1 \ldots \partial u_n} = \int\limits_0^1 \varphi_{\boldsymbol{\Sigma}}\left(z_1(u_1, s), \ldots, z_n(u_n, s)\right) \prod_{i=1}^n [w_i(s)]^{-1} ds / \prod_{i=1}^n f_{\nu_i}(x_i). \tag{9}$$

Here:

- $z_i(u_i, s) = t_{\nu_i}^{-1}(u_i)/w_i(s), \; i = 1, 2, \ldots, n$;

- $w_i(s) = G_{\nu_i}^{-1}(s)$;

- $\varphi_{\boldsymbol{\Sigma}}(z_1, \ldots, z_n) = \exp(-\frac{1}{2}\mathbf{z}'\boldsymbol{\Sigma}^{-1}\mathbf{z})/[(2\pi)^{n/2}(\det\boldsymbol{\Sigma})^{1/2}]$ is the multivariate normal density;

- $x_i = t_{\nu_i}^{-1}(u_i), \; i = 1, 2, \ldots, n$;

- $f_\nu(x) = (1 + x^2/\nu)^{-(\nu+1)/2} \Gamma((\nu + 1)/2)/[\Gamma(\nu/2)\sqrt{\nu\pi}]$ is the univariate t-density, where $\Gamma(\cdot)$ is a gamma function.

The multivariate density (9) involves a one-dimensional integration which makes the density calculation computationally more demanding than in the case of the standard t-copula, but still practical using available fast and accurate algorithms for the one-dimensional integration. If all the dof parameters are equal, i.e. $\nu_1 = \cdots = \nu_n = \nu$, then it is easy to show that the copula defined by (8) becomes the standard t-copula; see Luo and Shevchenko (2010) for a proof.

2.2 FX data and GARCH filtering

As a case study we consider modeling dependence between six FXs using t-copulas introduced in previous section. The daily foreign exchange rate data for the six FX majors in

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the period January 2004 to April 2008 (a total of 1092 trading days) were downloaded from the *Federal Reserve Statistical Release* (http://www.federalreserve.gov/releases). These daily data have been certified by the Federal Reserve Bank of New York as the noon buying rates in New York City. For our purpose, we study the six major currencies (AUD, CAD, CHF, EUR, GBP and JPY). Rates were converted to USD per currency unit in the present study, if not already in this convention. This unified convention allows a portfolio of currencies to be conveniently valued in terms of a single currency, the USD.

Following common practice (see McNeil *et al* 2005), we use the GARCH(1,1) model to standardize the log-returns of the exchange rates marginally. The GARCH(1,1) model calculates the current squared volatility σ_t^2 as

$$\sigma_t^2 = \omega + \alpha(x_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2, \quad \omega \geq 0, \ \alpha, \beta \geq 0, \ \alpha + \beta < 1, \tag{10}$$

where x_{t-1} denotes the log-return of an exchange rate on date $t-1$. GARCH parameters ω, α and β are estimated using the maximum likelihood method. Log-return was modeled as

$$x_t = \mu + \sigma_t\varepsilon^{(t)}, \tag{11}$$

where μ is the average historical return or drift for the asset and $\varepsilon^{(t)}$ is a sequence of iid random variables referred to as the residuals. The GARCH filtered residuals of the FX rates were then used to fit the t-copula models. Before the fitting the residuals were transformed to the (0,1) domain marginally using empirical distributions of the residuals.

2.3   Configuration of grouped t-copula

With six dimensions, the grouped t-copula can have a total of 201 possible combinations (not counting the standard t-copula and the $\tilde{t}_{\boldsymbol{\nu}}$-copula). In this study we concentrate on the class of configurations with two groups only, which is the next level of complexity compared with the standard t-copula. This reduces the number of possible grouped t-copula models to 31. These 31 grouped t-copula models are:

- 10 models from the complete subset of (3,3) configurations (with two groups and three members in each group).

- 15 models from the complete subset of (2,4) configurations (with two members in the first group and four members in the second group).

- 6 models from the complete subset of (1,5) configurations (with one member in the first group and five members in the second group).

Note, a (1,5) combination is the same as a (5,1) combination, and a (2,4) combination is the same as a (4,2) combination. So, altogether we have 33 competing models to choose from – the standard t-copula, the 31 two-grouped t-copula and the generalized t-copula ($\tilde{t}_{\boldsymbol{\nu}}$-copula).

Table 1 lists all 33 models for modeling the six FX majors, their grouping configurations and parameter notations. In column 2 of Table 1, each pair of parentheses define a subgroup configuration. The generalized grouped t-copula has six sub-groups with a single member in each sub-group, while the standard t-copula has one group containing all six members. Note that for the grouped t-copula, exchanging the two sub-groups makes no difference – these two configurations have exactly the same combinations of members, so no new models will emerge from this exchange.

2.4 Maximum likelihood calibration

Consider a random vector of data $\mathbf{Y} = (Y_1, \ldots, Y_n)'$. To estimate a parametric copula using observations $\mathbf{y}^{(j)}$, $j = 1, \ldots, K$, where K is the number of observations, the first step is to project the data to the $[0, 1]^n$ domain to obtain $\mathbf{u}^{(j)}$, using estimated marginal distributions. In our study the margins are modeled using empirical distributions but it can also be modeled using parametric distributions or a combination of these methods, e.g. empirical distribution for the body and a generalized Pareto distribution for the

tail of a marginal distribution (McNeil et al 2005, page 233). Given pseudo sample $\mathbf{u}^{(j)}$ constructed using the original data, the copula parameters can be estimated using, for example, the maximum likehood method or MCMC.

Accurate maximum likelihood estimates (MLEs) of the copula parameters should be obtained by fitting all unknown parameters jointly. In practice, to simplify the calibration procedure, correlation matrix coefficients for t-copulas are often calculated pair-wise using Kendall's *tau* rank correlation coefficients $\tau(Y_i, Y_j)$ via the formula (McNeil *et al* 2005)

$$\Sigma_{ij} = \sin\left(\tfrac{1}{2}\pi\tau(Y_i, Y_j)\right). \tag{12}$$

Then in a second stage the dof parameters $\nu_1, \ldots, \nu_n$ are estimated. Strictly speaking (12) is valid for bivariate case only, however in practice it works well for multivariate case too. It was noted in Daul *et al* (2003) that formula (12) is still highly accurate even when it is applied to find the correlation coefficients between risks from the different groups. McNeil *et al* (2005) observed that the estimated parameters using Kendall's *tau* are identical to those obtained by joint estimation to two significant digits, confirming good accuracy of the Kendall's *tau* simplification. It was also observed in Luo and Shevchenko (2010) that the difference in estimated parameters between the Kendall's *tau* approximation and the joint estimation was mostly in the third significant digit and was smaller than the standard errors for the MLEs. In addition, a study of small sample properties in Luo and Shevchenko (2010) showed that the bias introduced by the Kendall's *tau* approximation is very small even for a small sample size of 50. In the present work the data sample size is over 1000. The small bias of the Kendall's *tau* approximation is certainly insignificant when compared with the often large difference existing between dof parameters of different t-copula models. In other words, using (12) for the correlation coefficients should cause little material difference in the present model choice study where the difference is expected to come from different group configurations.

Because the Kendall's *tau* approximation is applied pair-wise, we have identical correla-

tion matrix for all the copula models to be considered. This simplification is computation-ally very significant for the grouped t-copula for which the calibration using density (9) is computationally demanding. By using the Kendall's *tau* approximation, the number of unknown parameters reduces from $M = n(n+1)/2$ to $M = n$ for the generalized grouped t-copula. With six-dimensions considered in this study, this amounts to a reduction from 21 parameters to only 6. For the grouped t-copula with two groups, this reduction is from 17 to 2, an even more dramatic reduction. A substantial saving of computing time is achieved in both cases.

Remark: An accurate calibration of grouped t-copula requires joint estimation of dof parameters. Sometimes in practice an approximate approach is taken where a grouped t-copula is calibrated marginally, i.e. each sub-group is calibrated separately using a standard t-copula. This approximation is not always justified; also it can not be applied to a generalized t-copula. For a proper and fair comparison between the grouped t-copula and the generalized grouped t-copula, in this study we perform joint calibration for both copulas. When the grouped t-copula is calibrated jointly, its density is given by the integral formula (9), the same as the generalized t-copula, so a proper joint calibration of the grouped t-copula is also computationally demanding when compared with the calibration of a standard t-copula.

Let $\boldsymbol{\nu}$ be the vector of n dof parameters $\nu_1, \ldots, \nu_n$ (the grouped t-copula is treated as a special case of $\tilde{t}_{\boldsymbol{\nu}}$-copula). Denote the density of the $\tilde{t}_{\boldsymbol{\nu}}$-copula evaluated at $\mathbf{u}^{(j)}$ as $c_{\boldsymbol{\nu}}(\mathbf{u}^{(j)})$, which can be obtained using (9). Then the MLEs for $\boldsymbol{\nu}$ are calculated by maximizing the

log-likelihood function

$$
\begin{aligned}
\ell_{\mathbf{U}}(\boldsymbol{\nu}) &= \ln \prod_{j=1}^{K} c_{\boldsymbol{\nu}}^{\Sigma}(\mathbf{u}^{(j)}) \\
&= \sum_{j=1}^{K} \ln \left(\int_{0}^{1} \varphi_{\Sigma}\left(z_{1}^{(j)}(s), \ldots, z_{n}^{(j)}(s) \right) \prod_{i=1}^{n} [w_{i}(s)]^{-1} ds \right) \\
&\quad + \sum_{j=1}^{K} \sum_{i=1}^{n} \left(\tfrac{1}{2}(\nu_{i}+1) \right) \ln[1 + (x_{i}^{(j)})^{2}/\nu_{i}] \\
&\quad + K \sum_{i=1}^{n} \left(\tfrac{1}{2} \ln(\nu_{i}\pi) + \ln[\Gamma(\tfrac{1}{2}\nu_{i})/\Gamma(\tfrac{1}{2}(\nu_{i}+1))] \right),
\end{aligned}
\tag{13}
$$

where $x_{i}^{(j)} = t_{\nu_{i}}^{-1}(u_{i}^{(j)})$, $z_{i}^{(j)}(s) = x_{i}^{(j)}/w_{i}(s)$, $i = 1, \ldots, n$, $j = 1, \ldots, K$. In this work we use the double precision *IMSL* function DQDAGS, a globally adaptive integration scheme documented in Piessens *et al* (1983) for the integration in (9). For the maximization of (13) the double precision *IMSL* function DBCPOL is used, which employs a direct search Simplex algorithm that does not require calculation of gradients.

3 Bayesian inference and MCMC

In this section we describe Bayesian approach and MCMC procedure to estimate t-copulas, and model selection criteria used to choose the t-copula model. Under the Bayesian approach, the model parameters $\boldsymbol{\theta}$ (in our case $\boldsymbol{\theta}$ is just the dof parameter $\boldsymbol{\nu}$) are treated as random variables. Given a *prior* distribution $\pi(\boldsymbol{\theta})$ and a conditional density of the data given $\boldsymbol{\theta}$ (i.e. likelihood) $\pi(\mathbf{y}|\boldsymbol{\theta})$, the joint density of data $\mathbf{Y}$ and the model parameters $\boldsymbol{\theta}$ is $\pi(\mathbf{y}, \boldsymbol{\theta}) = \pi(\mathbf{y}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})$. Having observed data $\mathbf{Y}$, the distribution of $\boldsymbol{\theta}$ conditional on $\mathbf{Y}$, the *posterior* distribution, is determined by Bayes' theorem

$$
\pi(\boldsymbol{\theta}|\mathbf{y}) = \frac{\pi(\mathbf{y}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int \pi(\mathbf{y}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}} \propto \pi(\mathbf{y}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}).
\tag{14}
$$

The posterior can then be used for predictive inference. There is a large number of useful texts on Bayesian inference; for a good introduction, see Berger (1985) and Robert (2001).

3.1 MCMC under Bayesian framework

The explicit evaluation of the normalization constant in (14) is often difficult especially in high dimensions. The complexity in our case is evident from the log-likelihood expression (13). The MCMC method provides a highly efficient alternative to traditional techniques by sampling from the posterior indirectly and performing the integration implicitly.

MCMC is especially suited to a Bayesian inference framework. It facilitates the quantification of parameter uncertainty and model risks. It also allows a unified estimation procedure that estimates parameters and latent variables. In the last case a special algorithm called data augmentation can be employed, see Tanner and Wong (1987). The Bayesian estimates of particular interest from MCMC are the maximum *a posterior* (MAP) estimate and the minimum mean square error (MMSE) estimate, defined as follows

$$\text{MAP}: \quad \hat{\boldsymbol{\theta}}^{MAP} = \arg\max_{\boldsymbol{\theta}}[\pi(\boldsymbol{\theta}|\mathbf{y})], \tag{15}$$

$$\text{MMSE}: \quad \hat{\boldsymbol{\theta}}^{MMSE} = E[\boldsymbol{\theta}|\mathbf{y}]. \tag{16}$$

The MAP and MMSE estimates are the posterior mode and mean respectively. If the prior $\pi(\boldsymbol{\theta})$ is constant and the parameter range includes the MLE, then the MAP of the posterior is the same as MLE.

3.2 Metropolis-Hastings algorithm

In our case study we use the Metropolis-Hastings algorithm first described by Hastings (1970) as a generalization of the Metropolis algorithm (Metropolis *et al* 1953). Denote the state vector at step t as $\boldsymbol{\theta}^{(t)}$ and we wish to update it to a new state $\boldsymbol{\theta}^{(t+1)}$. We generate a candidate $\boldsymbol{\theta}^*$ from density $q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})$, and accept this point as the new state of the chain with probability given by

$$\alpha(\boldsymbol{\theta}^{(t)}, \boldsymbol{\theta}^*) = \min\left\{1, \frac{\pi(\boldsymbol{\theta}^*)q(\boldsymbol{\theta}^{(t)}|\boldsymbol{\theta}^*)}{\pi(\boldsymbol{\theta}^{(t)})q(\boldsymbol{\theta}^*|\boldsymbol{\theta}^{(t)})}\right\}. \tag{17}$$

If the proposal is accepted, the new state $\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^*$, otherwise $\boldsymbol{\theta}^{(t+1)} = \boldsymbol{\theta}^{(t)}$. The single component Metropolis-Hastings is often more efficient in practice. Here the state variable $\boldsymbol{\theta}^{(t)}$ is partitioned into components $\boldsymbol{\theta}^{(t)} = (\theta_1^{(t)}, \theta_2^{(t)}, \ldots, \theta_n^{(t)})$ which are updated one by one or block by block. This was the framework for MCMC originally proposed by Metropolis *et al.* (1953), and is adapted in this study.

The likelihood is computed as $\pi(\mathbf{y}|\boldsymbol{\theta}) = \exp(\ell_{\boldsymbol{y}}(\boldsymbol{\theta}))$, where $\ell_{\boldsymbol{y}}(\boldsymbol{\theta})$ is the log-likelihood given by (13). In computer implementation, we take advantage of the fact that only one component is updated at each sub-step in the single component Metropolis-Hastings algorithm by saving and re-using any values not affected by the current updating. For example, each evaluation of (13) calls for the inverse of the t-distribution for all the data points and all the dof values. Saving and re-using these inverse values reduce the calculation by a factor of six for the six-dimensional MCMC computation.

3.3   Bayesian model selection using MCMC

Powerful MCMC methods such as the Gibbs sampler (Gelfand and Smith 1990) and the Metropolis-Hastings (MH) algorithm (Hastings 1970) enable direct estimation of the posterior and predictive quantities of interest, but do not lend themselves readily to estimation of the model probabilities. While one of the most common classical techniques is the Bayesian Information Criterion (BIC) (Schwarz 1978), many new approaches have been suggested in the literature.

The most widely used methods include the harmonic mean estimator of Newton and Raftery (1994), importance sampling (Fruhwirth-Schnatter 1995), the reciprocal importance sampling estimator (Gelfand and Dey 1994), and bridge sampling (Meng and Wong 1996, Fruhwirth-Schnatter 2004). A comprehensive review of some of these methods applied to Bayesian model selection can be found in Kass and Raftery (1995).

Consider model M with parameter vector $\boldsymbol{\theta}$. The model likelihood with data $\mathbf{y}$ can be

found by integrating out the parameter $\boldsymbol{\theta}$

$$\pi(\mathbf{y}|M) = \int \pi(\mathbf{y}|\boldsymbol{\theta}, M)\pi(\boldsymbol{\theta}|M)d\boldsymbol{\theta}, \tag{18}$$

where $\pi(\boldsymbol{\theta}|M)$ is the prior density of $\boldsymbol{\theta}$ in model M. Given a set of H competing models $\mathbf{M} = (M_1, M_2, \ldots, M_H)$, the Bayesian alternative to traditional hypothesis testing is to evaluate and compare the posterior probability ratio between the models. For model M_l $(1 \leq l \leq H)$, assuming we have some prior knowledge about the model probability $\pi(M_l)$, we can compute the posterior probabilities for all models using the model likelihoods

$$\pi(M_l|\mathbf{y}) = \frac{\pi(\mathbf{y}|M_l)\,\pi(M_l)}{\sum_{h=1}^{H}\pi(\mathbf{y}|M_h)\,\pi(M_h)}. \tag{19}$$

Consider two competing models M_1 and M_2, parameterized by $\boldsymbol{\theta}_{[1]}$ and $\boldsymbol{\theta}_{[2]}$ respectively. The choice between the two models can be based on the posterior model probability ratio, given by

$$\frac{\pi(M_1|\mathbf{y})}{\pi(M_2|\mathbf{y})} = \frac{\pi(\mathbf{y}|M_1)\,\pi(M_1)}{\pi(\mathbf{y}|M_2)\,\pi(M_2)} = \frac{\pi(M_1)}{\pi(M_2)}B_{12}, \tag{20}$$

where $B_{12} = \pi(\mathbf{y}|M_1)/\pi(\mathbf{y}|M_2)$ is the Bayes factor, the ratio of posterior odds of model M_1 to that of model M_2. As shown by Lavine and Scherrish (1999), an accurate interpretation of the Bayes factor is that the ratio B_{12} captures the change of the odds in favour of model M_1 as we move from prior to posterior. Jeffreys (1961) recommended a scale of evidence for interpreting Bayes factors, which was later modified by Wasserman (1997). A Bayes factor $B_{12} > 10$ is considered strong evidence in favour of M_1. For a detailed review of Bayes factors, see Kass and Raftery (1995).

Typically, the integral (18) required by the Bayes factor is not analytically tractable and sampling based methods must be used to obtain estimates of the model likelihoods. In the current study we choose three methods for model selection:

- direct estimation of the Bayes factor in (20) using Reciprocal Importance Sampling Estimation presented in Section 3.3.1;

- deviance information criterion (see Section 3.3.2);

- direct computation of the posterior model probabilities using formula presented in Section 3.3.3.

3.3.1 Reciprocal Importance Sampling Estimator

Given samples $\boldsymbol{\theta}^{(t)}$, $t = 1, \ldots, N$ from the posterior distribution obtained through MCMC, Gelfand and Dey (1994) proposed the *reciprocal importance sampling estimator* (RISE) to approximate the model likelihood as

$$\pi(\mathbf{y}|M) \approx \left[\frac{1}{N} \sum_{t=1}^{N} \frac{h(\boldsymbol{\theta}^{(t)}|M)}{\pi(\mathbf{y}|\boldsymbol{\theta}^{(t)}, M)\,\pi(\boldsymbol{\theta}^{(t)}|M)} \right]^{-1}, \tag{21}$$

where h plays the role of an importance sampling density roughly matching the posterior. Gelfand and Dey (1994) suggested a multivariate normal or t-distribution density with mean and covariance fitted to the posterior sample.

The RISE estimator can be regarded as a generalization of the *harmonic mean estimator* suggested by Newton and Raftery (1994). If $h = 1$ then (21) becomes the harmonic mean estimator. Other estimators include the *bridge sampling* proposed by Meng and Wong (1996), and the *Chib's candidate's estimator* (Chib 1995). In a recent comparison study by Miazhynskaia and Dorffner (2006), these estimators were employed as competing methods for Bayesian model selection on GARCH-type models, along with the reversible jump MCMC. It was demonstrated that the RISE estimator (either with normal or t importance sampling density), the bridge sampling method and the Chib's algorithm gave statistically equal performance in model selection, and their performance more or less matches the much more involved reversible jump MCMC.

3.3.2 Deviance Information Criterion

The deviance information criterion (DIC) is a generalization of the *Bayesian information criterion* (Schwarz 1978, Spiegelhalter et al 2002). For a given model M (for simplicity

we drop notation M in the formula below) the deviance is defined as

$$D(\boldsymbol{\theta}) = -2\log(\pi(\mathbf{y}|\boldsymbol{\theta})) + C, \tag{22}$$

where the constant C is common to all nested models. Then DIC is calculated as

$$DIC = 2E_{\boldsymbol{\theta}}[D(\boldsymbol{\theta})] - D(E_{\boldsymbol{\theta}}[\boldsymbol{\theta}]) = E_{\boldsymbol{\theta}}[D(\boldsymbol{\theta})] + (E_{\boldsymbol{\theta}}[D(\boldsymbol{\theta})] - D(E_{\boldsymbol{\theta}}[\boldsymbol{\theta}])), \tag{23}$$

where $E_{\boldsymbol{\theta}}[\cdot]$ is the expectation with respect to $\boldsymbol{\theta}$. The expectation $E_{\boldsymbol{\theta}}[D(\boldsymbol{\theta})]$ is a measure of how well the model fits the data; the smaller its value, the better the fit. The difference $E_{\boldsymbol{\theta}}[D(\boldsymbol{\theta})] - D(E_{\boldsymbol{\theta}}[\boldsymbol{\theta}])$ can be regarded as the effective number of parameters, the larger this term, the easier it is for the model to fit the data. So the DIC criterion favours the model with a better fit but at the same time penalizes the model with more parameters. Under this setting the model with the smallest DIC value is the preferred model.

3.3.3 Posterior model probabilities

A popular approach for model choice is based on Reversible Jump MCMC (Green 1995). Here we adopt an alternative proposed recently by Peters *et al* (2009) based on the work of Congdon (2006). In this procedure the posterior model probabilities $\pi(M_l|\mathbf{y})$ are estimated using the Markov chain in each model as

$$\pi(M_l|\mathbf{y}) = \sum_{t=1}^{N} \frac{L_{\mathbf{y}}(M_l, \boldsymbol{\theta}_{[l]}^{(t)})}{\sum_{h=1}^{H} L_{\mathbf{y}}(M_h, \boldsymbol{\theta}_{[h]}^{(t)})}, \tag{24}$$

where $\boldsymbol{\theta}_{[l]}^{(t)}$ is the MCMC posterior sample at Markov chain step t for model M_l, $L_{\mathbf{y}}(M_l, \boldsymbol{\theta}_{[l]}^{(t)})$ is the likelihood of $\mathbf{y}$ for a given model M_l with parameter vector $\boldsymbol{\theta}_{[l]}^{(t)}$, and N is the total number of MCMC steps after burn-in period. In (24), it is assumed that priors $\pi(\theta_{[l]}|M_l)$ and $\pi(M_l)$ are constant.

4 MCMC simulation results and analysis

Prior distributions. In all MCMC simulation runs, we assume a uniform prior for every model parameter. The only subjective judgement we bring to the prior is the support of the dof parameter. Denote the kth dof parameter of the hth t-copula model as $\nu_k^{(h)}$ (see Table 1). We impose a common lower and upper bounds for all dof components, specifically $1 = \nu_{\min} < \nu_k^{(h)} < \nu_{\max} = 100$. In our case study the support (1, 100) for dof parameter of the t-distribution should be sufficiently large to allow the posterior to be implied mainly by the observed data. To make sure the range is sufficiently large, we also tested a wider range of (1, 200) and found no material difference in the results.

MCMC procedure. The starting value for the Markov chain for each component is set to a uniform random number drawn independently from the support $(\nu_{\min}, \nu_{\max})$. In the single component Metropolis-Hastings algorithm, we adopt a truncated Gaussian distribution as the symmetric random walk proposal density for $q(\cdot|\cdot)$ in (17). For each component, the mean of the Gaussian density was set to the current state and the variance was pre-tuned so that the acceptance rate is close to the optimal level. For d-dimensional target distributions with iid components, the asymptotic optimal acceptance rate has been reported to be 0.234; see Gelman et al (1997) and Roberts and Rosenthal (2001). In pre-tuning the variances for all the components we set 0.234 as the target acceptance rate. In addition, the Gaussian density was truncated below $\nu_{\min}$ and above $\nu_{\max}$ to ensure each proposal was drawn within the support for the parameters. Specifically, for the kth component at chain step t, the proposal density is

$$q_k(\theta^*|\theta_k^{(t)}) = \frac{f_N(\theta^*; \theta_k^{(t)}, \sigma_k)}{F_N(\nu_{\max}; \theta_k^{(t)}, \sigma_k) - F_N(\nu_{\min}; \theta_k^{(t)}, \sigma_k)}, \tag{25}$$

where $f_N(\cdot; \mu, \sigma)$ and $F_N(\cdot; \mu, \sigma)$ are the Gaussian density and distribution functions respectively, with mean μ and standard deviation σ.

An independent Markov chain was run for each of the 33 models listed in Table 1.

Each run consists of three stages:

- Tuning - tune and adjust the proposal standard deviation to achieve optimal acceptance rate for each component.

- "Burn-in" - samples from this period are discarded.

- Posterior sampling - here the Markov chian is considered to have converged to the stationary target distribution and samples are used for model estimates.

Unless stated otherwise, we use a "burn-in" period of length $N_b = 20,000$. We then let the chain run for an additional $N = 100,000$ iterations to generate the posterior samples. Each step contained a complete update of all components.

MCMC convergence. Figure 1 shows the first 30,000 samples, taken *after* the burn-in period, of the dof component $\nu_1^{(0)}$ for model M_0 (i.e. the case of the generalized t-copula). Since M_0 has the highest parameter dimensions among all the candidate models, in general it requires the longest length of chains to converge to a stationary distribution. This figure shows that after the burn-in period the samples are mixing well over the support of the posterior distribution.

In addition to inspecting the sample paths, we also monitor the autocorrelation of the samples. Figure 2 shows the autocorrelations over multiple lags computed from the posterior samples for component $\nu_1^{(0)}$ of model M_0. A useful value to compute from these autocorrelations for each component is the autocorrelation time defined as

$$\tau_k = 1 + 2 \sum_{g=1}^{\infty} \rho_k(g), \tag{26}$$

where $\rho_k(g)$ is the autocorrelation at lag g for component θ_k. This autocorrelation is sometimes used to compute an "effective sample size" by dividing the number of samples by τ_k. The standard errors for the parameters can then be based on the effective sample size to compensate for the autocorrelation (see Ripley 1987, Neal 1993). In practice it

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is necessary to cut off the sum in (26) at $g = g_k^{\max}$ where the autocorrelations seem to have fallen to near zero, because including higher lags adds too much noise (for some interesting discussion on this issue, see Kass *et al.* 1998). As shown in Figure 2, in those well mixed MCMC samples the autocorrelation falls to near zero quickly and stays near zero at larger lags. For this study we have chosen a $g_k^{\max}$ for each component such that the autocorrelation at lag $g_k^{\max}$ has reduced to less than 0.01. That is, the autocorrelation time τ_k is estimated by

$$\hat{\tau}_k \approx 1 + 2\sum_{g=1}^{g_k^{\max}} \rho_k(g), \quad g_k^{\max} = \min\{g : \rho_k(g) < 0.01\}. \tag{27}$$

The $\hat{\tau}_k$ values estimated from MCMC output for model M_0 are shown in Table 2, along with the cut-off lag number $g_k^{\max}$. MCMC convergence characteristics for other components and for other models are very similar to those shown here for model M_0.

4.1 Bayesian estimates of parameters

This section presents results for posterior mean (MMSE), mode (MAP) and numerical error due to finite number of MCMC iterations.

4.1.1 Posterior mean and its numerical error

Table 3 shows values for the estimated mean from MCMC posterior samples for all 33 models. The standard errors (numerical error due to finite number of MCMC iterations) are shown in parentheses and the log-likelihoods corresponding to the estimated means are in the last column. Since the samples from MCMC are typically serially correlated, the usual formula for estimating the standard error of a sample mean (i.e. standard deviation divided by $\sqrt{N}$) will introduce significant under-estimation. Here, we use *batch sampling* for the standard error estimate of the MCMC posterior mean; see Gilks *et al.* (1996).

Consider a MCMC posterior sample $y_1, y_2, ..., y_N$ with length $N = Q \times L$, where L is

sufficiently large, so that the batch means

$$\bar{y}_q = \frac{1}{L} \sum_{t=(q-1)L+1}^{q \times L} y^{(t)} \quad , \quad q = 1, \ldots, Q \tag{28}$$

are considered approximately independent. Then $\bar{y} = (\bar{y}_1 + \cdots + \bar{y}_Q)/Q$ and the standard error of the posterior sample mean $\bar{y}$ can be approximated by

$$\sqrt{\text{Var}(\bar{y})} \approx \frac{1}{\sqrt{Q}} \sqrt{\frac{1}{Q-1} \sum_{q=1}^{Q} (\bar{y}_q - \bar{y})^2}, \tag{29}$$

Note that Q is the number of quasi-independent batches and $L = N/Q$ is the size of each batch.

4.1.2 Posterior mode and likelihood ratio tests

Values of the posterior mode taken from the MCMC samples for all the 33 models are shown in Table 4, along with the corresponding log-likelihood values. Using results of the maximum likelihood corresponding the posterior mode in Table 4, a classical likelihood ratio test can be performed to compare model likelihoods.

Consider the null hypothesis that the observed FX daily return data are from distribution described by the grouped t-copula model M_1, and the alternative hypothesis that the data are distributed according to the generalized t-copula model M_0. The likelihood ratio for the two models is simply $\Lambda = L_1/L_0$, where L_1 and L_0 are the maximum likelihood values (i.e. the likelihood value at the mode) for M_1 and M_0 respectively. The test statistic $-2\log(\Lambda)$ will be asymptotically χ^2 distributed with degrees of freedom equal to the difference in the number of dof parameters in M_0 and M_1, which is 4 in this case.

We can perform likelihood ratio tests on all other grouped t-copula models (M_h, $h = 2, \ldots, 31$) against the same alternative hypothesis of model M_0, the generalized t-copula. For the standard t-copula the difference in the number of parameters is 5. The test statistic and the associated p-value (χ^2 significance) are given in Table 4. Clearly, according to

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the p-value, all the null hypotheses should be rejected and the alternative hypothesis, the generalized t-copula model M_0, is statistically justified.

Excluding model M_0, among the other 32 t-copula models (M_h, $h = 1, \ldots, 32$), the one achieving the highest likelihood is M_{27}, which is one of the six (1,5) two-group configurations. The p-value of this best grouped t-copula model against M_0 is 0.0045 which is still very small, suggesting a rather strong rejection of the grouped t-copula (including the standard t-copula) in favour of the $\tilde{t}_\nu$-copula model M_0. Achieving the highest likelihood from the fifteen (2,4) configurations is model M_4. It is interesting to notice that both M_{27} and M_4 have three European currencies (CHF, EUR, GBP) in one group (see Table 1), perhaps reflecting a natural geopolitical and economic grouping.

4.2 Bayesian model choice

While the likelihood ratio test relies on a single point estimate, the Bayesian model choice makes decisions based on the entire posterior distribution. As discussed in Section 3.3, three Bayesian inference criteria were used to choose among the 33 t-copula models: RISE given by (21); DIC given by (23); and the posterior model probabilities (24).

The RISE calculation involves fitting the MCMC posterior samples to a multivariate normal or t-distribution and taking expectation of the reciprocal likelihood. The DIC calculation requires taking expectation of the likelihood and the parameters. Column 2 in Table 5 shows the RISE factor $B_{0h} = R_0/R_h$, $h = 1, \ldots, 32$, where R_h is the RISE value for model M_h. That is, B_{0h} ($1 \leq h \leq 32$) is a measure of strength in the argument that the generalized t-copula (model M_0) is the Bayesian choice. The very large Bayes factors ($B_{0h} > e^{11} > 5.9 \times 10^4$) shown in Table 5 overwhelmingly support the generalized t-copula, confirming the likelihood ratio tests discussed previously. Excluding model M_0, these Bayes factors also point to M_{27} as the most favoured model among the grouped t-copulas (M_h, $1 \leq h \leq 32$) confirming the likelihood ratio tests. The larger the Bayes

factor B_{0h}, the stronger the case against model M_h ($1 \leq h \leq 32$).

The DIC values for 33 models are shown in column 3 of Table 5. Since only relative DIC value matters, the common constant in (22) was set in such a way that the DIC value for model M_0 is zero. As shown in Table 5, the DIC value for all the other models is significantly positive, relative to that of M_0. Thus under the DIC criterion the model of choice is clearly M_0, i.e. the generalized t-copula. In addition, similar to the RISE based Bayes factors and the likelihood tests, the DIC values also pick M_{27} as the most likely grouped t-copula model after M_0, the same as the RISE factor and the likelihood ratio test. The larger the DIC value, the stronger the case against the model. It is interesting to observe that the magnitude of the DIC value is close to that of the logarithm of the Bayes factor based on the reciprocal importance sampling estimator, when both are evaluated relative to the same model M_0.

As shown by column 4 in Table 5, the results for posterior model probabilities (24) also agree with the RISE and DIC results, i.e. model M_0 has a very high probability of 88%, and model M_{27} has the second highest probability. If we exclude model M_0, then model M_{27} has a high probability of 68%. In summary, all three Bayesian choice criteria point to the same model M_0 as the best choice followed by model M_{27}, and these choices are in agreement with the classical likelihood ratio tests, as shown in Table 4.

5 Conditional Value-at-Risk

Consider a portfolio of six major currencies. Denote the exchange rates (USD per currency unit) for these currencies at time t by $S_i^{(t)}$, $i = 1, \ldots, 6$. Assume we hold λ_i units for the ith currency. The portfolio value at time t is then $V^{(t)} = \sum_{i=1}^{6} \lambda_i S_i^{(t)}$. The log-return for the ith currency at time $t+1$ is given by $x_i^{(t+1)} = \ln S_i^{(t+1)} - \ln S_i^{(t)}$. The portfolio loss for

one time step is then

$$
\begin{aligned}
-\delta V^{(t+1)} = V^{(t)} - V^{(t+1)} &= \sum_{i=1}^{6} \lambda_i S_i^{(t)} \left(1 - \exp\left(x_i^{(t+1)}\right)\right) \\
&= V^{(t)} \sum_{i=1}^{6} w_i \left(1 - \exp\left(x_i^{(t+1)}\right)\right), \qquad (30)
\end{aligned}
$$

where $w_i = \lambda_i S_i^{(t)}/V^{(t)}$ is the proportion of the portfolio value in currency i at time t, that is, it is the dollar weight of the ith currency. Now we wish to simulate the distribution of portfolio return

$$
Z = -\delta V^{(t+1)}/V^{(t)} = \sum_{i=1}^{6} w_i \left(1 - \exp\left(x_i^{(t+1)}\right)\right) \approx \sum_{i=1}^{6} -w_i x_i^{(t+1)}.
$$

In the present study we model the dependence of the log-returns $x_i^{(t+1)}$ by one of the t-copula models as described in the previous sections. Recall that the dof parameters and their posterior distributions are already obtained by Bayesian MCMC. To focus on the impact of copula models, we use the standard normal distribution for all the six marginals. We take the CVaR as our risk measure. Assume that a random variable Z has continuous density $f(\cdot)$ and distribution $F(\cdot)$. Given a threshold quantile level α, the CVaR above $F^{-1}(\alpha)$ is defined as

$$
CVaR_\alpha[Z] = E[Z|Z \geq F^{-1}(\alpha)] = \frac{1}{1-\alpha} \int_{F^{-1}(\alpha)}^{\infty} x f(x) dx, \qquad (31)
$$

which is the average of the losses exceeding $F^{-1}(\alpha)$. To demonstrate model impact on risk quantification, we compare CVaR of the two most likely models, M_0 and M_{27}, the best and the second best models of all 33 candidates. CVaR is calculated numerically using 10^7 Monte Carlo simulations with t-copula model parameters given in Table (4).

Table 6 shows $CVaR_{0.99}^{(M_0)}$ and $CVaR_{0.99}^{(M_{27})}$ predicted by models M_0 and M_{27} for two portfolios (defined by weights in Table 6). Note in both portfolios we have negative weights (selling the currency) and the weights in each portfolio add to 1.0. As shown in Table 6, model M_{27} underestimates the 0.99 CVaR by 16% for the first portfolio, and

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this underestimate reverses to a slight overestimate for the second portfolio, assuming the correct estimates are from model M_0. The second portfolio is only slightly different from the first – swapping the position of EUR and CHF (long/short position) in the first portfolio yields the second portfolio. The two portfolios are deliberately chosen to demonstrate that the model impact on risk quantification can be in either direction – it may be overestimation or it may be underestimation, depending on the portfolio. Thus it is important to choose the most suitable model statistically, such as by means of Bayesian model inference. Table 7 compares 0.99 CVaR prediction of model M_0 with that of model M_4, the most likely model from the (3,3) configuration, for the same two portfolios as those in Table 6. Here again the 0.99 CVaR for the first portfolio is underestimated by the incorrect model, and for the second portfolio it is overestimated.

6   Conclusion

This paper describes a Bayesian model choice methodology for t-copula models. As an illustration, altogether 33 t-copula models of six dimensions were considered: the generalized t-copula; the standard t-copula; and 31 grouped t-copula models from the complete subset of (3,3), (2,4) and (1,5) configurations. MCMC simulations under a Bayesian inference framework were performed to obtain the posterior distribution of dof parameters for all 33 t-copula models. Using historical data of foreign exchange rates as a case study, we found that Bayesian model choice based on the RISE, the DIC and the *posterior model probabilities* overwhelmingly favors the generalized t-copula model M_0. In addition, all three Bayesian choice criteria point to the same second most likely model M_{27}. These Bayesian choices are also in agreement with classical likelihood ratio tests.

The impact of model choice on the CVaR for two portfolios of six FX majors was observed to be significant.

For a comprehensive modeling of multivariate dependence in finance or insurance, there

are other issues in data analysis that should be addressed carefully, such as time-dependent correlation parameters and validation. These are not considered in the present study.

7   Acknowledgement

References

[1] Berger, J. O. (1985) *Statistical Decision Theory and Bayesian Analysis*. Springer-Verlag, New York.

[2] Breymann, W., Dias, A. and Embrechts, P. (2003) Dependence structures for multi-variate high-frequency data in finance. *Quantitative Finance* **3**, 1–14.

[3] Cairns, A. (2000) A discussion of of parameter and model uncertainty in insurance. *Insurance: Mathematics and Economics* **27**, 313–330.

[4] Chib, S. (1995) Marginal likelihood from the Gibbs output. *Journal of the American Statistical Association* **90**(432), 1313–1321.

[5] Congdon, P. (2006) *Bayesian Statistical Modelling*. John Wiley & Sons, Ltd, 2nd edn.

[6] Daul, S., De Giorgi, E., Lindskog, F. and McNeil, A. (2003) The grouped t-copula with an application to credit risk. *Risk* **16**, 73–76.

[7] Demarta, S. and McNeil, A. (2005) The t copula and related copulas. *International Statistical Review* **73**(1), 111–129.

[8] Embrechts, P., McNeil, A. and Straumann, D. (2001) Correlation and dependence in risk management: properties and pitfalls. Dempster, M. and Moffatt, H. (Eds.), *Risk Management: Value at Risk and Beyond*, pp. 176–223, Cambridge University Press.

[9] Fang, H., Fang, K. and Kotz, S. (2002) The meta-elliptical distributions with given marginals. *Journal of Multivariate Analysis* **82**, 1–16.

[10] Fruhwirth-Schnatter, S. (1995) Bayesian model discrimination and bayes factor for linear Gaussian state space models. *Journal of the Royal Statistical Society, series B* **57**, 237–246.

[11] Fruhwirth-Schnatter, S. (2004) Estimating marginal likelihoods for mixture and markov switching models using bridge sampling techniques. *The Econometrics Journal* **7**, 143–167.

[12] Gelfand, A. and Dey, D. (1994) Bayesian model choice: Asymptotic and exact calculations. *Journal of the Royal Statistical Society, series B* **56**, 501–514.

[13] Gelfand, A. and Smith, A. (1990) Sampling-based approaches to calculating marginal densities. *Journal of the American Statistical Association* **85**, 398–409.

[14] Gelman, A., Gilks, W. R. and Roberts, G. O. (1997) Weak convergence and optimal scaling of random walk Metropolis algorithm. *Annals of Applied Probability* **7**, 110–120.

[15] Gilks, W. R., Richardson, S. and Spiegelhalter, D. J. (1996) *Markov Chain Monte Carlo in practice*. Chapman & Hall, London.

[16] Green, P. (1995) Reversible jump MCMC computation and Bayesian model determination. *Biometrika* **82**, 711–732.

[17] Hastings, W. (1970) Monte Carlo sampling methods using Markov chains and their applications. *Biometrika* **57**, 97–109.

[18] Jeffreys, H. (1961) *Theory of Probability*. Oxford University Press, Oxford, 3 edn.

[19] Joe, H. (1997) *Multivariate Models and Dependence Concepts*. Chapman & Hall, London.

[20] Kass, R. and Raftery, A. (1995) Bayes factor. *Journal of the American Statistical Association* **90**, 773–792.

[21] Kass, R. E., Carlin, B. P., Gelman, A. and Neal, R. M. (1998) Markov chain Monte Carlo in practice: a roundtable discussion. *The American Statistician* **52**(2), 93–100.

[22] Lavine, M. and Schervish, M. J. (1999) Bayes factors: what they are and what they are not. *The American Statistician* **53**(2), 119–122.

[23] Luo, X. and Shevchenko, P. V. (2010) The t copula with multiple parameters of degrees of freedom: bivariate characteristics and application to risk management. *Quantitative Finance* **10**(9), 1039–1054.

[24] Luo, X. and Shevchenko, P. V. (2010) LGD credit risk model: estimation of capital with parameter uncertainty using MCMC. Preprint arXiv:1011.2827v3 available from http://arxiv.org.

[25] Mashal, R., Naldi, M. and Zeevi, A. (2003) On the dependence of equity and asset returns. *Risk* **16**, 83–87.

[26] McNeil, A. J., Frey, R. and Embrechts, P. (2005) *Quantitative Risk Management: Concepts, Techniques and Tools*. Princeton University Press, Princeton.

[27] Meng, X. and Wong, W. (1996) Simulating ratios of normalizing constants via a simple identity. *Statistical Sinica* **6**, 831–860.

[28] Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H. and Teller, E. (1953) Equations of state calculations by fast computing machines. *Journal of Chemical physics* **21**, 1087–1091.

[29] Miazhynskaia, T. and Dorffner, G. (2006) A comparison of Bayesian model selection based on MCMC with an application to GARCH-type models. *Statistical Papers* **47**, 525–549.

[30] Neal, R. M. (1993) Porbabilistic inference using Markov chain samplers. Technical report, Department of Computer Science, University of Toronto.

[31] Newton, M. and Raftery, A. (1994) Approximate Bayesian inference by the weighted likelihood bootstrap. *Journal of the Royal Statistical Society, series B* **56**, 1–48.

[32] Peters, G. W., Shevchenko, P. V. and Wüthrich, M. V. (2009) Model uncertainty in claims reserving within Tweedie's compound poisson models. *ASTIN Bulletin* **39**(1), 1–33.

[33] Peters, G. W., Wüthrich, M. V. and Shevchenko, P. V. (2010) Chain ladder method: Bayesian bootstrap versus classical bootstrap. *Insurance Mathematics and Economics* **47**(1), 36–51.

[34] Piessens, R., De Doncker-Kapenga, E., Überhuber, C. W. and Kahaner, D. K. (1983) *QUADPACK – a Subroutine Package for Automatic Integration*. Springer-Verlag.

[35] Ripley, B. D. (1987) *Stochastic Simulation*. Wiley, New York.

[36] Robert, C. P. (2001) *The Bayesian Choice*. Springer Verlag, New York.

[37] Roberts, G. O. and Rosenthal, J. S. (2001) Optimal scaling for various Metropolis-Hastings algorithms. *Statistical Science* **16**, 351–367.

[38] Schwarz, G. (1978) Estimation the dimension of a model. *Annals of Statistics* **6**, 461–464.

[39] Shevchenko, P. V. (2010) Implementing loss distribution approach for operational risk. *Applied Stochastic Models in Business and Industry* **26**(3), 277–307.

[40] Sklar, A. (1959) Fonctions de rpartition n dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris* **8**, 229–231.

[41] Spiegelhalter, D. J., Best, N. G., Carlin, B. P. and Van Der Linde, A. (2002) Bayesian measures of model complexity and fit. *Journal of the Royal Statistical Society: Series B* **64**(4), 583–639.

[42] Tanner, M. A. and Wong, W. H. (1987) The calculation of posterior distributions by data augmentation. *Journal of American Statistical Association* **82**(398), 528–540.

[43] Venter, G., Barnett, J., Kreps, R. and Major, J. (2007) Multivariate copulas for financial modeling. *Variance* **1**(1), 103–119.

[44] Wasserman, L. (1997) Bayesian model selection and model averaging. Technical report, Statistics Department, Carnegie Mellon University.
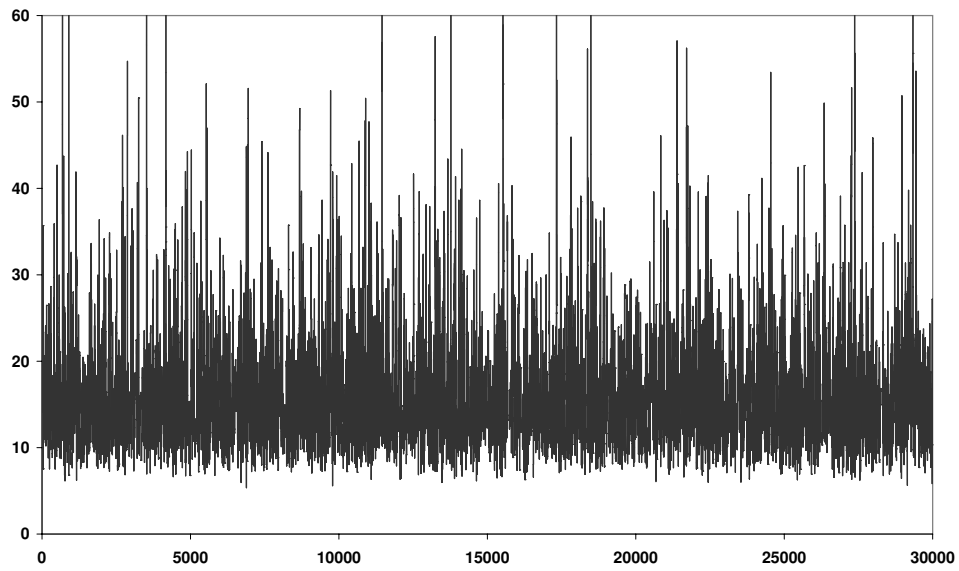
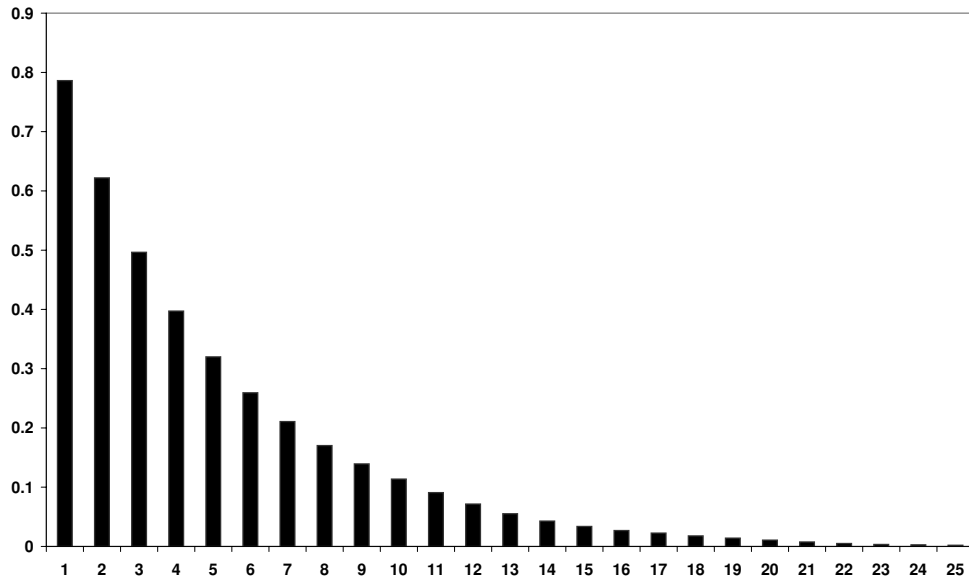Figure 1: Markov chain paths for parameter $\nu_1^{(0)}$ of model M_0.

Figure 2: Autocorrelation of Markov chain samples for dof parameter $\nu_1^{(0)}$ of model M_0.

Table 1: Group configurations and parameters for the 33 t-copula models.

Model	Group Configuration	Parameters
M_0	(AUD), (CAD), (CHF), (EUR), (GBP), (JPY)	$\nu_1^{(0)},\ \nu_2^{(0)},\ \nu_3^{(0)},\ \nu_4^{(0)},\ \nu_5^{(0)}, \nu_6^{(0)}$
M_1	(AUD, CAD, CHF), (EUR, GBP, JPY)	$\nu_1^{(1)},\quad \nu_2^{(1)}$
M_2	(AUD, CAD, EUR), (CHF, GBP, JPY)	$\nu_1^{(2)},\quad \nu_2^{(2)}$
M_3	(AUD, CAD, GBP), (CHF, EUR, JPY)	$\nu_1^{(3)},\quad \nu_2^{(3)}$
M_4	(AUD, CAD, JPY), (CHF, EUR, GBP)	$\nu_1^{(4)},\quad \nu_2^{(4)}$
M_5	(AUD, CHF, EUR), (CAD, GBP, JPY)	$\nu_1^{(5)},\quad \nu_2^{(5)}$
M_6	(AUD, CHF, GBP), (CAD, EUR, JPY)	$\nu_1^{(6)},\quad \nu_2^{(6)}$
M_7	(AUD, CHF, JPY), (CAD, EUR, GBP)	$\nu_1^{(7)},\quad \nu_2^{(7)}$
M_8	(AUD, EUR, GBP), (CAD, CHF, JPY)	$\nu_1^{(8)},\quad \nu_2^{(8)}$
M_9	(AUD, EUR, JPY), (CAD, CHF, GBP)	$\nu_1^{(9)},\quad \nu_2^{(9)}$
M_{10}	(AUD, GBP, JPY), (CAD, CHF, EUR)	$\nu_1^{(10)},\quad \nu_2^{(10)}$
M_{11}	(GBP, JPY), (AUD, CAD, CHF, EUR)	$\nu_1^{(11)},\quad \nu_2^{(11)}$
M_{12}	(AUD, CAD), (CHF, EUR, GBP, JPY)	$\nu_1^{(12)},\quad \nu_2^{(12)}$
M_{13}	(AUD, CHF), (CAD, EUR, GBP, JPY)	$\nu_1^{(13)},\quad \nu_2^{(13)}$
M_{14}	(AUD, EUR), (CAD, CHF, GBP, JPY)	$\nu_1^{(14)},\quad \nu_2^{(14)}$
M_{15}	(AUD, GBP), (CAD, CHF, EUR, JPY)	$\nu_1^{(15)},\quad \nu_2^{(15)}$
M_{16}	(AUD, JPY), (CAD, CHF, EUR, GBP)	$\nu_1^{(16)},\quad \nu_2^{(16)}$
M_{17}	(CAD, CHF), (AUD, EUR, GBP, JPY)	$\nu_1^{(17)},\quad \nu_2^{(17)}$
M_{18}	(CAD, EUR), (AUD, CHF, GBP, JPY)	$\nu_1^{(18)},\quad \nu_2^{(18)}$
M_{19}	(CAD, GBP), (AUD, CHF, EUR, JPY)	$\nu_1^{(19)},\quad \nu_2^{(19)}$
M_{20}	(CAD, JPY), (AUD, CHF, EUR, GBP)	$\nu_1^{(20)},\quad \nu_2^{(20)}$
M_{21}	(CHF, EUR), (AUD, CAD, GBP, JPY)	$\nu_1^{(21)},\quad \nu_2^{(21)}$
M_{22}	(CHF, GBP), (AUD, CAD, EUR, JPY)	$\nu_1^{(22)},\quad \nu_2^{(22)}$
M_{23}	(CHF, JPY), (AUD, CAD, GBP, EUR)	$\nu_1^{(23)},\quad \nu_2^{(23)}$
M_{24}	(EUR, GBP), (AUD, CAD, CHF, JPY)	$\nu_1^{(24)},\quad \nu_2^{(24)}$
M_{25}	(EUR, JPY), (AUD, CAD, GBP, CHF)	$\nu_1^{(25)},\quad \nu_2^{(25)}$
M_{26}	(AUD), (CAD, CHF, EUR, GBP, JPY)	$\nu_1^{(26)},\quad \nu_2^{(26)}$
M_{27}	(CAD), (AUD, CHF, EUR, GBP, JPY)	$\nu_1^{(27)},\quad \nu_2^{(27)}$
M_{28}	(CHF), (CAD, AUD, EUR, GBP, JPY)	$\nu_1^{(28)},\quad \nu_2^{(28)}$
M_{29}	(EUR), (CAD, CHF, AUD, GBP, JPY)	$\nu_1^{(29)},\quad \nu_2^{(29)}$
M_{30}	(GBP), (CAD, CHF, EUR, AUD, JPY)	$\nu_1^{(30)},\quad \nu_2^{(30)}$
M_{31}	(JPY), (CAD, CHF, EUR, GBP, AUD)	$\nu_1^{(31)},\quad \nu_2^{(31)}$
M_{32}	(AUD, CAD, CHF, EUR, GBP, JPY)	$\nu_1^{(32)}$

Table 2: Autocorrelation estimates and corresponding cut-off lag number.

Parameter	$\nu_1^{(0)}$	$\nu_2^{(0)}$	$\nu_3^{(0)}$	$\nu_4^{(0)}$	$\nu_5^{(0)}$	$\nu_6^{(0)}$
$\hat{\tau}_k$	8.79	2.23	23.5	23.1	8.41	8.69
$g_k^{\max}$	23	14	65	57	30	34

Table 3: MCMC output values of posterior mean, standard error and log-likelihood.

Model	Posterior Mean (Standard Error)			Log-likelihood
M_0	$\nu_1^{(0)} = 15.4(0.79),$ $\nu_4^{(0)} = 6.38(0.20),$	$\nu_2^{(0)} = 67.3(1.3),$ $\nu_5^{(0)} = 11.6(0.46),$	$\nu_3^{(0)} = 8.76(0.34),$ $\nu_6^{(0)} = 18.3(1.1)$	2353.1
M_1	$\nu_1^{(1)} = 15.1\ (0.29),$	$\nu_2^{(1)} = 9.37\ (0.14)$		2342.8
M_2	$\nu_1^{(2)} = 10.2\ (0.15),$	$\nu_2^{(2)} = 13.4\ (0.21)$		2338.6
M_3	$\nu_1^{(3)} = 18.2\ (0.28),$	$\nu_2^{(3)} = 8.56\ (0.09)$		2341.9
M_4	$\nu_1^{(4)} = 24.4\ (0.82),$	$\nu_2^{(4)} = 7.78\ (0.16)$		2343.8
M_5	$\nu_1^{(5)} = 8.51\ (0.13),$	$\nu_2^{(5)} = 18.7\ (0.49)$		2341.9
M_6	$\nu_1^{(6)} = 13.4\ (0.26),$	$\nu_2^{(6)} = 10.2\ (0.16)$		2338.6
M_7	$\nu_1^{(7)} = 13.6\ (0.27),$	$\nu_2^{(7)} = 10.3\ (0.17)$		2338.6
M_8	$\nu_1^{(8)} = 9.24(0.13),$	$\nu_2^{(8)} = 15.7\ (0.35)$		2343.2
M_9	$\nu_1^{(9)} = 8.76(0.13),$	$\nu_2^{(9)} = 14.0\ (0.32)$		2343.2
M_{10}	$\nu_1^{(10)} = 12.6(0.23),$	$\nu_2^{(10)} = 11.1\ (0.17)$		2336.7
M_{11}	$\nu_1^{(11)} = 27.9(4.87),$	$\nu_2^{(11)} = 8.6\ (0.09)$		2336.7
M_{12}	$\nu_1^{(12)} = 14.1(0.44),$	$\nu_2^{(12)} = 10.6\ (0.14)$		2336.7
M_{13}	$\nu_1^{(13)} = 8.56(0.11),$	$\nu_2^{(13)} = 13.8\ (0.25)$		2343.6
M_{14}	$\nu_1^{(14)} = 13.2(0.74),$	$\nu_2^{(14)} = 11.1\ (0.14)$		2336.7
M_{15}	$\nu_1^{(15)} = 13.3(0.94),$	$\nu_2^{(15)} = 11.2\ (0.17)$		2336.6
M_{16}	$\nu_1^{(16)} = 17.4(0.85),$	$\nu_2^{(16)} = 9.78\ (0.13)$		2343.0
M_{17}	$\nu_1^{(17)} = 9.78(0.14),$	$\nu_2^{(17)} = 12.9\ (0.2)$		2338.9
M_{18}	$\nu_1^{(18)} = 24.9(4.4),$	$\nu_2^{(18)} = 8.96\ (0.09)$		2342.5
M_{19}	$\nu_1^{(19)} = 32.5(7.99),$	$\nu_2^{(19)} = 8.76\ (0.08)$		2343.4
M_{20}	$\nu_1^{(20)} = 7.46(0.11),$	$\nu_2^{(20)} = 16.8\ (0.59)$		2343.0
M_{21}	$\nu_1^{(21)} = 14.2(0.43),$	$\nu_2^{(21)} = 10.5\ (0.14)$		2338.6
M_{22}	$\nu_1^{(22)} = 14.3(0.49),$	$\nu_2^{(22)} = 10.5\ (0.15)$		2338.4
M_{23}	$\nu_1^{(23)} = 8.64(0.11),$	$\nu_2^{(23)} = 14.2\ (0.32)$		2343.7
M_{24}	$\nu_1^{(24)} = 8.66(0.09),$	$\nu_2^{(24)} = 13.6\ (0.23)$		2343.4
M_{25}	$\nu_1^{(25)} = 12.8(0.50),$	$\nu_2^{(25)} = 11.2\ (0.13)$		2336.7
M_{26}	$\nu_1^{(26)} = 15.2(1.63),$	$\nu_2^{(26)} = 11.3\ (0.31)$		2336.4
M_{27}	$\nu_1^{(27)} = 64.7(3.01),$	$\nu_2^{(27)} = 9.26\ (0.24)$		2346.7
M_{28}	$\nu_1^{(28)} = 16.0(1.15),$	$\nu_2^{(28)} = 11.0\ (0.38)$		2338.4
M_{29}	$\nu_1^{(29)} = 7.94(0.32),$	$\nu_2^{(29)} = 12.9\ (0.48)$		2344.4
M_{30}	$\nu_1^{(30)} = 15.1(1.53),$	$\nu_2^{(30)} = 11.3\ (0.32)$		2336.5
M_{31}	$\nu_1^{(31)} = 11.4(0.32),$	$\nu_2^{(31)} = 1.81\ (0.17)$		2336.4
M_{32}	$\nu_1^{(32)} = 11.4\ (0.14)$			2336.8

Table 4: MCMC output values of posterior mode, corresponding log-likelihood, likelihood ratio Λ and p-value comparing M_h with M_0.

Model	MCMC posterior mode	Log-likelihood	$-2\log(\Lambda)$	p-value
M_0	$\boldsymbol{\nu}^{(0)} = (11.5, 82.4, 7.92, 5.81, 10.3, 14.3)$	2354.3	0	N/A
M_1	$\boldsymbol{\nu}^{(1)} = (14.0, \quad 8.96)$	2342.9	22.8	0.00014
M_2	$\boldsymbol{\nu}^{(2)} = (9.75, \quad 12.6)$	2338.7	31.2	<0.00001
M_3	$\boldsymbol{\nu}^{(3)} = (16.6, \quad 8.22)$	2342.1	24.4	<0.0001
M_4	$\boldsymbol{\nu}^{(4)} = (21.0, \quad 7.49)$	2344.1	20.4	0.00042
M_5	$\boldsymbol{\nu}^{(5)} = (8.17, \quad 16.9)$	2342.1	24.4	<0.0001
M_6	$\boldsymbol{\nu}^{(6)} = (12.6, \quad 9.73)$	2338.8	31.0	<0.00001
M_7	$\boldsymbol{\nu}^{(7)} = (12.8, \quad 9.79)$	2338.7	31.2	<0.00001
M_8	$\boldsymbol{\nu}^{(8)} = (8.84, \quad 14.4)$	2343.4	21.8	0.00022
M_9	$\boldsymbol{\nu}^{(9)} = (8.76, \quad 14.0)$	2343.3	22.0	0.00020
M_{10}	$\boldsymbol{\nu}^{(10)} = (11.8, \quad 10.5)$	2336.9	34.8	<0.000001
M_{11}	$\boldsymbol{\nu}^{(11)} = (22.6, \quad 8.68)$	2343.0	22.6	0.00015
M_{12}	$\boldsymbol{\nu}^{(12)} = (13.1, \quad 10.2)$	2338.6	31.5	<0.00001
M_{13}	$\boldsymbol{\nu}^{(13)} = (8.23, \quad 13.1)$	2343.7	21.3	0.00028
M_{14}	$\boldsymbol{\nu}^{(14)} = (11.9, \quad 10.7)$	2336.9	34.8	<0.000001
M_{15}	$\boldsymbol{\nu}^{(15)} = (11.7, \quad 10.8)$	2336.8	34.9	<0.000001
M_{16}	$\boldsymbol{\nu}^{(16)} = (15.7, \quad 9.39)$	2343.2	22.2	0.00018
M_{17}	$\boldsymbol{\nu}^{(17)} = (9.31, \quad 12.7)$	2339.0	30.6	<0.00001
M_{18}	$\boldsymbol{\nu}^{(18)} = (20.9, \quad 8.66)$	2342.7	23.1	0.00012
M_{19}	$\boldsymbol{\nu}^{(19)} = (25.2, \quad 8.48)$	2343.8	21.1	0.00030
M_{20}	$\boldsymbol{\nu}^{(20)} = (7.14, \quad 15.7)$	2343.1	22.3	0.00017
M_{21}	$\boldsymbol{\nu}^{(21)} = (13.2, \quad 10.1)$	2338.5	31.5	<0.00001
M_{22}	$\boldsymbol{\nu}^{(22)} = (13.3, \quad 10.1)$	2338.5	31.5	<0.00001
M_{23}	$\boldsymbol{\nu}^{(23)} = (8.27, \quad 13.4)$	2343.8	21.0	0.00031
M_{24}	$\boldsymbol{\nu}^{(24)} = (8.31, \quad 12.9)$	2343.5	21.7	0.00023
M_{25}	$\boldsymbol{\nu}^{(25)} = (11.7, \quad 10.8)$	2336.8	34.9	<0.000001
M_{26}	$\boldsymbol{\nu}^{(26)} = (11.8, \quad 10.9)$	2336.8	34.9	<0.000001
M_{27}	$\boldsymbol{\nu}^{(27)} = (68.3, \quad 9.03)$	2346.8	15.1	0.0045
M_{28}	$\boldsymbol{\nu}^{(28)} = (14.0, \quad 10.5)$	2338.6	31.4	<0.00001
M_{29}	$\boldsymbol{\nu}^{(29)} = (7.55, \quad 12.2)$	2344.5	19.6	0.00060
M_{30}	$\boldsymbol{\nu}^{(30)} = (12.0, \quad 10.9)$	2336.8	34.9	<0.000001
M_{31}	$\boldsymbol{\nu}^{(31)} = (11.4, \quad 11.0)$	2336.8	35.0	<0.000001
M_{32}	$\boldsymbol{\nu}^{(32)} = (11.1)$	2336.9	34.9	<0.00001

Table 5: Bayes factors B_{0i}, DIC and model probabilities of all candidates.

Model	Log(B_{0h})	DIC	Model prob. (%)	Model prob. (%) excl. M_0
M_0	0	0	88.5	N/A
M_1	16.7	15.8	0.15	1.28
M_2	20.1	24.0	< 0.1	< 0.1
M_3	16.6	17.5	< 0.1	0.54
M_4	16.4	13.0	0.45	3.89
M_5	18.7	17.4	< 0.1	0.55
M_6	19.3	24.1	< 0.1	< 0.1
M_7	21.5	24.3	< 0.1	< 0.1
M_8	14.5	15.0	0.12	1.07
M_9	18.9	15.1	0.22	1.89
M_{10}	21.9	27.8	< 0.1	< 0.1
M_{11}	18.0	15.6	0.15	1.26
M_{12}	24.9	24.7	< 0.1	< 0.1
M_{13}	14.9	14.3	0.31	2.68
M_{14}	23.2	27.9	< 0.1	< 0.1
M_{15}	25.5	28.0	< 0.1	< 0.1
M_{16}	14.2	15.3	0.20	1.73
M_{17}	21.7	23.7	< 0.1	< 0.1
M_{18}	15.3	16.1	0.12	1.03
M_{19}	15.1	14.1	0.29	2.52
M_{20}	15.1	15.3	0.19	1.64
M_{21}	21.5	24.3	< 0.1	< 0.1
M_{22}	20.6	24.6	< 0.1	< 0.1
M_{23}	17.1	14.1	0.36	3.10
M_{24}	15.4	14.7	0.25	2.22
M_{25}	22.6	27.9	< 0.1	< 0.1
M_{26}	22.2	28.0	< 0.1	< 0.1
M_{27}	11.0	6.73	7.8	68.3
M_{28}	21.0	24.6	< 0.1	< 0.1
M_{29}	16.3	12.7	0.7	6.18
M_{30}	28.0	28.0	< 0.1	< 0.1
M_{31}	22.8	28.1	< 0.1	< 0.1
M_{32}	22.3	28.1	< 0.1	< 0.1

Table 6: The 0.99 conditional Value-at-Risk ($CVaR_{0.99}$) predicted by model M_0 and M_{27}, for two portfolios of six major currencies. $\delta = (CVaR_{0.99}^{M_{27}} - CVaR_{0.99}^{M_0})/CVaR_{0.99}^{M_0}$ is the relative difference of CVaR between M_{27} and M_0. Standard errors are in parentheses.

Portfolio asset weights w_i (AUD, CAD, CHF, EUR, GBP, JPY)	$CVaR_{0.99}^{(M_0)}$	$CVaR_{0.99}^{(M_{27})}$	δ
(0.25, 0.25, 0.8, -0.8, 0.25, 0.25)	1.707 (0.004)	1.425 (0.003)	-16.5%
(0.25, 0.25, -0.8, 0.8, 0.25, 0.25)	1.737 (0.003)	1.782 (0.003)	2.6%

Table 7: The 0.99 conditional Value-at-Risk ($CVaR_{0.99}$) predicted by model M_0 and M_4, for two portfolios of six major currencies. $\delta = (CVaR_{0.99}^{M_4} - CVaR_{0.99}^{M_0})/CVaR_{0.99}^{M_0}$ is the relative difference of CVaR between M_4 and M_0. Standard errors are in parentheses.

Portfolio asset weights w_i (AUD, CAD, CHF, EUR, GBP, JPY)	$CVaR_{0.99}^{(M_0)}$	$CVaR_{0.99}^{(M_4)}$	δ
(0.25, 0.25, 0.8, -0.8, 0.25, 0.25)	1.571 (0.004)	1.366 (0.003)	-13.0%
(0.25, 0.25, -0.8, 0.8, 0.25, 0.25)	1.608 (0.003)	1.732 (0.003)	7.7%