

Quantum Theory, Noncommutativity and Heuristics

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Abstract

Noncommutative field theories are a class of theories beyond the standard model of elementary particle physics. Their importance may be summarized in two facts. Firstly as field theories on noncommutative spacetimes they come with natural regularization parameters. Secondly they are related in a natural way to theories of quantum gravity which typically give rise to noncommutative spacetimes. Therefore noncommutative field theories can shed light on the problem of quantizing gravity. An attractive aspect of noncommutative field theories is that they can be formulated so as to preserve spacetime symmetries and to avoid the introduction of irrelevant degrees freedom and so they provide models of consistent fundamental theories.

In these notes we review the formulation of symmetry aspects of noncommutative field theories on the simplest type of noncommutative spacetime, the Moyal plane. We discuss violations of Lorentz, P, CP, PT and CPT symmetries as well as causality. Some experimentally detectable signatures of these violations involving Planck scale physics of the early universe and linear response finite temperature field theory are also presented.

Contents

1	Introduction	12
1.1	Quantum Theory	15
1.1.1	Quantum mechanics	15
1.1.2	Quantum field theory	23
1.2	Quantization of spacetime	28
1.3	Motivation for noncommutative field theory	30
1.3.1	Phase space in quantum mechanics	31
1.3.2	Superspace in supersymmetric field theory	31
1.3.3	The center of motion of an electron in a magnetic field	31
1.3.4	Phase space of a Landau problem with a strong magnetic field	32
1.3.5	Fundamental strings and D-branes	32
1.3.6	Myers Effect	33
2	Introduction to Noncommutative geometry	38
2.1	Introduction	38
2.2	Noncommutative Spacetime	41
2.2.1	A Little Bit of History	41
2.2.2	Spacetime Uncertainties	41

2.2.3	The Groenewold-Moyal Plane	42
2.3	The Star Products	43
2.3.1	Deforming an Algebra	43
2.3.2	The Voros and Moyal Star Products	45
2.3.2.1	Coherent States	45
2.3.2.2	The Coherent State or Voros $*$ -product on the GM Plane	46
2.3.2.3	The Moyal Product on the GM Plane	50
2.3.3	Properties of the $*$ -Products	51
2.3.3.1	Cyclic Invariance	51
2.3.3.2	A Special Identity for the Weyl Star	52
2.3.3.3	Equivalence of $*_C$ and $*_W$	53
2.3.3.4	Integration and Tracial States	54
2.3.3.5	The θ -Expansion	55
2.4	Spacetime Symmetries on Noncommutative Plane	56
2.4.1	The Deformed Poincaré Group Action	57
2.4.2	The Twisted Statistics	62
2.4.3	Statistics of Quantum Fields	64
2.4.4	From Twisted Statistics to Noncommutative Spacetime . . .	70
2.4.5	Violation of the Pauli Principle	71
2.4.6	Statistical Potential	72
2.5	Matter Fields, Gauge Fields and Interactions	76
2.5.1	Pure Matter Fields	76
2.5.2	Covariant Derivatives of Quantum Fields	78
2.5.3	Matter fields with gauge interactions	79
2.5.4	Causality and Lorentz Invariance	82

2.6	Discrete Symmetries - C , P , T and CPT	85
2.6.1	Transformation of Quantum Fields Under C , P and T	85
2.6.1.1	Charge conjugation C	86
2.6.1.2	Parity P	87
2.6.1.3	Time reversal T	88
2.6.1.4	CPT	89
2.6.2	CPT in Non-Abelian Gauge Theories	90
2.6.2.1	Matter fields coupled to gauge fields	90
2.6.2.2	Pure Gauge Fields	92
2.6.2.3	Matter and Gauge Fields	92
2.6.3	On Feynman Graphs	93
3	CMB Power Spectrum and Anisotropy	98
3.1	INTRODUCTION	99
3.2	Noncommutative Spacetime and Deformed Poincaré Symmetry	101
3.3	Quantum Fields in Noncommutative Spacetime	104
3.4	Cosmological Perturbations and (Direction-Independent) Power Spectrum for $\theta^{\mu\nu} = 0$	109
3.5	Direction-Dependent Power Spectrum	114
3.6	Signature of Noncommutativity in the CMB Radiation	117
3.7	Non-causality and Noncommutative Fluctuations	121
3.8	Non-Gaussianity from noncommutativity	123
3.9	Conclusions: Chapter 3	126
4	Constraint from the CMB, Causality	128
4.1	Introduction	128

4.2	Likelihood Analysis for Noncomm. CMB	130
4.3	Non-causality from Noncommutative Fluctuations	139
4.4	Conclusions: Chapter 4	143
5	Finite Temperature Field Theory	146
5.1	INTRODUCTION	147
5.2	Review of standard theory: Sinha-Sorkin results	149
5.3	Quantum Fields on Commutative Spacetime	154
	5.3.0.1 Spacelike Disturbances	155
	5.3.0.2 Timelike Disturbances	156
5.4	Quantum Fields on the Moyal Plane	161
	5.4.1 An exact expression for susceptibility	165
	5.4.2 Zeros and Oscillations in $\tilde{\chi}_\theta^{(j)}$	167
5.5	Finite temperature Lehmann representation	168
5.6	Conclusions: Chapter 5	172
6	Conclusions	173
A	Some physical concepts	175
A.1	Motion of an electron in constant magnetic field	175
A.2	Symmetries and the least action principle	176
	A.2.1 Use of symmetries	176
	A.2.2 Analogy and least action principle	178
A.3	Renormalizability	179
A.4	Rules for writing probability amplitudes of physical processes	181
B	Quantization	183

B.1	Canonical quantization, deformation quantization and noncommu- tative geometry	183
B.1.1	Star products and regularization	188
B.2	The quantum field	195
B.3	The algebra of quantum fields	198
B.3.1	Operator product ordering and physical correlations	202
B.3.2	From Weyl or symmetric ordering to normal or classical or- dering	204
B.4	Hamilton-Jacobi theory	209
B.5	(Orbital) angular momentum and spherical functions	212
B.5.1	∂^2 in a minimally coupled system?	213
B.5.2	Spherical eigenfunctions	214
C	Variation principle and classic symmetries	221
C.1	Division of spaces	221
C.1.1	Spectrum of a group algebra	222
C.2	Gauge symmetry and Noether's theorem	223
C.3	Symmetry breaking/violation	223
C.4	Action/on-shell symmetries	225
C.5	Noether's theorem and Ward-Takahashi identities	226
C.5.1	Dynamics using differential forms	231
C.6	Faddeev-Popov gauge fixing method	232
D	Geometry and Symmetries	236
D.1	Manifold structure	236
D.2	Relativity or Observer Symmetry	237

D.3	Hopf symmetry transformations	242
	D.3.0.1 Example	245
D.3.1	Quasi-triangular Hopf algebras and R-matrix	246
D.3.2	Action	246
D.3.3	Duality and integration	247
E	Some math concepts	249
E.1	Groups, Rings (Algebras), Fields, Vector spaces, Modules	249
E.2	Set commutant algebra	250
E.3	Projector algebra	253
E.4	Matrix-valued functions and BCH formula	254
	E.4.1 Limits	254
	E.4.2 Matrix functions	255
	E.4.3 Symmetric ordered extension	258
	E.4.4 Baker-Campbell-Hausdorff (BCH) formula	259
E.5	Complex analytic transforms	260
	E.5.1 Laurent series	264
	E.5.2 Fourier series and other derived transforms	265
	E.5.3 Groups of invertible functions and related transforms	268
	E.5.4 Several variables	270
E.6	Some inequalities	271
	E.6.1 Young's inequality	271
	E.6.2 Holder's inequality	272
	E.6.3 Minkowski's inequality	273
E.7	Map continuity	273

E.7.1	Uniform continuity in terms of sets	276
E.8	Sequences and series	277
E.9	Connectedness and convexity	280
E.10	Some topology	280
E.11	More on compactness and separability	284
E.12	On the realization of compact spaces	285
E.13	Metric topology of \mathbb{R}	288
E.14	On Measures I	289
E.14.1	Measurability	292
E.15	On Measures II	292
E.15.1	Haar measure: existence and uniqueness	296
E.15.2	Invariant linear maps	299
F	C^*-algebras	301
F.1	Cauchy-Schwarz inequality	301
F.2	Hilbert space and operator norm	302
F.3	Convex subspaces	309
F.3.1	States of a $*$ -algebra	310
F.4	Spectral theory	311
F.4.1	Gelfand-Mazur theorem	313
F.4.2	Gelfand-Naimark theorem	313
F.5	Ideals and Identities	315
F.6	GNS construction	319
F.7	Algebra Homomorphisms (Representations)	320
F.8	Geometry/algebra dictionary	321

G Sets and Physical Logic	322
G.1 Exclusive sets	322
G.1.1 Conditional algebra	324
G.1.2 Maps and bundling	325
G.1.3 Counting isomorphisms ?	328
G.2 Nonexclusive sets: Generalizations	328
G.2.1 G1	329
G.2.2 G2	331
G.3 Physics: The logic of quantum theory	336
G.3.1 Coordinate types	344
G.3.2 On Gravity	346
G.3.3 Projectors on Self Hilbert Spaces	348
G.4 Primitivity: The logic of human society	349
 Bibliography	 350

List of Figures

2.1	Statistical potential $v(r)$ measured in units of $k_B T$. An irrelevant additive constant has been set zero. The upper two curves represent the fermionic cases and the lower curves the bosonic cases. The solid line shows the noncommutative result and the dashed line the commutative case. The curves are drawn for the value $\frac{\theta}{\lambda^2} = 0.3$. The separation r is measured in units of the thermal length λ . [63]	75
2.2	A Feynman diagram in QCD with non-trivial θ -dependence. The twist of $\mathcal{H}_{I_0}^{M,G}$ changes the gluon propagator. The propagator is different from the usual one by its dependence on terms of the form $\vec{\theta}^0 \cdot \mathbf{P}_{in}$, where $(\vec{\theta}^0)_i = \theta^{0i}$ and \mathbf{P}_{in} is the total momentum of the incoming particles. Such a frame-dependent modification violates Lorentz invariance.	82
2.3	CPT violating processes on GM plane. (1) shows quark-gluon scattering with a three-gluon vertex. (2) shows a gluon-loop contribution to quark-quark scattering.	94
4.1	Transfer function Δ_l for $l = 10$ as a function of k . It peaks around $k = 0.001 \text{ Mpc}^{-1}$	133

4.2	Transfer function Δ_l for $l = 800$ as a function of k . It peaks around $k = 0.06 \text{ Mpc}^{-1}$	134
4.3	CMB power spectrum of ΛCDM model (solid curve) compared to the WMAP data (points with error bars).	135
4.4	The values of k which maximize $\Delta_l(k)$, as a function of l	136
4.5	χ^2 versus $H\theta$ for ACBAR data	137
4.6	The amount of causality violation with respect to the relative orientation between the vectors $\vec{\theta}^0$ and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. It is maximum when the angle between the two vectors is zero. Notice that the minima do not occur when the two vectors are orthogonal to each other. This plot is generated using the Cuba integrator [117].	142

Chapter 1

Introduction

Quantum theory and the theory of general relativity do not appear to be compatible at very short distance scales due to the following argument. One generally expects that at very short length scales the general relativistic theory of gravity needs to become a quantum field theory due to the high energies that are required to probe such short distances. However, standard quantization methods do not suffice because the quantization of classical gravity theories results in quantum theories lacking in renormalizability which is one of the requirements for a consistent fundamental quantum field theory.

In quantum field theory (QFT) renormalization is an attempt to understand the physical reasons for the UV or short distance divergences that occur in the naturally expected contributions of energetically unrestricted intermediate processes to the potential or probability amplitude of a given energetically restricted physical process in spacetime. Renormalization procedures naturally start with some kind of regulator, a set of regularizing parameters, followed by the isolation of regulator dependent contributions into finite and purely divergent pieces. A theory is said to

be renormalizable if the divergences can be understood with a finite regulator; one containing a manageable number of regularizing parameters, without the need of introducing or allowing an arbitrarily large number of extra fundamental degrees of freedom, otherwise the theory is inconsistent and is said to be nonrenormalizable.

A physical process, an isolation of part of the course of dynamics of a physical system, is one whose potential survives any induced and intrinsic isomorphic transformations, ie. symmetries, of both spacetime and the spaces of configurations or auxiliary variables of the system in spacetime. The potential depends on the configuration variables and on the way these variables couple in the classical action that describes the dynamics of the system through a least action principle. The way the variables couple is in turn determined by symmetries. Our configuration variables shall be fields which include matter or half integer spin fields and gauge or integer spin fields which are thought to mediate fundamental interactions between the matter fields. Symmetries may be separated into nonlocal and local symmetries. Nonlocal symmetries are homogeneous in that the value of the transformation parameter is the same at each point or infinitesimal region of spacetime and/or spaces of configurations. The definition of a gauge field allows it to have some physically irrelevant components. Gauge symmetries are special local symmetries often used as a tool or standard for tracking the number of irrelevant components in a gauge field in addition to their normal use as symmetries; to determine how gauge fields couple among themselves, and to other fields, in the classical action. The use of gauge symmetries to determine the coupling of gauge fields is due to the assumption that they should correspond to some global symmetry when their transformation parameters are made homogeneous and vice versa.

A finite regulator may or may not survive all of the symmetries of a quan-

tum theory. Anomalies are unexpected (nonsymmetric) contributions, from the intermediate processes, which are found to be due to the nonexistence of a finite regulator that can survive all symmetries of the action for the underlying theory. The anomalies can be presented as the failure of a conserved (Noether) current of a symmetry of the classical action to remain conserved after quantization. The underlying theory in this case is said to be anomalous. Renormalization by definition must also account for the anomalies as well. Following symmetries anomalies may be global or local. Global anomalies do not introduce any extra degrees of freedom and so do not spoil renormalizability. However the theory will be nonrenormalizable if the gauge anomalies from all possible intermediate processes do not sum to zero. This is because the unphysical degrees of freedom that the gauge symmetry represents will contribute to a supposedly physical intermediate process implying an inconsistency. The nonrenormalizability of the quantized version of any classically successful theory such as the theory of gravity indicates that such a theory is only an effective theory that can be obtained in the classical limit of a more fundamental quantum theory. Theories on noncommutative spacetime come with a natural symmetry surviving regulator and can therefore serve as bases for testing consistent quantum theories of gravity.

We will review quantum theory and quantization of spacetime in this chapter. In chapter 2, mostly [119] with minor changes, we will review quantum field theory on the Moyal plane and some of its physical implications including results of investigations on discrete spacetime symmetries and locality. Chapter 3, mostly [99], involves a theoretical model for a possible effect of noncommutativity on the CMB power spectrum meanwhile chapter 4, mostly [98], presents results on the analysis of possible effects of noncommutativity from anisotropy in the CMB radiation. Investigations on causality violating effects in finite temperature field theory appear

in chapter 5, mostly [122]. Chapter 6 is the concluding chapter and the appendices contain indispensable information that is mostly in heuristic form.

1.1 Quantum Theory

A classical theory, in the description of a physical system, assumes that any underlying characteristic of the physical system can undergo only (deterministic) continuous changes. Noncontinuous changes (which can be nondeterministic) are attributed to statistically averaged characteristics, of a given physical system placed, in an ensemble (ie. a large collection) of physical systems.

Quantum theory involves extensions, of the classical theoretical description of a physical system, in which some of the underlying characteristics of the physical system instead undergo noncontinuous changes (which may be deterministic, nondeterministic or partially deterministic). The classical description can be obtained from the quantum description in the limit where the noncontinuous changes are small enough to be approximately considered as continuous changes.

The effects of noncontinuous changes are expected to be observed when the system is involved in high energy interactions, where dissociations are most likely to occur.

1.1.1 Quantum mechanics

Mechanics describes the characteristic changes of a given mechanical system (any physical system involved in mostly nondestructive interactions). Quantum mechanics focuses on an extension of the classical mechanical description to include also those underlying characteristics (electrical charge, radiative energy, angular

momentum, etc) of the mechanical system that undergo noncontinuous changes.

Quantum mechanics resulted from efforts that either predicted or explained observed phenomena such as the energy distribution in a black body's spectrum, the photoelectric effect, the Compton effect, electron diffraction, atomic spectra, etc. Early quantization ideas were presented by Planck, Einstein, Bohr, De Broglie, Hiesenberg and Schrodinger.

Planck had to assume that the *blackbody* consisted of oscillators that could emit or absorb energy only in fixed amounts ε that needed to depend linearly on the frequency only. That is $\varepsilon = hf$, where h is a constant. Similarly Einstein in order to explain the *photoelectric effect* (the ejection of electrons from the light-illuminated surface of a metal, with the kinetic energy of the electrons depending linearly on frequency but not on the intensity of the light) assumed that the energy of light was quantized (distributed in space as localized lumps each of which can be produced, transported or absorbed only as a whole) so that the energy of a particle of light may be written as $E = hf$ and hence deduced a corresponding momentum with $|\vec{p}| = \frac{h}{c}f = \frac{h}{\lambda}$. Thus the wave phase of light could then be rewritten as $e^{2\pi i(ft - \vec{k} \cdot \vec{x})} = e^{i\frac{2\pi}{h}(Et - \vec{p} \cdot \vec{x})}$, in terms of its particles' states (\vec{x}, \vec{p}) ,

$\vec{p} = h\vec{k} = \frac{h}{\lambda}\hat{v} = \frac{h}{\lambda}\frac{\vec{x}}{|\vec{x}|}$, or $(x^\mu, p^\mu) \equiv (ct, \vec{x}, \frac{E}{c}, \vec{p})$. Since the energy and momentum of a massive free particle are related by $E^2 = \vec{p}^2c^2 + m^2c^4$ the particle of light is therefore a massless free particle. De Broglie postulated that the wave phase relation be applied also to massive free particles $E^2 = \vec{p}^2c^2 + m^2c^4$ in which case these particles should also display wave-like properties with

$f = \frac{E}{h}$, $\lambda = \frac{h}{|\vec{p}|}$, $\vec{p} = \frac{E}{c^2}\vec{v}$. This was confirmed in *electron diffraction* experiments. It was then straightforward to write down "wave" or "field" equations (eg. the nonrelativistic Schrodinger equation $(i\partial_t + \frac{1}{2m}\partial^2 - V(\vec{x}, t))\psi(\vec{x}, t) = 0$) for massive particles in an external potential $V(\vec{x}, t)$, where a "field" $\psi(\vec{x}, t)$ is a superposition

or linear sum of “waves”. Light quantization also explains the *Compton effect*: the observed shift in wavelength of light when it scatters off free electrons.

On the other hand, it was realized by Bohr and others that it is not possible to map out a clear path or orbit for the electron in an atom. In the continuum theory, the Fourier transform of the electron’s electric dipole moment eq predicted a continuous frequency spectrum for radiation with the Fourier coefficients of eq giving the intensities associated with each radiated frequency. However, the observed frequencies were discrete implying that the Fourier representation was not an appropriate way to represent eq . A matrix representation was finally chosen by Heisenberg and others as an appropriate representation for eq , where the components of the matrix may be interpreted as “transition probabilities” among the discrete frequencies in analogy to the classical Fourier coefficients which were normally interpreted as radiation intensities associated with the continuous frequencies.

Empirical results in *atomic spectroscopy*, eg. Rydberg’s wavelength formula $\frac{1}{\lambda_{ij}} = \frac{R}{n_i} - \frac{R}{n_j}$ where n_i, n_j are integers and R a constant, indicate that the energy levels of an electron in a physical atom may be represented by the eigenvalues of a matrix called Hamiltonian H . The Hamiltonian H is a matrix-valued “function” of equally matrix-valued ¹ observable ² quantities q, p that represent the canonical position and momentum from classical Hamiltonian mechanics. The relations may be expressed as follows

¹instead of a Fourier sequence

²In “observable” or “measurable”, measurement of a quantity U refers to an assignment of a number to the quantity U . An observable will randomly take on one value of its spectrum each time it is measurement.

$$H = H(p, q, t) = (H_{mn}), \quad (1.1.1)$$

$$\frac{dF}{dt} = \frac{i}{\hbar}(HF - FH) + \frac{\partial F}{\partial t}, \quad F = F(p, q, t) = (F_{mn}), \quad (1.1.2)$$

$$qp - pq = i\hbar, \quad q = q(t) = (q_{mn}), \quad p = p(t) = (p_{mn}), \quad (1.1.3)$$

$$H\psi_\nu = \hbar\nu \psi_\nu \quad (\text{Eigenvalue problem for the matrix } H), \quad (1.1.4)$$

$$H = S\Lambda S^{-1}, \quad \Lambda_{mn} = \hbar\omega_m\delta_{mn}, \quad \omega_m = 2\pi\nu_m, \quad (1.1.5)$$

where the commutator $[H, F] = HF - FH$ may be interpreted as a quantum mechanical analogue of the classical Poisson bracket $\{h, f\} = \partial_q h \partial_p f - \partial_p h \partial_q f$.

The equations above come from an empirically deduced form for the coordinate q given by

$$\begin{aligned} q_{mn}(t) &= q_{mn}^0 e^{i\omega_{mn}t}, \quad \omega_{mn} = 2\pi(\nu_m - \nu_n), \\ \frac{dq_{mn}(t)}{dt} &= \frac{i}{\hbar}(\Lambda q - q\Lambda)_{mn} = i\omega_{mn} q_{mn}(t). \end{aligned} \quad (1.1.6)$$

where $\nu_{mn} = \nu_m - \nu_n$ is the frequency of a photon emitted by an electron that “drops” from a higher energy level m to a lower energy level n (the energy of the photon is $h\nu$). The canonical quantization conditions

$[q_i, p_j] = i\hbar\delta_{ij}$, $[p_i, p_j] = [q_i, q_j] = 0$ are an extension (see eqn (B.1.9) of appendix B.1) of the Bohr-Sommerfeld quantization³ condition

$$\oint_C p_i dq^i = nh. \quad (1.1.7)$$

³This model considers (planar) elliptical rather than (planar) circular orbits of Bohr’s model for the electronic orbits of Hydrogen. This quantization condition is merely an additional constraint (to the usual classical equations of motion) imposed in order to obtain a discrete rather than a continuous set of orbits, energies, angular momenta and related quantities. It may also be written as $\oint_C (p_j dq^j - q_j dp^j) = 2nh$ or as $\oint_C \bar{z}_j dz^j = 2n\hbar i$, $z_j = q_j + ip_j$.

The time evolution equation (1.1.2) generates a one parameter time translation group $\{e^{iHt}\}$ with the Hamiltonian H as the sole generator. The spectrum (from the eigenvalue problem $H\psi_\nu = h\nu\psi_\nu$ for H) of H is preserved by this time translation symmetry and consequently each atom has a unique emission or absorption spectrum that characterizes (or serves as a thumbprint for) the type of chemical element the atoms of that type produce. The eigenvalue problem for $H = H(\vec{q}, \vec{p})$ may be seen as the problem of finding the irreducible representations of the one parameter time translation group and so each frequency represents an irreducible or elementary attributes (a single excitation, or energy, level of an electron of the atom) of a non-rotating atomic electron system. Naturally, the electron system can be free to rotate around or relative to the nucleus in which case we have invariance under the time translation plus rotation group whose irreducible representations would give the elementary attributes of the system. The canonical quantization condition for a system with several canonical degrees of freedom is $[q_i, p_j] = i\delta_{ij}\hbar$, $[q_i, q_j] = [p_i, p_j] = 0$. For a system with Hamiltonian $H = H(\vec{p}^2, \vec{p} \cdot \vec{q}, \vec{q}^2)$ and angular momentum $L_{ij} = \frac{1}{2}(q_i p_j - q_j p_i)$, H commutes with L_{ij} and $\{H, L^2 = L_{ij}L_{ij}\}$ generate the center of the algebra of the symmetry group.

$$[L_{ij}, L_{kl}] = -\frac{1}{2}(\delta_{ik}L_{jl} + \delta_{jl}L_{ik}) + \frac{1}{2}(\delta_{il}L_{jk} + \delta_{jk}L_{il}). \quad (1.1.8)$$

All parts of the atomic system can also be displaced by the same amount in “free” space without disturbing the spectrum of the atomic system. Thus one needs to consider a Hamiltonian of the form $H = \sum_{ab} h(\vec{p}_a \cdot \vec{p}_b, \vec{p}_a(\vec{q}_a - \vec{q}_b), (\vec{q}_a - \vec{q}_b)^2)$ where a, b label the various pieces or particles of the system. Then H also commutes with the total momentum operator $\vec{P} = \sum_a \vec{p}_a$ which is the generator of spatial

translations. The canonical commutation relations are

$$[q_a^i, p_b^j] = i\delta^{ij}\delta_{ab}\hbar, \quad [q_a^i, q_b^j] = [p_a^i, p_b^j] = 0 \quad (1.1.9)$$

and the angular momentum operator will be the sum of the individual ones:

$$\begin{aligned} L^{ij} &= \sum_a L_a^{ij} = \sum_a \frac{1}{2}(q_a^i p_a^j - q_a^j p_a^i) = \frac{1}{2}(Q^i P^j - Q^j P^i), \\ \vec{Q} &= \sum_a \vec{q}_a. \end{aligned} \quad (1.1.10)$$

The center of the algebra of the symmetry group of the atomic system is now generated by (H, L^2, \vec{P}) . At this point one realizes that the problem of quantizing the atomic system includes the problem of finding the irreducible representations of its symmetry group (or equivalently of the algebra of the symmetry group) generated by H, P^i, L^{ij} . To include relativistic effects, one needs to replace the (spatial rotation plus spatial translation plus time translation) group with the Poincare group (spacetime rotation plus translation group). Then relativistic quantum mechanics involves the problem of finding the spectrum of the center of the group generated

by the operators $\mathcal{P}^\mu, J^{\mu\nu}$ which have the canonical representation

$$\begin{aligned}
\mathcal{P}^\mu &= (\mathcal{P}^0(H), \vec{\mathcal{P}}(\vec{P})), \quad \mathcal{Q}^\mu = (\mathcal{Q}^0(Q^0), \vec{\mathcal{Q}}(\vec{Q})), \\
\mathcal{P}^0(H) &= H = H(\gamma^0, \vec{\gamma}, t, \vec{Q}, \vec{P}), \quad \vec{\mathcal{P}}(\vec{P}) = \vec{P}, \\
\mathcal{Q}^0(Q^0) &= Q^0, \quad \vec{\mathcal{Q}}(\vec{Q}) = \vec{Q}, \\
J^{\mu\nu} &= \frac{1}{2}(\mathcal{Q}^\mu \mathcal{P}^\nu - \mathcal{Q}^\nu \mathcal{P}^\mu) + \frac{i}{4}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) + \dots \equiv L^{\mu\nu} \otimes 1_S + 1_L \otimes S^{\mu\nu} + \dots, \\
&= J_L^{\mu\nu} + J_S^{\mu\nu}, \\
[J_{\mu\nu}, J_{\alpha\beta}] &= -\frac{1}{2}(\eta_{\mu\alpha} J_{\nu\beta} + \eta_{\nu\beta} J_{\mu\alpha}) + \frac{1}{2}(\eta_{\mu\beta} J_{\nu\alpha} + \eta_{\nu\alpha} J_{\mu\beta}), \\
[J_L^{\mu\nu}, J_S^{\alpha\beta}] &= 0, \\
[P^\mu, Q^\nu] &= i\eta^{\mu\nu}, \quad Q^\mu = (Q^0, \vec{Q}), \\
\{\gamma^\mu, \gamma^\nu\} &= 2\eta^{\mu\nu}.
\end{aligned} \tag{1.1.11}$$

In the Schrodinger representation $Q^\mu \rightarrow \mu_{x^\mu}$, $P_\mu \rightarrow i\frac{\partial}{\partial x^\mu}$ (here μ_{x^μ} denotes ordinary multiplication by the spacetime coordinates x^μ), one then has the consistency condition

$$i\frac{\partial}{\partial t} = H(\gamma^0, \vec{\gamma}, t, \vec{x}, i\frac{\partial}{\partial \vec{x}}) \tag{1.1.12}$$

on the space of sections $E/\mathbb{R}^{d+1} = \{\psi : \mathbb{R}^{d+1} \rightarrow E \simeq O(\mathbb{C}^M \otimes \mathbb{C}^N) \times (\mathbb{C}^M \otimes \mathbb{C}^N)\}$ of a vector bundle E over \mathbb{R}^{d+1} where $\psi = \psi_L \otimes \psi_S$ is the product of the orbital and spin angular momentum wavefunctions and $O(\mathbb{C}^M \otimes \mathbb{C}^N)$ is the space of linear operators on $\mathbb{C}^M \otimes \mathbb{C}^N$.

Even though it is not possible to say precisely where the atomic electron's orbit is, it is however possible to say that it is mostly around the nucleus of the atom; that is, the electron's orbit is localized in the region around the nucleus. A basic quantity introduced for the study of localization⁴ was Schrodinger's wave-

⁴A system is localized in a certain region \mathcal{D} at a particular time if the total probability of

function in wave mechanics which is any function satisfying the consistency condition (1.1.12). Schrodinger's wave mechanics is equivalent to Heisenberg's matrix mechanics which was discussed earlier. In general the wave function is a complex-valued function(al) of the quantized configuration variables such as canonical coordinate in quantum mechanics or fields in quantum field theory, whose absolute value can be interpreted as a joint probability density function for the quantized canonical variables on which it depends.

When the quantum, ie. quantized classical, configuration variables are represented as elements of an algebra $\mathcal{O}(\mathcal{H})$ of operators on a Hilbert space⁵ \mathcal{H} then the wavefunction would be the value of a chosen linear functional⁶ on the quantum configuration variable in question. Thus the time evolution equation may also be written either in terms of the wavefunction or in terms of a corresponding vector in the Hilbert space \mathcal{H} . The time evolution equation in terms of the wavefunction is known as Schrodinger's equation. More specifically the sole irreducible representation, up to unitary equivalence, of the relations (1.1.1) through (1.1.5) on a Hilbert space is known as Schrodinger's representation.

finding it in \mathcal{D} at that time is 1. Alternatively, the region \mathcal{D} is dense in the support of the probability density function of the system.

⁵A Hilbert space is a vector space completed into a metric space by a norm that is induced by an inner product measure defined on the vector space.

⁶Wavefunctions of physical systems and probability amplitudes for various physical processes are examples of (values of) linear functionals on $\mathcal{O}(\mathcal{H})$. The wavefunction for a physical system is a time-dependent linear functional whose value on a given quantum configuration is the probability amplitude for finding the system in that quantum configuration and it satisfies Schrodinger's equation.

1.1.2 Quantum field theory

Quantum field theory is a relativistic quantum theory of systems with arbitrary numbers and types of degrees of freedom. Quantum mechanics treats a system of N (interacting) particles using a fixed number and type of N (coupled) equations. However not all interacting systems have a fixed number and species of particles. Particle transformations and relativistic quantum effects such as particle creation and annihilation may occur. Particles of a kind are now regarded as localizable disturbances (ie. perturbations or fluctuations) in a field of that kind. In particular the field description treats elementary particles as (Fourier) modes of the oscillatory part of an associated field in direct analogy to the electromagnetic field, the modes of whose oscillatory part correspond to the various frequencies of the electromagnetic spectrum. One has an analog of the canonical quantization condition;

$$\begin{aligned}
\vec{q}_n(t) &\rightarrow q_p^\alpha(t) = \sum_{\vec{x}} \psi^\alpha(\vec{x}, t) u_p(\vec{x}, t), \\
\vec{p}_n(t) &\rightarrow \pi_p^\alpha(t) = \sum_{\vec{x}} \Pi^\alpha(\vec{x}, t) u_p(\vec{x}, t), \\
q_p^\alpha(t) \pi_{p'}^\beta(t) - (-1)^{2s} \pi_{p'}^\beta(t) q_p^\alpha(t) &= i\hbar \delta^{\alpha\beta} \delta_{pp'}, \\
\sum_p u_p^*(\vec{x}, t) u_p(\vec{y}, t) &= \delta^3(\vec{x} - \vec{y}), \\
\Pi^\alpha(\vec{x}, t) \psi^\beta(\vec{y}, t) - (-1)^{2s} \psi^\beta(\vec{y}, t) \Pi^\alpha(\vec{x}, t) &= i\hbar \delta^{\alpha\beta} \delta(\vec{x} - \vec{y}), \\
\Pi(\vec{x}, t) &= \frac{\partial \mathcal{L}}{\partial \partial_t \psi}(\vec{x}, t), \\
S[\psi] &= \int \mathcal{L}(x, dx, \psi, d\psi), \tag{1.1.13}
\end{aligned}$$

where n is a discrete label for a collection of particles and the value of x needs to be chosen in such a way as to obtain a consistent theory for the field ψ . For example

Pauli's exclusion principle⁷ requires that s be a half integer for matter fields and integer for interaction mediation fields. p is a characteristic or typical value (an eigenvalue for a corresponding momentum operator as a Noether charge associated with translational invariance) for the momentum of an individual mode. This is because \vec{q}_1 and \vec{q}_2 (corresponding to $q_{p_1}^\alpha$, $q_{p_2}^\alpha$) denote different positions in space. α is a "spin" index which is an extension of the spatial vector index. The differential action or Lagrangian $\mathcal{L}(x, dx, \psi, d\psi)$ is a differential form on spacetime.

Thus an individual mode is described by the triple $(q_p^\alpha(t), \pi_p^\alpha(t), u_p(\vec{x}, t))$, where $|u_p(\vec{x}, t)|^2 d^3x$ is the probability of finding the mode in an infinitesimal neighborhood of \vec{x} of volume d^3x at any given time t . This means that the role of the point \vec{x} is now being played by the linear functional

$u_p : (q_p^\alpha(t), \psi_\alpha(\vec{x}, t)) \mapsto u_p(\vec{x}, t) = \langle q_p^\alpha(t) \psi_\alpha(\vec{x}, t) \rangle$. The field ψ^α can also be directly interpreted as the particle coordinate, where the particle is constrained to move along a time-parametrized path $\vec{q} : [0, 1] \rightarrow \mathcal{C}$ of the configuration space

$\mathcal{C} = \bigcup_n \mathcal{C}_n \equiv \bigcup_n \{\vec{q}_n\}$ in many particle quantum mechanics meanwhile the particle is constrained to move along a spacetime-parametrized hypersurface

$\psi^\alpha : \mathcal{M} \simeq ([0, 1]^4, g) \rightarrow \mathcal{U} = \bigcup_{x \in \mathcal{M}} \{\psi^\alpha(x)\} \subset \mathcal{C}$ of the configuration space

$\mathcal{C} = \bigcup_p \{q_p^\alpha\}$ in quantum field theory and similarly the particle is constrained to move along a (σ, τ) -parametrized two dimensional surface

⁷The exclusion principle associates the shell structure of atomic electron systems, space occupying/shape forming properties of matter, stability of astronomical objects such as neutron stars, etc to the difficulty for two elementary matter systems to have exactly the same set of fundamental quantum labels. Electromagnetic fields for example and other force fields do not appear to exhibit these properties. The exclusion principle is connected to the idea of spin angular momentum by the requirement that the probability amplitude of a composite physical process must be a rotationally invariant/covariant functional of the probability amplitudes for the individual elementary processes of which it is composed.

$X^\alpha : ([0, 1]^2, h) \rightarrow \mathcal{C} = \mathbb{R}^{d+1}$ in string theory.

In quantum field theory the role originally played by the Hamiltonian H alone in quantum mechanics is now played by the 4-momentum operator

$P_\mu = (H, \vec{P})$ (A component $T^{\mu 0}$ of the Energy-momentum tensor

$T^{\mu\nu} = \int d^3x \mathcal{T}^{\mu\nu}$, $\partial_\mu \mathcal{T}^{\mu\nu} = 0$, a Noether charge corresponding to spacetime translation symmetry). The eigen-value problem for H , and any other quantities that commute with H , is replaced by the problem of finding the solutions U of the equation

$U(\Lambda_1, b_1)U(\Lambda_2, b_2) = U(\Lambda_1\Lambda_2, b_1 + \Lambda_1 b_2)$ which is Wigner's method of classifying elementary particle states. That is, finding the irreducible representations of the Lorentz-Poincare transformation

$LP : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^{3+1}$, $x \mapsto \Lambda x + b$, $\Lambda^T = \Lambda^{-1}$, $x = (x_\mu) = (x_0, \vec{x})$, the automorphism or symmetry group of the spacetime \mathbb{R}^{3+1} . The irreducible representations correspond to free elementary point particles that can be localized in \mathbb{R}^{3+1} . In addition to reparametrization symmetry the Lorentz-Poincare transformation is a symmetry and thus a canonical transformation⁸ of the relativistic point particle action

$$S[x, \Gamma] = m \int_\Gamma \sqrt{\eta_{\mu\nu} dx^\mu dx^\nu} \quad (1.1.14)$$

since this Lagrangian involves only the metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ which is the defining structure of the Minkowski spacetime.

Thus given any space \mathcal{S} , one can also consider the problem of finding the irreducible representations of the automorphism group $G(\mathcal{S}) : \mathcal{S} \rightarrow \mathcal{S}$ of \mathcal{S} so as to be able to characterize/classify all the possible elementary physical systems that can be localized in \mathcal{S} . Examples of spaces include topological metric spaces, man-

⁸Section B.1

ifolds (which also include Lie groups), fiber bundles, etc and other spaces derived from these using various mathematical constructs. Here it is also important to note that the symmetry group of a topologically nontrivial⁹ space (as compared to the flat spacetime \mathbb{R}^{3+1}) is “enlarged” mainly due to additional discrete transformation channels leading to various periodicity types and therefore one expects additional distinct physical properties induced on the elementary systems in \mathcal{S} by its nontrivial topology. Conversely, if the elementary systems in \mathcal{S} are observed to display unexpected additional properties, say through experimentation, that do not seem to depend on the geometry, ie. shape/size structures, on \mathcal{S} then they may be investigated by introducing nontrivial topology. Ways of introducing nontrivial topology include employing nondynamical constraints (like quotienting of a [topologically trivial] space by the actions of [discrete] transformation groups) as well as dynamical constraints such as postulating the presence of unknown forms of “elementary” systems that can couple to the known elementary systems in a way that can explain the additional properties and also gives possible explanations as to whether the unknown forms of elementary systems could be experimentally detectable or not.

For example, consider the variational problem for an electron (considered as the less physically realistic case, a point particle, so that it can only trace 1-dimensional paths) with action $S[q] = \int_0^1 L(t, dt, q, dq)$, $q : [0, 1] \rightarrow \mathbb{R}^{3+1}$. If there is a very strong magnetic field confined in a thin infinitely long tube through the space \mathbb{R}^{3+1} , then since any electron (and hence its path) with insufficient energy cannot

⁹A space is topologically nontrivial if any two of its subspaces cannot always be continuously deformed into each other. Topology is the study of invariance under continuous shape change or deformation (ie. geometry) transformations. Physically interesting geometries would be the fixed points of these geometry transformations.

penetrate this tube, it means that for such an electron the variational problem will have more than one solution as a path on one side of the magnetic tube cannot be continuously varied to a path on the opposite side of the tube. For the same reason a path that wraps around the tube n times cannot be varied to a path that wraps around it any m times in the opposite sense or $m \neq n$ times in the same sense. Hence to every path is associated an integer parameter labeling the number of times and sense in which its path winds around the tube. Therefore if identical electrons of insufficient energy are produced at some point and later interact then one expects to observe the effect of the difference in the topological charges (winding numbers) they gained during their individual journeys. This effect may be included in the action by adding a non trivial but smooth path deformation independent term

$$\begin{aligned}
\nu[q] &= \int_0^1 B_i dq^i, \\
\delta\nu[q] &= \delta\left(\int_0^1 B_i dq^i\right) = \int_0^1 (\delta B_i dq^i + B_i \delta dq^i) = \int_0^1 (\delta q^i \partial_i B_j dq^j + B_i \partial_j (\delta q^i) dq^j) \\
&= \int_0^1 \delta q^i (\partial_i B_j - \partial_j B_i) dq^j + \int_0^1 \partial_i (B_j \delta q^j) dq^i \\
&= \int_0^1 \delta q^i (\partial_i B_j - \partial_j B_i) dq^j + [B_j \delta q^j]_0^1 = 0, \tag{1.1.15}
\end{aligned}$$

where the path $q(t)$ can be smoothly deformed to the path $q(t) + \delta q(t)$. That is, smooth path deformation independence requires $dB = 0$ in the region between between any two paths, with common end points, that can be continuously deformed into each other, $\oint_{\Gamma} B = n(\Gamma) \in \mathbb{Z}$, $B = B_i dq^i$. Alternatively, let Γ_+ (Γ_-) be the path oriented from $t = 0$ to $t = 1$ ($t = 1$ to $t = 0$), $(\Gamma + \delta\Gamma)_+$ be the varied (with end fixed $\delta q(0) = 0 = \delta q(1)$) path oriented from $t = 1$ to $t = 0$ and Γ be the

closed path $(\Gamma + \delta\Gamma)_+ + \Gamma_-$. Then Stokes' theorem implies that

$$\begin{aligned} \delta\nu[q] &= \nu_{(\Gamma+\delta\Gamma)_+}[q] - \nu_{\Gamma_+}[q] = \nu_{(\Gamma+\delta\Gamma)_+}[q] + \nu_{\Gamma_-}[q] = \nu_{(\Gamma+\delta\Gamma)_+ + \Gamma_-}[q] \\ &= \nu_{\Gamma}[q] = \oint_{\Gamma} B = \int_{\text{int}(\Gamma)} dB. \end{aligned} \tag{1.1.16}$$

Therefore, $\delta\nu[q] = 0$ unless the variation takes the path across the tube since $dB|_{\mathbb{R}^{3+1} \setminus \text{tube}} = 0$. B may be normalized so that any non-zero contribution from $\nu[q]$ is an integer.

One notes obviously that B (as well as the tube) can also be a dynamical field. The configuration space of the electron is $\mathcal{S} \simeq \mathbb{R}^{3+1} \setminus \text{tube}$ instead of \mathbb{R}^{3+1} . This same analysis can be carried out for the physically more realistic systems such as strings, p -branes, and fields in general as well; which can be sensitive to several other kinds of topologies. The additional terms $\nu(q)$ are known as Wess-Zumino terms and their gauge non-invariance can be adapted to cancel gauge anomalies and so they may be used to define gauge invariant functional integrals in quantum field theory¹⁰.

1.2 Quantization of spacetime

It is estimated [9] that in order to satisfy the uncertainty principle in quantum theory and also prevent the undesirable phenomenon of blackhole formation in the general relativistic theory of gravity during a high energy experiment, the length scales being probed by the experiment must not be much smaller than the Planck length $l_p = 10^{-35}m$. Information will be lost if blackholes are allowed to form during the experiment. It follows that one cannot, by such careful experiments,

¹⁰See for example [3] for a review of quantum field theory.

distinguish between two local events in spacetime whose separation is much smaller than l_p .

Thus the physical spacetime is expected to be quantized with cells of size of the order of l_p . Nonrenormalizable field theories, including the theory of gravity, are expected to be regularized in the physical spacetime. Here the minimum length scale naturally provides the UV cutoff needed to regulate otherwise divergent integrals encountered in the computation of probability amplitudes of certain scattering processes.

Various methods of quantizing spacetime include the following.

1. *Lattice regularization* methods. Space time is given the structure of a lattice with a lattice constant of the order l_p . These methods preserve gauge symmetries but break spacetime symmetries.
2. *Canonical quantization* methods. Here the local coordinates of spacetime become noncommutative in such a way that their spectra¹¹ are still invariant under the original classical symmetry group of the spacetime.
3. *Deformation quantization* methods. The local coordinates of spacetime become noncommutative and the symmetry group of spacetime is also modified so that it can preserve the spectra of the coordinates. Because of the observation that the elementary physical systems that live in a space \mathcal{S} are given by the irreducible representations of the symmetry group of \mathcal{S} , one could rather directly quantize/deform the group algebras of the symmetry group of \mathcal{S} and study their consequences. Mathematically these deformed groups

¹¹See section F.4. If a describes a physically measurable quantity then its spectrum would contain the possible values that one can obtain when the quantity is measured.

or quantum groups may be considered as belonging to a certain class of Hopf algebras¹².

4. *Noncommutative geometry*¹³. The noncommutative algebra of spacetime coordinates is first introduced. One then constructs any possible C^* -algebras¹⁴ from the universal algebra generated by the algebra of coordinates.

Noncommutative geometry involves extensions of classical geometric structures to an arbitrary $*$ -algebra \mathcal{A} and its dual space of linear functionals $\mathcal{A}^* = \{\phi : \mathcal{A} \rightarrow \mathbb{C}\}$. The extensions are based on duality between a classical space¹⁵ X and the point-wise product algebra $\mathcal{A} = (\mathcal{C}(X), +, \text{pt-wise})$ of complex classical functions $\mathcal{C}(X) = \{f : X \rightarrow \mathbb{C}\}$ on X . The algebra \mathcal{A} plays the role of a (non)commutative space of functions on the dual space \mathcal{A}^* . The symmetry groups of noncommutative spaces are known as quantum groups. Noncommutative geometry provides a unifying framework for various methods of quantizing spacetime.

1.3 Motivation for noncommutative field theory

Apart from the fact that spacetime quantization arose historically due to the need for regularization in quantum field theory, noncommutative spaces also arise naturally in various physical and mathematical theories. This fact lends support to

¹²See [30, 32, 33]

¹³[4] for example gives an informal introduction to noncommutative geometry.

¹⁴See equation (F.2.4)

¹⁵A classical space is a set of points (ie. imaginary objects) with one or more classical structures, such as special mapping or transformation structures, group structures, topological or continuity structures, differential structures and so on, defined on it.

the construction of general spacetime quantization schemes and noncommutative geometry. We will now list some of the noncommutative spaces that are often encountered in physics.

1.3.1 Phase space in quantum mechanics

In quantum mechanics the canonical quantization conditions

$[q_i, p_j] = i\hbar\delta_{ij}$, $[p_i, p_j] = [q_i, q_j] = 0$ imply that the quantum phase space is a naturally noncommutative space.

1.3.2 Superspace in supersymmetric field theory

Geometrically, supersymmetric theories are theories on a noncommutative space (known as superspace) with graded coordinates $x_I = (x_\mu, \theta_a)$

$$\begin{aligned} x_I x_J &= (-1)^{|x_I||x_J|} x_J x_I, & |x_\mu| &= 0, & |\theta_a| &= 1, \\ |x_I x_J| &= |x_I| + |x_J|. \end{aligned} \tag{1.3.1}$$

1.3.3 The center of motion of an electron in a magnetic field

When an electron moves in a constant magnetic field the coordinates of the center of its circular motion (ie. guiding center) become noncommutative when the system is quantized canonically. The solution to

$$m \frac{d\vec{v}}{dt} = e\vec{v} \times \vec{B}, \quad \vec{v} = \frac{d\vec{x}}{dt} \tag{1.3.2}$$

(see appendix A.1) shows that the center of circular motion is

$$\vec{x}_c(t) = \vec{x}_0 + \frac{m}{eB} \hat{B} \times \vec{v}_0 + \hat{B} (\hat{B} \cdot \vec{v}_0)(t - t_0), \quad (1.3.3)$$

which upon quantization (see appendix A.1) satisfies the relation

$$[x_c^i(t), x_c^j(t)] = i\theta^{ij} = i\frac{\hbar}{eB} \varepsilon^{ikj} \hat{B}^k \quad \forall t. \quad (1.3.4)$$

1.3.4 Phase space of a Landau problem with a strong magnetic field

Consider the problem where an electron moves in a plane $\vec{x} = (x, y, 0)$ subject to a constant magnetic field $\vec{B} = (0, 0, B)$ perpendicular to the plane, then the Lagrangian is

$$L = \frac{1}{2}m\vec{v}^2 - e\vec{v} \cdot \vec{A}, \quad \vec{v} = (\dot{x}, \dot{y}, 0), \quad \vec{A} = -\frac{1}{2}\vec{x} \times \vec{B}. \quad (1.3.5)$$

When the magnetic field is strong, ie. $eB \gg mc$, we have $\mathcal{L} \simeq -e\vec{v} \cdot \vec{A}$ giving $p_x = \frac{eB}{2}y$ and $p_y = -\frac{eB}{2}x$ so that canonical quantization yields

$$[x, y] \simeq i\frac{\hbar}{eB}. \quad (1.3.6)$$

1.3.5 Fundamental strings and D-branes

Consider the open string sigma model given by

$$S = \frac{1}{2\pi\lambda} \left\{ \int_{\mathcal{D}} \frac{1}{2} (g_{\mu\nu} dx^\mu \vee dx^\nu + i\lambda B_{\mu\nu} dx^\mu \wedge dx^\nu) + i \int_{\partial\mathcal{D}} dx^\mu A_\mu \right\}, \quad (1.3.7)$$

where \mathcal{D} is the string's worldsheet and $g_{\mu\nu}$, $B_{\mu\nu}$ are constant. Then the second term is a surface term and so the noncommutativity that arises will be on a D-

brane at the end of the string. In string theory¹⁶ a D -brane is the spacetime hypersurface on which the end of an open string can move freely (ie. the end of the string is confined to this hypersurface) as allowed by a nontrivial choice of boundary conditions in the variational principle that determines the dynamics of the string. More precisely, a Dp -brane (or Dirichlet p -brane) is the hypersurface of dimensions p (or $p + 1$ when the time direction is included) defined by Dirichlet boundary conditions

$$\delta X^{r\sigma_0}(\tau, \sigma_0) = 0, \quad \sigma_0 \in \{0, \pi\}, \quad r_{\sigma_0} : \{0, 1, 2, \dots, D\} \rightarrow \{1, 2, \dots, D - p\}$$

or $\{1, 2, \dots, D - p - 1\}$.

One may write the Fourier expansion

$$x^\mu(\sigma, \tau) = \sum_k x_k^\mu(\tau) e^{-ik\sigma}, \quad (1.3.8)$$

where the modes $x_k^\mu(\tau)$ may be regarded as individual particles. Then one has a Landau problem for each mode and noncommutativity of the coordinates $x_k^\mu(\tau)$ of the type (1.3.6) arises when $B_{\mu\nu}$ is large.

1.3.6 Myers Effect

The action principle for a collection of N $D0$ -branes in the presence of background fields leads to Lie algebra-type noncommutativity

$$[x^i, x^j] = f^{ij}_k x^k \quad (1.3.9)$$

¹⁶String theory is a quantum field theory in which elementary particles states arise as dynamical fluctuations of the trajectory of a one dimensional object, known as the fundamental string, in superspacetime. The fundamental strings as well as the elementary particle states can interact and/or condense to produce charged p dimensional classical or bound states, known as p -branes, the existence of some of which is a prediction of the theory. The charged p -branes interact with one another as well as with the elementary particle states of the fundamental string.

for the coordinates of the system of $D0$ -branes in space-time. This corresponds to static configurations ($\dot{x}^i = 0$) of the $D0$ -brane system that extremize the action as required by the least action principle.

The coordinates x^i are $N \times N$ matrices in the adjoint representation of $U(N)$. In the absence of the background field f one has $[x^i, x^j] = 0$ and these matrices can be simultaneously diagonalized and the N eigen-values represent the positions of the N $D0$ -branes [5, 6].

Summary of Chapter 2

1. Noncommutativity of spacetime \mathbb{R}^{d+1} coordinates is implied by a noncommutative spacetime function algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1}) = (\mathcal{F}(\mathbb{R}^{d+1}), *)$; with multiplication or self action $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, $(f, h) \mapsto f * g$ which induces left, right multiplicative self representations $\mu^L, \mu^R : \mathcal{A} \rightarrow O(V(\mathcal{A}))$ on \mathcal{A} regarded as a vector space $V(\mathcal{A}) = \mathcal{A}$. Group action needs to preserve the noncommutative product structure μ ; ie.

$$g \circ \mu = \mu \circ \Delta(g), \quad g \in G \tag{1.3.10}$$

(where $\Delta(g) = g \otimes g$, or $\Delta(e^{\alpha T}) = e^{\alpha \Delta(T)}$, $\Delta(T) = T \otimes 1 + 1 \otimes T$, in the undeformed case) which requires a **deformed group action** and hence a **deformed group algebra**.

2. The group action equally needs to preserve the spectrum

$$\mathcal{A}_\tau^* = \{\tau^* : \mathcal{A}_\tau \rightarrow \mathbb{C}, \tau^*(ab) = \tau^*(a)\tau^*(b), \tau^*(a+b) = \tau^*(a) + \tau^*(b)\} \tag{1.3.11}$$

of the algebra \mathcal{A}_τ of the statistics or particle interchange operators τ_i . It suffices for the action of g to commute with each $\tau \in \mathcal{A}_\tau$

$$\Delta(g) \circ \tau = \tau \circ \Delta(g), \quad \forall \tau \in \mathcal{A}_\tau \quad (1.3.12)$$

which implies a **deformed statistics operator**. Here

$$\mathcal{A}_\tau = \mathcal{A}\{\tau_i, i \in \mathbb{N}; \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}, \tau_i \tau_i = 1\} \quad (1.3.13)$$

is the group algebra of the permutation group (a subalgebra of the automorphism algebra of any tensor product algebra just as the permutation group is a subgroup of the automorphism group of any homogeneous *tensor product* algebra, so named after a homogeneous *polynomial* algebra).

$$\begin{aligned} \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} &\Rightarrow \tau^*(\tau_i) = \tau^*(\tau_{i+1}), \\ \tau_i^2 = e &\Rightarrow \tau_\pm^*(\tau_i) = \pm \tau_\pm^*(e) \end{aligned} \quad (1.3.14)$$

and

$$e^2 = e \Rightarrow \tau_\pm^*(e) = 1 \Rightarrow \tau_\pm^*(\tau_i) = \pm 1 \quad \forall i. \quad (1.3.15)$$

3. Let $\mathcal{A}_\tau \rightarrow O(\mathcal{H})$ be a representation of \mathcal{A}_τ as an algebra of operators $O(\mathcal{H})$ on a Hilbert space $\mathcal{H} = \mathcal{T}^*(\Phi)$ (the dual space of the tensor algebra $\mathcal{T}(\Phi)$ of quantum fields $\Phi = \{\phi\}$). Since the associated spectrum of eigenfunctions (fermion/boson or pure identical many-particle wavefunctions $\{\psi : \mathcal{T}(\Phi) \rightarrow \mathbb{C}^N\}$) must be preserved in the same manner, a **deformed algebra of quantum operators** in the quantum fields is required in addition to the star-product deformation of the localization functions of the fields.

4. The above process is reversible in that a deformed algebra of quantum fields necessarily leads to a noncommutative algebra of functions.
5. Consequences of deformed statistics of quantum fields include
 - 1) Modification of the statistical interparticle force and hence degeneracy pressure that determines the fate of galactic nuclei after fuel burning ceases.
 - 2) Pauli forbidden transitions may be observable,
 - 3) Lorentz, P, PT, CP, CPT and causality violations can occur.
6. In scattering theory the S -operator contains time ordering T and only interaction terms. Therefore the twist factor $e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$ does not always drop out directly as surface terms in the action S_I . However it can be checked that the twist factor drops out from all terms in the expansion of the S -operator in abelian gauge theories with or without matter fields as well as in pure nonabelian gauge theories.
7. In the (Schrodinger) representation of the noncommutative Moyal algebra $\mathcal{A}_\theta(\mathbb{R}^D) = (\mathcal{F}(\mathbb{R}^D), *)$ as an algebra of multiplication operators

$$m(\mathcal{H}_\theta) = \{\mu_f : \mathcal{H}_\theta \rightarrow \mathcal{H}_\theta, f \in \mathcal{A}_\theta(\mathbb{R}^D), \xi \rightarrow \mu_f \xi\},$$

on the Schrodinger Hilbert space $\mathcal{H}_\theta = (\mathcal{A}_\theta(\mathbb{R}^D), \langle \rangle)$, with the coordinates \hat{x}^μ acting as multiplication operators and the momenta \hat{p} acting as derivations, one encounters two possible independent multiplication representations μ^L , $\mu_f^L \mu_g^L = \mu_{f*g}^L$ and μ^R , $\mu_f^R \mu_g^R = \mu_{g*f}^R$, $[\mu_f^L, \mu_g^R] = 0$, corresponding to left and right multiplication

$$\mu_f^L \xi = f * \xi, \quad \mu_f^R \xi = \xi * f, \quad (1.3.16)$$

and this is the case for any noncommutative algebra. The Moyal case is special in that a commutative representation $\mu_{\hat{x}^\mu}^c = \frac{1}{2}(\mu_{\hat{x}^\mu}^L + \mu_{\hat{x}^\mu}^R)$ can be found for the algebra of the coordinates as one can check that

$$[\mu_f^L \pm \mu_f^R, \mu_g^L \pm \mu_g^R] = \mu_{f*g-g*f}^L \pm \mu_{g*f-f*g}^R \quad \forall f, g \in \mathcal{A}_\theta(\mathbb{R}^D).$$

Thus some of the fields in physics can be associated with the commutative sector $\mathcal{A}_0(\mathbb{R}^D)$ generated by the commutative coordinates $\{\hat{x}_c^\mu\}$ which are defined by

$\mu_{\hat{x}^\mu}^c = \mu_{\hat{x}_c^\mu}$. Owing to the commutativity of momenta

($\mu_{\hat{p}_\mu} = \frac{1}{2}\theta_{\mu\nu}^{-1}\text{ad}_{\hat{x}^\nu} \equiv \frac{1}{2}\theta_{\mu\nu}^{-1}(\mu_{\hat{x}^\nu}^L - \mu_{\hat{x}^\nu}^R)$) and the principle of minimal coupling, gauge fields (including Yang-Mills and Gravity fields) may be associated with the commutative sector $\mathcal{A}_0(\mathbb{R}^D)$.

If gauge fields are commutative while matter fields are noncommutative then the matter-gauge interaction terms will inherit (via the choice of covariant derivative D_μ) a twist factor from the matter sector meanwhile the pure gauge interaction terms will lack this factor leading to

$S = T(e^{-iS_I} e^{-\frac{i}{2}\overleftarrow{\text{ad}}_{P_0}\theta^{0i}P_i^{\text{in}}}) \neq S_0$, where P_i^{in} represents the anticipated total incident momentum when the matrix elements $\langle f|S|i\rangle$ of S are finally taken.

This will lead to P, CPT noninvariance of the S -operator.

8. The direct Poincare transformation of products of deformed or twisted quantum fields $\phi = \phi_0 e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}$ takes into account the use of the coproduct to transform local products of fields. The S operator $S = T e^{-i \int \mathcal{H}_I}$ is invariant under this transformation even though the causality or locality condition, which is required for Lorentz invariance in commutative theories, does not hold:

$$[\mathcal{H}_I(x), \mathcal{H}_I(y)] \neq 0 \quad \text{for} \quad (x_0 - y_0)^2 < (\vec{x} - \vec{y})^2. \quad (1.3.17)$$

Chapter 2

Introduction to Noncommutative geometry

We give an introductory review of quantum physics on the noncommutative space-time called the Groenewold-Moyal plane. Basic ideas like star products, twisted statistics, second quantized fields and discrete symmetries are discussed. We also outline some of the recent developments in these fields and mention where one can search for experimental signals.

2.1 Introduction

Quantum electrodynamics is not free from divergences. The calculation of Feynman diagrams involves a cut-off Λ on the momentum variables in the integrands. In this case, the theory will not see length scales smaller than Λ^{-1} . The theory fails to explain physics in the regions of spacetime volume less than Λ^{-4} .

Heisenberg proposed in the 1930's that an effective cut-off can be introduced

in quantum field theories by introducing an effective lattice structure for the underlying spacetime. A lattice structure of spacetime takes care of the divergences in quantum field theories, but a lattice breaks Lorentz invariance.

Heisenberg’s proposal to obtain an effective lattice structure was to make the spacetime noncommutative. The noncommutative spacetime structure is pointless on small length scales. Noncommuting spacetime coordinates introduce a fundamental length scale. This fundamental length can be taken to be of the order of the Planck length. The notion of point below this length scale has no operational meaning.

We can explain Heisenberg’s ideas by recalling the quantization of a classical system. The point of departure from classical to quantum physics is the algebra of functions on the phase space. The classical phase space, a symplectic manifold M , consists of “points” forming the pure states of the system. Every observable physical quantity on this manifold M is specified by a function f . The Hamiltonian H is a function on M , which measures energy. The evolution of f on the manifold is specified by H by the equation

$$\dot{f} = \{f, H\} \tag{2.1.1}$$

where $\dot{f} = df/dt$ and $\{ , \}$ is the Poisson bracket.

The quantum phase space is a “noncommutative space” where the algebra of functions is replaced by the algebra of linear operators. The algebra $\mathcal{F}(T^*Q)$ of functions on the classical phase space T^*Q , associated with a given spacetime Q , is a commutative algebra. According to Dirac, quantization can be achieved by replacing a function f in this algebra by an operator \hat{f} and equating $i\hbar$ times the Poisson bracket between functions to the commutator between the corresponding operators. In classical physics, the functions f commute, so $\mathcal{F}(T^*Q)$ is a com-

mutative algebra. But the corresponding quantum algebra $\hat{\mathcal{F}}$ is not commutative. Dynamics is on $\hat{\mathcal{F}}$. So quantum physics is *noncommutative dynamics*.

A particular aspect of this dynamics is *fuzzy phase space* where we cannot localize points, and which has an attendant effective ultraviolet cutoff. A fuzzy phase space can still admit the action of a continuous symmetry group such as the spatial rotation group as the automorphism group [7]. For example, one can quantize functions on a sphere S^2 to obtain a fuzzy sphere [8]. It admits $SO(3)$ as an automorphism group. The fuzzy sphere can be identified with the algebra M_n of $n \times n$ complex matrices. The volume of phase space in this case becomes finite. Semiclassically there are a finite number of cells on the fuzzy sphere, each with a finite area [7].

Thus in quantum physics, the commutative algebra of functions on phase space is deformed to a noncommutative algebra, leading to a “noncommutative phase space”. Such deformations, characteristic of quantization, are now appearing in different approaches to fundamental physics. Examples are the following:

- 1.) Noncommutative geometry has made its appearance as a method for regularizing quantum field theories (qft’s) and in studies of deformation quantization.
- 2.) It has turned up in string physics as quantized D -branes.
- 3.) Certain approaches to canonical gravity [64] have used noncommutative geometry with great effectiveness.
- 4.) There are also plausible arguments based on the uncertainty principle [9] that indicate a noncommutative spacetime in the presence of gravity.
- 5.) It has been conjectured by ‘t Hooft [10] that the horizon of a black hole should have a fuzzy 2-sphere structure to give a finite entropy.
- 6.) A noncommutative structure emerges naturally in quantum Hall effect [11].

2.2 Noncommutative Spacetime

2.2.1 A Little Bit of History

The idea that spacetime geometry may be noncommutative is old and goes back as far as the 30's. In 1947 Snyder used the noncommutative structure of spacetime to introduce a small length scale cut-off in field theory without breaking Lorentz invariance [12]. In the same year, Yang [13] also published a paper on quantized spacetime, extending Snyder's work. The term 'noncommutative geometry' was introduced by von Neumann [7]. He used it to describe in general a geometry in which the algebra of noncommuting linear operators replaces the algebra of functions.

Snyder's idea was forgotten with the successful development of the renormalization program. Later, in the 1980's Connes [14] and Woronowicz [15] revived noncommutative geometry by introducing a differential structure in the noncommutative framework.

2.2.2 Spacetime Uncertainties

It is generally believed that the picture of spacetime as a manifold of points breaks down at distance scales of the order of the Planck length: Spacetime events cannot be localized with an accuracy given by Planck length.

The following argument can be found in Doplicher *et al.* [9]. In order to probe physics at a fundamental length scale L close to the Planck scale, the Compton wavelength $\frac{\hbar}{Mc}$ of the probe must fulfill

$$\frac{\hbar}{Mc} \leq L \quad \text{or} \quad M \geq \frac{\hbar}{Lc} \simeq \text{Planck mass.} \quad (2.2.1)$$

Such high mass in the small volume L^3 will strongly affect gravity and can cause black holes and their horizons to form. This suggests a fundamental length limiting spatial localization. That is, there is a space-space uncertainty,

$$\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \gtrsim L^2 \quad (2.2.2)$$

Similar arguments can be made about time localization. Observation of very short time scales requires very high energies. They can produce black holes and black hole horizons will then limit spatial resolution suggesting

$$\Delta x_0 (\Delta x_1 + \Delta x_2 + \Delta x_3) \geq L^2. \quad (2.2.3)$$

The above uncertainty relations suggest that spacetime ought to be described as a noncommutative manifold just as classical phase space is replaced by noncommutative phase space in quantum physics which leads to Heisenberg's uncertainty relations. The points on the classical commutative manifold should then be replaced by states on a noncommutative algebra.

2.2.3 The Groenewold-Moyal Plane

The noncommutative Groenewold-Moyal (GM) spacetime is a deformation of ordinary spacetime in which the spacetime coordinate functions \hat{x}_μ do not commute [16, 17, 18, 19]:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} = \text{constants}, \quad (2.2.4)$$

where the coordinate functions \hat{x}_μ give Cartesian coordinates x_μ of (flat) spacetime:

$$\hat{x}_\mu(x) = x_\mu. \quad (2.2.5)$$

The deformation matrix θ is taken to be a real and antisymmetric constant matrix [20]. Its elements have the dimension of (length)², thus a scale for the smallest patch of area in the $\mu - \nu$ plane. They also give a measure of the strength of noncommutativity. One cannot probe spacetime with a resolution below this scale. That is, spacetime is “fuzzy” [21] below this scale. In the limit $\theta_{\mu\nu} \rightarrow 0$, one recovers ordinary spacetime.

2.3 The Star Products

In this part we will go into more details of the GM plane. The GM plane incorporates spacetime uncertainties. Such an introduction of spacetime noncommutativity replaces point-by-point multiplication of two fields by a type of “smeared” product. This type of product is called a star product.

2.3.1 Deforming an Algebra

There is a general way of deforming the algebra of functions on a manifold M [22]. The GM plane, $\mathcal{A}_\theta(\mathbb{R}^{d+1})$, associated with spacetime \mathbb{R}^{d+1} is an example of such a deformed algebra.

Consider a Riemannian manifold (M, g) with metric g . If the group \mathcal{R}^N ($N \geq 2$) acts as a group of isometries on M , then it acts on the Hilbert space $L^2(M, d\mu_g)$ of square integrable functions on M . The volume form $d\mu_g$ for the scalar product on $L^2(M, d\mu_g)$ is induced from g .

If $\{\lambda = (\lambda_1, \dots, \lambda_N)\}$ denote the unitary irreducible representations (UIR’s) of \mathcal{R}^N , then we can write

$$L^2(M, d\mu_g) = \bigoplus_{\lambda} \mathcal{H}^{(\lambda)}, \quad (2.3.1)$$

where \mathcal{R}^N acts by the UIR λ on $\mathcal{H}^{(\lambda)}$.

We choose λ such that

$$\lambda : a \longrightarrow e^{i\lambda a} \quad (2.3.2)$$

where $a = (a_1, a_2, \dots, a_N) \in \mathbb{R}^N$.

Choose two smooth functions f_λ and $f_{\lambda'}$ in $\mathcal{H}^{(\lambda)}$ and $\mathcal{H}^{(\lambda')}$. Then under the pointwise multiplication

$$f_\lambda \otimes f_{\lambda'} \rightarrow f_\lambda f_{\lambda'} \quad (2.3.3)$$

where, if p is a point on M ,

$$(f_\lambda f_{\lambda'})(p) = f_\lambda(p) f_{\lambda'}(p). \quad (2.3.4)$$

Also

$$f_\lambda f_{\lambda'} \in \mathcal{H}^{(\lambda+\lambda')} \quad (2.3.5)$$

where we have taken the group law as addition.

Let $\theta^{\mu\nu}$ be an antisymmetric constant matrix in the space of UIR's of \mathbb{R}^N . The above algebra with pointwise multiplication can be deformed into a new deformed algebra. The pointwise product becomes a θ dependent ‘‘smeared’’ product $*_\theta$ in the deformed algebra,

$$f_\lambda *_\theta f_{\lambda'} = f_\lambda f_{\lambda'} e^{-\frac{i}{2} \lambda_\mu \theta^{\mu\nu} \lambda'_\nu}. \quad (2.3.6)$$

This deformed algebra is also associative because of eqn. (2.3.5). The GM plane, $\mathcal{A}_\theta(\mathbb{R}^{d+1})$, is a special case of this algebra.

In the case of the GM plane, the group \mathbb{R}^{d+1} acts on $\mathcal{A}_\theta(\mathbb{R}^{d+1}) \{= \mathcal{C}^\infty(\mathbb{R}^{d+1}) \text{ as a set} \}$ by translations leaving the flat Euclidean metric

invariant. The IRR's are labelled by the “momenta” $\lambda = p = (p^0, p^1, \dots, p^d)$. A basis for the Hilbert space $\mathcal{H}^{(p)}$ is formed by plane waves e_p with $e_p(x) = e^{-ip_\mu x^\mu}$, $x = (x^0, x^1, \dots, x^d)$ being a point of \mathbb{R}^{d+1} . The $*$ -product for the GM plane follows from eqn. (2.3.6),

$$e_p *_{\theta} e_q = e_p e_q e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu} . \quad (2.3.7)$$

This $*$ -product defines the Moyal plane $\mathcal{A}_\theta(\mathbb{R}^{d+1})$.

In the limit $\theta_{\mu\nu} \rightarrow 0$, the operators e_p and e_q become commutative functions on \mathbb{R}^N .

2.3.2 The Voros and Moyal Star Products

This section is based on the book [8].

The algebra \mathcal{A}_0 of smooth functions on a manifold M under point-wise multiplication is a commutative algebra. In the previous section we saw that \mathcal{A}_0 can be deformed into a new algebra \mathcal{A}_θ in which the point-wise product is deformed to a noncommutative (but still associative) product called the $*$ -product.

Such deformations were studied by Weyl, Wigner, Groenewold and Moyal [24, 25, 26]. The $*$ -product has a central role in many discussions of noncommutative geometry. It appears in other branches of physics like quantum optics.

The $*$ -product can be obtained from the algebra of creation and annihilation operators. It is explained below.

2.3.2.1 Coherent States

The dynamics of a quantum harmonic oscillator most closely resembles that of a classical harmonic oscillator when the oscillator quantum state is a coherent

state. Consider a quantum oscillator with annihilation and creation operators a , a^\dagger , $aa^\dagger = a^\dagger a + 1$. The coherent states $|z\rangle$ defined by

$$a|z\rangle = z|z\rangle \quad (2.3.8)$$

are given by

$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle = e^{-\frac{1}{2}|z|^2} e^{za^\dagger}|0\rangle, \quad z \in \mathbb{C}.$$

They also have the property

$$\langle z'|z\rangle = e^{\frac{1}{2}|z-z'|^2}. \quad (2.3.9)$$

The coherent states are overcomplete, with the resolution of identity

$$\mathbf{1} = \int \frac{d^2z}{\pi} |z\rangle\langle z|, \quad d^2z = dx_1 dx_2, \quad (2.3.10)$$

where

$$z = \frac{x_1 + ix_2}{\sqrt{2}}.$$

Consider an operator \hat{A} . The “symbol” (or “representation”) of \hat{A} is a function A on \mathbb{C} with values $A(z, \bar{z}) = \langle z|\hat{A}|z\rangle$. A central property of coherent states is that an operator \hat{A} is determined just by its diagonal matrix elements, that is, by the symbol A of \hat{A} .

2.3.2.2 The Coherent State or Voros *-product on the GM Plane

As indicated above, we can map an operator \hat{A} to a function A using coherent states as follows:

$$\hat{A} \longrightarrow A, \quad A(z, \bar{z}) = \langle z|\hat{A}|z\rangle. \quad (2.3.11)$$

This is a bijective linear map and induces a product $*_C$ on functions (C indicating “coherent state”). With this product, we get an algebra $(C^\infty(\mathbb{C}), *_C)$ of functions. Since the map $\hat{A} \rightarrow A$ has the property $(\hat{A})^* \rightarrow A^* \equiv \bar{A}$, this map is a $*$ -morphism from operators to $(C^\infty(\mathbb{C}), *_C)$ where $*$ on functions is complex conjugation.

Let us get familiar with this new function algebra.

The image of a is the function α where $\alpha(z, \bar{z}) = z$. The image of a^n has the value z^n at (z, \bar{z}) , so by definition,

$$(\alpha *_C \alpha \dots *_C \alpha)(z, \bar{z}) = z^n. \quad (2.3.12)$$

The image of $a^* \equiv a^\dagger$ is $\bar{\alpha}$ where $\bar{\alpha}(z, \bar{z}) = \bar{z}$ and that of $(a^*)^n$ is $\bar{\alpha} *_C \bar{\alpha} \dots *_C \bar{\alpha}$ where

$$\bar{\alpha} *_C \bar{\alpha} \dots *_C \bar{\alpha}(z, \bar{z}) = \bar{z}^n. \quad (2.3.13)$$

Since $\langle z|a^*a|z\rangle = \bar{z}z$ and $\langle z|aa^*|z\rangle = \bar{z}z + 1$, we get

$$\bar{\alpha} *_C \alpha = \bar{\alpha}\alpha, \quad \alpha *_C \bar{\alpha} = \alpha\bar{\alpha} + \mathbf{1}, \quad (2.3.14)$$

where $\bar{\alpha}\alpha = \alpha\bar{\alpha}$ is the pointwise product of α and $\bar{\alpha}$, and $\mathbf{1}$ is the constant function with value 1 for all z .

For general operators \hat{f} , the construction proceeds as follows. Consider

$$: e^{\xi a^\dagger - \bar{\xi} a} : \quad (2.3.15)$$

where the normal ordering symbol $:\dots:$ means as usual that a^\dagger 's are to be put to the left of a 's. Thus

$$\begin{aligned} : aa^\dagger a^\dagger a : &= a^\dagger a^\dagger aa, \\ : e^{\xi a^\dagger - \bar{\xi} a} : &= e^{\xi a^\dagger} e^{-\bar{\xi} a}. \end{aligned}$$

Hence

$$\langle z | : e^{\xi a^\dagger - \bar{\xi} a} : | z \rangle = e^{\xi \bar{z} - \bar{\xi} z}. \quad (2.3.16)$$

Writing \hat{f} as a Fourier transform,

$$\hat{f} = \int \frac{d^2 \xi}{\pi} : e^{\xi a^\dagger - \bar{\xi} a} : \tilde{f}(\xi, \bar{\xi}), \quad \tilde{f}(\xi, \bar{\xi}) \in \mathbb{C}, \quad (2.3.17)$$

its symbol is seen to be

$$f = \int \frac{d^2 \xi}{\pi} e^{\xi \bar{z} - \bar{\xi} z} \tilde{f}(\xi, \bar{\xi}). \quad (2.3.18)$$

This map is invertible since f determines \tilde{f} . Consider also the second operator

$$\hat{g} = \int \frac{d^2 \eta}{\pi} : e^{\eta a^\dagger - \bar{\eta} a} : \tilde{g}(\eta, \bar{\eta}), \quad (2.3.19)$$

and its symbol

$$g = \int \frac{d^2 \eta}{\pi} e^{\eta \bar{z} - \bar{\eta} z} \tilde{g}(\eta, \bar{\eta}). \quad (2.3.20)$$

The task is to find the symbol $f *_C g$ of $\hat{f} \hat{g}$. Let us first find

$$e^{\xi \bar{z} - \bar{\xi} z} *_C e^{\eta \bar{z} - \bar{\eta} z}. \quad (2.3.21)$$

We have

$$: e^{\xi a^\dagger - \bar{\xi} a} : : e^{\eta a^\dagger - \bar{\eta} a} : = : e^{\xi a^\dagger - \bar{\xi} a} e^{\eta a^\dagger - \bar{\eta} a} : e^{-\bar{\xi} \eta} \quad (2.3.22)$$

and hence

$$\begin{aligned} e^{\xi \bar{z} - \bar{\xi} z} *_C e^{\eta \bar{z} - \bar{\eta} z} &= e^{-\bar{\xi} \eta} e^{\xi \bar{z} - \bar{\xi} z} e^{\eta \bar{z} - \bar{\eta} z} \\ &= e^{\xi \bar{z} - \bar{\xi} z} e^{\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}}} e^{\eta \bar{z} - \bar{\eta} z}. \end{aligned} \quad (2.3.23)$$

The bidifferential operators $(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}})^k$, ($k = 1, 2, \dots$) have the definition

$$\alpha(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}})^k \beta(z, \bar{z}) = \frac{\partial^k \alpha(z, \bar{z})}{\partial z^k} \frac{\partial^k \beta(z, \bar{z})}{\partial \bar{z}^k}. \quad (2.3.24)$$

The exponential in (2.3.23) involving them can be defined using the power series.

The coherent state $*$ -product $f *_C g$ follows from (2.3.23):

$$f *_C g(z, \bar{z}) = (f e^{\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}}} g)(z, \bar{z}). \quad (2.3.25)$$

We can explicitly introduce a deformation parameter $\theta > 0$ in the discussion by changing (2.3.25) to

$$f *_C g(z, \bar{z}) = (f e^{\theta \overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}}} g)(z, \bar{z}). \quad (2.3.26)$$

After rescaling $z' = \frac{z}{\sqrt{\theta}}$, (2.3.26) gives (2.3.25). As z' and \bar{z}' after quantization become a, a^\dagger , z and \bar{z} become the scaled oscillators $a_\theta, a_\theta^\dagger$

$$[a_\theta, a_\theta] = [a_\theta^\dagger, a_\theta^\dagger] = 0, \quad [a_\theta, a_\theta^\dagger] = \theta. \quad (2.3.27)$$

Equation (2.3.27) is associated with the Moyal plane with Cartesian coordinate functions x_1, x_2 . If $a_\theta = \frac{x_1 + ix_2}{\sqrt{2}}$, $a_\theta^\dagger = \frac{x_1 - ix_2}{\sqrt{2}}$,

$$[x_i, x_j] = i\theta \varepsilon_{ij}, \quad \varepsilon_{ij} = -\varepsilon_{ji}, \quad \varepsilon_{12} = 1. \quad (2.3.28)$$

The Moyal plane is the plane \mathbb{R}^2 , but with its function algebra deformed in accordance with eqn. (2.3.28). The deformed algebra has the product eqn. (2.3.26) or equivalently the Moyal product derived below.

2.3.2.3 The Moyal Product on the GM Plane

We get this by changing the map $\hat{f} \rightarrow f$ from operators to functions. For a given function f , the operator \hat{f} is thus different for the coherent state and Moyal $*$'s. The $*$ -product on two functions is accordingly also different.

Let us introduce the Weyl map and the Weyl symbol. The Weyl map of the operator

$$\hat{f} = \int \frac{d^2\xi}{\pi} \tilde{f}(\xi, \bar{\xi}) e^{\xi a^\dagger - \bar{\xi} a} \quad (2.3.29)$$

to the function f is defined by

$$f(z, \bar{z}) = \int \frac{d^2\xi}{\pi} \tilde{f}(\xi, \bar{\xi}) e^{\xi \bar{z} - \bar{\xi} z}. \quad (2.3.30)$$

Equation (2.3.30) makes sense since \tilde{f} is fully determined by \hat{f} as follows:

$$\langle z | \hat{f} | z \rangle = \int \frac{d^2\xi}{\pi} \tilde{f}(\xi, \bar{\xi}) e^{-\frac{1}{2}\xi\bar{\xi}} e^{\xi\bar{z} - \bar{\xi}z}.$$

\tilde{f} can be calculated from here by Fourier transformation.

The map is invertible since \tilde{f} follows from f by the Fourier transform of eqn. (2.3.30) and \tilde{f} fixes \hat{f} by eqn. (2.3.29). f is called the *Weyl symbol* of \hat{f} .

As the Weyl map is bijective, we can find a new $*$ product, call it $*_W$, between functions by setting $f *_W g = \text{Weyl symbol of } \hat{f}\hat{g}$.

For

$$\hat{f}(\xi, \bar{\xi}) = e^{\xi a^\dagger - \bar{\xi} a}, \quad \hat{g}(\eta, \bar{\eta}) = e^{\eta a^\dagger - \bar{\eta} a},$$

to find $f *_W g$, we first rewrite $\hat{f}\hat{g}$ according to

$$\hat{f}\hat{g} = e^{\frac{1}{2}(\xi\bar{\eta} - \bar{\xi}\eta)} e^{(\xi+\eta)a^\dagger - (\bar{\xi}+\bar{\eta})a}.$$

Hence

$$\begin{aligned} f *_W g(z, \bar{z}) &= e^{\xi\bar{z}-\bar{\xi}z} e^{\frac{1}{2}(\xi\bar{\eta}-\bar{\xi}\eta)} e^{\eta\bar{z}-\bar{\eta}z} \\ &= f e^{\frac{1}{2}(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)} g(z, \bar{z}). \end{aligned} \quad (2.3.31)$$

Multiplying by \tilde{f}, \tilde{g} and integrating, we get eqn. (2.3.31) for arbitrary functions:

$$f *_W g(z, \bar{z}) = \left(f e^{\frac{1}{2}(\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)} g \right)(z, \bar{z}). \quad (2.3.32)$$

Note that

$$\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z = i(\overleftarrow{\partial}_1 \overrightarrow{\partial}_2 - \overleftarrow{\partial}_2 \overrightarrow{\partial}_1) = i\varepsilon_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j.$$

Introducing also θ , we can write the $*_W$ -product as

$$f *_W g = f e^{i\frac{\theta}{2}\varepsilon_{ij} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g. \quad (2.3.33)$$

By eqn. (2.3.28), $\theta\varepsilon_{ij} = \omega_{ij}$ fixes the Poisson brackets, or the Poisson structure on the Moyal plane. Eqn. (2.3.33) is customarily written as

$$f *_W g = f e^{i\frac{\omega_{ij}}{2} \overleftarrow{\partial}_i \overrightarrow{\partial}_j} g$$

using the Poisson structure. (But we have not cared to position the indices so as to indicate their tensor nature and to write ω^{ij} .)

2.3.3 Properties of the $*$ -Products

A $*$ -product without a subscript indicates that it can be either a $*_C$ or a $*_W$.

2.3.3.1 Cyclic Invariance

The trace of operators, $\text{Tr} : \hat{A} \mapsto \int \frac{d^2z}{\pi} \langle z | \hat{A} | z \rangle$, has the fundamental property $\text{Tr} \hat{A} \hat{B} = \text{Tr} \hat{B} \hat{A}$, which leads to the general cyclic identities

$$\text{Tr} \hat{A}_1 \dots \hat{A}_n = \text{Tr} \hat{A}_n \hat{A}_1 \dots \hat{A}_{n-1}. \quad (2.3.34)$$

We now show that

$$Tr \hat{A}\hat{B} = \int \frac{d^2z}{\pi} A * B(z, \bar{z}), \quad * = *_C \quad \text{or} \quad *_W. \quad (2.3.35)$$

(The functions on the right hand side are different for $*_C$ and $*_W$ if \hat{A}, \hat{B} are fixed). From this follows the analogue of (2.3.34):

$$\int \frac{d^2z}{\pi} (A_1 * A_2 * \cdots * A_n)(z, \bar{z}) = \int \frac{d^2z}{\pi} (A_n * A_1 * \cdots * A_{n-1})(z, \bar{z}) \quad (2.3.36)$$

For $*_C$, eqn. (2.3.35) follows from eqn. (2.3.10). The coherent state image of $e^{\xi a^\dagger - \bar{\xi} a}$ is the function with value

$$e^{\xi \bar{z} - \bar{\xi} z} e^{-\frac{1}{2} \bar{\xi} \xi} \quad (2.3.37)$$

at z , with a similar correspondence if $\xi \rightarrow \eta$. So

$$Tr e^{\xi a^\dagger - \bar{\xi} a} e^{\eta a^\dagger - \bar{\eta} a} = \int \frac{d^2z}{\pi} \left(e^{\xi \bar{z} - \bar{\xi} z} e^{-\frac{1}{2} \bar{\xi} \xi} \right) \left(e^{\eta \bar{z} - \bar{\eta} z} e^{-\frac{1}{2} \bar{\eta} \eta} \right) e^{-\bar{\xi} \eta}$$

The integral produces the δ -function

$$\prod_i 2\delta(\xi_i + \eta_i), \quad \xi_i = \frac{\xi_1 + \xi_2}{\sqrt{2}}, \quad \eta_i = \frac{\eta_1 + \eta_2}{\sqrt{2}}.$$

We can hence substitute $e^{-\left(\frac{1}{2} \bar{\xi} \xi + \frac{1}{2} \bar{\eta} \eta + \bar{\xi} \eta\right)}$ by $e^{\frac{1}{2}(\xi \bar{\eta} - \bar{\xi} \eta)}$ and get eqn. (2.3.35) for Weyl $*$ for these exponentials and so for general functions by using eqn. (2.3.29).

2.3.3.2 A Special Identity for the Weyl Star

The above calculation also gives the identity

$$\int \frac{d^2z}{\pi} A *_W B(z, \bar{z}) = \int \frac{d^2z}{\pi} A(z, \bar{z}) B(z, \bar{z}).$$

That is because

$$\prod_i \delta(\xi_i + \eta_i) e^{\frac{1}{2}(\xi \bar{\eta} - \bar{\xi} \eta)} = \prod_i \delta(\xi_i + \eta_i).$$

In eqn. (2.3.36), A and B in turn can be Weyl $*$ -products of other functions. Thus in integrals of Weyl $*$ -products of functions, one $*_W$ can be replaced by the pointwise (commutative) product:

$$\begin{aligned} & \int \frac{d^2 z}{\pi} (A_1 *_W A_2 *_W \cdots A_K) *_W (B_1 *_W B_2 *_W \cdots B_L) (z, \bar{z}) \\ &= \int \frac{d^2 z}{\pi} (A_1 *_W A_2 *_W \cdots A_K) (B_1 *_W B_2 *_W \cdots B_L) (z, \bar{z}). \end{aligned}$$

This identity is frequently useful.

2.3.3.3 Equivalence of $*_C$ and $*_W$

For the operator

$$\hat{A} = e^{\xi a^\dagger - \bar{\xi} a}, \quad (2.3.38)$$

the coherent state function A_C has the value (2.3.37) at z , and the Weyl symbol A_W has the value

$$A_W(z, \bar{z}) = e^{\xi \bar{z} - \bar{\xi} z}.$$

As both $(C^\infty(\mathbb{R}^2), *_C)$ and $(C^\infty(\mathbb{R}^2), *_W)$ are isomorphic to the operator algebra, they too are isomorphic. The isomorphism is established by the maps

$$A_C \longleftrightarrow A_W$$

and their extension via Fourier transform to all operators and functions $\hat{A}, A_{C,W}$.

Clearly

$$A_W = e^{-\frac{1}{2}\partial_z \partial_{\bar{z}}} A_C, \quad A_C = e^{\frac{1}{2}\partial_z \partial_{\bar{z}}} A_W,$$

$$A_C *_C B_C \longleftrightarrow A_W *_W B_W.$$

The mutual isomorphism of these three algebras is a $*$ -isomorphism since $(\hat{A}\hat{B})^\dagger \longrightarrow \bar{B}_{C,W} *_C *_W \bar{A}_{C,W}$.

2.3.3.4 Integration and Tracial States

This is a good point to introduce the ideas of a state and a tracial state on a $*$ -algebra \mathcal{A} with unity $\mathbf{1}$.

A state ω is a linear map from \mathcal{A} to \mathbb{C} , $\omega(a) \in \mathbb{C}$ for all $a \in \mathcal{A}$ with the following properties:

$$\begin{aligned}\omega(a^*) &= \overline{\omega(a)}, \\ \omega(a^*a) &\geq 0, \\ \omega(\mathbf{1}) &= 1.\end{aligned}$$

If \mathcal{A} consists of operators on a Hilbert space and ρ is a density matrix, it defines a state ω_ρ via

$$\omega_\rho(a) = \text{Tr}(\rho a). \quad (2.3.39)$$

If $\rho = e^{-\beta H} / \text{Tr}(e^{-\beta H})$ for a Hamiltonian H , it gives a Gibbs state via eqn. (2.3.39).

Thus the concept of a state on an algebra \mathcal{A} generalizes the notion of a density matrix. There is a remarkable construction, the Gel'fand- Naimark-Segal (GNS) construction, which shows how to associate any state with a rank-1 density matrix [27].

A state is *tracial* if it has cyclic invariance:

$$\omega(ab) = \omega(ba). \quad (2.3.40)$$

The Gibbs state is not tracial, but fulfills an identity generalizing eqn. (2.3.40). It is a Kubo-Martin-Schwinger (KMS) state [27].

A positive map ω' is in general an unnormalized state: It must fulfill all the conditions that a state fulfills, but is not obliged to fulfill the condition $\omega'(\mathbf{1}) = 1$.

Let us define a positive map ω' on $(C^\infty(\mathbb{R}^2), *)$ ($*$ = $*_C$ or $*_W$) using integration:

$$\omega'(A) = \int \frac{d^2z}{\pi} \hat{A}(z, \bar{z}).$$

It is easy to verify that ω' fulfills the properties of a positive map. A *tracial* positive map ω' also has the cyclic invariance, eqn. (2.3.40).

The cyclic invariance (2.3.40) of $\omega'(A * B)$ means that it is a tracial positive map.

2.3.3.5 The θ -Expansion

On introducing θ , we have (2.3.26) and

$$f *_W g(z, \bar{z}) = f e^{\frac{\theta}{2} (\overleftarrow{\partial}_z \overrightarrow{\partial}_{\bar{z}} - \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z)} g(z, \bar{z}).$$

The series expansion in θ is thus

$$f *_C g(z, \bar{z}) = fg(z, \bar{z}) + \theta \frac{\partial f}{\partial z}(z, \bar{z}) \frac{\partial g}{\partial \bar{z}}(z, \bar{z}) + \mathcal{O}(\theta^2),$$

$$f *_W g(z, \bar{z}) = fg(z, \bar{z}) + \frac{\theta}{2} \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} \right) (z, \bar{z}) + \mathcal{O}(\theta^2).$$

Introducing the notation

$$[f, g]_* = f * g - g * f, \quad * = *_C \quad \text{or} \quad *_W, \quad (2.3.41)$$

we see that

$$[f, g]_{*_C} = \theta \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} \right) (z, \bar{z}) + \mathcal{O}(\theta^2),$$

$$[f, g]_{*_W} = \theta \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} - \frac{\partial f}{\partial \bar{z}} \frac{\partial g}{\partial z} \right) (z, \bar{z}) + \mathcal{O}(\theta^2).$$

We thus see that

$$[f, g]_* = i\theta \{f, g\}_{P.B.} + \mathcal{O}(\theta^2), \quad (2.3.42)$$

where $\{f, g\}$ is the Poisson bracket of f and g and the $\mathcal{O}(\theta^2)$ term depends on $*_{C,W}$. Thus the $*$ -product is an associative product that to leading order in the deformation parameter (“Planck’s constant”) θ is compatible with the rules of quantization of Dirac. We can say that with the $*$ -product, we have deformation quantization of the classical commutative algebra of functions.

But it should be emphasized that even to leading order in θ , $f *_C g$ and $f *_W g$ do not agree. Still the algebras $(C^\infty(\mathbb{R}^2, *_C))$ and $(C^\infty(\mathbb{R}^2, *_W))$ are $*$ -isomorphic.

If a Poisson structure on a manifold M with Poisson bracket $\{.,.\}$ is given, then one can have a $*$ -product $f * g$ as a formal power series in θ such that eqn. (2.3.42) holds [28].

2.4 Spacetime Symmetries on Noncommutative Plane

In this section we address how to implement spacetime symmetries on the noncommutative spacetime algebra $\mathcal{A}_\theta(\mathbb{R}^N)$, where functions are multiplied by a $*$ -product. In section 2, we modelled the spacetime noncommutativity using the commutation relations given by eqn. (2.2.4). Those relations are clearly not invariant under naive Lorentz transformations. That is, the noncommutative structure we have modelled breaks Lorentz symmetry. Fortunately, there is a way to overcome this difficulty: one can interpret these relations in a Lorentz-invariant way by implementing a deformed Lorentz group action [29].

2.4.1 The Deformed Poincaré Group Action

The single particle states in quantum mechanics can be identified with the carrier (or representation) space of the one-particle unitary irreducible representations (UIRR's) of the identity component of the Poincaré group, P_+^\uparrow or rather its two-fold cover \bar{P}_+^\uparrow . Let $U(g)$, $g \in \bar{P}_+^\uparrow$, be the UIRR for a spinless particle of mass m on a Hilbert space \mathcal{H} . Then \mathcal{H} has the basis $\{|k\rangle\}$ of momentum eigenstates, where $k = (k_0, \mathbf{k})$, $k_0 = |\sqrt{\mathbf{k}^2 + m^2}|$. $U(g)$ transforms $|k\rangle$ according to

$$U(g)|k\rangle = |gk\rangle. \quad (2.4.1)$$

Then conventionally \bar{P}_+^\uparrow acts on the two-particle Hilbert space $\mathcal{H} \otimes \mathcal{H}$ in the following way:

$$U(g) \otimes U(g) |k\rangle \otimes |q\rangle = |gk\rangle \otimes |gq\rangle. \quad (2.4.2)$$

There are similar equations for multiparticle states.

Note that we can write $U(g) \otimes U(g) = [U \otimes U](g \times g)$.

Thus while defining the group action on multi-particle states, we see that we have made use of the isomorphism $G \rightarrow G \times G$ defined by $g \rightarrow g \times g$. This map is essential for the group action on multi-particle states. It is said to be a coproduct on G . We denote it by Δ :

$$\Delta : G \rightarrow G \times G, \quad (2.4.3)$$

$$\Delta(g) = g \times g. \quad (2.4.4)$$

The coproduct exists in the algebra level also. Tensor products of representations of an algebra are in fact determined by Δ [30, 31]. It is a homomorphism

from the group algebra (generalization of the Fourier transform, the group algebra of the group \mathbb{R}^n) G^* to $G^* \otimes G^*$. A coproduct map need not be unique: Not all choices of Δ are equivalent. In particular the Clebsch-Gordan coefficients, that occur in the reduction of group representations, can depend upon Δ . Examples of this sort occur for \bar{P}_+^\uparrow . In any case, it must fulfill

$$\Delta(g_1)\Delta(g_2) = \Delta(g_1g_2), \quad g_1, g_2 \in G \quad (2.4.5)$$

Note that eqn. (2.4.5) implies the coproduct on the group algebra G^* by linearity. If $\alpha, \beta : G \rightarrow \mathbb{C}$ are smooth compactly supported functions on G , then the group algebra G^* contains the generating elements

$$\int d\mu(g)\alpha(g)g, \quad \int d\mu(g')\alpha(g')g', \quad (2.4.6)$$

where $d\mu$ is the measure in G . The coproduct action on G^* is then

$$\begin{aligned} \Delta : G^* &\rightarrow G^* \otimes G^* \\ \int d\mu(g)\alpha(g)g &\rightarrow \int d\mu(g)\alpha(g)\Delta(g). \end{aligned} \quad (2.4.7)$$

The representations U_k of G^* on \mathcal{H}_k ($k = i, j$),

$$U_k : \int d\mu(g)\alpha(g)g \rightarrow \int d\mu(g)\alpha(g)U_k(g) \quad (2.4.8)$$

induced by those of G also extend to the representation $U_i \otimes U_j$ on $\mathcal{H}_i \otimes \mathcal{H}_j$:

$$U_i \otimes U_j : \int d\mu(g)\alpha(g)g \rightarrow \int d\mu(g)\alpha(g)(U_i \otimes U_j)\Delta(g). \quad (2.4.9)$$

Thus the action of a symmetry group on the tensor product of representation spaces carrying any two representations ρ_1 and ρ_2 is determined by Δ :

$$g \triangleright (\alpha \otimes \beta) = (\rho_1 \otimes \rho_2)\Delta(g)(\alpha \otimes \beta). \quad (2.4.10)$$

If the representation space is itself an algebra \mathcal{A} , we have a rule for taking products of elements of \mathcal{A} that involves the multiplication map m :

$$m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad (2.4.11)$$

$$\alpha \otimes \beta \rightarrow m(\alpha \otimes \beta) = \alpha\beta, \quad (2.4.12)$$

where $\alpha, \beta \in \mathcal{A}$.

It is now essential that Δ be compatible with m . That is

$$m\left[(\rho \otimes \rho)\Delta(g)(\alpha \otimes \beta)\right] = \rho(g)m(\alpha \otimes \beta), \quad (2.4.13)$$

where ρ is a representation of the group acting on the algebra.

The compatibility condition (2.4.13) is encoded in the commutative diagram:

$$\begin{array}{ccc} \alpha \otimes \beta & \xrightarrow{g^\triangleright} & (\rho \otimes \rho)\Delta(g)\alpha \otimes \beta \\ m \downarrow & & \downarrow m \end{array} \quad (2.4.14)$$

$$m(\alpha \otimes \beta) \xrightarrow{g^\triangleright} \rho(g)m(\alpha \otimes \beta)$$

If such a Δ can be found, G is an automorphism of \mathcal{A} . In the absence of such a Δ , G does not act on \mathcal{A} .

Let us consider the action of P_+^\uparrow on the noncommutative spacetime algebra (GM plane) $\mathcal{A}_\theta(\mathbb{R}^{d+1})$. The algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ consists of smooth functions on \mathbb{R}^{d+1} with the multiplication map

$$m_\theta : \mathcal{A}_\theta(\mathbb{R}^{d+1}) \otimes \mathcal{A}_\theta(\mathbb{R}^{d+1}) \rightarrow \mathcal{A}_\theta(\mathbb{R}^{d+1}). \quad (2.4.15)$$

For two functions α and β in the algebra \mathcal{A}_θ , the multiplication map is not a point-wise multiplication, it is the $*$ -multiplication:

$$m_\theta(\alpha \otimes \beta)(x) = (\alpha * \beta)(x). \quad (2.4.16)$$

Explicitly the $*$ -product between two functions α and β is written as

$$(\alpha * \beta)(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}\right) \alpha(x)\beta(y) \Big|_{x=y}. \quad (2.4.17)$$

Before implementing the Poincaré group action on \mathcal{A}_θ , we write down a useful expression for m_θ in terms of the commutative multiplication map m_0 ,

$$m_\theta = m_0 \mathcal{F}_\theta, \quad (2.4.18)$$

where

$$\mathcal{F}_\theta = \exp\left(-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta\right), \quad P_\alpha = -i\partial_\alpha \quad (2.4.19)$$

is called the ‘‘Drinfel’d twist’’ or simply the ‘‘twist’’. The indices here are raised or lowered with the Minkowski metric with signature $(+, -, -, -)$.

It is easy to show from this equation that the Poincaré group action through the coproduct $\Delta(g)$ on the noncommutative algebra of functions is not compatible with the $*$ -product. That is, P_+^\dagger does not act on $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ in the usual way. There is a way to implement Poincaré symmetry on noncommutative algebra. Using the twist element, the coproduct of the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra can be deformed in such a way that it is compatible with the above $*$ -multiplication. The deformed coproduct, denoted by Δ_θ is:

$$\Delta_\theta = \mathcal{F}_\theta^{-1} \Delta \mathcal{F}_\theta \quad (2.4.20)$$

We can check compatibility of the twisted coproduct Δ_θ with the twisted multiplication m_θ as follows

$$\begin{aligned} m_\theta((\rho \otimes \rho)\Delta_\theta(g)(\alpha \otimes \beta)) &= m_0(\mathcal{F}_\theta(\mathcal{F}_\theta^{-1}\rho(g) \otimes \rho(g)\mathcal{F}_\theta)\alpha \otimes \beta) \\ &= \rho(g)(\alpha * \beta), \quad \alpha, \beta \in \mathcal{A}_\theta(\mathbb{R}^{d+1}) \end{aligned} \quad (2.4.21)$$

as required. This compatibility is encoded in the commutative diagram

$$\begin{array}{ccc}
\alpha \otimes \beta & \xrightarrow{g^\triangleright} & (\rho \otimes \rho)\Delta_\theta(g)\alpha \otimes \beta \\
m_\theta \downarrow & & \downarrow m_\theta \\
\alpha * \beta & \xrightarrow{g^\triangleright} & \rho(g)(\alpha * \beta)
\end{array} \tag{2.4.22}$$

Thus G is an automorphism of \mathcal{A}_θ if the coproduct is Δ_θ .

It is easy to see that the coproduct for the generators P_α of the Lie algebra of the translation group are not deformed,

$$\Delta_\theta(P_\alpha) = \Delta(P_\alpha) \tag{2.4.23}$$

while the coproduct for the generators of the Lie algebra of the Lorentz group are deformed:

$$\begin{aligned}
\Delta_\theta(M_{\mu\nu}) &= 1 \otimes M_{\mu\nu} + M_{\mu\nu} \otimes 1 - \frac{1}{2} \left[(P \cdot \theta)_\mu \otimes P_\nu - P_\nu \otimes (P \cdot \theta)_\mu - (\mu \leftrightarrow \nu) \right], \\
(P \cdot \theta)_\lambda &= P_\rho \theta_\lambda^\rho.
\end{aligned} \tag{2.4.24}$$

The idea of twisting the coproduct in noncommutative spacetime algebra is due to [29, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 64]. But its origins can be traced back to Drinfel'd [32] in mathematics. This Drinfel'd twist leads naturally to deformed R -matrices and statistics for quantum groups, as discussed by Majid [33]. Subsequently, Fiore and Schupp [35] and Watts [38, 40] explored the significance of the Drinfel'd twist and R -matrices while Fiore [36, 37] and Fiore and Schupp [34], Oeckl [39] and Grosse *et al.* [41] studied the importance of R -matrices for statistics. Oeckl [39] and Grosse *et al.* [41] also developed quantum field theories using different and apparently inequivalent approaches, the first on the Moyal plane and

the second on the q -deformed fuzzy sphere. In [64, 42] the authors focused on the diffeomorphism group \mathcal{D} and developed Riemannian geometry and gravity theories based on Δ_θ , while [29] focused on the Poincaré subgroup \mathcal{P} of \mathcal{D} and explored the consequences of Δ_θ for quantum field theories. Twisted conformal symmetry was discussed by [43]. Recent work, including ours [44, 84, 85, 86, 48, 49, 50], has significant overlap with the earlier literature.

2.4.2 The Twisted Statistics

In the previous section, we discussed how to implement the Poincaré group action in the noncommutative framework. We changed the ordinary coproduct to a twisted coproduct Δ_θ to make it compatible with the multiplication map m_θ . This very process of twisting the coproduct has an impact on statistics. In this section we discuss how the deformed Poincaré symmetry leads to a new kind of statistics for the particles.

Consider a two-particle system in quantum mechanics for the case $\theta^{\mu\nu} = 0$. A two-particle wave function is a function of two sets of variables, and lives in $\mathcal{A}_0 \otimes \mathcal{A}_0$. It transforms according to the usual coproduct Δ . Similarly in the noncommutative case, the two-particle wave function lives in $\mathcal{A}_\theta \otimes \mathcal{A}_\theta$ and transforms according to the twisted coproduct Δ_θ .

In the commutative case, we require that the physical wave functions describing identical particles are either symmetric (bosons) or antisymmetric (fermions), that is, we work with either the symmetrized or antisymmetrized tensor product,

$$\phi \otimes_S \chi \equiv \frac{1}{2} (\phi \otimes \chi + \chi \otimes \phi), \quad (2.4.25)$$

$$\phi \otimes_A \chi \equiv \frac{1}{2} (\phi \otimes \chi - \chi \otimes \phi). \quad (2.4.26)$$

which satisfies

$$\phi \otimes_S \chi = +\chi \otimes_S \phi, \quad (2.4.27)$$

$$\phi \otimes_A \chi = -\chi \otimes_A \phi. \quad (2.4.28)$$

These relations have to hold in all frames of reference in a Lorentz-invariant theory. That is, symmetrization and antisymmetrization must commute with the Lorentz group action.

Since $\Delta(g) = g \times g$, we have

$$\tau_0(\rho \otimes \rho)\Delta(g) = (\rho \times \rho)\Delta(g)\tau_0, \quad g \in P_+^\uparrow \quad (2.4.29)$$

where τ_0 is the flip operator:

$$\tau_0(\phi \otimes \chi) = \chi \otimes \phi. \quad (2.4.30)$$

Since

$$\phi \otimes_{S,A} \chi = \frac{1 \pm \tau_0}{2} \phi \otimes \chi, \quad (2.4.31)$$

we see that Lorentz transformations preserve symmetrization and anti-symmetrization.

The twisted coproduct action of the Lorentz group is not compatible with the usual symmetrization and anti-symmetrization. The origin of this fact can be traced to the fact that the coproduct is not cocommutative except when $\theta^{\mu\nu} = 0$. That is,

$$\tau_0 \mathcal{F}_\theta = \mathcal{F}_\theta^{-1} \tau_0, \quad (2.4.32)$$

$$\tau_0(\rho \otimes \rho)\Delta_\theta(g) = (\rho \otimes \rho)\Delta_{-\theta}(g)\tau_0 \quad (2.4.33)$$

One can easily construct an appropriate deformation τ_θ of the operator τ_0 using the twist operator \mathcal{F}_θ and the definition of the twisted coproduct, such that

it commutes with Δ_θ . Since $\Delta_\theta(g) = \mathcal{F}_\theta^{-1}\Delta(g)\mathcal{F}_\theta$, it is

$$\tau_\theta = \mathcal{F}_\theta^{-1}\tau_0\mathcal{F}_\theta. \quad (2.4.34)$$

It has the property,

$$(\tau_\theta)^2 = \mathbf{1} \otimes \mathbf{1}. \quad (2.4.35)$$

The states constructed according to

$$\phi \otimes_{S_\theta} \chi \equiv \left(\frac{1 + \tau_\theta}{2} \right) (\phi \otimes \chi), \quad (2.4.36)$$

$$\phi \otimes_{A_\theta} \chi \equiv \left(\frac{1 - \tau_\theta}{2} \right) (\phi \otimes \chi) \quad (2.4.37)$$

form the physical two-particle Hilbert spaces of (generalized) bosons and fermions obeying twisted statistics.

2.4.3 Statistics of Quantum Fields

The very act of implementing Poincaré symmetry on a noncommutative spacetime algebra leads to twisted fermions and bosons. In this section we look at the second quantized version of the theory and we encounter another surprise on the way.

We can connect an operator in Hilbert space and a quantum field in the following way. A quantum field on evaluation at a spacetime point gives an operator-valued distribution acting on a Hilbert space. A quantum field at a spacetime point x_1 acting on the vacuum gives a one-particle state centered at x_1 . Similarly we can construct a two-particle state in the Hilbert space. The product of two quantum fields at spacetime points x_1 and x_2 when acting on the vacuum generates a two-particle state where one particle is centered at x_1 and the other at x_2 .

In the commutative case, a free spin-zero quantum scalar field $\varphi_0(x)$ of mass m has the mode expansion

$$\varphi_0(x) = \int d\mu(p) (c_{\mathbf{p}} e_p(x) + d_{\mathbf{p}}^\dagger e_{-p}(x)) \quad (2.4.38)$$

where

$$e_p(x) = e^{-i p \cdot x}, \quad p \cdot x = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}, \quad d\mu(p) = \frac{1}{(2\pi)^3} \frac{d^3 p}{2p_0}, \quad p_0 = \sqrt{\mathbf{p}^2 + m^2} > 0.$$

The annihilation-creation operators $c_{\mathbf{p}}$, $c_{\mathbf{p}}^\dagger$, $d_{\mathbf{p}}$, $d_{\mathbf{p}}^\dagger$ satisfy the standard commutation relations,

$$c_{\mathbf{p}} c_{\mathbf{q}}^\dagger \pm c_{\mathbf{q}}^\dagger c_{\mathbf{p}} = 2p_0 \delta^3(\mathbf{p} - \mathbf{q}) \quad (2.4.39)$$

$$d_{\mathbf{p}} d_{\mathbf{q}}^\dagger \pm d_{\mathbf{q}}^\dagger d_{\mathbf{p}} = 2p_0 \delta^3(\mathbf{p} - \mathbf{q}). \quad (2.4.40)$$

The remaining commutators involving these operators vanish.

If $c_{\mathbf{p}}$ is the annihilation operator of the second-quantized field $\varphi_0(x)$, an elementary calculation tells us that

$$\langle 0 | \varphi_0(x) c_{\mathbf{p}}^\dagger | 0 \rangle = e_p(x) = e^{-i p \cdot x}.$$

$$\begin{aligned} \frac{1}{2} \langle 0 | \varphi_0(x_1) \varphi_0(x_2) c_{\mathbf{q}}^\dagger c_{\mathbf{p}}^\dagger | 0 \rangle &= \left(\frac{\mathbf{1} \pm \tau_0}{2} \right) (e_p \otimes e_q)(x_1, x_2) \\ &\equiv (e_p \otimes_{S_0, A_0} e_q)(x_1, x_2) \\ &\equiv \langle x_1, x_2 | p, q \rangle_{S_0, A_0}. \end{aligned} \quad (2.4.41)$$

where we have used the commutation relation

$$c_{\mathbf{p}}^\dagger c_{\mathbf{q}}^\dagger = \pm c_{\mathbf{q}}^\dagger c_{\mathbf{p}}^\dagger. \quad (2.4.42)$$

From the previous section we have learned that the two-particle states in non-commutative spacetime should be constructed in such a way that they obey twisted

symmetry. That is,

$$|p, q\rangle_{S_0, A_0} \rightarrow |p, q\rangle_{S_\theta, A_\theta}. \quad (2.4.43)$$

This can happen only if we modify the quantum field $\varphi_0(x)$ in such a way that the analogue of eqn. (2.4.41) in the noncommutative framework gives us $|p, q\rangle_{S_\theta, A_\theta}$.

Let us denote the modified quantum field by φ_θ . It has a mode expansion

$$\varphi_\theta(x) = \int d\mu(p) (a_{\mathbf{p}} e_p(x) + b_{\mathbf{p}}^\dagger e_{-p}(x)) \quad (2.4.44)$$

Noncommutativity of spacetime does not change the dispersion relation for the quantum field in our framework. It will definitely change the operator coefficients of the plane wave basis. Here we denote the new θ -deformed annihilation-creation operators by $a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger, b_{\mathbf{p}}, b_{\mathbf{p}}^\dagger$. Let us try to connect the quantum field in noncommutative spacetime with its counterpart in commutative spacetime, keeping in mind that they should coincide in the limit $\theta^{\mu\nu} \rightarrow 0$.

The two-particle state $|p, q\rangle_{S_\theta, A_\theta}$ for bosons and fermions obeying deformed statistics is constructed as follows:

$$\begin{aligned} |p, q\rangle_{S_\theta, A_\theta} &\equiv |p\rangle \otimes_{S_\theta, A_\theta} |q\rangle = \left(\frac{1 \pm \tau_\theta}{2}\right) (|p\rangle \otimes |q\rangle) \\ &= \frac{1}{2} \left(|p\rangle \otimes |q\rangle \pm e^{-iq_\mu \theta^{\mu\nu} p_\nu} |q\rangle \otimes |p\rangle \right). \end{aligned} \quad (2.4.45)$$

Exchanging p and q in the above, one finds

$$|p, q\rangle_{S_\theta, A_\theta} = \pm e^{ip_\mu \theta^{\mu\nu} q_\nu} |q, p\rangle_{S_\theta, A_\theta}. \quad (2.4.46)$$

In Fock space the above two-particle state is constructed from the modified second-quantized field φ_θ according to

$$\begin{aligned} \frac{1}{2} \langle 0 | \varphi_\theta(x_1) \varphi_\theta(x_2) a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger | 0 \rangle &= \left(\frac{1 \pm \tau_\theta}{2}\right) (e_p \otimes e_q)(x_1, x_2) \\ &= (e_p \otimes_{S_\theta, A_\theta} e_q)(x_1, x_2) \\ &= \langle x_1, x_2 | p, q \rangle_{S_\theta, A_\theta}. \end{aligned} \quad (2.4.47)$$

On using eqn. (3.3.7), this leads to the relation

$$a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger = \pm e^{ip_\mu \theta^{\mu\nu} q_\nu} a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger. \quad (2.4.48)$$

It implies

$$a_{\mathbf{p}} a_{\mathbf{q}} = \pm e^{ip_\mu \theta^{\mu\nu} q_\nu} a_{\mathbf{q}} a_{\mathbf{p}}. \quad (2.4.49)$$

Thus we have a new type of bilinear relations reflecting the deformed quantum symmetry.

This result shows that while constructing a quantum field theory on noncommutative spacetime, we should twist the creation and annihilation operators in addition to the $*$ -multiplication between the fields.

In the limit $\theta^{\mu\nu} = 0$, the twisted creation and annihilation operators should match with their counterparts in the commutative case. There is a way to connect these operators in the two cases. The transformation connecting the twisted operators, $a_{\mathbf{p}}$, $b_{\mathbf{p}}$, and the untwisted operators, $c_{\mathbf{p}}$, $d_{\mathbf{p}}$, is called the “dressing transformation” [51, 52]. It is defined as follows:

$$a_{\mathbf{p}} = c_{\mathbf{p}} e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \quad b_{\mathbf{p}} = d_{\mathbf{p}} e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \quad (2.4.50)$$

where P_μ is the four-momentum operator,

$$P_\mu = \int \frac{d^3 p}{2p_0} (c_{\mathbf{p}}^\dagger c_{\mathbf{p}} + d_{\mathbf{p}}^\dagger d_{\mathbf{p}}) p_\mu. \quad (2.4.51)$$

The Grosse-Faddeev-Zamolodchikov algebra is the above twisted or dressed algebra [51, 52]. (See also [53] in this connection.)

Note that the four-momentum operator P_μ can also be written in terms of the twisted operators:

$$P_\mu = \int \frac{d^3 p}{2p_0} (a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{\mathbf{p}}) p_\mu. \quad (2.4.52)$$

That is because $p_\mu \theta^{\mu\nu} P_\nu$ commutes with any of the operators for momentum p . For example

$$[P_\mu, a_{\mathbf{p}}] = -p_\mu a_{\mathbf{p}}, \quad (2.4.53)$$

so that

$$[p_\nu \theta^{\nu\mu} P_\mu, a_{\mathbf{p}}] = p_\nu \theta^{\nu\mu} p_\mu = 0, \quad (2.4.54)$$

θ being antisymmetric.

The antisymmetry of $\theta^{\mu\nu}$ allows us to write

$$c_{\mathbf{p}} e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c_{\mathbf{p}}, \quad (2.4.55)$$

$$c_{\mathbf{p}}^\dagger e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} = e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} c_{\mathbf{p}}^\dagger. \quad (2.4.56)$$

Hence the ordering of factors here is immaterial.

It should also be noted that the map from the c - to the a -operators is invertible,

$$c_{\mathbf{p}} = a_{\mathbf{p}} e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \quad d_{\mathbf{p}} = b_{\mathbf{p}} e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu},$$

where P_μ is written as in eqn. (2.4.52).

The \star -product between the modified (twisted) quantum fields is

$$(\varphi_\theta \star \varphi_\theta)(x) = \varphi_\theta(x) e^{\frac{i}{2} \overleftarrow{\partial} \wedge \overrightarrow{\partial}} \varphi_\theta(y)|_{x=y}, \quad (2.4.57)$$

$$\overleftarrow{\partial} \wedge \overrightarrow{\partial} := \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu.$$

The twisted quantum field φ_θ differs from the untwisted quantum field φ_0 in two ways:

$$i.) e_p \in \mathcal{A}_\theta(\mathbb{R}^{d+1})$$

and

ii.) $a_{\mathbf{p}}$ is twisted by statistics.

The twisted statistics can be accounted by writing [86]

$$\varphi_\theta = \varphi_0 e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}, \quad (2.4.58)$$

where P_μ is the total momentum operator. From this follows that the \star -product of an arbitrary number of fields $\varphi_\theta^{(i)}$ ($i = 1, 2, 3, \dots$) is

$$\varphi_\theta^{(1)} \star \varphi_\theta^{(2)} \star \dots = (\varphi_0^{(1)} \varphi_0^{(2)} \dots) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}. \quad (2.4.59)$$

Similar deformations occur for all tensorial and spinorial quantum fields.

In [54], a noncommutative cosmic microwave background (CMB) power spectrum is calculated by promoting the quantum fluctuations φ_0 of the scalar field driving inflation (the inflaton) to a twisted quantum field φ_θ . The power spectrum becomes direction-dependent, breaking the statistical anisotropy of the CMB. Also, n -point correlation functions become non-Gaussian when the fields are noncommutative, assuming that they are Gaussian in their commutative limits. These effects can be tested experimentally.

In this chapter we discuss field theory with spacetime noncommutativity. It should also be noted that there is another approach in which noncommutativity is encoded in the degrees of freedom of the fields while keeping spacetime commutative [55, 56]. Such noncommutativity can also be interpreted in terms of twisted statistics. In [53] a noncommutative black body spectrum is calculated using this approach (which is based on [55, 56]). Also, a noncommutative-gas driven inflation is considered in [53] along this formulation.

2.4.4 From Twisted Statistics to Noncommutative Spacetime

Noncommutative spacetime leads to twisted statistics. It is also possible to start from a twisted statistics and end up with a noncommutative spacetime [22, 57]. Consider the commutative version φ_0 of the above quantum field φ_θ . The creation and annihilation operators of this field fulfill the standard commutation relations as given in eqn. (2.4.39).

Let us twist statistics by deforming the creation-annihilation operators $c_{\mathbf{p}}$ and $c_{\mathbf{p}}^\dagger$ to

$$a_{\mathbf{p}} = c_{\mathbf{p}} e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \quad a_{\mathbf{p}}^\dagger = c_{\mathbf{p}}^\dagger e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu} \quad (2.4.60)$$

Now statistics is twisted since a 's and a^\dagger 's no longer fulfill standard relations. They obey the relations given in eqn. (2.4.48) and eqn. (2.4.49). This twist affects the usual symmetry of particle interchange. The n -particle wave function $\psi_{k_1 \dots k_n}$,

$$\psi_{k_1, \dots, k_n}(x_1, \dots, x_n) = \langle 0 | \varphi(x_1) \varphi(x_2) \dots \varphi(x_n) a_{\mathbf{k}_n}^\dagger a_{\mathbf{k}_{n-1}}^\dagger \dots a_{\mathbf{k}_1}^\dagger | 0 \rangle \quad (2.4.61)$$

is no longer symmetric under the interchange of k_i . It fulfills a twisted symmetry given by

$$\psi_{k_1 \dots k_i k_{i+1} \dots k_n} = \exp\left(-ik_i^\mu \theta_{\mu\nu} k_{i+1}^\nu\right) \psi_{k_1 \dots k_{i+1} k_i \dots k_n} \quad (2.4.62)$$

showing that statistics is twisted. We can show that this in fact leads to a noncommutative spacetime if we require Poincaré invariance. It is explained below.

In the commutative case, the elements g of P_+^\dagger acts on $\psi_{k_1 \dots k_n}$ by the representative $U(g) \otimes U(g) \otimes \dots \otimes U(g)$ (n factors) compatibly with the symmetry of $\psi_{k_1 \dots k_n}$. This action is based on the coproduct

$$\Delta(g) = g \times g. \quad (2.4.63)$$

But for $\theta^{\mu\nu} \neq 0$, and for $g \neq$ identity, already for the case $n = 2$,

$$\begin{aligned}
\Delta(g)\psi_{p,q} &= \psi_{gp,gq} \\
&= e^{-ip_\mu\theta^{\mu\nu}q_\nu} \Delta(g)\psi_{q,p} \\
&= e^{-ip_\mu\theta^{\mu\nu}q_\nu} \psi_{gq,gp} \\
&\neq e^{-i(gp)_\mu\theta^{\mu\nu}(gq)_\nu} \psi_{gq,gp}.
\end{aligned} \tag{2.4.64}$$

Thus the usual coproduct Δ_0 is not compatible with the statistics (2.4.62). It has to be twisted to

$$\Delta_\theta(g) = \mathcal{F}_\theta^{-1}\Delta(g)\mathcal{F}_\theta, \quad \Delta(g) = (g \times g) \tag{2.4.65}$$

to be compatible with the new statistics. At this point $\Delta_\theta(g)$ is not compatible with m_0 , the commutative (point-wise) multiplication map. So we are forced to change the multiplication map to m_θ ,

$$m_\theta = m_0 \mathcal{F}_\theta \tag{2.4.66}$$

for this compatibility. Since

$$m_\theta(\alpha \otimes \beta) = \alpha * \beta, \tag{2.4.67}$$

we end up with noncommutative spacetime. Thus twisted statistics can lead to spacetime noncommutativity.

2.4.5 Violation of the Pauli Principle

In section 4.3, we wrote down the twisted commutation relations. In the fermionic sector, these relations read

$$a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger + e^{ip_\mu\theta^{\mu\nu}q_\nu} a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger = 0 \tag{2.4.68}$$

$$a_{\mathbf{p}} a_{\mathbf{q}}^\dagger + e^{-ip_\mu\theta^{\mu\nu}q_\nu} a_{\mathbf{q}}^\dagger a_{\mathbf{p}} = 2q_0\delta^3(\mathbf{p} - \mathbf{q}). \tag{2.4.69}$$

In the commutative case, above relations read

$$c_{\mathbf{p}}^{\dagger}c_{\mathbf{q}}^{\dagger} + c_{\mathbf{q}}^{\dagger}c_{\mathbf{p}}^{\dagger} = 0 \quad (2.4.70)$$

$$c_{\mathbf{p}}c_{\mathbf{q}}^{\dagger} + c_{\mathbf{q}}^{\dagger}c_{\mathbf{p}} = 2q_0\delta^3(\mathbf{p} - \mathbf{q}). \quad (2.4.71)$$

The phase factor appearing in eqn (2.4.68) and eqn. (2.4.69) while exchanging the operators has a nontrivial physical consequence which forces us to reconsider the Pauli exclusion principle. A modification of Pauli principle compatible with the twisted statistics can lead to Pauli forbidden processes and they can be subjected to stringent experimental tests.

For example, there are results from SuperKamiokande [58] and Borexino [59] putting limits on the violation of Pauli exclusion principle in nucleon systems. These results are based on non-observed transition from Pauli-allowed states to Pauli-forbidden states with β^{\pm} decays or γ , p , n emission. A bound for θ as strong as 10^{11} Gev is obtained from these results [60].

2.4.6 Statistcal Potential

Twisting the statistics can modify the spatial correlation functions of fermions and bosons and thus affect the statistical potential existing between any two particles.

Consider a canonical ensemble, a system of N indistinguishable, non-interacting particles confined to a three-dimensional cubical box of volume V , characterized by the inverse temperature β . In the coordinate representation, we write down the density matrix of the system [61]

$$\langle \mathbf{r}_1, \dots, \mathbf{r}_N | \hat{\rho} | \mathbf{r}'_1, \dots, \mathbf{r}'_N \rangle = \frac{1}{Q_N(\beta)} \langle \mathbf{r}_1, \dots, \mathbf{r}_N | e^{-\beta \hat{H}} | \mathbf{r}'_1, \dots, \mathbf{r}'_N \rangle, \quad (2.4.72)$$

where $Q_N(\beta)$ is the partition function of the system given by

$$Q_N(\beta) = \text{Tr}(e^{-\beta \hat{H}}) = \int d^{3N} r \langle \mathbf{r}_1, \dots, \mathbf{r}_N | e^{-\beta \hat{H}} | \mathbf{r}'_1, \dots, \mathbf{r}'_N \rangle. \quad (2.4.73)$$

Since the particles are non-interacting, we may write down the eigenfunctions and eigenvalues of the system in terms of the single-particle wave functions and single-particle energies.

For free non-relativistic particles, we have the energy eigenvalues

$$E = \frac{\hbar^2}{2m} \sum_{i=1}^N k_i^2 \quad (2.4.74)$$

where k_i is the magnitude of the wave vector of the i -th particle. Imposing periodic boundary conditions, we write down the normalized single-particle wave function

$$u_{\mathbf{k}}(\mathbf{r}) = V^{-1/2} e^{i\mathbf{k}\cdot\mathbf{r}} \quad (2.4.75)$$

with $\mathbf{k} = 2\pi V^{-1/3} \mathbf{n}$ and \mathbf{n} is a three-dimensional vector whose components take values $0, \pm 1, \pm 2, \dots$.

Following the steps given in [61], we write down the diagonal elements of the density matrix for the simplest relevant case with $N = 2$,

$$\langle \mathbf{r}_1, \mathbf{r}_2 | \hat{\rho} | \mathbf{r}_1, \mathbf{r}_2 \rangle \approx \frac{1}{V^2} (1 \pm \exp(-2\pi r_{12}^2 / \lambda^2)) \quad (2.4.76)$$

where the plus and the minus signs indicate bosons and fermions respectively, $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ and λ is the mean thermal wavelength,

$$\lambda = \hbar \sqrt{\frac{2\pi\beta}{m}}, \quad \beta = \frac{1}{k_B T}. \quad (2.4.77)$$

Note that eqn. (2.4.76) is obtained under the assumption that the mean interparticle distance $(V/N)^{1/3}$ in the system is much larger than the mean thermal wavelength λ . Eqn. (2.4.76) indicates that spatial correlations are non-zero even when the particles are non-interacting. These correlations are purely due to statistics: They emerge from the symmetrization or anti-symmetrization of the wave

functions describing the particles. Particles obeying Bose statistics give a positive spatial correlation and particles obeying Fermi statistics give a negative spatial correlation.

We can express spatial correlations between particles by introducing a statistical potential $v_s(r)$ and thus treat the particles classically [62]. The statistical potential corresponding to the spatial correlation given in eqn. (2.4.76) is

$$v_s(r) = -k_B T \ln \left(1 \pm \exp(-2\pi r_{12}^2 / \lambda^2) \right) \quad (2.4.78)$$

From this equation, it follows that two bosons always experience a “statistical attraction” while two fermions always experience a “statistical repulsion”. In both cases, the potential decays rapidly when $r > \lambda$.

So far our discussion focussed on particles in commutative spacetime. We can derive an expression for the statistical potential between two particles living in a noncommutative spacetime. The results [63] are interesting. In a noncommutative spacetime with 2+1 dimensions and for the case $\theta^{0i} = 0$, we write down the answer for the spatial correlation between two non-interacting particles from [63]

$$\langle \mathbf{r}_1, \mathbf{r}_2 | \hat{\rho} | \mathbf{r}_1, \mathbf{r}_2 \rangle_\theta \approx \frac{1}{A^2} \left(1 \pm \frac{1}{1 + \frac{\theta^2}{\lambda^4}} e^{-2\pi r_{12}^2 / (\lambda^2(1 + \frac{\theta^2}{\lambda^4}))} \right) \quad (2.4.79)$$

Here A is the area of the system. This result can be generalized to higher dimensions by replacing θ^2 by an appropriate sum of $(\theta^{ij})^2$ [63]. It reduces to the standard (untwisted) result given in eqn. (2.4.76) in the limit $\theta \rightarrow 0$. Notice that the spatial correlation function for fermions does not vanish in the limit $r \rightarrow 0$ (See Fig. 2.1). That means that there is a finite probability that fermions may come very close to each other. This probability is determined by the noncommutativity parameter θ . Also notice that the assumptions made in [63] are valid for low temperature and low density limits. At high temperature and high density

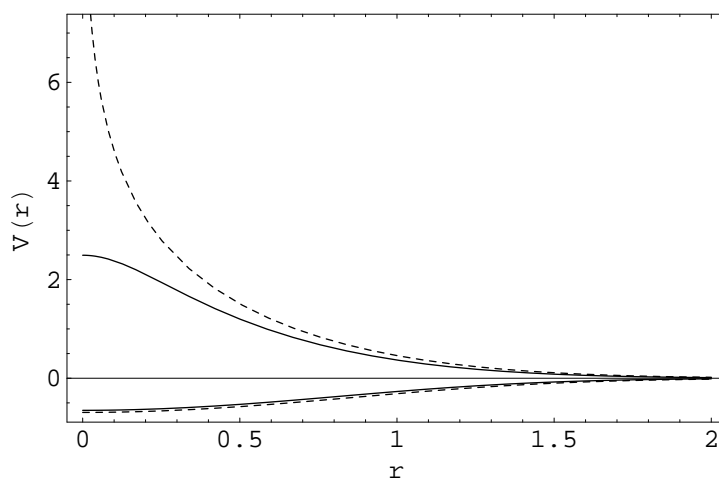


Figure 2.1: Statistical potential $v(r)$ measured in units of $k_B T$. An irrelevant additive constant has been set zero. The upper two curves represent the fermionic cases and the lower curves the bosonic cases. The solid line shows the noncommutative result and the dashed line the commutative case. The curves are drawn for the value $\frac{\theta}{\lambda^2} = 0.3$. The separation r is measured in units of the thermal length λ . [63]

limits a much more careful analysis is required to investigate the noncommutative effects.

2.5 Matter Fields, Gauge Fields and Interactions

In section 4, we discussed the statistics of quantum fields by taking a simple example of a massive, spin-zero quantum field. In this section, we discuss how matter and gauge fields are constructed in the noncommutative formulation and their interactions. We also explain some interesting results which can be verified experimentally.

2.5.1 Pure Matter Fields

Consider a second quantized real Hermitian field of mass m ,

$$\Phi = \Phi^- + \Phi^+ \tag{2.5.1}$$

where the creation and annihilation fields are constructed from the creation and annihilation operators:

$$\Phi^-(x) = \int d\mu(p) e^{ipx} a_{\mathbf{p}}^\dagger \tag{2.5.2}$$

$$\Phi^+(x) = \int d\mu(p) e^{-ipx} a_{\mathbf{p}} \tag{2.5.3}$$

The deformed quantum field Φ can be written in terms of the un-deformed quantum field Φ_0 ,

$$\Phi(x) = \Phi_0(x) e^{\frac{1}{2} \overleftarrow{\partial}^\mu \theta_{\mu\nu} P^\nu} \tag{2.5.4}$$

where the creation and annihilation fields of the un-deformed quantum field is constructed from the usual creation and annihilation operators

$$\Phi_0^-(x) = \int d\mu(p) e^{ipx} c_{\mathbf{p}}^\dagger, \quad (2.5.5)$$

$$\Phi_0^+(x) = \int d\mu(p) e^{-ipx} c_{\mathbf{p}} \quad (2.5.6)$$

When evaluating the product of Φ 's at the same point, we must take $*$ -product of the e_p 's since $e_p \in \mathcal{A}_\theta(\mathbb{R}^N)$. We can make use of eqn. (2.5.4) to simplify the $*$ -product of Φ 's at the same point to a commutative (point-wise) product of Φ_0 's. For the $*$ -product of n Φ 's,

$$\Phi(x) * \Phi(x) * \dots * \Phi(x) = \left(\Phi_0(x) \right)^n e^{\frac{1}{2} \overleftarrow{\partial}^\mu \theta_{\mu\nu} P^\nu} \quad (2.5.7)$$

This is a very important result. Using this result, we can prove that there is no UV-IR mixing in a noncommutative field theory with matter fields and no gauge interactions [39, 85].

The interaction Hamiltonian density is built out of quantum fields. It transforms like a single scalar field in the noncommutative theory also. (This is the case only when we choose a $*$ -product between the fields to write down the Hamiltonian density.) Thus a generic interaction Hamiltonian density \mathcal{H}_I involving only Φ 's (for simplicity) is given by

$$\mathcal{H}_I(x) = \Phi(x) * \Phi(x) * \dots * \Phi(x) \quad (2.5.8)$$

This form of the Hamiltonian and the twisted statistics of the fields is all that is required to show that there is no UV-IR mixing in this theory. This happens because the S -matrix becomes independent of $\theta^{\mu\nu}$.

We illustrate this result for the first nontrivial term $S^{(1)}$ in the expansion of

the S -matrix. It is

$$S^{(1)} = \int d^4x \mathcal{H}_I(x). \quad (2.5.9)$$

Using eqn. (2.5.4) we write down the interaction Hamiltonian density given in eqn. (2.5.8) as

$$\mathcal{H}_I(x) = \left(\Phi_0(x)\right)^n e^{\frac{1}{2}\overleftarrow{\partial}^{\mu}\theta_{\mu\nu}P^{\nu}} \quad (2.5.10)$$

Assuming that the fields behave “nicely” at infinity, the integration over x gives

$$\int d^4x \left(\Phi_*(x)\right)^n = \int d^4x \left(\Phi_0(x)\right)^n e^{\frac{1}{2}\overleftarrow{\partial}^{\mu}\theta_{\mu\nu}P^{\nu}} = \int d^4x \left(\Phi_0(x)\right)^n. \quad (2.5.11)$$

Thus $S^{(1)}$ is independent of $\theta^{\mu\nu}$. By similar calculations we can show that the S -operator is independent of $\theta^{\mu\nu}$ to all orders [84, 85, 86, 48].

2.5.2 Covariant Derivatives of Quantum Fields

In this section we briefly discuss how to choose appropriate covariant derivatives D_μ of a quantum field associated with $\mathcal{A}_\theta(\mathbb{R}^{3+1})$.

To define the desirable properties of covariant derivatives D_μ , let us first look at ways of multiplying the field Φ_θ by a function $\alpha_0 \in \mathcal{A}_0(\mathbb{R}^{3+1})$. There are two possibilities [86]:

$$\Phi \rightarrow (\Phi_0\alpha_0)e^{\frac{1}{2}\overleftarrow{\partial}^{\mu}\wedge P} \equiv T_0(\alpha_0)\Phi, \quad (2.5.12)$$

$$\Phi \rightarrow (\Phi_0 *_\theta \alpha_0)e^{\frac{1}{2}\overleftarrow{\partial}^{\mu}\wedge P} \equiv T_\theta(\alpha_0)\Phi \quad (2.5.13)$$

where T_0 gives a representation of the commutative algebra of functions and T_θ gives that of a $*$ -algebra.

A D_μ that can qualify as the covariant derivative of a quantum field associated with $\mathcal{A}_0(\mathbb{R}^{3+1})$ should preserve statistics, Poincaré and gauge invariance and must obey the Leibnitz rule

$$D_\mu(T_0(\alpha_0)\Phi) = T_0(\alpha_0)(D_\mu\Phi) + T_0(\partial_\mu\alpha_0)\Phi \quad (2.5.14)$$

The requirement given in eqn. (2.5.14) reflects the fact that D_μ is associated with the commutative algebra $\mathcal{A}_0(\mathbb{R}^{3+1})$.

There are two immediate choices for $D_\mu\Phi$:

$$1. \quad D_\mu\Phi = ((D_\mu)_0\Phi_0)e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}, \quad (2.5.15)$$

$$2. \quad D_\mu\Phi = ((D_\mu)_0e^{\frac{1}{2}\overleftarrow{\partial}\wedge P})(\Phi_0)e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \quad (2.5.16)$$

where $(D_\mu)_0 = \partial_\mu + (A_\mu)_0$ and $(A_\mu)_0$ is the commutative gauge field, a function only of the commutative coordinates x_c .

Both the choices preserve statistics, Poincaré and gauge invariance, but the second choice does not satisfy eqn. (2.5.14). Thus we identify the correct covariant derivative in our formalism as the one given in the first choice, eqn. (2.5.15).

2.5.3 Matter fields with gauge interactions

We assume that gauge (and gravity) fields are commutative fields, which means that they are functions only of x_c^μ . For Aschieri et al. [64], instead, they are associated with $\mathcal{A}_\theta(\mathbb{R}^{3+1})$. Matter fields on $\mathcal{A}_\theta(\mathbb{R}^{3+1})$ must be transported by the connection compatibly with eqn. (2.5.4), so from the previous section, we see that the natural choice for covariant derivative is

$$D_\mu\Phi = (D_\mu^c\Phi_0)e^{\frac{i}{2}\overleftarrow{\partial}\wedge P}, \quad (2.5.17)$$

where

$$D_\mu^c \Phi_0 = \partial_\mu \Phi_0 + A_\mu \Phi_0, \quad (2.5.18)$$

P_μ is the total momentum operator for all the fields and the fields A_μ and Φ_0 are multiplied point-wise,

$$A_\mu \Phi_0(x) = A_\mu(x) \Phi_0(x). \quad (2.5.19)$$

Having identified the correct covariant derivative, it is simple to write down the Hamiltonian for gauge theories. The commutator of two covariant derivatives gives us the curvature. On using eqn. (2.5.17),

$$[D_\mu, D_\nu] \Phi = \left([D_\mu^c, D_\nu^c] \Phi_0 \right) e^{\frac{i}{2} \overleftarrow{\partial} \wedge P} \quad (2.5.20)$$

$$= \left(F_{\mu\nu}^c \Phi_0 \right) e^{\frac{i}{2} \overleftarrow{\partial} \wedge P}. \quad (2.5.21)$$

As $F_{\mu\nu}^c$ is the standard $\theta^{\mu\nu} = 0$ curvature, our gauge field is associated with $\mathcal{A}_0(\mathbb{R}^{3+1})$. Thus pure gauge theories on the GM plane are identical to their counterparts on commutative spacetime. (For Aschieri et al. [64] the curvature would be the \star -commutator of D_μ 's.)

The gauge theory formulation we adopt here is fully explained in [86]. It differs from the formulation of Aschieri et al. [64] (where covariant derivative is defined using star product) and has the advantage of being able to accommodate any gauge group and not just $U(N)$ gauge groups and their direct products. The gauge theory formulation we adopt here thus avoids multiplicity of fields that the expression for covariant derivatives with \star product entails.

In the single-particle sector (obtained by taking the matrix element of eqn. (2.5.17) between vacuum and one-particle states), the P term can be dropped and we get for a single particle wave function f of a particle associated with Φ ,

$$D_\mu f(x) = \partial_\mu f(x) + A_\mu(x) f(x). \quad (2.5.22)$$

Note that we can also write $D_\mu \Phi$ using \star -product:

$$D_\mu \Phi = \left(D_\mu^c e^{\frac{i}{2} \overleftarrow{\partial} \wedge P} \right) \star \left(\Phi_0 e^{\frac{i}{2} \overleftarrow{\partial} \wedge P} \right). \quad (2.5.23)$$

Our choice of covariant derivative allows us to write the interaction Hamiltonian density for pure gauge fields as follows:

$$\mathcal{H}_{I\theta}^G = \mathcal{H}_{I0}^G. \quad (2.5.24)$$

For a theory with matter and gauge fields, the interaction Hamiltonian density splits into two parts,

$$\mathcal{H}_{I\theta} = \mathcal{H}_{I\theta}^{M,G} + \mathcal{H}_{I\theta}^G, \quad (2.5.25)$$

where

$$\begin{aligned} \mathcal{H}_{I\theta}^{M,G} &= \mathcal{H}_{I0}^{M,G} e^{\frac{i}{2} \overleftarrow{\partial} \wedge P}, \\ \mathcal{H}_{I\theta}^G &= \mathcal{H}_{I0}^G. \end{aligned} \quad (2.5.26)$$

The matter-gauge field couplings are also included in $\mathcal{H}_{I\theta}^{M,G}$.

In quantum electrodynamics (*QED*), $\mathcal{H}_{I\theta}^G = 0$. Thus the S -operator for the twisted *QED* is the same for the untwisted *QED*:

$$S_\theta^{QED} = S_0^{QED}. \quad (2.5.27)$$

In a non-abelian gauge theory, $\mathcal{H}_\theta^G = \mathcal{H}_0^G \neq 0$, so that in the presence of nonsinglet matter fields [86],

$$S_\theta^{M,G} \neq S_0^{M,G}, \quad (2.5.28)$$

because of the cross-terms between $\mathcal{H}_{I\theta}^{M,G}$ and $\mathcal{H}_{I\theta}^G$. In particular, this inequality happens in QCD. One such example is the quark-gluon scattering through a gluon exchange. The Feynman diagram for this process is given in Fig. 2.2.

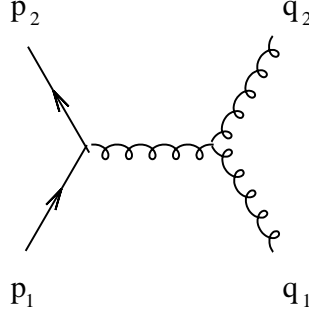


Figure 2.2: A Feynman diagram in QCD with non-trivial θ -dependence. The twist of $\mathcal{H}_{I0}^{M,G}$ changes the gluon propagator. The propagator is different from the usual one by its dependence on terms of the form $\vec{\theta}^0 \cdot \mathbf{P}_{in}$, where $(\vec{\theta}^0)_i = \theta^{0i}$ and \mathbf{P}_{in} is the total momentum of the incoming particles. Such a frame-dependent modification violates Lorentz invariance.

2.5.4 Causality and Lorentz Invariance

The very process of replacing the point-wise multiplication of functions at the same point by a $*$ -multiplication makes the theory non-local. The $*$ -product contains an infinite number of space-time derivatives and this in turn affects the fundamental causal structure on which all local, point-like quantum field theories are built upon.

Let \mathcal{H}_I be the interaction Hamiltonian density in the interaction representation. The interaction representation S -matrix is

$$S = T \exp\left(-i \int d^4x \mathcal{H}_I(x)\right). \quad (2.5.29)$$

In a commutative theory, the interaction Hamiltonian density \mathcal{H}_I satisfies the Bogoliubov - Shirkov [65] causality

$$[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0, \quad x \sim y \quad (2.5.30)$$

where $x \sim y$ means x and y are space-like separated.

This causality relation plays a crucial role in maintaining the Lorentz invariance in all the local, point-like quantum field theories. Weinberg [66, 67] has discussed the fundamental significance of this equation in connection with the relativistic invariance of the S -matrix. If eqn. (2.5.30) fails, S cannot be relativistically invariant.

To see why this is the case, we consider the lowest term $S^{(2)}$ of the S -matrix containing non-trivial time ordering. It is $S^{(2)} = -\frac{1}{2} \int d^4x d^4y T(\mathcal{H}_I(x)\mathcal{H}_I(y))$, where

$$\begin{aligned} T(\mathcal{H}_I(x)\mathcal{H}_I(y)) &:= \theta(x^0 - y^0)\mathcal{H}_I(x)\mathcal{H}_I(y) + \theta(y^0 - x^0)\mathcal{H}_I(y)\mathcal{H}_I(x) \\ &= \mathcal{H}_I(x)\mathcal{H}_I(y) + (\theta(x^0 - y^0) - 1)\mathcal{H}_I(x)\mathcal{H}_I(y) + \theta(y^0 - x^0)\mathcal{H}_I(y)\mathcal{H}_I(x) \\ &= \mathcal{H}_I(x)\mathcal{H}_I(y) - \theta(y^0 - x^0)[\mathcal{H}_I(x), \mathcal{H}_I(y)]. \end{aligned} \quad (2.5.31)$$

If $U(\Lambda)$ is the unitary operator on the quantum Hilbert space for implementing the Lorentz transformation Λ connected to the identity, that is, $\Lambda \in P_+^\uparrow$, then

$$U(\Lambda)T(\mathcal{H}_I(x)\mathcal{H}_I(y))U(\Lambda)^{-1} = \mathcal{H}_I(\Lambda x)\mathcal{H}_I(\Lambda y) - \theta(y^0 - x^0)[\mathcal{H}_I(\Lambda x), \mathcal{H}_I(\Lambda y)].$$

If this is equal to $T(\mathcal{H}_I(\Lambda x)\mathcal{H}_I(\Lambda y))$, that is, if

$$\theta(y^0 - x^0)[\mathcal{H}_I(\Lambda x), \mathcal{H}_I(\Lambda y)] = \theta((\Lambda y)^0 - (\Lambda x)^0)[\mathcal{H}_I(\Lambda x), \mathcal{H}_I(\Lambda y)],$$

then $S^{(2)}$ is invariant under $\Lambda \in P_+^\uparrow$. It is clearly invariant under translations. Hence the invariance of $S^{(2)}$ under P_+^\uparrow requires that either $\theta(y^0 - x^0)$ is invariant or that $[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0$.

When $x \approx y$, the time step function $\theta(y^0 - x^0)$ is invariant under P_+^\uparrow since $\Lambda \in P_+^\uparrow$ cannot reverse the direction of time.

However, when $x \sim y$, $\Lambda \in P_+^\uparrow$ can reverse the direction of time and so $\theta(y^0 - x^0)$ is not invariant. One therefore requires that $[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0$ if $x \sim y$. Therefore a commonly imposed condition for the invariance of $S^{(2)}$ under P_+^\uparrow is

$$[\mathcal{H}_I(x), \mathcal{H}_I(y)] = 0 \quad \text{whenever} \quad x \sim y. \quad (2.5.32)$$

One can show by similar arguments that it is natural to impose the causality condition (2.5.32) to maintain the P_+^\uparrow invariance of of the general term

$$S^{(n)} = \frac{(-i)^n}{n!} \int d^4x_1 d^4x_2 \dots d^4x_n T(\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) \dots \mathcal{H}_I(x_n)),$$

in S . Here

$$\begin{aligned} & T(\mathcal{H}_I(x_1) \dots \mathcal{H}_I(x_n)) \\ &= \sum_{p \in S_n} \theta(x_{p(1)}^0 - x_{p(2)}^0) \theta(x_{p(2)}^0 - x_{p(3)}^0) \dots \theta(x_{p(n-1)}^0 - x_{p(n)}^0) \mathcal{H}_I(x_{p(1)}) \dots \mathcal{H}_I(x_{p(n)}). \end{aligned}$$

In a noncommutative theory, due to twisted statistics, the interaction Hamiltonian density might not satisfy (2.5.32) but S can still be Lorentz-invariant. For example, consider the interaction Hamiltonian density for the electron-photon system

$$\mathcal{H}_I(x) = ie (\bar{\psi} * \gamma^\rho A_\rho \psi)(x). \quad (2.5.33)$$

For simplicity, we consider the case where $\theta^{0i} = 0$ and $\theta^{ij} \neq 0$. We write down the S -matrix

$$S = \text{T exp} \left(-i \int d^3x \mathcal{H}_I(x) \right) \quad (2.5.34)$$

where $\mathcal{H}_I(x) = ie (\bar{\psi}\gamma^\rho A_\rho\psi)(x)$. Here we have used the property of the Moyal product to remove the $*$ in \mathcal{H}_I while integrating over the spatial variables. The fields ψ and $\bar{\psi}$ are still noncommutative as their oscillator modes contain $\theta^{\mu\nu}$.

We can write down $\mathcal{H}_I(x)$ in the form

$$\mathcal{H}_I(x) = \mathcal{H}_I^{(0)}(x)e^{\frac{1}{2}\overleftarrow{\partial}\wedge\vec{P}} \quad (2.5.35)$$

where $\mathcal{H}_I^{(0)}$ gives the interaction Hamiltonian for $\theta^{\mu\nu} = 0$ and satisfies the causality condition (2.5.32). It follows that \mathcal{H}_I does not fulfill the causality condition (2.5.32). Still, as shown in [86], S is Lorentz invariant. (For further discussion, see [86].)

2.6 Discrete Symmetries - C, P, T and CPT

So far our discussion was centered around the identity component P_+^\dagger of the Lorentz group P . In this section we investigate the symmetries of our noncommutative theory under the action of discrete symmetries - parity **P**, time reversal **T**, charge conjugation **C** and their combined operation **CPT**. The **CPT** theorem [68, 69] is very fundamental in nature and all local relativistic quantum field theories are **CPT** invariant. Quantum field theories on the GM plane are non-local and so it is important to investigate the validity of the **CPT** theorem in these theories.

2.6.1 Transformation of Quantum Fields Under C, P and T

Under **C**, the Poincaré group P_+^\dagger , the creation and annihilation operators $c_{\mathbf{k}}, c_{\mathbf{k}}^\dagger, d_{\mathbf{k}}, d_{\mathbf{k}}^\dagger$ of a second quantized field transform in the same way as their counterparts

in an untwisted theory [86]. Using the dressing transformation [51, 52], we can then deduce the transformation laws for $a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger, b_{\mathbf{k}}, b_{\mathbf{k}}^\dagger$, and the quantum fields. They automatically imply the appropriate twisted coproduct in the matter sector (and of course the untwisted coproduct for gauge fields.) It then implies the transformation laws for the fields under the full group generated by \mathbf{C} and \mathcal{P} by the group properties of that group: they are all induced from those of $c_{\mathbf{k}}, c_{\mathbf{k}}^\dagger, d_{\mathbf{k}}, d_{\mathbf{k}}^\dagger$ in the above fashion. (We always try to preserve such group properties.) We make use of this observation when we discuss the transformation properties of quantum fields under discrete symmetries.

So far we have not mentioned the transformation property of the noncommutativity parameter $\theta^{\mu\nu}$. The matrix $\theta^{\mu\nu}$ is a constant antisymmetric matrix. In the approach using the twisted coproduct for the Poincaré group, $\theta^{\mu\nu}$ is *not* transformed by Poincaré transformations or in fact by any other symmetry: they are truly constants. Nevertheless Poincaré invariance and other symmetries can be certainly recovered for interactions invariant under the twisted symmetry actions at the level of classical theory and also for Wightman functions [32, 48, 64, 70].

We discuss the transformation of quantum fields under the action of discrete symmetries below.

2.6.1.1 Charge conjugation \mathbf{C}

The charge conjugation operator is not a part of the Lorentz group and commutes with P_μ (and in fact with the full Poincaré group). This implies that the coproduct [29, 64] for the charge conjugation operator \mathbf{C} in the twisted case is the same as the coproduct for \mathbf{C} in the untwisted case. So, we write

$$\Delta_\theta(\mathbf{C}) = \Delta_0(\mathbf{C}) = \mathbf{C} \otimes \mathbf{C}, \tag{2.6.1}$$

with the understanding that \mathbf{C} is an element of the group algebra G^* , where $G = \{\mathbf{C}\} \times P_+^\uparrow$. (This is why we use \otimes and not \times in (2.6.1).)

Under charge conjugation,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{C}} d_{\mathbf{k}}, \quad a_{\mathbf{k}} \xrightarrow{\mathbf{C}} b_{\mathbf{k}} \quad (2.6.2)$$

where the twisted operators are related to the untwisted ones by the dressing transformation [51, 52]: $a_{\mathbf{k}} = c_{\mathbf{k}} e^{-\frac{i}{2}k \wedge P}$ and $b_{\mathbf{k}} = d_{\mathbf{k}} e^{-\frac{i}{2}k \wedge P}$.

It follows that

$$\varphi_\theta \xrightarrow{\mathbf{C}} \varphi_0^{\mathbf{C}} e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}, \quad \varphi_0^{\mathbf{C}} = \mathbf{C} \varphi_0 \mathbf{C}^{-1}. \quad (2.6.3)$$

while the \star -product of two such fields φ_θ and χ_θ transforms according to

$$\begin{aligned} \varphi_\theta \star \chi_\theta &= (\varphi_0 \chi_0) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \\ &\xrightarrow{\mathbf{C}} (\mathbf{C} \varphi_0 \chi_0 \mathbf{C}^{-1}) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \\ &= (\varphi_0^{\mathbf{C}} \chi_0^{\mathbf{C}}) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}. \end{aligned} \quad (2.6.4)$$

2.6.1.2 Parity \mathbf{P}

Parity is a unitary operator on $\mathcal{A}_0(\mathbb{R}^{3+1})$. But parity transformations do not induce automorphisms of $\mathcal{A}_\theta(\mathbb{R}^{3+1})$ [44] if its coproduct is

$$\Delta_0(\mathbf{P}) = \mathbf{P} \otimes \mathbf{P}. \quad (2.6.5)$$

That is, this coproduct is not compatible with the \star -product. Hence the coproduct for parity is not the same as that for the $\theta^{\mu\nu} = 0$ case.

But the twisted coproduct Δ_θ , where

$$\Delta_\theta(\mathbf{P}) = \mathcal{F}_\theta^{-1} \Delta_0(\mathbf{P}) \mathcal{F}_\theta, \quad (2.6.6)$$

is compatible with the \star -product. So, for \mathbf{P} as well, compatibility with the \star -product fixes the coproduct [84].

Under parity,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{P}} c_{-\mathbf{k}}, \quad d_{\mathbf{k}} \xrightarrow{\mathbf{P}} d_{-\mathbf{k}} \quad (2.6.7)$$

and hence

$$a_{\mathbf{k}} \xrightarrow{\mathbf{P}} a_{-\mathbf{k}} e^{i(k_0\theta^{0i}P_i - k_i\theta^{i0}P_0)}, \quad b_{\mathbf{k}} \xrightarrow{\mathbf{P}} b_{-\mathbf{k}} e^{i(k_0\theta^{0i}P_i - k_i\theta^{i0}P_0)}. \quad (2.6.8)$$

By an earlier remark [86], eqns. (2.6.7) and (2.6.8) imply the transformation law for twisted scalar fields. A twisted complex scalar field φ_θ transforms under parity as follows,

$$\varphi_\theta = \varphi_0 e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \xrightarrow{\mathbf{P}} \mathbf{P}\left(\varphi_0 e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}\right)\mathbf{P}^{-1} = \varphi_0^{\mathbf{P}} e^{\frac{1}{2}\overleftarrow{\partial}\wedge(P_0, -\vec{P})}, \quad (2.6.9)$$

where $\varphi_0^{\mathbf{P}} = \mathbf{P}\varphi_0\mathbf{P}^{-1}$ and $\overleftarrow{\partial}\wedge(P_0, -\vec{P}) := -\overleftarrow{\partial}_0\theta^{0i}P_i - \overleftarrow{\partial}_i\theta^{ij}P_j + \overleftarrow{\partial}_i\theta^{i0}P_0$.

The product of two such fields φ_θ and χ_θ transforms according to

$$\varphi_\theta \star \chi_\theta = (\varphi_0\chi_0) e^{\frac{1}{2}\overleftarrow{\partial}\wedge P} \xrightarrow{\mathbf{P}} (\varphi_0^{\mathbf{P}}\chi_0^{\mathbf{P}}) e^{\frac{1}{2}\overleftarrow{\partial}\wedge(P_0, -\vec{P})} \quad (2.6.10)$$

Thus fields transform under \mathbf{P} with an extra factor $e^{-(\overleftarrow{\partial}_0\theta^{0i}P_i + \partial_i\theta^{ij}P_j)} = e^{-\overleftarrow{\partial}_\mu\theta^{\mu j}P_j}$ when $\theta^{\mu\nu} \neq 0$.

2.6.1.3 Time reversal \mathbf{T}

Time reversal \mathbf{T} is an anti-linear operator. Due to antilinearity, \mathbf{T} induces automorphisms on $\mathcal{A}_\theta(\mathbb{R}^{3+1})$ for the coproduct

$$\Delta_0(T) = T \otimes T \quad \text{if } \theta^{ij} = 0,$$

but not otherwise.

Under time reversal,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{T}} c_{-\mathbf{k}}, \quad d_{\mathbf{k}} \xrightarrow{\mathbf{T}} d_{-\mathbf{k}} \quad (2.6.11)$$

$$a_{\mathbf{k}} \xrightarrow{\mathbf{T}} a_{-\mathbf{k}} e^{-i(k_i \theta^{ij} P_j)}, \quad b_{\mathbf{k}} \xrightarrow{\mathbf{T}} b_{-\mathbf{k}} e^{-i(k_i \theta^{ij} P_j)}. \quad (2.6.12)$$

When $\theta^{\mu\nu} \neq 0$, compatibility with the \star -product fixes the coproduct for \mathbf{T} to be

$$\Delta_{\theta}(\mathbf{T}) = \mathcal{F}_{\theta}^{-1} \Delta_0(\mathbf{T}) \mathcal{F}_{\theta}. \quad (2.6.13)$$

This coproduct is also required in order to maintain the group properties of \mathcal{P} , the full Poincaré group.

A twisted complex scalar field φ_{θ} hence transforms under time reversal as follows,

$$\varphi_{\theta} = \varphi_0 e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \xrightarrow{\mathbf{T}} \varphi_0^{\mathbf{T}} e^{\frac{1}{2} \overleftarrow{\partial} \wedge (P_0, -\vec{P})}, \quad (2.6.14)$$

where $\varphi_0^{\mathbf{T}} = T\varphi_0 T^{-1}$, while the product of two such fields φ_{θ} and χ_{θ} transforms according to

$$\varphi_{\theta} \star \chi_{\theta} = (\varphi_0 \chi_0) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \xrightarrow{\mathbf{T}} (\varphi_0^{\mathbf{T}} \chi_0^{\mathbf{T}}) e^{\frac{1}{2} \overleftarrow{\partial} \wedge (P_0, -\vec{P})} \quad (2.6.15)$$

Thus the time reversal operation as well induces an extra factor $e^{-\overleftarrow{\partial}_i \theta^{ij} P_j}$ in the transformation property of fields when $\theta^{\mu\nu} \neq 0$.

2.6.1.4 CPT

When **CPT** is applied,

$$c_{\mathbf{k}} \xrightarrow{\mathbf{CPT}} d_{\mathbf{k}}, \quad d_{\mathbf{k}} \xrightarrow{\mathbf{CPT}} c_{\mathbf{k}}, \quad (2.6.16)$$

$$a_{\mathbf{k}} \xrightarrow{\mathbf{CPT}} b_{\mathbf{k}} e^{i(k \wedge P)}, \quad b_{\mathbf{k}} \xrightarrow{\mathbf{CPT}} a_{\mathbf{k}} e^{i(k \wedge P)}. \quad (2.6.17)$$

The coproduct for **CPT** is of course

$$\Delta_{\theta}(\mathbf{CPT}) = \mathcal{F}_{\theta}^{-1} \Delta_0(\mathbf{CPT}) \mathcal{F}_{\theta}. \quad (2.6.18)$$

A twisted complex scalar field φ_{θ} transforms under **CPT** as follows,

$$\begin{aligned} \varphi_{\theta} &= \varphi_0 e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \\ &\xrightarrow{\mathbf{CPT}} \mathbf{CPT} \left(\varphi_0 e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \right) (\mathbf{CPT})^{-1} \\ &= \varphi_0^{\mathbf{CPT}} e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}, \end{aligned} \quad (2.6.19)$$

while the product of two such fields φ_{θ} and χ_{θ} transforms according to

$$\begin{aligned} \varphi_{\theta} \star \chi_{\theta} &= (\varphi_0 \chi_0) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \\ &\xrightarrow{\mathbf{CPT}} (\varphi_0^{\mathbf{CPT}} \chi_0^{\mathbf{CPT}}) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}. \end{aligned} \quad (2.6.20)$$

2.6.2 CPT in Non-Abelian Gauge Theories

The standard model, a non-abelian gauge theory, is **CPT** invariant, but it is not invariant under **C**, **P**, **T** or products of any two of them. So we focus on discussing just **CPT** for its S -matrix when $\theta^{\mu\nu} \neq 0$. The discussion here can be easily adapted to any other non-abelian gauge theory.

2.6.2.1 Matter fields coupled to gauge fields

The interaction representation S -matrix is

$$S_{\theta}^{M,G} = \text{T exp} \left[-i \int d^4x \mathcal{H}_{I\theta}^{M,G}(x) \right] \quad (2.6.21)$$

where $\mathcal{H}_{I\theta}^{M,G}$ is the interaction Hamiltonian density for matter fields (including also matter-gauge field couplings). Under **CPT**,

$$\mathcal{H}_{I\theta}^{M,G}(x) \xrightarrow{\text{CPT}} \mathcal{H}_{I\theta}^{M,G}(-x) e^{\overleftarrow{\partial} \wedge P} \quad (2.6.22)$$

where $\overleftarrow{\partial}$ has components $\frac{\overleftarrow{\partial}}{\partial x_\mu}$. We write $\mathcal{H}_{I\theta}^{M,G}$ as

$$\mathcal{H}_{I\theta}^{M,G} = \mathcal{H}_{I0}^{M,G} e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}. \quad (2.6.23)$$

Thus we can write the interaction Hamiltonian density after **CPT** transformation in terms of the untwisted interaction Hamiltonian density:

$$\begin{aligned} \mathcal{H}_{I\theta}^{M,G}(x) &\xrightarrow{\text{CPT}} \mathcal{H}_{I\theta}^{M,G}(-x) e^{\overleftarrow{\partial} \wedge P} \\ &= \mathcal{H}_{I0}^{M,G}(-x) e^{-\frac{1}{2} \overleftarrow{\partial} \wedge P} e^{\overleftarrow{\partial} \wedge P} \\ &= \mathcal{H}_{I0}^{M,G}(-x) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}. \end{aligned} \quad (2.6.24)$$

Hence under **CPT**,

$$\begin{aligned} S_\theta^{M,G} &= \text{T exp} \left[-i \int d^4x \mathcal{H}_{I0}^{M,G}(x) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \right] \rightarrow \text{T exp} \left[i \int d^4x \mathcal{H}_{I0}^{M,G}(x) e^{-\frac{1}{2} \overleftarrow{\partial} \wedge P} \right] \\ &= (S_{-\theta}^{M,G})^{-1}. \end{aligned}$$

But it has been shown elsewhere that $S_\theta^{M,G}$ is independent of θ [85]. Hence also $S_\theta^{M,G}$ is independent of θ .

Therefore a quantum field theory with no pure gauge interaction is **CPT** “invariant” on $\mathcal{A}_\theta(\mathbb{R}^{3+1})$. In particular quantum electrodynamics (*QED*) preserves **CPT**.

2.6.2.2 Pure Gauge Fields

The interaction Hamiltonian density for pure gauge fields is independent of $\theta^{\mu\nu}$ in the approach of [86]:

$$\mathcal{H}_{I\theta}^G = \mathcal{H}_{I0}^G . \quad (2.6.25)$$

Hence also the S becomes θ -independent,

$$S_\theta^G = S_0^G , \quad (2.6.26)$$

and **CPT** holds as a good “symmetry” of the theory.

2.6.2.3 Matter and Gauge Fields

All interactions of matter and gauge fields can be fully discussed by writing the S -operator as

$$\mathbf{S}_\theta^{M,G} = \text{T exp} \left[-i \int d^4x \mathcal{H}_{I\theta}(x) \right], \quad (2.6.27)$$

$$\mathcal{H}_{I\theta} = \mathcal{H}_{I\theta}^{M,G} + \mathcal{H}_{I\theta}^G, \quad (2.6.28)$$

where

$$\mathcal{H}_{I\theta}^{M,G} = \mathcal{H}_{I0}^{M,G} e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}$$

and

$$\mathcal{H}_{I\theta}^G = \mathcal{H}_{I0}^G .$$

In *QED*, $\mathcal{H}_{I\theta}^G = 0$. Thus the S -operator \mathbf{S}_θ^{QED} is the same as for the $\theta^{\mu\nu} = 0$. That is,

$$\mathbf{S}_\theta^{QED} = \mathbf{S}_0^{QED} . \quad (2.6.29)$$

Hence **C**, **P**, **T** and **CPT** are good “symmetries” for *QED* on the GM plane.

For a non-abelian gauge theory with non-singlet matter fields, $\mathcal{H}_{I\theta}^G = \mathcal{H}_{I0}^G \neq 0$ so that if $\mathbf{S}_\theta^{M,G}$ is the *S*-matrix of the theory,

$$\mathbf{S}_\theta^{M,G} \neq \mathbf{S}_0^{M,G}. \quad (2.6.30)$$

The *S*-operator $\mathbf{S}_\theta^{M,G}$ depends only on θ^{0i} in a non-abelian theory, that is, $\mathbf{S}_\theta^{M,G} = \mathbf{S}_\theta^{M,G}|_{\theta^{ij}=0}$. Applying **C**, **P** and **T** on $\mathbf{S}_\theta^{M,G}$ we can see that **C** and **T** do not affect θ^{0i} while **P** changes its sign. Thus a non-zero θ^{0i} contributes to **P** and **CPT** violation. For further analysis see [23].

2.6.3 On Feynman Graphs

This section uses the results of [86] and [71] where Feynman rules are fully developed and field theories are analyzed further.

In non-abelian gauge theories, $\mathcal{H}_{I\theta}^G = \mathcal{H}_{I0}^G$ is not zero as gauge fields have self-interactions. The preceding discussions show that the effects of $\theta^{\mu\nu}$ can show up only in Feynman diagrams which are sensitive to products of $\mathcal{H}_{I\theta}^{M,G}$'s with \mathcal{H}_{I0}^G 's. Fig. (2.3) shows two such diagrams.

As an example, consider the first diagram in Fig. (2.3) To lowest order, it depends on θ^{0i} .

We can substitute eqn. (2.6.23) for $\mathcal{H}_{I\theta}^{M,G}$ and integrate over \mathbf{x} . That gives,

$$\mathbf{S}^{(2)} = -\frac{1}{2} \int d^4x d^4y \text{T} \left(\mathcal{H}_{I0}^{M,G}(x) e^{\frac{1}{2} \overleftarrow{\partial}_0 \theta^{0i} P_i} \mathcal{H}_{I0}^G(y) \right)$$

where $\overleftarrow{\partial}_0$ acts *only* on $\mathcal{H}_{I0}^{M,G}(x)$ (and not on the step functions in time entering in the definition of T.)

Now P_i , being components of spatial momentum, commutes with

$$\int d^3y \mathcal{H}_{I0}^G(y)$$

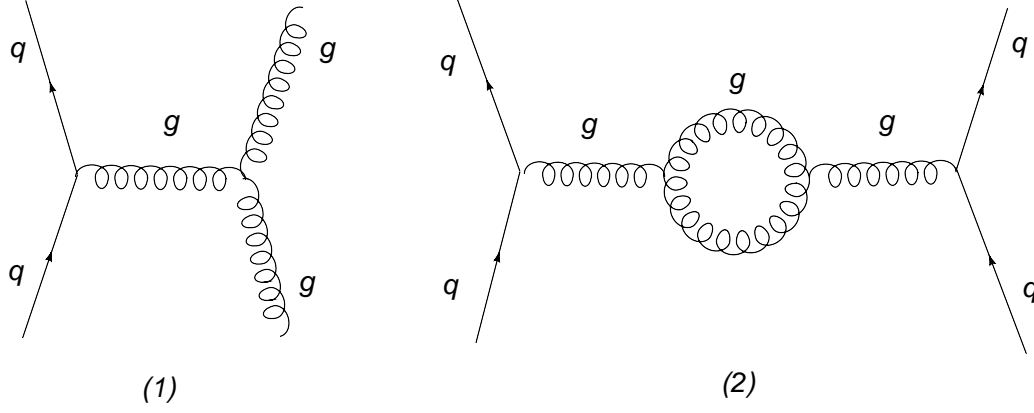


Figure 2.3: **CPT** violating processes on GM plane. (1) shows quark-gluon scattering with a three-gluon vertex. (2) shows a gluon-loop contribution to quark-quark scattering.

and hence for computing the matrix element defining the process (1) in Fig. (2.3), we can substitute \vec{P}_{in} for \vec{P} , \vec{P}_{in} being the total incident spatial momentum:

$$\mathbf{S}^{(2)} = -\frac{1}{2} \int d^4x d^4y \text{T} \left(\mathcal{H}_{I_0}^{M,G}(x) e^{\frac{1}{2} \overleftarrow{\partial}_0 \theta^{0i} P_i^{\text{in}}} \mathcal{H}_{I_0}^G(y) \right). \quad (2.6.31)$$

Thus $\mathbf{S}^{(2)}$ depends on θ^{0i} unless

$$\theta^{0i} P_i^{\text{in}} = 0. \quad (2.6.32)$$

This will happen in the center-of-mass system or more generally if $\vec{\theta}^0 = (\theta^{01}, \theta^{02}, \theta^{03})$ is perpendicular to \vec{P}^{in} .

Under **P** and **CPT**, $\theta^{0i} \rightarrow -\theta^{0i}$. This shows clearly that in a general frame, θ^{0i} contributes to **P** violation and causes **CPT** violation.

The dependence of $S^{(2)}$ on the incident total spatial momentum shows that the scattering matrix is not Lorentz invariant. This noninvariance is caused by the nonlocality of the interaction Hamiltonian density: if we evaluate it at two spacelike

separated points, the resultant operators do not commute. Such a violation of causality can lead to Lorentz-noninvariant S -operators [86].

The reasoning that reduced $e^{\frac{1}{2}\overleftarrow{\partial}\wedge P}$ to $e^{\frac{1}{2}\overleftarrow{\partial}_0\theta^{0i}P_i^{\text{in}}}$ is valid to all such factors in an arbitrary order in the perturbation expansion of the S -matrix and for arbitrary processes, \vec{P}^{in} being the total incident spatial momentum. As $\theta^{\mu\nu}$ occur only in such factors, this leads to an interesting conclusion: if scattering happens in the center-of-mass frame, or any frame where $\theta^{0i}P_i^{\text{in}} = 0$, then the θ -dependence goes away from the S -matrix. That is, P and CPT remain intact if $\theta^{0i}P_i^{\text{in}} = 0$. The theory becomes P and CPT violating in all other frames.

Terms with products of $\mathcal{H}_{I\theta}^{M,G}$ and $\mathcal{H}_{I\theta}^G$ are θ -dependent and they violate **CPT**. Electro-weak and QCD processes will thus acquire dependence on θ . This is the case when a diagram involves products of $\mathcal{H}_{I\theta}^{M,G}$ and $\mathcal{H}_{I\theta}^G$. For example quark-gluon and quark-quark scattering on the GM plane become θ -dependent **CPT** violating processes (See Fig. (2.3)).

These effects can be tested experimentally.

Summary of Chapter 3

1. Tiny (small scale) nonuniformities (inhomogeneities and anisotropies) in the CMB radiation suggest the existence of temperature fluctuations (ie. nonequilibrium) in the early universe just before photon-baryon decoupling. These are reflected in the distribution of large scale structures such as galaxy clusters in the universe today.
2. There are problems in the standard model of cosmology: The theory of inflation attempts to explain the high causal connectedness or correlation in the CMB radiation (High isotropy of CMB implies that radiation from two opposite points in the sky must have been in causal contact before decoupling.

Decoupling happened in the “far past”, too close to the big bang singularity, and so such causal contact is not possible with the homogeneous and isotropic metric of standard big bang cosmology), flatness or small curvature of the present universe, absence of primordial or early phase transition byproducts such as monopoles and cosmic strings and the origin of tiny nonuniformities in the highly (large-scale) uniform CMB radiation. A scalar field (inflaton) could have caused a fast expansion of the early universe thus neutralizing accausal, curvature and phase transition byproduct effects and quantum corrections to its dynamics would be responsible for tiny nonuniformities in the CMB radiation.

Other cosmological problems susceptible to noncommutativity include dark matter (associated with inconsistencies involving excesses in the observed motion of galaxies and clusters), dark energy (associated with observed redshifts which suggest an accelerated expansion of the universe) and the fact that only four spacetime dimensions are observed even though physical theories predict more than four dimensions for spacetime.

3. Quantum theory predicts a noncommutative structure for spacetime at small scales. Therefore noncommutativity of spacetime will contribute to the tiny nonuniformities of the CMB radiation through its naturally expected effects on the quantum dynamics (taking place precisely at such small scales) of the inflaton.
4. During inflation, metric fluctuations are negligible compared to inflaton fluctuations. However, at the end of inflation the quantum fluctuations of the inflaton become a *source* of fluctuations in the metric of spacetime as well as of radiation and matter. The power spectrum or Fourier transform of

the (metric) two-point correlation amplitude or potential will depend on the spacetime noncommutativity parameter. Using nonequilibrium dynamics one can find the fluctuations in temperature, and corresponding temperature correlations, induced by the metric fluctuations. These temperature fluctuations will then show up in the CMB radiation.

5. One gets a noncommutativity dependent power spectrum, noncommutativity-induced causality violation and a non-Gaussian probability distribution.

Chapter 3

CMB Power Spectrum and Anisotropy

Modern cosmology has now emerged as a testing ground for theories beyond the standard model of particle physics. In this paper, we consider quantum fluctuations of the inflaton scalar field on certain noncommutative spacetimes and look for noncommutative corrections in the cosmic microwave background (CMB) radiation. Inhomogeneities in the distribution of large scale structure and anisotropies in the CMB radiation can carry traces of noncommutativity of the early universe. We show that its power spectrum becomes direction-dependent when spacetime is noncommutative. (The effects due to noncommutativity can be observed experimentally in the distribution of large scale structure of matter as well.) Furthermore, we have shown that the probability distribution determining the temperature fluctuations is not Gaussian for noncommutative spacetimes.

3.1 INTRODUCTION

The CMB radiation shows how the universe was like when it was only 400,000 years old. If photons and baryons were in equilibrium before they decoupled from each other, then the CMB radiation we observe today should have a black body spectrum indicating a smooth early universe. But in 1992, the Cosmic Background Explorer (COBE) satellite detected anisotropies in the CMB radiation, which led to the conclusion that the early universe was not smooth: There were small perturbations in the photon-baryon fluid.

The theory of inflation was introduced [72, 73, 74] to resolve the fine tuning problems associated with the standard Big Bang cosmology. An important property of inflation is that it can generate irregularities in the universe, which may lead to the formation of structure. Inflation is assumed to be driven by a classical scalar field that accelerates the observed universe towards a perfect homogeneous state. But we live in a quantum world where perfect homogeneity is never attained. The classical scalar field has quantum fluctuations around it and these fluctuations act as seeds for the primordial perturbations over the smooth universe. Thus according to these ideas, the early universe had inhomogeneities and we observe them today in the distribution of large scale structure and anisotropies in the CMB radiation.

Physics at Planck scale could be radically different. It is the regime of string theory and quantum gravity. Inflation stretches a region of Planck size into cosmological scales. So, at the end of inflation, physics at Planck region should leave its signature on the cosmological scales too.

There are indications both from quantum gravity and string theory that space-time is noncommutative with a length scale of the order of Planck length. In this paper we explore the consequences of such noncommutativity for CMB radiation

in the light of recent developments in the field of noncommutative quantum field theories relating to deformed Poincaré symmetry.

The early universe and CMB in the noncommutative framework have been addressed in many places [75, 76, 77, 78, 79, 53, 80, 81]. In [75], the noncommutative parameter $\theta_{\mu\nu} = -\theta_{\nu\mu} = \text{constants}$ with $\theta_{0i} = 0$, ($\mu, \nu = 0, 1, 2, 3$, with 0 denoting time direction), characterizing the Moyal plane is scale dependent, while [77, 79, 78] have considered noncommutativity based on stringy space-time uncertainty relations. Our approach differs from these authors since our quantum fields obey twisted statistics, as implied by the deformed Poincaré symmetry in quantum theories.

We organize the paper as follows: In section II, we discuss how noncommutativity breaks the usual Lorentz invariance and indicate how this breaking can be interpreted as invariance under a deformed Poincaré symmetry. In section III, we write down an expression for a scalar quantum field in the noncommutative framework and show how its two-point function is modified. We review the theory of cosmological perturbations and (direction-independent) power spectrum for $\theta_{\mu\nu} = 0$ in section IV. In section V, we derive the power spectrum for the noncommutative Groenewold-Moyal plane \mathcal{A}_θ and show that it is direction-dependent and breaks statistical isotropy. In section VI, we compute the angular correlations using this power spectrum and show that there are nontrivial $\mathcal{O}(\theta^2)$ corrections to the CMB temperature fluctuations. Next, in section VII, we discuss the modifications of the n -point functions for any n brought about by a non-zero $\theta^{\mu\nu}$ and show in particular that the underlying probability distribution is not Gaussian. The paper concludes with section VIII.

3.2 Noncommutative Spacetime and Deformed Poincaré Symmetry

At energy scales close to the Planck scale, the quantum nature of spacetime is expected to become important. Arguments based on Heisenberg's uncertainty principle and Einstein's theory of classical gravity suggest that spacetime has a noncommutative structure at such length scales [9]. We can model such spacetime noncommutativity by the commutation relations [16, 17, 18, 19]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \quad (3.2.1)$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are constants and \hat{x}_μ are the coordinate functions of the chosen coordinate system:

$$\hat{x}_\mu(x) = x_\mu. \quad (3.2.2)$$

The above relations depend on choice of coordinates. The commutation relations given in eqn. (3.2.1) only hold in special coordinate systems and will look quite complicated in other coordinate systems. Therefore, it is important to know in which coordinate system the above simple form for the commutation relations holds. For cosmological applications, it is natural to assume that eqn. (3.2.1) holds in a comoving frame, the coordinates in which galaxies are freely falling. Not only does it make the analysis and comparison with the observation easier, but also make the time coordinate the proper time for us (neglecting the small local accelerations).

The relations (3.2.1) are not invariant under naive Lorentz transformations either. But they are invariant under a deformed Lorentz Symmetry [29], in which the coproduct on the Lorentz group is deformed while the group structure is kept intact, as we briefly explain below.

The Lie algebra \mathcal{P} of the Poincaré group has generators (basis) $M_{\alpha\beta}$ and P_μ . The subalgebra of infinitesimal generators P_μ is abelian and we can make use of this fact to construct a twist element \mathcal{F}_θ of the underlying quantum group theory [32, 82, 83]. Using this twist element, the coproduct of the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra can be deformed in such a way that it is compatible with the above commutation relations.

The coproduct Δ_0 appropriate for $\theta_{\mu\nu} = 0$ is a symmetric map from $\mathcal{U}(\mathcal{P})$ to $\mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$. It defines the action of \mathcal{P} on the tensor product of representations. In the case of the generators X of \mathcal{P} , this standard coproduct is

$$\Delta_0(X) = 1 \otimes X + X \otimes 1. \quad (3.2.3)$$

The twist element is

$$\mathcal{F}_\theta = \exp\left(-\frac{i}{2}\theta^{\alpha\beta} P_\alpha \otimes P_\beta\right), \quad P_\alpha = -i\partial_\alpha. \quad (3.2.4)$$

(The Minkowski metric with signature $(-, +, +, +)$ is used to raise and lower the indices.)

In the presence of the twist, the coproduct Δ_0 is modified to Δ_θ where

$$\Delta_\theta = \mathcal{F}_\theta^{-1} \Delta_0 \mathcal{F}_\theta. \quad (3.2.5)$$

It is easy to see that the coproduct for translation generators are not deformed,

$$\Delta_\theta(P_\alpha) = \Delta_0(P_\alpha) \quad (3.2.6)$$

while the coproduct for Lorentz generators are deformed:

$$\begin{aligned} \Delta_\theta(M_{\mu\nu}) &= 1 \otimes M_{\mu\nu} + M_{\mu\nu} \otimes 1 - \frac{1}{2} \left[(P \cdot \theta)_\mu \otimes P_\nu - P_\nu \otimes (P \cdot \theta)_\mu - (\mu \leftrightarrow \nu) \right], \\ (P \cdot \theta)_\lambda &= P_\rho \theta_\lambda^\rho. \end{aligned} \quad (3.2.7)$$

The algebra \mathcal{A}_0 of functions on the Minkowski space \mathcal{M}^4 is commutative with the commutative multiplication m_0 :

$$m_0(f \otimes g)(x) = f(x)g(x). \quad (3.2.8)$$

The Poincaré algebra acts on \mathcal{A}_0 in a well-known way

$$P_\mu f(x) = -i\partial_\mu f(x), \quad M_{\mu\nu} f(x) = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)f(x). \quad (3.2.9)$$

It acts on tensor products $f \otimes g$ using the coproduct $\Delta_0(X)$.

This commutative multiplication is changed in the Groenewold-Moyal algebra \mathcal{A}_θ to m_θ :

$$m_\theta(f \otimes g)(x) = m_0 \left[e^{-\frac{i}{2}\theta^{\alpha\beta}P_\alpha \otimes P_\beta} f \otimes g \right](x) = (f \star g)(x). \quad (3.2.10)$$

Equation (3.2.1) is a consequence of this \star -multiplication:

$$[\widehat{x}_\mu, \widehat{x}_\nu]_\star = m_\theta (\widehat{x}_\mu \otimes \widehat{x}_\nu - \widehat{x}_\nu \otimes \widehat{x}_\mu) = i\theta_{\mu\nu}. \quad (3.2.11)$$

The Poincaré algebra acts on functions $f \in \mathcal{A}_\theta$ in the usual way while it acts on tensor products $f \otimes g \in \mathcal{A}_\theta \otimes \mathcal{A}_\theta$ using the coproduct $\Delta_\theta(X)$ [29, 64].

Quantum field theories can be constructed on the noncommutative spacetime \mathcal{A}_θ by replacing ordinary multiplication between the fields by \star -multiplication and deforming statistics as we discuss below [84, 85, 87, 86]. These theories are invariant under the deformed Poincaré action [29, 64, 87, 86] under which $\theta_{\mu\nu}$ is invariant. It is thus possible to observe $\theta_{\mu\nu}$ without violating deformed Poincaré symmetry. But of course they are not invariant under the standard undeformed action of the Poincaré group as shown for example by the observability of $\theta_{\mu\nu}$.

3.3 Quantum Fields in Noncommutative Space-time

It can be shown immediately that the action of the deformed coproduct is not compatible with standard statistics [87]. Thus for $\theta^{\mu\nu} = 0$, we have the axiom in quantum theory that the statistics operator τ_0 defined by

$$\tau_0(\phi \otimes \chi) = \chi \otimes \phi \quad (3.3.1)$$

is superselected. In particular, the Lorentz group action must and *does* commute with the statistics operator,

$$\tau_0 \Delta_0(\Lambda) = \Delta_0(\Lambda) \tau_0, \quad (3.3.2)$$

where $\Lambda \in \mathcal{P}_+^\uparrow$, the connected component of the Poincaré group.

Also all the states in a given superselection sector are eigenstates of τ_0 with the same eigenvalue. Given an element $\phi \otimes \chi$ of the tensor product, the physical Hilbert spaces can be constructed from the elements

$$\left(\frac{1 \pm \tau_0}{2}\right)(\phi \otimes \chi). \quad (3.3.3)$$

Now since $\tau_0 \mathcal{F}_\theta = \mathcal{F}_\theta^{-1} \tau_0$, we have that

$$\tau_0 \Delta_\theta(\Lambda) \neq \Delta_\theta(\Lambda) \tau_0 \quad (3.3.4)$$

showing that the use of the usual statistics operator is not compatible with the deformed coproduct.

But the new statistics operator

$$\tau_\theta \equiv \mathcal{F}_\theta^{-1} \tau_0 \mathcal{F}_\theta, \quad \tau_\theta^2 = 1 \otimes 1 \quad (3.3.5)$$

does commute with the deformed coproduct.

The two-particle state $|p, q\rangle_{S_\theta, A_\theta}$ for bosons and fermions obeying deformed statistics is constructed as follows:

$$\begin{aligned} |p, q\rangle_{S_\theta, A_\theta} &= |p\rangle \otimes_{S_\theta, A_\theta} |q\rangle = \left(\frac{1 \pm \tau_\theta}{2}\right) (|p\rangle \otimes |q\rangle) \\ &= \frac{1}{2} \left(|p\rangle \otimes |q\rangle \pm e^{-ip_\mu \theta^{\mu\nu} q_\nu} |q\rangle \otimes |p\rangle \right). \end{aligned} \quad (3.3.6)$$

Exchanging p and q in the above, one finds

$$|p, q\rangle_{S_\theta, A_\theta} = \pm e^{-ip_\mu \theta^{\mu\nu} q_\nu} |q, p\rangle_{S_\theta, A_\theta}. \quad (3.3.7)$$

In Fock space, the above two-particle state is constructed from a second-quantized field φ_θ according to

$$\begin{aligned} \frac{1}{2} \langle 0 | \varphi_\theta(x_1) \varphi_\theta(x_2) a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger | 0 \rangle &= \left(\frac{1 \pm \tau_\theta}{2}\right) (e_p \otimes e_q)(x_1, x_2) \\ &= (e_p \otimes_{S_\theta, A_\theta} e_q)(x_1, x_2) \\ &= \langle x_1, x_2 | p, q \rangle_{S_\theta, A_\theta} \end{aligned} \quad (3.3.8)$$

where φ_0 is a boson(fermion) field associated with $|p, q\rangle_{S_0}$ ($|p, q\rangle_{A_0}$).

On using eqn. (3.3.7), this leads to the commutation relation

$$a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger = \pm e^{ip_\mu \theta^{\mu\nu} q_\nu} a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger. \quad (3.3.9)$$

Let P_μ be the Fock space momentum operator. (It is the representation of the translation generator introduced previously. We use the same symbol for both.)

Then the operators $a_{\mathbf{p}}$, $a_{\mathbf{p}}^\dagger$ can be written as follows:

$$a_{\mathbf{p}} = c_{\mathbf{p}} e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \quad a_{\mathbf{p}}^\dagger = c_{\mathbf{p}}^\dagger e^{\frac{i}{2} p_\mu \theta^{\mu\nu} P_\nu}, \quad (3.3.10)$$

$c_{\mathbf{p}}$'s being $\theta^{\mu\nu} = 0$ annihilation operators.

The map from $c_{\mathbf{p}}, c_{\mathbf{p}}^\dagger$ to $a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger$ in eqn. (3.3.10) is known as the “dressing transformation” [51, 52].

In the noncommutative case, a free spin-zero quantum scalar field of mass m has the mode expansion

$$\varphi_\theta(x) = \int \frac{d^3p}{(2\pi)^3} (a_{\mathbf{p}} e_p(x) + a_{\mathbf{p}}^\dagger e_{-p}(x)) \quad (3.3.11)$$

where

$$e_p(x) = e^{-i p \cdot x}, \quad p \cdot x = p_0 x_0 - \mathbf{p} \cdot \mathbf{x}, \quad p_0 = \sqrt{\mathbf{p}^2 + m^2} > 0.$$

The deformed quantum field φ_θ differs from the undeformed quantum field φ_0 in two ways: *i.*) e_p belongs to the noncommutative algebra of \mathcal{M}^4 and *ii.*) $a_{\mathbf{p}}$ is deformed by statistics. The deformed statistics can be accounted for by writing [88]

$$\varphi_\theta = \varphi_0 e^{\frac{1}{2} \overleftarrow{\partial} \wedge P} \quad (3.3.12)$$

where

$$\overleftarrow{\partial} \wedge P \equiv \overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu. \quad (3.3.13)$$

It is easy to write down the n -point correlation function for the deformed quantum field $\varphi_\theta(x)$ in terms of the undeformed field $\varphi_0(x)$:

$$\begin{aligned} & \langle 0 | \varphi_\theta(x_1) \varphi_\theta(x_2) \cdots \varphi_\theta(x_n) | 0 \rangle \\ &= \langle 0 | \varphi_0(x_1) \varphi_0(x_2) \cdots \varphi_0(x_n) | 0 \rangle e^{(-\frac{i}{2} \sum_{J=2}^n \sum_{I=1}^{J-1} \overleftarrow{\partial}_{x_I} \wedge \overleftarrow{\partial}_{x_J})}. \end{aligned}$$

On using

$$\varphi_\theta(x) = \varphi_\theta(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \Phi_\theta(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (3.3.14)$$

we find for the vacuum expectation values, in momentum space

$$\begin{aligned}
\langle 0 | \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \cdots \Phi_\theta(\mathbf{k}_n, t_n) | 0 \rangle &= e^{(\frac{i}{2} \sum_{J>I} \mathbf{k}_I \wedge \mathbf{k}_J)} \times \\
\langle 0 | \Phi_0(\mathbf{k}_1, t_1 + \frac{\vec{\theta}^0 \cdot \mathbf{k}_2 + \vec{\theta}^0 \cdot \mathbf{k}_3 + \cdots + \vec{\theta}^0 \cdot \mathbf{k}_n}{2}) \Phi_0(\mathbf{k}_2, t_2 + \frac{-\vec{\theta}^0 \cdot \mathbf{k}_1 + \vec{\theta}^0 \cdot \mathbf{k}_3 + \cdots + \vec{\theta}^0 \cdot \mathbf{k}_n}{2}) \\
\cdots \Phi_0(\mathbf{k}_n, t_n + \frac{-\vec{\theta}^0 \cdot \mathbf{k}_1 - \vec{\theta}^0 \cdot \mathbf{k}_2 - \cdots - \vec{\theta}^0 \cdot \mathbf{k}_{n-1}}{2}) | 0 \rangle &
\end{aligned} \tag{3.3.15}$$

where

$$\vec{\theta}^0 = (\theta^{01}, \theta^{02}, \theta^{03}). \tag{3.3.16}$$

Since the underlying Friedmann-Lemaître-Robertson-Walker (FLRW) space-time has spatial translational invariance,

$$\mathbf{k}_1 + \mathbf{k}_2 + \cdots + \mathbf{k}_n = 0,$$

the n -point correlation function in momentum space becomes

$$\begin{aligned}
&\langle 0 | \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \cdots \Phi_\theta(\mathbf{k}_n, t_n) | 0 \rangle \\
&= e^{(\frac{i}{2} \sum_{J>I} \mathbf{k}_I \wedge \mathbf{k}_J)} \langle 0 | \Phi_0(\mathbf{k}_1, t_1 - \frac{\vec{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \vec{\theta}^0 \cdot \mathbf{k}_1 - \frac{\vec{\theta}^0 \cdot \mathbf{k}_2}{2}) \\
&\cdots \Phi_0(\mathbf{k}_n, t_n - \vec{\theta}^0 \cdot \mathbf{k}_1 - \vec{\theta}^0 \cdot \mathbf{k}_2 - \cdots - \vec{\theta}^0 \cdot \mathbf{k}_{n-1} - \frac{\vec{\theta}^0 \cdot \mathbf{k}_n}{2}) | 0 \rangle.
\end{aligned} \tag{3.3.17}$$

In particular, the two-point correlation function is

$$\langle 0 | \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) | 0 \rangle = \langle 0 | \Phi_0(\mathbf{k}_1, t_1 - \frac{\vec{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \frac{\vec{\theta}^0 \cdot \mathbf{k}_1}{2}) | 0 \rangle, \tag{3.3.18}$$

since it vanishes unless $\mathbf{k}_1 + \mathbf{k}_2 = 0$ and hence $e^{(\frac{i}{2} \sum_{J>I} k_I \wedge k_J)} = 1$.

We emphasize that eqns. (27), (29) and (30) come from eqn. (20) which implies eqns. (21), (23) and (25). They are exclusively due to deformed statistics. The

*-product is still mandatory when taking products of φ_θ evaluated at the same point.

In standard Hopf algebra theory, the exchange operation is to be performed using the \mathcal{R} -matrix times the flip operator σ [30, 31]. It is easy to check that $\mathcal{R}\sigma$ acts as identity on any pair of factors in eqns. (27) and (29).

One can also explicitly show that the n -point functions are invariant under the twisted Poincaré group while those of the conventional theory are not. Hence the requirement of twisted Poincaré invariance fixes the structure of n -point functions. These points are discussed further in [87].

It is interesting to note that the two-point correlation function is nonlocal in time in the noncommutative frame work. Also note the following: Assuming that $\theta^{\mu\nu}$ is non-degenerate, we can write it as

$$\begin{aligned}\theta^{\mu\nu} &= \alpha \epsilon_{ab} e_a^\mu e_b^\nu + \beta \epsilon_{ab} f_a^\mu f_b^\nu, \\ \alpha, \beta &\neq 0, \quad \epsilon_{ab} = -\epsilon_{ba}, \quad a, b = 1, 2\end{aligned}$$

where e_a, e_b, f_a, f_b are orthonormal real vectors. Thus $\theta^{\mu\nu}$ defines two distinguished two-planes in \mathcal{M}^4 , namely those spanned by e_a and by f_a . For simplicity we have assumed that one of these planes contains the time direction, say $e_1 : e_1^\mu = \delta_0^\mu$. The θ^{0i} part then can be regarded as defining a spatial direction $\vec{\theta}^0$ as given by eqn. (3.3.16).

We will make use of the modified two-point correlation functions given by eqn. (3.3.18) when we define the power spectrum for inflaton field perturbations in the noncommutative frame work.

3.4 Cosmological Perturbations and (Direction-Independent) Power Spectrum for $\theta^{\mu\nu} = 0$

In this section we briefly review how fluctuations in the inflaton field cause inhomogeneities in the distribution of matter and radiation following [89].

The scalar field ϕ driving inflation can be split into a zeroth order homogeneous part and a first order perturbation:

$$\phi(\mathbf{x}, t) = \phi^{(0)}(t) + \delta\phi(\mathbf{x}, t) \quad (3.4.1)$$

The energy-momentum tensor for ϕ is

$$\mathcal{T}^\alpha{}_\beta = g^{\alpha\nu} \frac{\partial\phi}{\partial x^\nu} \frac{\partial\phi}{\partial x^\beta} - g^\alpha{}_\beta \left[\frac{1}{2} g^{\mu\nu} \frac{\partial\phi}{\partial x^\mu} \frac{\partial\phi}{\partial x^\nu} + V(\phi) \right] \quad (3.4.2)$$

We assume a spatially flat, homogeneous and isotropic (FLRW) background with the metric

$$ds^2 = dt^2 - a^2(t) d\mathbf{x}^2 \quad (3.4.3)$$

where a is the cosmological scale factor, and nonvanishing Γ 's

$$\Gamma^0{}_{ij} = \delta_{ij} a^2 H \quad \text{and} \quad \Gamma^i{}_{0j} = \Gamma^i{}_{j0} = \delta_j^i H$$

where H is the Hubble parameter.

In conformal time η where $d\eta = \frac{dt}{a(t)}$, $-\infty < \eta < 0$, the metric becomes

$$ds^2 = a^2(\eta)(d\eta^2 - d\mathbf{x}^2), \quad (3.4.4)$$

where a is the cosmological scale factor now regarded as a function of conformal time. Using this metric we write the equation for the zeroth order part of ϕ [89],

$$\ddot{\phi}^{(0)} + 2aH\dot{\phi}^{(0)} + a^2 V' \phi^{(0)} = 0, \quad (3.4.5)$$

where overdots denote derivatives with respect to conformal time η and V' is the derivative of V with respect to the field $\phi^{(0)}$. Notice that in conformal time η we have $\frac{da(\eta)}{d\eta} = a^2(\eta)H$ while in cosmic time t we have $\frac{da(t)}{dt} = aH$.

The equation for $\delta\phi$ can be obtained from the first order perturbation of the energy-momentum tensor conservation equation:

$$\mathcal{T}^\mu{}_{\nu;\mu} = \frac{\partial \mathcal{T}^\mu{}_\nu}{\partial x^\mu} + \Gamma^\mu{}_{\alpha\mu} \mathcal{T}^\alpha{}_\nu - \Gamma^\alpha{}_{\nu\mu} \mathcal{T}^\mu{}_\alpha = 0. \quad (3.4.6)$$

The perturbed part of the energy-momentum tensor $\delta T^\mu{}_\nu$ satisfies the following conservation equation in momentum space [89]:

$$\frac{\partial \delta T^0{}_0}{\partial t} + ik_i \delta T^i{}_0 + 3H \delta T^0{}_0 - H \delta T^i{}_i = 0, \quad (3.4.7)$$

where

$$T^{\mu\nu}(\mathbf{k}, t) = \int d^3x \mathcal{T}^{\mu\nu}(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}. \quad (3.4.8)$$

Let $\phi(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \tilde{\phi}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}}$. Writing down the perturbations to the energy-momentum tensor in terms of $\tilde{\phi}(\mathbf{k}, t)$,

$$\begin{aligned} \delta T^i{}_0 &= \frac{ik_i \dot{\tilde{\phi}}^{(0)}}{a^3} \delta \tilde{\phi}, \\ \delta T^0{}_0 &= \frac{-\dot{\tilde{\phi}}^{(0)} \delta \tilde{\phi}}{a^2} - V'(\tilde{\phi}^{(0)}) \delta \tilde{\phi}, \\ \delta T^i{}_j &= \delta_{ij} \left(\frac{\dot{\tilde{\phi}}^{(0)} \delta \tilde{\phi}}{a^2} - V'(\tilde{\phi}^{(0)}) \delta \tilde{\phi} \right), \end{aligned}$$

the conservation equation becomes

$$\delta \ddot{\tilde{\phi}} + 2aH \delta \dot{\tilde{\phi}} + k^2 \delta \tilde{\phi} = 0. \quad (3.4.9)$$

Eliminating the middle Hubble damping term by a change of variable $\zeta(\mathbf{k}, \eta) = a(\eta) \delta \tilde{\phi}(\mathbf{k}, \eta)$, the above equation becomes

$$\zeta \ddot{\zeta}(\mathbf{k}, \eta) + \omega_k^2(\eta) \zeta(\mathbf{k}, \eta) = 0, \quad \omega_k^2(\eta) \equiv \left(k^2 - \frac{\ddot{a}(\eta)}{a(\eta)} \right). \quad (3.4.10)$$

The mode functions u associated with the quantum operator $\hat{\zeta}$ satisfy

$$\ddot{u}(\mathbf{k}, \eta) + \left(k^2 - \frac{\ddot{a}(\eta)}{a(\eta)} \right) u(\mathbf{k}, \eta) = 0 \quad (3.4.11)$$

with the initial conditions $u(\mathbf{k}, \eta_i) = \frac{1}{\sqrt{2\omega_k(\eta_i)}}$ and $\dot{u}(\mathbf{k}, \eta_i) = i\sqrt{\omega_k(\eta_i)}$. Notice that these initial conditions have meaning only when $\omega_k(\eta_i) > 0$.

We can immediately write down the quantum operator associated with the variable ζ ,

$$\hat{\zeta}(\mathbf{k}, \eta) = u(\mathbf{k}, \eta)\hat{a}_{\mathbf{k}} + u^*(\mathbf{k}, \eta)\hat{a}_{\mathbf{k}}^\dagger, \quad (3.4.12)$$

with the bosonic commutation relations $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}^\dagger, \hat{a}_{\mathbf{k}'}^\dagger] = 0$ and $[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}')$.

During inflation we have scale factor $a(\eta) \simeq -(\eta H)^{-1}$. Thus eqn. (3.4.11) takes the form [89]

$$\ddot{u} + \left(k^2 - \frac{2}{\eta^2} \right) u = 0. \quad (3.4.13)$$

When the perturbation modes are well within the horizon, $k|\eta| \gg 1$, one can obtain a properly normalized solution $u(\mathbf{k}, \eta)$ from the conditions imposed on it at very early times during inflation. Such a solution is [89, 90]

$$u(\mathbf{k}, \eta) = \frac{1}{\sqrt{2k}} \left(1 - \frac{i}{k\eta} \right) e^{-ik(\eta - \eta_i)}. \quad (3.4.14)$$

The variances involving $\hat{\zeta}$ and $\hat{\zeta}^\dagger$ are

$$\begin{aligned} \langle 0 | \hat{\zeta}(\mathbf{k}, \eta) \hat{\zeta}(\mathbf{k}', \eta) | 0 \rangle &= 0, \\ \langle 0 | \hat{\zeta}^\dagger(\mathbf{k}, \eta) \hat{\zeta}^\dagger(\mathbf{k}', \eta) | 0 \rangle &= 0, \\ \langle 0 | \hat{\zeta}^\dagger(\mathbf{k}, \eta) \hat{\zeta}(\mathbf{k}', \eta) | 0 \rangle &= (2\pi)^3 |u(\mathbf{k}, \eta)|^2 \delta^3(\mathbf{k} - \mathbf{k}') \\ &\equiv (2\pi)^3 P_\zeta(\mathbf{k}, \eta) \delta^3(\mathbf{k} - \mathbf{k}') \end{aligned} \quad (3.4.15)$$

where P_ζ is the power spectrum of $\hat{\zeta}$. Eqn. (3.4.15) can be treated as a general definition of power spectrum.

In the case when spacetime is commutative ($\theta^{\mu\nu} = 0$), the power spectrum in eqn. (3.4.15) is

$$\langle 0 | \hat{\zeta}^\dagger(\mathbf{k}, \eta) \hat{\zeta}(\mathbf{k}', \eta) | 0 \rangle = (2\pi)^3 P_\zeta(k, \eta) \delta^3(\mathbf{k} - \mathbf{k}'). \quad (3.4.16)$$

The Dirac delta function in eqns. (3.4.15) and (3.4.16) shows that perturbations with different wave numbers are uncoupled as a consequence of the translational invariance of the underlying spacetime. Rotational invariance of the underlying (commutative) spacetime constraints the power spectrum $P_\zeta(k, \eta)$ to depend only on the magnitude of \mathbf{k} .

Towards the end of inflation, $k|\eta|$ ($-\infty < \eta < 0$) becomes very small. In that case the small argument limit of eqn. (3.4.14),

$$\lim_{k|\eta| \rightarrow 0} u(\mathbf{k}, \eta) = \frac{1}{\sqrt{2k}} \frac{-i}{k\eta} e^{-ik(\eta - \eta_i)}, \quad (3.4.17)$$

gives the power spectrum $P_\zeta(k, \eta) = |u(\mathbf{k}, \eta)|^2$. On using $\zeta(\mathbf{k}, \eta) = a(\eta) \delta\tilde{\phi}(\mathbf{k}, \eta)$, we write the power spectrum $P_{\delta\tilde{\phi}}$ for the scalar field perturbations [89]:

$$P_{\delta\tilde{\phi}}(k, \eta) = \frac{|u(\mathbf{k}, \eta)|^2}{a(\eta)^2} = \frac{1}{2k^3} \frac{1}{a(\eta)^2 \eta^2}. \quad (3.4.18)$$

In terms of the Hubble parameter H during inflation ($H \simeq -\frac{1}{a(\eta)\eta}$), the power spectrum becomes

$$P_{\delta\tilde{\phi}}(k, \eta) = \frac{1}{2k^3} H^2. \quad (3.4.19)$$

We are interested in the post-inflation power spectrum for the scalar metric perturbations since they couple to matter and radiation and give rise to inhomogeneities and anisotropies in their respective distributions which we observe. This

spectrum comes from the inflaton field since the inflaton field perturbations get transferred to the scalar part of the metric.

We write the perturbed metric in the longitudinal gauge [91],

$$ds^2 = a^2(\eta) \left[(1 + 2\chi(\mathbf{x}, \eta)) d\eta^2 - (1 - 2\Psi(\mathbf{x}, \eta)) \gamma^{ij}(\mathbf{x}, \eta) dx_i dx_j \right], \quad (3.4.20)$$

where χ and Ψ are two physical metric degrees of freedom describing the scalar metric perturbations and γ^{ij} is the metric of the unperturbed spatial hypersurfaces.

In our model, as in the case of most simple cosmological models, in the absence of anisotropic stress ($\delta T^i_j = 0$ for $i \neq j$), the two scalar metric degrees of freedom χ and Ψ coincide upto a sign:

$$\Psi = -\chi. \quad (3.4.21)$$

The remaining metric perturbation Ψ can be expressed in terms of the inflaton field fluctuation $\delta\tilde{\phi}$ at horizon crossing [89],

$$\tilde{\Psi} \Big|_{\text{post inflation}} = \frac{2}{3} aH \frac{\delta\tilde{\phi}}{\dot{\tilde{\phi}}(0)} \Big|_{\text{horizon crossing}} \quad (3.4.22)$$

where $\tilde{\Psi}$ is the Fourier coefficient of Ψ .

On using the general definition of power spectrum as in eqn. (3.4.16), the power spectra for $P_{\tilde{\Psi}}$ and $P_{\delta\tilde{\phi}}$ can be connected when a mode k crosses the horizon, i.e. when $a(\eta)H = k$, say for $\eta = \eta_0$:

$$P_{\tilde{\Psi}}(\mathbf{k}, \eta) = \frac{4}{9} \left(\frac{a(\eta)H}{\dot{\tilde{\phi}}(0)} \right)^2 P_{\delta\tilde{\phi}} \Big|_{a(\eta_0)H=k}. \quad (3.4.23)$$

From eqn. (3.4.19), eqn. (3.4.21) and using

$$aH/\dot{\tilde{\phi}}(0) = \sqrt{4\pi G/\epsilon} \quad (3.4.24)$$

at horizon crossing, where G is Newton's gravitational constant and ϵ is the slow-roll parameter in the single field inflation model [89], we have the power spectrum (defined as in eqn. (3.4.16)) for the scalar metric perturbation at horizon crossing,

$$P_{\tilde{\Psi}}(k, \eta(t)) = P_{\Phi_0}(k, \eta(t)) = \frac{16\pi G}{9\epsilon} \frac{H^2}{2k^3} \Big|_{a(\eta_0)H=k}, \quad (3.4.25)$$

Here we wrote Φ_0 for $\tilde{\chi}$.

Note that the Hubble parameter H is (nearly) constant during inflation and also it is the same in conformal time η and cosmic time t . Since the time dependence of the power spectrum is through the Hubble parameter in eqn. (3.4.25), we have

$$P_{\Phi_0}(k, \eta(t)) = P_{\Phi_0}(k, t) \equiv P_{\Phi_0}(k) = \text{constant in time.} \quad (3.4.26)$$

The power spectrum in eqn. (3.4.25) is for commutative spacetime and it depends on the magnitude of \mathbf{k} and not on its direction. In the next section, we will show that the power spectrum becomes direction-dependent when we make spacetime noncommutative.

3.5 Direction-Dependent Power Spectrum

The two-point function in noncommutative spacetime, using eqn. (3.3.18), takes the form

$$\langle 0 | \Phi_\theta(\mathbf{k}, \eta) \Phi_\theta(\mathbf{k}', \eta) | 0 \rangle = \langle 0 | \Phi_0(\mathbf{k}, \eta^-) \Phi_0(\mathbf{k}', \eta^-) | 0 \rangle, \quad (3.5.1)$$

where $\eta^- = \eta(t - \frac{\theta^0 \cdot \mathbf{k}}{2})$.

In the commutative case, the reality of the two-point correlation function (since the density fields Φ_0 are real) is obtained by imposing the condition

$$\langle \Phi_0(\mathbf{k}, \eta) \Phi_0(\mathbf{k}', \eta) \rangle^* = \langle \Phi_0(-\mathbf{k}, \eta) \Phi_0(-\mathbf{k}', \eta) \rangle. \quad (3.5.2)$$

But this condition is not correct when the fields are deformed. That is because even if Φ_θ is self-adjoint, $\Phi_\theta(\mathbf{x}, t)\Phi_\theta(\mathbf{x}', t') \neq \Phi_\theta(\mathbf{x}', t')\Phi_\theta(\mathbf{x}, t)$ for space-like separations. A simple and natural modification (denoted by subscript M) of the correlation function that ensures reality involves ‘‘symmetrization’’ of the product of φ_θ ’s or keeping its self-adjoint part. That involves replacing the product of ϕ_θ ’s by half its anti-commutator,

$$\frac{1}{2}[\varphi_\theta(\mathbf{x}, \eta), \varphi_\theta(\mathbf{y}, \eta)]_+ = \frac{1}{2}\left(\varphi_\theta(\mathbf{x}, \eta)\varphi_\theta(\mathbf{y}, \eta) + \varphi_\theta(\mathbf{y}, \eta)\varphi_\theta(\mathbf{x}, \eta)\right). \quad (3.5.3)$$

(We emphasize that this procedure for ensuring reality is a matter of choice)

For the Fourier modes Φ_θ , this procedure gives :

$$\langle \Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta) \rangle_M = \frac{1}{2}\left(\langle \Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta) \rangle + \langle \Phi_\theta(-\mathbf{k}, \eta)\Phi_\theta(-\mathbf{k}', \eta) \rangle^*\right) \quad (3.5.4)$$

After the modification of the correlation function, the power spectrum for scalar metric perturbation takes the form

$$\langle \Phi_\theta(\mathbf{k}, \eta)\Phi_\theta(\mathbf{k}', \eta) \rangle_M = (2\pi)^3 P_{\Phi_\theta}(\mathbf{k}, \eta)\delta^3(\mathbf{k} + \mathbf{k}'). \quad (3.5.5)$$

Using eqns. (3.4.18), (3.4.23), (3.5.1) and (3.5.4) we write down the modified power spectrum:

$$P_{\Phi_\theta}(\mathbf{k}, \eta) = \frac{1}{2}\left[\frac{4}{9}\left(\frac{a(\eta)H}{\dot{\phi}(0)}\right)^2 \frac{1}{a(\eta)^2}\left(|u(\mathbf{k}, \eta^-)|^2 + |u(-\mathbf{k}, \eta^+)|^2\right)\right]. \quad (3.5.6)$$

where $\eta^\pm = \eta(t \pm \frac{\theta^0 \cdot \mathbf{k}}{2})$. Notice that here the argument of the scale factor $a(\eta)$ is not shifted, since it is not deformed by noncommutativity.

It is easy to show that

$$u(\mathbf{k}, \eta^\pm) = \frac{e^{-ik\eta^\pm}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta^\pm}\right) \quad (3.5.7)$$

are also solutions of eqn. (3.4.13).

Thus on using eqn. (3.4.24) and the limit $k\eta^\pm \rightarrow 0$ of eqn. (3.5.7), the modified power spectrum is found to be

$$\begin{aligned}
P_{\Phi_\theta}(\mathbf{k}, \eta) &= \frac{1}{2} \left[\frac{16\pi G}{9\epsilon} \frac{1}{a(\eta)^2} \left(|u(\mathbf{k}, \eta^-)|^2 + |u(-\mathbf{k}, \eta^+)|^2 \right) \right] \\
&= \frac{1}{2} \left[\frac{16\pi G}{9\epsilon} \frac{1}{a(\eta)^2} \left(\frac{1}{2k^3(\eta^-)^2} + \frac{1}{2k^3(\eta^+)^2} \right) \right] \\
&= \frac{8\pi G}{9\epsilon} \frac{1}{2k^3 a(\eta)^2} \left(\frac{1}{(\eta^-)^2} + \frac{1}{(\eta^+)^2} \right). \tag{3.5.8}
\end{aligned}$$

Assuming that the Hubble parameter H is nearly a constant during inflation, the conformal time [89]

$$\eta(t) \simeq \frac{-1}{Ha_0} e^{-Ht}. \tag{3.5.9}$$

gives an expression for η^\pm :

$$\eta^\pm = \eta(t) e^{\mp \frac{1}{2} H \vec{\theta}^0 \cdot \mathbf{k}}. \tag{3.5.10}$$

On using eqn. (3.5.10) in eqn. (3.5.8) we can easily write down an analytic expression for the modified primordial power spectrum at horizon crossing,

$$P_{\Phi_\theta}(\mathbf{k}) = P_{\Phi_0}(k) \cosh(H\vec{\theta}^0 \cdot \mathbf{k}) \tag{3.5.11}$$

where $P_{\Phi_0}(k)$ is given by eqn. (3.4.25). Note that the modified power spectrum also respects the $\mathbf{k} \rightarrow -\mathbf{k}$ parity symmetry.

This power spectrum depends on both the magnitude and direction of \mathbf{k} and clearly breaks rotational invariance. In the next section we will connect this power spectrum to the two-point temperature correlations in the sky and obtain an expression for the amount of deviation from statistical isotropy due to noncommutativity.

3.6 Signature of Noncommutativity in the CMB Radiation

We are interested in quantifying the effects of noncommutative scalar perturbations on the cosmic microwave background fluctuations. We assume homogeneity of temperature fluctuations observed in the sky. Hence it is a function of a unit vector giving the direction in the sky and can be expanded in spherical harmonics:

$$\frac{\Delta T(\hat{n})}{T} = \sum_{lm} a_{lm} Y_{lm}(\hat{n}), \quad (3.6.1)$$

Here \hat{n} is the direction of incoming photons.

The coefficients of spherical harmonics contain all the information encoded in the temperature fluctuations. For $\theta^{\mu\nu} = 0$, they can be connected to the primordial scalar metric perturbations Φ_0 ,

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Delta_l(k) \Phi_0(\mathbf{k}, \eta) Y_{lm}^*(\hat{k}), \quad (3.6.2)$$

where $\Delta_l(k)$ are called *transfer functions*. They describe the evolutions of scalar metric perturbations Φ_0 from horizon crossing epoch to a time well into the radiation dominated epoch.

The two-point temperature correlation function can be expanded in spherical harmonics:

$$\left\langle \frac{\Delta T(\hat{n})}{T} \frac{\Delta T(\hat{n}')}{T} \right\rangle = \sum_{lm'l'm'} \langle a_{lm} a_{l'm'}^* \rangle Y_{lm}^*(\hat{n}) Y_{l'm'}(\hat{n}'). \quad (3.6.3)$$

The variance of a_{lm} 's is nonzero. For $\theta^{\mu\nu} = 0$, we have

$$\langle a_{lm} a_{l'm'}^* \rangle = C_l \delta_{ll'} \delta_{mm'}. \quad (3.6.4)$$

Using eqn. (3.4.16) and eqn. (3.6.2), we can derive the expression for C_l 's for $\theta^{\mu\nu} = 0$:

$$\begin{aligned}
& \langle a_{lm} a_{l'm'}^* \rangle \\
&= 16\pi^2 (-i)^{l-l'} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \Delta_l(k) \Delta_{l'}(k') \langle \Phi_0(\mathbf{k}, \eta) \Phi_0^*(\mathbf{k}', \eta) \rangle Y_{lm}^*(\hat{k}) Y_{l'm'}(\hat{k}') \\
&= 16\pi^2 (-i)^{l-l'} \int \frac{d^3 k}{(2\pi)^3} \Delta_l(k) \Delta_{l'}(k) P_{\Phi_0}(k) Y_{lm}^*(\hat{k}) Y_{l'm'}(\hat{k}) \\
&= \frac{2}{\pi} \int dk k^2 (\Delta_l(k))^2 P_{\Phi_0}(k) \delta_{ll'} \delta_{mm'} = C_l \delta_{ll'} \delta_{mm'}, \tag{3.6.5}
\end{aligned}$$

where $P_{\Phi_0}(k)$ is given by eqn. (3.4.25).

When the fields are noncommutative, the two-point temperature correlation function clearly depends on $\theta^{\mu\nu}$. We can still write the two-point temperature correlation as in eqn. (3.6.3):

$$\left\langle \frac{\Delta T(\hat{n})}{T} \frac{\Delta T(\hat{n}')}{T} \right\rangle_\theta = \sum_{lm'l'm'} \langle a_{lm} a_{l'm'}^* \rangle_\theta Y_{lm}(\hat{n}) Y_{l'm'}^*(\hat{n}'). \tag{3.6.6}$$

This gives

$$\begin{aligned}
& \langle a_{lm} a_{l'm'}^* \rangle_\theta \\
&= 16\pi^2 (-i)^{l-l'} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \Delta_l(k) \Delta_{l'}(k') \langle \Phi_\theta(\mathbf{k}, \eta) \Phi_\theta^\dagger(\mathbf{k}', \eta) \rangle_M Y_{lm}^*(\hat{k}) Y_{l'm'}(\hat{k}'). \tag{3.6.7}
\end{aligned}$$

The two-point correlation function in eqn. (3.6.7) is calculated during the horizon crossing of the mode \mathbf{k} . Once a mode crosses the horizon, it becomes independent of time, so that we can rewrite the two-point function as

$$\langle \Phi_\theta(\mathbf{k}, \eta) \Phi_\theta^\dagger(\mathbf{k}', \eta) \rangle_M = (2\pi)^3 P_{\Phi_\theta}(\mathbf{k}) \delta^3(\mathbf{k} - \mathbf{k}') \tag{3.6.8}$$

where $P_{\Phi_\theta}(\mathbf{k})$ is given by eqn. (3.5.11).

Thus we write the noncommutative angular correlation function as follows:

$$\langle a_{lm} a_{l'm'}^* \rangle_\theta = 16\pi^2 (-i)^{l-l'} \int \frac{d^3k}{(2\pi)^3} \Delta_l(k) \Delta_{l'}(k) P_{\Phi_\theta}(\mathbf{k}) Y_{lm}^*(\hat{k}) Y_{l'm'}(\hat{k}). \quad (3.6.9)$$

The regime in which the transfer functions act is well above the noncommutative length scale, so that it is perfectly legitimate to assume that the transfer functions are the same as in the commutative case.

Assuming that the $\vec{\theta}^0$ is along the z -axis, we have the expansion

$$e^{\pm H \vec{\theta}^0 \cdot \mathbf{k}} = \sum_{l=0}^{\infty} i^l \sqrt{4\pi(2l+1)} j_l(\mp i\theta k H) Y_{l0}(\cos\vartheta) \quad (3.6.10)$$

where $\vec{\theta}^0 \cdot \mathbf{k} = \theta k \cos\vartheta$ and j_l is the spherical Bessel function.

On using eqn. (3.6.10) and the identities $j_l(-z) = (-1)^l j_l(z)$ and $j_l(iz) = i^l i_l(z)$, where i_l is the modified spherical Bessel function, we can write eqn. (3.5.11) as

$$P_{\Phi_\theta}(\mathbf{k}) = P_{\Phi_0}(k) \sum_{l=0, l:\text{even}}^{\infty} \sqrt{4\pi(2l+1)} i_l(\theta k H) Y_{l0}(\cos\vartheta). \quad (3.6.11)$$

Using eqns. (3.6.9) and (3.6.11), we rewrite eqn. (3.6.9) as,

$$\begin{aligned} \langle a_{lm} a_{l'm'}^* \rangle_\theta &= \frac{2}{\pi} \int dk \sum_{l''=0, l'':\text{even}}^{\infty} (i)^{l-l'} (-1)^m (2l''+1) k^2 \Delta_l(k) \Delta_{l'}(k) P_{\Phi_0}(k) i_{l''}(\theta k H) \\ &\quad \times \sqrt{(2l+1)(2l'+1)} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & 0 \end{pmatrix}, \end{aligned} \quad (3.6.12)$$

the Wigner's 3-j symbols in eqn. (4.2.4) being related to the integrals of spherical

harmonics:

$$\begin{aligned} & \int d\Omega_k Y_{l,-m}(\hat{k}) Y_{l'm'}(\hat{k}) Y_{l''0}(\hat{k}) \\ &= \sqrt{(2l+1)(2l'+1)(2l''+1)/4\pi} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & 0 \end{pmatrix}. \end{aligned} \quad (3.6.13)$$

We can also get a simplified form of eqn. (4.2.4) by expanding the modified power spectrum in eqn. (3.5.11) in powers of θ up to the leading order:

$$P_{\Phi_\theta}(\mathbf{k}) \simeq P_{\Phi_0}(k) \left[1 + \frac{H^2}{2} (\vec{\theta}^0 \cdot \mathbf{k})^2 \right]. \quad (3.6.14)$$

A modified power spectrum of this form has been considered in [92], where the rotational invariance is broken by introducing a (small) nonzero vector. In our case, the vector that breaks rotational invariance is $\vec{\theta}^0$ and it emerges naturally in the framework of field theories on the noncommutative Groenewold-Moyal spacetime. We have also an exact expression for $P_{\Phi_\theta}(\mathbf{k})$ in eqn. (3.5.11).

Work is in progress to find a best fit for the data available and thereby to determine the length scale of noncommutativity.

The direction-dependent primordial power spectrum discussed in [92] is considered in a model independent way in [93] to compute minimum-variance estimators for the coefficients of direction-dependence. A test for the existence of a preferred direction in the primordial perturbations using full-sky CMB maps is performed in a model independent way in [94]. Imprints of cosmic microwave background anisotropies from a non-standard spinor field driven inflation is considered in [95]. Anisotropic dark energy equation of state can also give rise to a preferred direction in the universe [96].

3.7 Non-causality and Noncommutative Fluctuations

In the noncommutative frame work, the expression for the two-point correlation function for the field φ_θ contains real and imaginary parts. We identified the real part with the observed temperature correlations which are real. This gave us the modified power spectrum

$$P_{\Phi_\theta}(\mathbf{k}) = P_{\Phi_0}(k) \cosh(H\vec{\theta} \cdot \mathbf{k}). \quad (3.7.1)$$

In this section we discuss the imaginary part of the two-point correlation function for the field φ_θ . In position space, the imaginary part of the two-point correlation function is obtained from the “anti-symmetrization” of the fields for a space-like separation:

$$\frac{1}{2}[\varphi_\theta(\mathbf{x}, \eta), \varphi_\theta(\mathbf{y}, \eta)]_- = \frac{1}{2}(\varphi_\theta(\mathbf{x}, \eta)\varphi_\theta(\mathbf{y}, \eta) - \varphi_\theta(\mathbf{y}, \eta)\varphi_\theta(\mathbf{x}, \eta)). \quad (3.7.2)$$

The commutator of deformed fields, in general, is nonvanishing for space-like separations. This type of non-causality is an inherent property of noncommutative field theories constructed on the Groenewold-Moyal spacetime [97].

To study this non-causality, we consider two smeared fields localized at \mathbf{x}_1 and \mathbf{x}_2 . (The expression for non-causality diverges for conventional choices for P_{Φ_0} if we do not smear the fields. See after eqn. (4.3.10).) We write down smeared fields at \mathbf{x}_1 and \mathbf{x}_2 .

$$\varphi(\alpha, \mathbf{x}_1) = \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3x \varphi_\theta(\mathbf{x}) e^{-\alpha(\mathbf{x}-\mathbf{x}_1)^2}, \quad (3.7.3)$$

$$\varphi(\alpha, \mathbf{x}_2) = \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3x \varphi_\theta(\mathbf{x}) e^{-\alpha(\mathbf{x}-\mathbf{x}_2)^2}, \quad (3.7.4)$$

where α determines the amount of smearing of the fields. We have

$$\lim_{\alpha \rightarrow \infty} \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3x \varphi_\theta(\mathbf{x}) e^{-\alpha(\mathbf{x}-\mathbf{x}_1)^2} = \varphi_\theta(\mathbf{x}_1). \quad (3.7.5)$$

The scale α can be thought of as the width of a wave packet which is a measure of the size of the spacetime region over which an experiment is performed.

We can now write down the uncertainty relation for the fields $\varphi(\alpha, \mathbf{x}_1)$ and $\varphi(\alpha, \mathbf{x}_2)$ coming from eqn. (4.3.3):

$$\Delta\varphi(\alpha, \mathbf{x}_1)\Delta\varphi(\alpha, \mathbf{x}_2) \geq \frac{1}{2} \left| \langle 0 | [\varphi(\alpha, \mathbf{x}_1), \varphi(\alpha, \mathbf{x}_2)] | 0 \rangle \right| \quad (3.7.6)$$

This equation is an expression for the violation of causality due to noncommutativity.

Notice that, in momentum space, we can rewrite the commutator in terms of the primordial power spectrum $P_{\Phi_0}(k)$ at horizon crossing using the discussion following eqn. (3.5.4):

$$\frac{1}{2} \langle 0 | [\Phi_\theta(\mathbf{k}, \eta), \Phi_\theta(\mathbf{k}', \eta)]_- | 0 \rangle \Big|_{\text{horizon crossing}} = (2\pi)^3 P_{\Phi_0}(k) \sinh(H\bar{\theta}^0 \cdot \mathbf{k}) \delta^3(\mathbf{k} + \mathbf{k}') \quad (3.7.7)$$

We can calculate the right hand side of eqn. (4.3.7)

$$\begin{aligned} & \langle 0 | [\varphi(\alpha, \mathbf{x}_1), \varphi(\alpha, \mathbf{x}_2)] | 0 \rangle \\ &= \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y \langle 0 | [\varphi_\theta(\mathbf{x}), \varphi_\theta(\mathbf{y})] | 0 \rangle e^{-\alpha(\mathbf{x}-\mathbf{x}_1)^2} e^{-\alpha(\mathbf{y}-\mathbf{x}_2)^2} \\ &= \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \langle 0 | [\Phi_\theta(\mathbf{k}), \Phi_\theta(\mathbf{q})] | 0 \rangle e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}} e^{-\alpha[(\mathbf{x}-\mathbf{x}_1)^2+(\mathbf{y}-\mathbf{x}_2)^2]} \\ &= \frac{2}{(2\pi)^3} \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y d^3k d^3q P_{\Phi_0}(k) \sinh(H\bar{\theta}^0 \cdot \mathbf{k}) \delta^3(\mathbf{k} + \mathbf{q}) \times \\ &\quad e^{-i\mathbf{k}\cdot\mathbf{x}-i\mathbf{q}\cdot\mathbf{y}} e^{-\alpha[(\mathbf{x}-\mathbf{x}_1)^2+(\mathbf{y}-\mathbf{x}_2)^2]} \\ &= \frac{2}{(2\pi)^3} \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y d^3k P_{\Phi_0}(k) \sinh(H\bar{\theta}^0 \cdot \mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\alpha[(\mathbf{x}-\mathbf{x}_1)^2+(\mathbf{y}-\mathbf{x}_2)^2]} \\ &= \frac{2}{(2\pi)^3} \left(\frac{\alpha}{\pi}\right)^3 \int d^3k P_{\Phi_0}(k) \sinh(H\bar{\theta}^0 \cdot \mathbf{k}) \int dx dy e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\alpha[(\mathbf{x}-\mathbf{x}_1)^2+(\mathbf{y}-\mathbf{x}_2)^2]} \\ &= \frac{2}{(2\pi)^3} \int d^3k P_{\Phi_0}(k) \sinh(H\bar{\theta}^0 \cdot \mathbf{k}) e^{-\frac{k^2}{2\alpha}-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}. \end{aligned} \quad (3.7.8)$$

This gives for eqn. (4.3.7),

$$\Delta\varphi(\alpha, \mathbf{x}_1)\Delta\varphi(\alpha, \mathbf{x}_2) \geq \left| \frac{1}{(2\pi)^3} \int d^3k P_{\Phi_0}(k) \sinh(H\bar{\theta}^0 \cdot \mathbf{k}) e^{-\frac{k^2}{2\alpha} - i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \right| \quad (3.7.9)$$

The right hand side of eqn. (4.3.10) is divergent for conventional asymptotic behaviours of P_{Φ_0} (such as P_{Φ_0} vanishing for large k no faster than some inverse power of k) when $\alpha \rightarrow \infty$ and thus the Gaussian width becomes zero. This is the reason for introducing smeared fields.

Notice that the amount of causality violation given in eqn. (4.3.10) is direction-dependent.

The uncertainty relation given in eqn. (4.3.10) is purely due to spacetime noncommutativity as it vanishes for the case $\theta^{\mu\nu} = 0$. It is an expression of causality violation.

3.8 Non-Gaussianity from noncommutativity

In this section, we briefly explain how n -point correlation functions become non-Gaussian when the fields are noncommutative, assuming that they are Gaussian in their commutative limits.

Consider a noncommutative field $\varphi_\theta(\mathbf{x}, t)$. Its first moment is obviously zero:

$$\langle \varphi_\theta(\mathbf{x}, t) \rangle = \langle \varphi_0(\mathbf{x}, t) \rangle = 0.$$

The information about noncommutativity is contained in the higher moments of φ_θ . We show that the n -point functions cannot be written as sums of products of two-point functions. That proves that the underlying probability distribution is non-Gaussian.

The n -point correlation function is

$$C_n(x_1, x_2, \dots, x_n) = \langle \varphi_\theta(\mathbf{x}_1, t_1) \cdots \varphi_\theta(\mathbf{x}_n, t_n) \rangle \quad (3.8.1)$$

Since φ_0 is assumed to be Gaussian and φ_θ is given in terms of φ_0 by eqn. (3.3.12), all the odd moments of φ_θ vanish.

But the even moments of φ_θ need not vanish and do not split into sums of products of its two-point functions in a familiar way.

Non-Gaussianity cannot be seen at the level of two-point functions. Consider the two-point function C_2 . We write this in momentum space in terms of Φ_0 :

$$C_2 = \langle \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \rangle = e^{-\frac{i}{2}(\mathbf{k}_2 \wedge \mathbf{k}_1)} \left\langle \Phi_0(\mathbf{k}_1, t_1 + \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \Phi_0(\mathbf{k}_2, t_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \right\rangle. \quad (3.8.2)$$

where $\mathbf{k}_i \wedge \mathbf{k}_j \equiv k_i \theta^{ij} k_j$.

Making use of the translation invariance $\mathbf{k}_1 + \mathbf{k}_2 = 0$, the above equation becomes

$$\langle \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \rangle = \left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \right\rangle. \quad (3.8.3)$$

Non-Gaussianity can be seen in all the n -point functions for $n \geq 4$ and even n . Still they can all be written in terms of correlation functions of Φ_0 . For example, let us consider the four-point function C_4 :

$$\begin{aligned} C_4 &= \langle \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \Phi_\theta(\mathbf{k}_3, t_3) \Phi_\theta(\mathbf{k}_4, t_4) \rangle \\ &= e^{-\frac{i}{2}(\mathbf{k}_3 \wedge \mathbf{k}_2 + \mathbf{k}_3 \wedge \mathbf{k}_1 + \mathbf{k}_2 \wedge \mathbf{k}_1)} \left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \times \right. \\ &\quad \left. \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \Phi_0(\mathbf{k}_4, t_4 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \bar{\theta}^0 \cdot \mathbf{k}_3 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_4}{2}) \right\rangle \end{aligned}$$

Here we have used translational invariance, which implies that $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0$. Using this equation once more to eliminate \mathbf{k}_4 , we find

$$\begin{aligned} C_4 &= e^{-\frac{i}{2}(\mathbf{k}_3 \wedge \mathbf{k}_2 + \mathbf{k}_3 \wedge \mathbf{k}_1 + \mathbf{k}_2 \wedge \mathbf{k}_1)} \left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \times \right. \\ &\quad \left. \times \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1 + \bar{\theta}^0 \cdot \mathbf{k}_2 + \bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \end{aligned}$$

Assuming Gaussianity for the field Φ_0 and denoting $\Phi_0(\mathbf{k}_i, t_i)$ by $\Phi_0^{(i)}$, we have,

$$\begin{aligned} \langle \Phi_0^{(1)} \Phi_0^{(2)} \dots \Phi_0^{(i)} \Phi_0^{(i+1)} \dots \Phi_0^{(n)} \rangle &= \langle \Phi_0^{(1)} \Phi_0^{(2)} \rangle \langle \Phi_0^{(3)} \Phi_0^{(4)} \rangle \dots \langle \Phi_0^{(i)} \Phi_0^{(i+1)} \rangle \dots \langle \Phi_0^{(n-1)} \Phi_0^{(n)} \rangle \\ &+ \text{permutations (for } n \text{ even)} \end{aligned} \quad (3.8.4)$$

and

$$\langle \Phi_0^{(1)} \Phi_0^{(2)} \dots \Phi_0^{(i)} \Phi_0^{(i+1)} \dots \Phi_0^{(n)} \rangle = 0 \quad (\text{for } n \text{ odd}). \quad (3.8.5)$$

Therefore C_4 is

$$\begin{aligned} &\langle \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \Phi_\theta(\mathbf{k}_3, t_3) \Phi_\theta(\mathbf{k}_4, t_4) \rangle \\ &= e^{-\frac{i}{2}(\mathbf{k}_3 \wedge \mathbf{k}_2 + \mathbf{k}_3 \wedge \mathbf{k}_1 + \mathbf{k}_2 \wedge \mathbf{k}_1)} \left(\left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \right\rangle \right. \\ &\left\langle \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1 + \bar{\theta}^0 \cdot \mathbf{k}_2 + \bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \times \\ &+ \left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \times \\ &\left\langle \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1 + \bar{\theta}^0 \cdot \mathbf{k}_2 + \bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \\ &+ \left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1 + \bar{\theta}^0 \cdot \mathbf{k}_2 + \bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \times \\ &\left. \left\langle \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \right). \end{aligned} \quad (3.8.6)$$

Using spatial translational invariance for each two-point function, we have

$$\begin{aligned} &\langle \Phi_\theta(\mathbf{k}_1, t_1) \Phi_\theta(\mathbf{k}_2, t_2) \Phi_\theta(\mathbf{k}_3, t_3) \Phi_\theta(\mathbf{k}_4, t_4) \rangle \\ &= \left[\left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_2, t_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \right\rangle \left\langle \Phi_0(\mathbf{k}_3, t_3 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_3}{2}) \right\rangle \right] \\ &+ e^{-i\mathbf{k}_2 \wedge \mathbf{k}_1} \left[\left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_2 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \right\rangle \right. \\ &\left. \left\langle \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \right\rangle \right] \\ &+ \left[\left\langle \Phi_0(\mathbf{k}_1, t_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \Phi_0(\mathbf{k}_4, t_4 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_1}{2}) \right\rangle \times \right. \\ &\left. \left\langle \Phi_0(\mathbf{k}_2, t_2 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \Phi_0(\mathbf{k}_3, t_3 - \bar{\theta}^0 \cdot \mathbf{k}_1 - \frac{\bar{\theta}^0 \cdot \mathbf{k}_2}{2}) \right\rangle \right]. \end{aligned} \quad (3.8.7)$$

Notice that the second term has a non-trivial phase which depends on the spatial momenta \mathbf{k}_1 and \mathbf{k}_2 and the noncommutative parameter θ . As C_4 cannot be written as sums of products of C_2 's in a standard way, we see that the noncommutative probability distribution is non-Gaussian. Also it should be noted that we still cannot achieve Gaussianity of n -point functions even if we modify them by imposing the reality condition as we did for the two-point case.

Non-Gaussianity affects the CMB distribution and also the large scale structure (the large scale distribution of matter in the universe). We have not considered the latter. An upper bound to the amount of non-Gaussianity coming from non-commutativity can be set by extracting the four-point function from the data.

3.9 Conclusions: Chapter 3

In this chapter, we have shown that the introduction of spacetime noncommutativity gives rise to nontrivial contributions to the CMB temperature fluctuations. The two-point correlation function in momentum space, called the power spectrum, becomes direction-dependent. Thus spacetime noncommutativity breaks the rotational invariance of the CMB spectrum. That is, CMB radiation becomes statistically anisotropic. This can be measured experimentally to set bounds on the noncommutative parameter. The next chapter (see [98]) presents numerical fits to the available CMB data to put bounds on θ .

We have also shown that the probability distribution governing correlations of fields on the Groenewold-Moyal algebra \mathcal{A}_θ are non-Gaussian. This affects the correlation functions of temperature fluctuations. By measuring the amount of non-Gaussianity from the four-point correlation function data for temperature fluctuations, we can thus set further limits on θ .

We have also discussed the signals of non-causality of non-commutative field theories in the temperature fluctuations of the CMB spectrum. It will be very interesting to test the data for such signals.

Summary of Chapter 4

- The noncommutativity parameter is not constrained by WMAP data, however ACBAR and CBI data restrict the lower bound of its energy scale to be around 10 TeV
- Upper bound for the noncommutativity parameter: $\sqrt{\theta} < 1.36 \times 10^{-19}\text{m}$. This corresponds to a 10 TeV lower bound for the energy scale.
- Amount of non-causality coming from spacetime noncommutativity for the fields of primordial scalar perturbations that are space-like separated

$$\Delta\varphi(\alpha, \mathbf{x}_1)\Delta\varphi(\alpha, \mathbf{x}_2) \geq \left| \frac{1}{(2\pi)^3} \int d^3k P_{\Phi_0}(k) \sinh(H\vec{\theta} \cdot \mathbf{k}) e^{-\frac{k^2}{2\alpha} - i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)} \right|.$$

Chapter 4

Constraint from the CMB, Causality

We try to constrain the noncommutativity length scale of the theoretical model given in [99] using the observational data from ACBAR, CBI and five year WMAP. The noncommutativity parameter is not constrained by WMAP data, however ACBAR and CBI data restrict the lower bound of its energy scale to be around 10 TeV. We also derive an expression for the amount of non-causality coming from spacetime noncommutativity for the fields of primordial scalar perturbations that are space-like separated. The amount of causality violation for these field fluctuations are direction dependent.

4.1 Introduction

In 1992, the Cosmic Background Explorer (COBE) satellite detected anisotropies in the CMB radiation, which led to the conclusion that the early universe was not

smooth: there were small density perturbations in the photon-baryon fluid before they decoupled from each other. Quantum corrections to the inflaton field generate perturbations in the metric and these perturbations could have been carried over to the photon-baryon fluid as density perturbations. We then observe them today in the distribution of large scale structure and anisotropies in the CMB radiation.

Inflation [100, 101, 72, 73, 74] stretches a region of Planck size into cosmological scales. So, at the end of inflation, physics at the Planck scale can leave its signature on cosmological scales too. Physics at the Planck scale is better described by models of quantum gravity or string theory. There are indications from considerations of either quantum gravity or string theory that spacetime is noncommutative with a length scale of the order of Planck length. CMB radiation, which consists of photons from the last scattering surface of the early universe can carry the signature of spacetime noncommutativity. With these ideas in mind, in this paper, we look for a constraint on the noncommutativity length scale from the WMAP5 [102, 103, 104], ACBAR [105, 106, 107] and CBI [108, 109, 110, 111, 112] observational data.

In a noncommutative spacetime, the commutator of quantum fields at space-like separations does not in general vanish, leading to violation of causality. This type of violation of causality in the context of the fields for the primordial scalar perturbations is also discussed in this paper. It is shown that the expression for the amount of causality violation is direction-dependent.

In [113], it was shown that causality violation coming from noncommutative spacetimes leads to violation of Lorentz invariance in certain scattering amplitudes. Measurements of these violations would be another way to put limits on the amount of spacetime noncommutativity.

This paper is a sequel to an earlier work [99]. The latter explains the the-

oretical basis of the formulae used in this paper. In [53] another approach of noncommutative inflation is considered based on target space noncommutativity of fields [53].

4.2 Likelihood Analysis for Noncomm. CMB

The CMBEasy [114] program calculates CMB power spectra based on a set of parameters and a cosmological model. It works by calculating the transfer functions Δ_l for multipole l for scalar perturbations at the present conformal time η_0 as [115]

$$\Delta_l(k, \eta = \eta_0) = \int_0^{\eta_0} d\eta S(k, \eta) j_l[k(\eta_0 - \eta)], \quad (4.2.1)$$

where S is a known “source” term and j_l is the spherical Bessel function. (Here “scalar perturbations” mean the scalar part of the primordial metric fluctuations. Primordial metric fluctuations can be decomposed into scalar, vector and second rank tensor fluctuations according to their transformation properties under spatial rotations [116]. They evolve independently in a linear theory. Scalar perturbations are most important as they couple to matter inhomogeneities. Vector perturbations are not important as they decay away in an expanding background cosmology. Tensor perturbations are less important than scalar ones, they do not couple to matter inhomogeneities at linear order. In the following discussion we denote the amplitudes of scalar and tensor perturbations by A_s and A_T respectively.) The lower limit of the time integral in eq. (4.2.1) is taken as a time well into the radiation dominance epoch. Eq. (4.2.1) shows that for each mode \mathbf{k} , the source term should be integrated over time η .

The transfer functions for scalar perturbations are then integrated over \mathbf{k} to

obtain the power spectrum for multipole moment l ,

$$C_l^{(0)} = (4\pi^2) \int dk k^2 P_{\Phi_0}(k) |\Delta_l(k, \eta = \eta_0)|^2, \quad (4.2.2)$$

where P_{Φ_0} is the initial power spectrum of scalar perturbations (cf. Ref. [99].), taken to be $P_{\Phi_0}(k) = A_s k^{-3+(n_s-1)}$ with a spectral index n_s .

The coordinate functions \hat{x}_μ on the noncommutative Moyal plane obey the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \quad \theta_{\mu\nu} = -\theta_{\nu\mu} = \text{const.} \quad (4.2.3)$$

We set $\vec{\theta}^0 \equiv (\theta^{01}, \theta^{02}, \theta^{03})$ to be in the third direction. In that case, $\vec{\theta}^0 = \theta \hat{\theta}^0$ where the unit vector $\hat{\theta}^0$ is $(0, 0, 1)$.

We now write down eq. (79) of [99],

$$\begin{aligned} \langle a_{lm} a_{l'm'}^* \rangle_\theta &= \frac{2}{\pi} \int dk \sum_{l''=0, l'':\text{even}}^{\infty} i^{l-l'} (-1)^m (2l''+1) k^2 \Delta_l(k) \Delta_{l'}(k) P_{\Phi_0}(k) i_{l''}(\theta k H) \\ &\times \sqrt{(2l+1)(2l'+1)} \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ -m & m' & 0 \end{pmatrix}, \end{aligned} \quad (4.2.4)$$

where i_l is the modified spherical Bessel function and H is the Hubble parameter during inflation. In the limit when $\theta = 0$ eq. (4.2.4) leads to the usual C_l 's [89]:

$$C_l = \frac{1}{2l+1} \sum_m \langle a_{lm} a_{lm}^* \rangle_0 = (4\pi^2) \int dk k^2 P_{\Phi_0}(k) |\Delta_l(k, \eta = \eta_0)|^2. \quad (4.2.5)$$

Our goal is to compare theory with the observational data from WMAP5, ACBAR and CBI. These data sets are only available for the diagonal terms $l = l'$ of eq. (4.2.4), and for the average over m for each l , so we consider only this case. Taking the average over m of eq. (4.2.4), for $lm = l'm'$ the sum collapses to

$$C_l^{(\theta)} \equiv \frac{1}{2l+1} \sum_m \langle a_{lm} a_{lm}^* \rangle_\theta = \int dk k^2 P_{\Phi_0}(k) |\Delta_l(k, \eta = \eta_0)|^2 i_0(\theta k H), \quad (4.2.6)$$

$$C_l^{(0)} = C_l. \quad (4.2.7)$$

The CMBEasy integrator was modified to include the additional i_0 code and the Monte Carlo Markov-chain (MCMC) facility of the program was used to find best-fit values for θH along with the other parameters of the standard Λ CDM cosmology.

In the first run the parameters were fit using a joint likelihood derived from the WMAP5, ACBAR and CBI data. The outcome of this analysis was inconclusive, as the resulting value was unphysically large. This result can be understood by examining the WMAP5 data alone and considering a χ^2 goodness-of-fit test, using

$$\chi^2 = \sum_l \left(\frac{C_l^{(\theta)} - C_{l,data}}{\sigma_l} \right)^2, \quad (4.2.8)$$

where $C_{l,data}$ is the power spectrum and σ_l is the standard deviation for each l as reported by WMAP observation.

We expect noncommutativity to have a negligible effect on most of the parameters of the standard Λ CDM cosmology. We therefore consider the effect on the CMB power spectrum of varying only the new parameter $H\theta$. To determine its effect, we consider the shape of the transfer functions $\Delta_l(k)$ as calculated by CMBEasy. The graphs of two such functions are shown in Figs. 4.1 and 4.2. As can be seen, these functions drop off rapidly with k , but extend to higher k with increasing l . (For example, in Fig. 4.1, the transfer function for $l = 10$, Δ_{10} , peaks around $k = 0.001 \text{ Mpc}^{-1}$ while in Fig. 4.2, the transfer function for $l = 800$, Δ_{800} , peaks around $k = 0.06 \text{ Mpc}^{-1}$.) As i_0 is a monotonically increasing function of k starting at $i_0(0) = 1$, this means that transfer functions of higher multipoles will feel the effect of noncommutativity first.

The spectrum from the WMAP observation is shown in Fig. 4.3. Note in

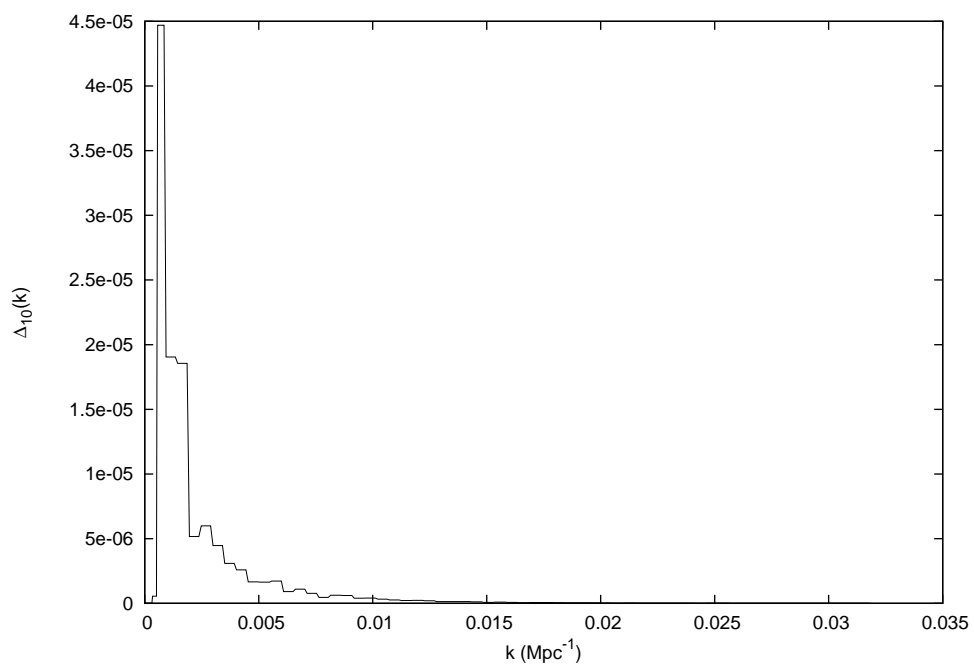


Figure 4.1: Transfer function Δ_l for $l = 10$ as a function of k . It peaks around $k = 0.001 \text{ Mpc}^{-1}$.

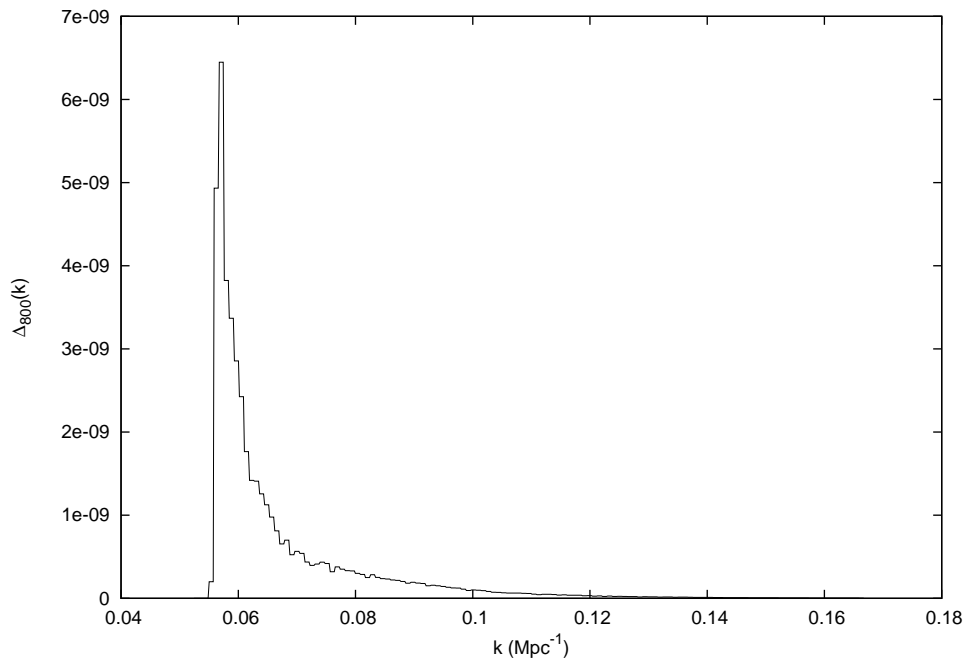


Figure 4.2: Transfer function Δ_l for $l = 800$ as a function of k . It peaks around $k = 0.06 \text{ Mpc}^{-1}$.

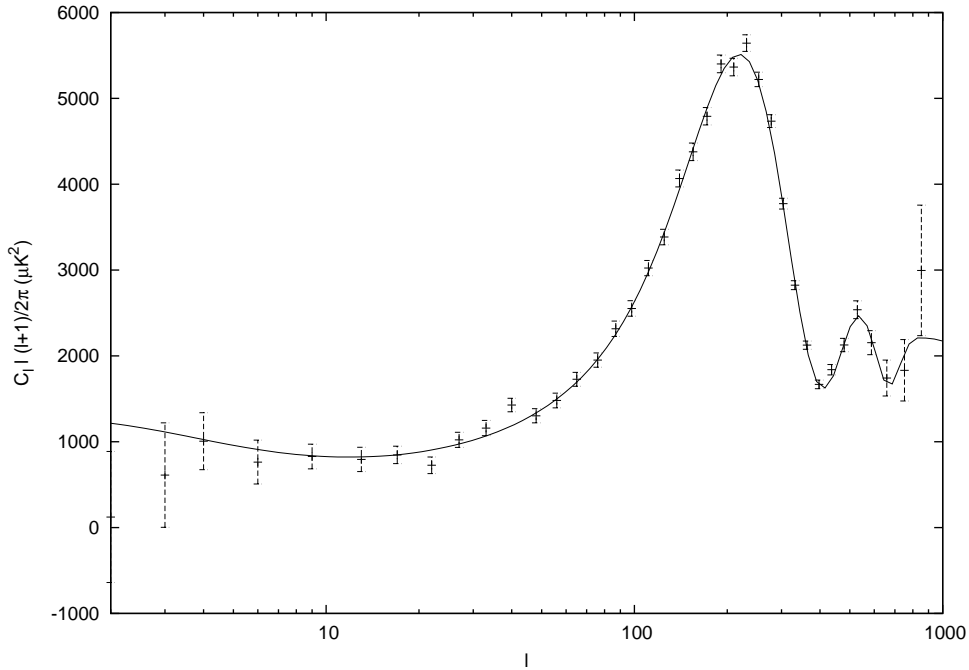


Figure 4.3: CMB power spectrum of Λ CDM model (solid curve) compared to the WMAP data (points with error bars).

particular that the last data point, corresponding to $l = 839$ falls significantly above the theoretical curve. This means that χ^2 can be lowered by a significant amount by using an unphysical value of $H\theta$ to fit this last point, so long as doing so does not also raise adjacent points too far outside their error bars. Performing the calculation shows that is indeed what happens. We therefore conclude that the WMAP data do not constrain $H\theta$.

Fig. 4.4 shows the values of k which maximize $\Delta_l(k)$, as a function of l , which in turn gives a rough estimate of the region over which the transfer functions contribute the most to the integral in eq. (4), and hence the region over which changes in $i_0(H\theta k)$ will most change the corresponding C_l . Thus to improve the

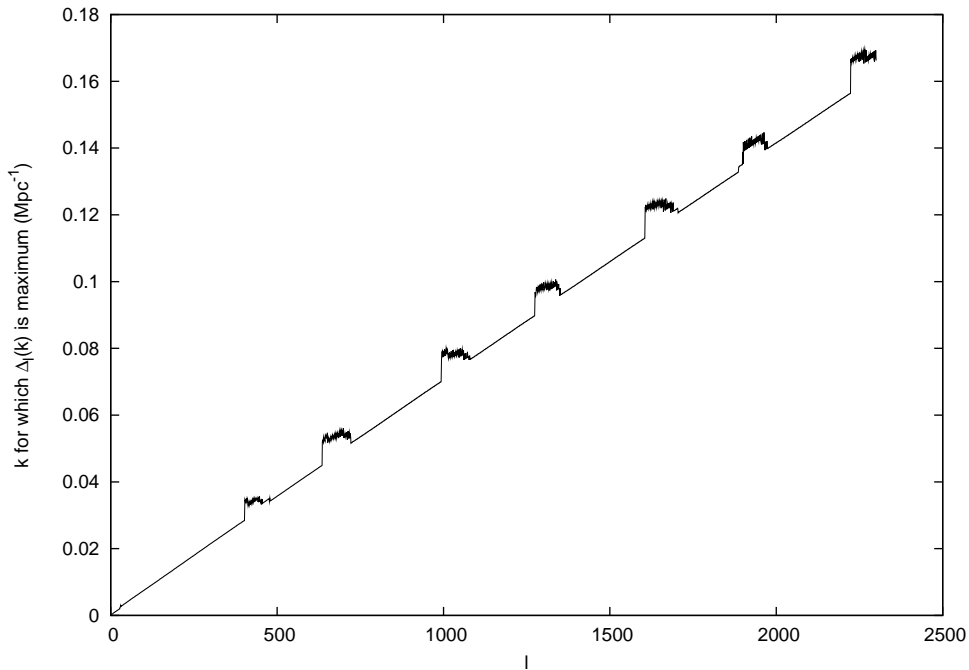


Figure 4.4: The values of k which maximize $\Delta_l(k)$, as a function of l

bound on $H\theta$, we need data at higher l ($l > 839$). In addition, tighter error bars at these higher l will, of course, also help constrain the new parameter.

Based on this analysis we performed a second run of CMBEasy excluding the WMAP data. This run resulted in a smaller, but still unphysically large, value of $H\theta$. To see why this happens, we again consider the effect of varying only the new parameter $H\theta$ and examine the behavior of χ^2 .

ACBAR and CBI are CMB data on small-scales (ACBAR and CBI give CMB power spectrum for multipoles up to $l = 2985$ and $l = 3500$ respectively) and hence may be better suited to determination of $H\theta$. A plot of χ^2 versus $H\theta$ for ACBAR+CBI data is shown in Fig. 4.5. The plateau between $H\theta = 0$ Mpc and $H\theta = 0.01$ Mpc is not physical, it results from limited numerical precision.

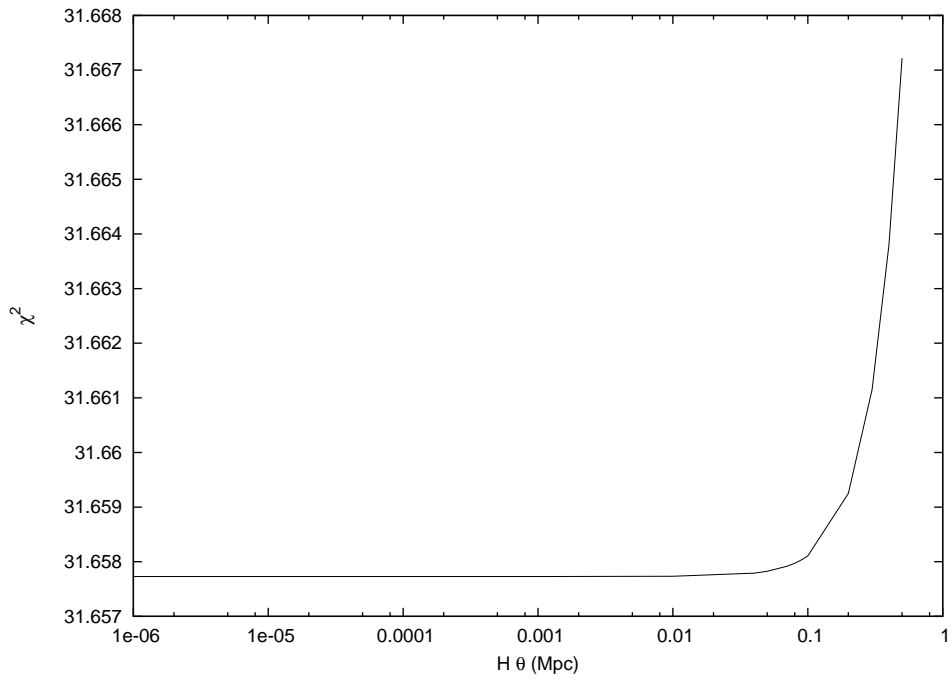


Figure 4.5: χ^2 versus $H\theta$ for ACBAR data

Therefore, likelihoods calculated in this range only restrict $H\theta < 0.01$ Mpc and hence cannot indicate whether the best fit is at $H\theta = 0$ Mpc or some small non-zero value.

However, it is possible to put a constraint on the energy scale of spacetime noncommutativity from $H\theta < 0.01$ Mpc. We discuss this below.

We can use the ACBAR+WMAP3 constraint on the amplitude of scalar power spectrum $A_s \simeq 2.15 \times 10^{-9}$ and the slow-roll parameter $\epsilon < 0.043$ [105] to find the Hubble parameter during inflation. The expression for the amplitude of the scalar power spectrum

$$A_s = \frac{1}{\pi\epsilon} \left(\frac{H}{M_p} \right)^2, \quad (4.2.9)$$

where M_p is the Planck mass, gives an upper limit on Hubble parameter:

$$H < 1.704 \times 10^{-5} M_p. \quad (4.2.10)$$

On using this upper limit for H in the relation $H\theta < 0.01$ Mpc, we have $\theta < 1.84 \times 10^{-9} m^2$.

We are interested to know the noncommutativity parameter at the end of inflation. That is, we should know the value of the cosmological scale factor a when inflation ended. Most of the single field slow-roll inflation models work at an energy scale of 10^{15} GeV or larger [89]. Assuming that the reheating temperature of the universe was close to the GUT energy scale (10^{16} GeV), we have for the scale factor at the end of inflation the value $a \simeq 10^{-29}$ [89]. Thus we have for the noncommutativity parameter, $\sqrt{\theta} < (1.84 a \times 10^{-9})^{1/2} = 1.36 \times 10^{-19} m$. This corresponds to a lower bound for the energy scale of 10 TeV.

4.3 Non-causality from Noncommutative Fluctuations

In the noncommutative frame work, the expression for the two-point correlation function for the field φ_θ for the scalar metric perturbations contains hermitian and anti-hermitian parts [99]. Taking the hermitian part, we obtained the modified power spectrum

$$P_{\Phi_\theta}(\mathbf{k}) = P_{\Phi_0}(k) \cosh(H\vec{\theta}^0 \cdot \mathbf{k}), \quad (4.3.1)$$

where $P_{\Phi_0}(k)$ is the power spectrum for the scalar metric perturbations in the commutative case (as discussed in [99]), H is the Hubble parameter during inflation. The constant spatial vector $\vec{\theta}^0$ is a measure of noncommutativity. The parameter θ is related to $\vec{\theta}^0$ by $\vec{\theta}^0 = \theta \hat{z}$ if we choose the z -axis in the direction of $\vec{\theta}^0$, \hat{z} being a unit vector. Also,

$$\Phi_\theta(\mathbf{k}, t) = \int d^3x \varphi_\theta(\mathbf{x}, t) e^{-i\mathbf{k}\cdot\mathbf{x}}. \quad (4.3.2)$$

This modified power spectrum was used to calculate the CMB angular power spectrum for the two-point temperature correlations.

In this section ¹, we discuss the imaginary part of the two-point correlation function for the field φ_θ . In position space, the imaginary part of the two-point correlation function is obtained from the “anti-symmetrization” (taking the anti-hermitian part) of the product of fields for a space-like separation:

$$\frac{1}{2}[\varphi_\theta(\mathbf{x}, \eta), \varphi_\theta(\mathbf{y}, \eta)]_- = \frac{1}{2}(\varphi_\theta(\mathbf{x}, \eta)\varphi_\theta(\mathbf{y}, \eta) - \varphi_\theta(\mathbf{y}, \eta)\varphi_\theta(\mathbf{x}, \eta)). \quad (4.3.3)$$

¹This section is based on the work of four of us with Sang Jo. It has been described in [99], but not published.

The commutator of deformed fields, in general, is nonvanishing for space-like separations. This type of non-causality is an inherent property of noncommutative field theories constructed on the Groenewold-Moyal spacetime [113].

To study this non-causality, we consider two smeared fields localized at \mathbf{x}_1 and \mathbf{x}_2 . (The expression for non-causality diverges for conventional choices for P_{Φ_0} if we do not smear the fields. See after eq. (4.3.10).) We write down smeared fields at \mathbf{x}_1 and \mathbf{x}_2 .

$$\varphi(\alpha, \mathbf{x}_1) = \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3x \varphi_\theta(\mathbf{x}) e^{-\alpha(\mathbf{x}-\mathbf{x}_1)^2}, \quad (4.3.4)$$

$$\varphi(\alpha, \mathbf{x}_2) = \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3x \varphi_\theta(\mathbf{x}) e^{-\alpha(\mathbf{x}-\mathbf{x}_2)^2}, \quad (4.3.5)$$

where α determines the amount of smearing of the fields. We have

$$\lim_{\alpha \rightarrow \infty} \left(\frac{\alpha}{\pi}\right)^{3/2} \int d^3x \varphi_\theta(\mathbf{x}) e^{-\alpha(\mathbf{x}-\mathbf{x}_1)^2} = \varphi_\theta(\mathbf{x}_1). \quad (4.3.6)$$

The scale $1/\sqrt{\alpha}$ can be thought of as the width of a wave packet which is a measure of the size of the spacetime region over which an experiment is performed.

We can now write down the uncertainty relation for the fields $\varphi(\alpha, \mathbf{x}_1)$ and $\varphi(\alpha, \mathbf{x}_2)$ coming from eq. (4.3.3):

$$\Delta\varphi(\alpha, \mathbf{x}_1)\Delta\varphi(\alpha, \mathbf{x}_2) \geq \frac{1}{2} \left| \langle 0 | [\varphi(\alpha, \mathbf{x}_1), \varphi(\alpha, \mathbf{x}_2)] | 0 \rangle \right| \quad (4.3.7)$$

This equation is an expression for the violation of causality due to noncommutativity.

We can connect the power spectrum for the field Φ_0 at horizon crossing with the commutator of the fields given in eq. (4.3.3):

$$\frac{1}{2} \langle 0 | [\Phi_\theta(\mathbf{k}, \eta), \Phi_\theta(\mathbf{k}', \eta)]_- | 0 \rangle \Big|_{\text{horizon crossing}} = (2\pi)^3 P_{\Phi_0}(k) \sinh(H\vec{\theta}^0 \cdot \mathbf{k}) \delta^3(\mathbf{k} + \mathbf{k}'). \quad (4.3.8)$$

Here we followed the same derivation given in [99], using a commutator for the fields to start with, instead of an anticommutator of the fields, to obtain the above result.

The right hand side of eq. (4.3.7) can be calculated as follows:

$$\begin{aligned}
\langle 0 | [\varphi(\alpha, \mathbf{x}_1), \varphi(\alpha, \mathbf{x}_2)] | 0 \rangle &= \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y \langle 0 | [\varphi_\theta(\mathbf{x}), \varphi_\theta(\mathbf{y})] | 0 \rangle e^{-\alpha(\mathbf{x}-\mathbf{x}_1)^2} e^{-\alpha(\mathbf{y}-\mathbf{x}_2)^2} \\
&= \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \langle 0 | [\Phi_\theta(\mathbf{k}), \Phi_\theta(\mathbf{q})] | 0 \rangle e^{-i\mathbf{k}\cdot\mathbf{x} - i\mathbf{q}\cdot\mathbf{y}} e^{-\alpha[(\mathbf{x}-\mathbf{x}_1)^2 + (\mathbf{y}-\mathbf{x}_2)^2]} \\
&= \frac{2}{(2\pi)^3} \left(\frac{\alpha}{\pi}\right)^3 \int d^3x d^3y d^3k P_{\Phi_0}(k) \sinh(H\vec{\theta}^0 \cdot \mathbf{k}) e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-\alpha[(\mathbf{x}-\mathbf{x}_1)^2 + (\mathbf{y}-\mathbf{x}_2)^2]} \\
&= \frac{2}{(2\pi)^3} \int d^3k P_{\Phi_0}(k) \sinh(H\vec{\theta}^0 \cdot \mathbf{k}) e^{-\frac{k^2}{2\alpha} - i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)}. \tag{4.3.9}
\end{aligned}$$

This gives for eq. (4.3.7),

$$\Delta\varphi(\alpha, \mathbf{x}_1)\Delta\varphi(\alpha, \mathbf{x}_2) \geq \left| \frac{1}{(2\pi)^3} \int d^3k P_{\Phi_0}(k) \sinh(H\vec{\theta}^0 \cdot \mathbf{k}) e^{-\frac{k^2}{2\alpha} - i\mathbf{k}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} \right|. \tag{4.3.10}$$

The right hand side of eq. (4.3.10) is divergent for conventional asymptotic behaviours of P_{Φ_0} (such as P_{Φ_0} vanishing for large k no faster than some inverse power of k) when $\alpha \rightarrow \infty$ and thus the Gaussian width becomes zero. This is the reason for introducing smeared fields.

Notice that the amount of causality violation given in eq. (4.3.10) is direction-dependent.

The uncertainty relation given in eq. (4.3.10) is purely due to spacetime non-commutativity as it vanishes for the case $\theta^{\mu\nu} = 0$. It is an expression of causality violation.

This amount of causality violation may be expressed in terms of the CMB temperature fluctuation $\Delta T/T$. We have the relation connecting the temperature

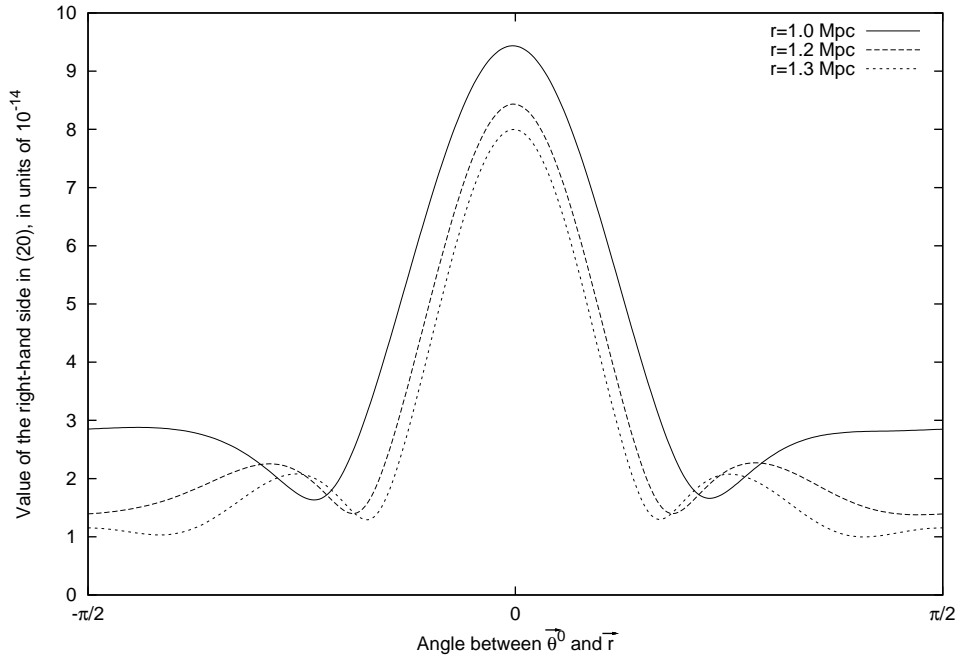


Figure 4.6: The amount of causality violation with respect to the relative orientation between the vectors $\vec{\theta}^0$ and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. It is maximum when the angle between the two vectors is zero. Notice that the minima do not occur when the two vectors are orthogonal to each other. This plot is generated using the Cuba integrator [117].

fluctuation we observe today and the primordial scalar perturbation Φ_θ ,

$$\begin{aligned} \frac{\Delta T(\hat{n}, \eta_0)}{T} &= \sum_{lm} a_{lm}(\eta_0) Y_{lm}(\hat{n}), \\ a_{lm}(\eta_0) &= 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Delta_l(k, \eta_0) \Phi_\theta(\mathbf{k}) Y_{lm}^*(\hat{k}), \end{aligned} \quad (4.3.11)$$

where \hat{n} is the direction of incoming photons and the transfer functions Δ_l take the primordial field perturbations to the present time η_0 . We can rewrite the commutator of the fields in terms of temperature fluctuations $\Delta T/T$ using eq. (4.3.11), but the corresponding correlator differs from the one for the CMB temperature anisotropy. It is not encoded in the two-point temperature correlation functions which as we have seen are given by the correlators of the anti-commutator of the fields.

In Fig. 4.6, we show the dependence of the amount of non-causality on the relative orientation of the vectors $\vec{\theta}^0$ and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$. The amount of causality violation is maximum when the two vectors are aligned.

4.4 Conclusions: Chapter 4

The power spectrum becomes direction dependent in the presence of spacetime noncommutativity, indicating a preferred direction in the universe. We tried a best-fit of the theoretical model in [99] with the WMAP data and saw that to improve the bound on $H\theta$, we need data at higher l . (The last data point for WMAP is at $l = 839$.) We therefore conclude that the WMAP data do not constrain $H\theta$. We also see that tighter error bars at these higher l will also help constrain the noncommutativity parameter. The small-scale CMB data like ACBAR and CBI give the CMB power spectrum for larger multipoles and hence may be better suited

for the determination of $H\theta$. ACBAR+CBI data only restrict $H\theta$ to $H\theta < 0.01$ Mpc and do not indicate whether the best fit is at $H\theta = 0$ Mpc or some small non-zero value. However, this restriction corresponds to a lower bound for the energy of θ of around 10 TeV.

Further work is needed before rejecting the initial hypothesis that the other parameters of the Λ CDM cosmology are unaffected by noncommutativity. It requires performing a full MCMC study of all seven parameters.

Also, we have shown the existence and direction-dependence of non-causality coming from spacetime noncommutativity for the fields describing the primordial scalar perturbations when they are space-like separated. We see that the amount of causality violation is maximum when the two vectors, $\vec{\theta}^0$ and $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, are aligned. Here \mathbf{r} is the relative spatial coordinate of the fields at spatial locations \mathbf{x}_1 and \mathbf{x}_2 .

Summary of Chapter 5

- Deformed Lorentz invariance leads to noncausal correlations which “correspond” to corrections $\delta\chi_\theta$ to susceptibility χ in linear response theory.
- Linear response theory involves determination of the linear dependence $\langle\delta q\rangle \approx \chi f$ of the expectation value $\langle\delta q\rangle$ of the change δq in a dynamical variable or coordinate q of a physical system when the Hamiltonian H of the system is perturbed $H \rightarrow H + qf$ by applying a weak external force f to the system.
- There are acausal corrections $\delta\chi_\theta$ to susceptibility χ due to spacetime noncommutativity. For input with a single frequency ω_0 the momentum dependence $\widetilde{\delta\chi_\theta}(\vec{k}, \omega)$ of the corrections $\delta\chi_\theta$ to the output due to noncommutativity

display zeroes and oscillations which are potential experimental signals for noncommutativity.

Chapter 5

Finite Temperature Field Theory

In this paper, we initiate the study of finite temperature quantum field theories (QFT's) on the Moyal plane. Such theories violate causality which influences the properties of these theories. In particular, causality influences the fluctuation-dissipation theorem: as we show, a disturbance in a spacetime region M_1 creates a response in a spacetime region M_2 spacelike with respect to M_1 ($M_1 \times M_2$). The relativistic Kubo formula with and without noncommutativity is discussed in detail, and the modified properties of relaxation time and the dependence of mean square fluctuations on time are derived. In particular, the Sinha-Sorkin result [118] on the logarithmic time dependence of the mean square fluctuations is discussed in our context.

We derive an exact formula for the noncommutative susceptibility in terms of the susceptibility for the corresponding commutative case. It shows that noncommutative corrections in the four-momentum space have remarkable periodicity properties as a function of the four-momentum k . They have direction dependence as well and vanish for certain directions of the spatial momentum. These

are striking observable signals for noncommutativity.

The Lehmann representation is also generalized to any value of the noncommutativity parameter $\theta^{\mu\nu}$ and finite temperatures.

5.1 INTRODUCTION

The Moyal plane is the algebra $\mathcal{A}_\theta(\mathbb{R}^d)$ of functions on \mathbb{R}^d with the $*$ -product given by

$$(f * g)(x) = f(x) e^{\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu} g(x) \equiv f(x) e^{\frac{i}{2} \overleftarrow{\partial} \wedge \overrightarrow{\partial}} g(x), \quad f, g \in \mathcal{A}_\theta(\mathbb{R}^d),$$

$$\theta_{\mu\nu} = -\theta_{\nu\mu} = \text{constant}. \quad (5.1.1)$$

If \hat{x}_μ are coordinate functions, $\hat{x}_\mu(x) = x_\mu$, then (5.1.1) implies that

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}. \quad (5.1.2)$$

Thus $\mathcal{A}_\theta(\mathbb{R}^d)$ is a deformation of $\mathcal{A}_0(\mathbb{R}^d)$ [119].

There is an action of a Poincaré-Hopf algebra with a "twisted" coproduct on $\mathcal{A}_\theta(\mathbb{R}^d)$. Its physical implication is that QFT's can be formulated on $\mathcal{A}_\theta(\mathbb{R}^d)$ compatibly with the Poincaré invariance of Wightman functions [32, 119]. There is also a map of untwisted to twisted fields corresponding to $\theta_{\mu\nu} = 0$ and $\theta_{\mu\nu} \neq 0$ ("the dressing transformation" [51, 52]). For matter fields, if these are φ_0 and φ_θ ,

$$\varphi_\theta(x) = \varphi_0(x) e^{\frac{1}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu} \equiv \varphi_0(x) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}, \quad (5.1.3)$$

$$P_\mu = \text{Total momentum operator}. \quad (5.1.4)$$

While there is no twist factor $e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}$ for gauge fields, the gauge field interactions of a matter current with a gauge field are twisted as well:

$$\mathcal{H}_I^\theta(x) = \mathcal{H}_I^0(x) e^{\frac{1}{2} \overleftarrow{\partial} \wedge P}, \quad (5.1.5)$$

where \mathcal{H}_I^0 can be the standard interaction $J^{0\mu}A_\mu$ of an untwisted matter current to the untwisted gauge field A_μ .

The twisted fields φ_θ and \mathcal{H}_I^θ are not causal (local). Thus even if φ_0 and \mathcal{H}_I^0 are causal fields,

$$[\varphi_0(x), \varphi_0(y)] = 0, \quad (5.1.6)$$

$$[\mathcal{H}_I^0(x), \mathcal{H}_I^0(y)] = 0, \quad (5.1.7)$$

$$[\mathcal{H}_I^0(x), \varphi_0(y)] = 0, \quad x \times y \quad (5.1.8)$$

($x \times y$ means that x and y are relatively spacelike), that is not the case for the corresponding twisted fields. For example,

$$[\varphi_\theta(x), \mathcal{H}_I^\theta(y)] = e^{-\frac{i}{2} \frac{\partial}{\partial x^\mu} \theta^{\mu\nu} \frac{\partial}{\partial y^\nu}} \varphi_0(x) \mathcal{H}_I^0(y) - e^{-\frac{i}{2} \frac{\partial}{\partial y^\mu} \theta^{\mu\nu} \frac{\partial}{\partial x^\nu}} \mathcal{H}_I^0(y) \varphi_0(x) \neq 0, \quad x \times y. \quad (5.1.9)$$

Thus acausality leads to correlation between events in spacelike regions. The study of these correlations at finite temperatures at the level of linear response theory (Kubo formula) is the central focus of this paper. We will also formulate the Lehmann representation for relativistic fields at finite temperature for $\theta_{\mu\nu} \neq 0$. It is possible that some of our results for $\theta_{\mu\nu} = 0$ and $\theta_{\mu\nu} \neq 0$ are known [120].

In section 3, we review the standard linear response theory [120] and the striking work of Sinha and Sorkin [118]. We also discuss the linear response theory for relativistic QFT's at finite temperature for $\theta_{\mu\nu} = 0$. It leads to a natural lower bound on relaxation time, a modification of the result “ $(\Delta r)^2 \approx \text{constant} \times \Delta t$ ” of Einstein and its generalization “ $(\Delta r)^2 \approx \text{constant} \times \log \Delta t$ ” to the “quantum regime” by Sinha and Sorkin [118].

Section 4 contains the linear response theory for the twisted QFT's for $\theta_{\mu\nu} \neq 0$. A striking result we find is the existence of correlations between spacelike events: A disturbance in a spacetime region M_2 evokes a fluctuation in a spacetime region M_1 spacelike with respect to M_2 ($M_1 \times M_2$). Noncommutative corrections in four-momentum space also have striking periodicity properties and zeros as a function of the four-momentum k . They are also direction-dependent and vanish in certain directions of the spatial momentum \vec{k} . All these results are discussed in this section.

The results of this section have a bearing on the homogeneity problem in cosmology. It is a problem in causal theories [121]. The noncommutative theories are not causal and hence can contribute to its resolution.

In section 5, we derive the finite temperature Lehmann representation for $\theta_{\mu\nu} = 0$ and generalize it to $\theta_{\mu\nu} \neq 0$. The Lehmann representation is known to be useful for the investigation of QFT's. The concluding remarks are in section 6.

5.2 Review of standard theory: Sinha-Sorkin results

Let H_0 be the Hamiltonian of a system in equilibrium at temperature T . It is described by the Gibbs state ω_β which gives for the mean value $\omega_\beta(A)$ of an observable A ,

$$\omega_\beta(A) = \frac{\text{Tr } e^{-\beta H_0} A}{\text{Tr } e^{-\beta H_0}}. \quad (5.2.1)$$

We assume that H_0 has no explicit time dependence, otherwise it is arbitrary and can describe an interacting system.

We now perturb the system by an interaction $H'(t)$ so that the Hamiltonian

becomes

$$H(t) = H_0 + H'(t). \quad (5.2.2)$$

When H' is treated as a perturbation, the change $\omega_\beta(\delta A(t))$ in the expectation value of an observable $A(t)$ in the Heisenberg picture at time t is

$$\omega_\beta(\delta A(t)) = \omega_\beta(U_I^{-1}(t) A U_I(t)) - \omega_\beta(A), \quad (5.2.3)$$

where

$$U_I(t) = \mathcal{T} e^{-\frac{i}{\hbar} \int_{-\infty}^t d\tau H_I(\tau)} \quad (5.2.4)$$

$$H_I(\tau) = e^{\frac{i}{\hbar} H_0 \tau} H'(\tau) e^{-\frac{i}{\hbar} H_0 \tau}. \quad (5.2.5)$$

Hence to leading order,

$$\omega_\beta(\delta A(t)) = -\frac{i}{\hbar} \int_{-\infty}^t d\tau \omega_\beta([A, H_I(\tau)]) \quad (5.2.6)$$

$$= -\frac{i}{\hbar} \int_{-\infty}^{\infty} d\tau \theta(t - \tau) \omega_\beta([A, H_I(\tau)]). \quad (5.2.7)$$

The linear response theory is based on this formula. It is completely general and applies equally well to quantum mechanics and QFT's. But in the latter case, the spatial dependence of the observable should also be specified.

For illustration of known results, we now specialize to quantum mechanics with one degree of freedom and to a dynamical variable $A(t) = x(t) = x(t)^\dagger$ and $H'(t) = x(t)f(t)$ where f is a weak external force. Then,

$$\omega_\beta(\delta x(t)) = -\frac{i}{\hbar} \int_{-\infty}^{\infty} d\tau \theta(t - \tau) \omega_\beta([x(t), x(\tau)]) f(\tau) \quad (5.2.8)$$

$$= \int_{-\infty}^{\infty} \chi(t - \tau) f(\tau), \quad (5.2.9)$$

where χ is the susceptibility:

$$\chi(t) = -\frac{i}{\hbar}\theta(t)\omega_\beta([x(t), x(0)]). \quad (5.2.10)$$

We have the following expressions:

$$W(t) = \omega_\beta(x(t)x(0)) = S(t) + iA(t), \quad (5.2.11)$$

$$S(t) = \frac{1}{2}\omega_\beta(\{x(t), x(0)\}), \quad A(t) = -\frac{i}{2}\omega_\beta([x(t), x(0)]),$$

$$\chi(t) = \frac{2}{\hbar}\theta(t)A(t). \quad (5.2.12)$$

The significant properties of these correlation functions are as follows:

1. Unitarity:

$$H_0^\dagger = H_0, \quad x(t)^\dagger = x(t) \quad \Rightarrow \quad \overline{S(t)} = S(t), \quad \overline{A(t)} = A(t).$$

2. Time translation invariance:

$$S(-t) = S(t), \quad A(-t) = -A(t) \quad \Rightarrow \quad \overline{W(t)} = W(-t)$$

from time independence of H_0 .

3. The KMS condition: (with $\hbar = 1$.)

$$W(-t - i\beta) = W(t). \quad (5.2.13)$$

Denoting the Fourier transform of these functions, including χ , by a tilde $\widetilde{}$, as for instance

$$\widetilde{W}(\omega) = \int dt e^{i\omega t} W(t), \quad (5.2.14)$$

one finds

$$\widetilde{W}(\omega) = e^{\beta\omega}\widetilde{W}(-\omega), \quad (5.2.15)$$

$$\text{Im}\widetilde{\chi}(\omega) = -\frac{1}{2}(1 - e^{-\beta\omega})\widetilde{W}(\omega), \quad (5.2.16)$$

$$\widetilde{S}(\omega) = -\coth\frac{\beta\omega}{2}\text{Im}\widetilde{\chi}(\omega). \quad (5.2.17)$$

The important aspect of these relations is that the dissipative part $\text{Im}\widetilde{\chi}$ of the (Fourier transform of) susceptibility χ completely determines all the two point correlations, and hence also the real part $\text{Re}\widetilde{\chi}$ of $\widetilde{\chi}$.

$\text{Re}\widetilde{\chi}$ can also be determined from $\text{Im}\widetilde{\chi}$ by the Kramers-Kronig relation [120].

Following an argument, presented in [118], which exploits the properties of the Heaviside function θ , we can write

$$\text{Im}\widetilde{\chi}(\omega) = -\frac{i}{2}\widetilde{\chi}'(\omega), \quad (5.2.18)$$

where

$$\begin{aligned} \chi'(t) &:= \text{sgn}(t) \chi(|t|), \\ \text{sgn}(t) &= \theta(t) - \theta(-t). \end{aligned} \quad (5.2.19)$$

Therefore, (5.2.17) becomes

$$\widetilde{S}(\omega) = \frac{i}{2} \coth\frac{\beta\omega}{2} \widetilde{\chi}'(\omega). \quad (5.2.20)$$

The Fourier transform of (5.2.20) gives

$$S(t) = \frac{1}{2\beta} P \int_{-\infty}^{\infty} dt' \text{sgn}(t' - t) \chi(|t' - t|) \coth\frac{\pi t'}{\beta}, \quad (5.2.21)$$

where P denotes the principal value of \coth . $\text{Re}\widetilde{\chi}$ does not contribute to (5.2.21).

This equation has important physics. In time Δt , the operator changes by $\Delta x(t) = x(t + \Delta t) - x(t)$. With $t = 0$, the square displacement due to equilibrium fluctuations is thus

$$\omega_\beta(\Delta x(0)^2) = 2[S(0) - S(\Delta t)] \quad (5.2.22)$$

so that we obtain the Sinha-Sorkin formula

$$\begin{aligned} & \frac{1}{2}\omega_\beta(\Delta x(0)^2) \\ &= \frac{i}{2\beta} P \int_0^\infty dt' \chi(t') [2 \coth(\Omega t') - \coth(\Omega(t' + \Delta t)) - \coth(\Omega(t' - \Delta t))], \quad \Omega = \frac{\pi}{\beta}. \end{aligned} \quad (5.2.23)$$

Sinha and Sorkin [118] have analyzed this equation for the (realistic) ansatz

$$\chi(t) = \mu[1 - e^{-\frac{t}{\tau}}]\theta(t) \xrightarrow{t \gg \tau} \mu \theta(t - \tau), \quad (5.2.24)$$

where τ is the relaxation time.

In that case,

$$\frac{1}{2}\omega_\beta(\Delta x(0)^2) = \frac{\mu\hbar}{\pi} \ln \frac{[\sinh(\Omega|\Delta t - \tau|) \sinh(\Omega|\Delta t + \tau|)]^{\frac{1}{2}}}{\sinh(\Omega\tau)}, \quad (5.2.25)$$

where we have restored \hbar .

Sinha and Sorkin [118] observed that (5.2.25) gives Einstein's relation in the classical regime:

$$\beta\hbar \ll \tau \ll \Delta t : \quad \frac{1}{2}\omega_\beta(\Delta x(0)^2) \approx \frac{\mu}{\beta} \Delta t. \quad (5.2.26)$$

But in addition they found a *logarithmic* dependence of Δt in the "quantum" regime:

$$\tau \ll \Delta t \ll \beta\hbar : \quad \frac{1}{2}\omega_\beta(\Delta x(0)^2) = \frac{\mu\hbar}{\pi} \ln \frac{\Delta t}{\tau}. \quad (5.2.27)$$

They have emphasized that this behavior can be tested experimentally.

They also discuss a regime between the classical and quantum extremes which interpolates (5.2.26) and (5.2.27).

5.3 Quantum Fields on Commutative Spacetime

Hereafter, we set $\hbar = c = 1$.

We now specialize to QFT's for $\theta_{\mu\nu} = 0$. For simplicity, we take

$$H'(t) = e \int d^3y N_0(y)\varphi_0(y), \quad (5.3.1)$$

where $N_0(y)$ is the number density of a charged spinor field ψ_0 ,

$$N_0(y) = \psi_0^\dagger(y)\psi_0(y). \quad (5.3.2)$$

φ_0 is the externally imposed scalar potential and the subscript denotes that $\theta_{\mu\nu} = 0$ for these fields. Again for simplicity, we choose A as well to be the number density at a spacetime point x . Then

$$\omega_\beta(\delta N_0(x)) = -\frac{ie}{\hbar} \int d^4y \theta(x_0 - y_0)\omega_\beta([N_0(x), N_0(y)])\varphi_0(y). \quad (5.3.3)$$

The natural definition of susceptibility in this case is

$$\chi_\beta(x, y) = -i\frac{e}{\hbar}\theta(x_0 - y_0)\omega_\beta([N_0(x), N_0(y)]). \quad (5.3.4)$$

With this definition,

$$\omega_\beta(\delta N_0(x)) = \int d^4y \chi_\beta(x, y)\varphi_0(y). \quad (5.3.5)$$

We will now analyze this formula.

The Kubo formulae

The susceptibility χ_β is related to the Wightman function

$$W_0^\beta(x, y) = \frac{i}{\hbar}\omega_\beta(N_0(x)N_0(y)) \quad (5.3.6)$$

and the autocorrelation and commutator functions

$$\begin{aligned}
W_0^\beta(x, y) &= S_0^\beta(x, y) + iA_0^\beta(x, y), \\
S_0^\beta(x, y) &= \frac{1}{2\hbar}\omega_\beta(N_0(x)N_0(y) + N_0(y)N_0(x)), \\
A_0^\beta(x, y) &= \frac{-i}{2\hbar}\omega_\beta([N_0(x), N_0(y)]), \\
\chi_\beta(x, y) &= 2e\theta(x_0 - y_0)A_0^\beta(x, y).
\end{aligned} \tag{5.3.7}$$

There are more nontrivial conditions coming from the KMS condition which we now discuss.

By assumption, H_0 commutes with spacetime translations and rotations as dictated by the Poincaré algebra. So ω_β enjoys these symmetries and $W_0^\beta(x, y)$, $S_0^\beta(x, y)$, $A_0^\beta(x, y)$ depend only on $x_0 - y_0$ and $(\vec{x} - \vec{y})^2$. Hence they are even in $\vec{x} - \vec{y}$:

$$W_0^\beta(x_0, \vec{x}_0 ; y_0, \vec{y}) = W_0^\beta(x_0, \vec{y}_0 ; y_0, \vec{x}) \text{ etc.} \tag{5.3.8}$$

$$= \hat{W}_0^\beta(x_0 - y_0 ; (\vec{x}_0 - \vec{y})^2). \tag{5.3.9}$$

As $\hat{W}_0^\beta(x_0 - y_0 ; (\vec{x}_0 - \vec{y})^2)$ can contain terms with $\theta(x_0 - y_0)$, we cannot always claim that it is even in $x_0 - y_0$ as well. The same goes for S_0^β and A_0^β .

5.3.0.1 Spacelike Disturbances

If x and y are relatively spacelike, $[N_0(x), N_0(y)] = 0$ because of causality (locality).

So if $\varphi_0 = 0$ outside the spacetime region D_2 and we observe the fluctuation in a spacetime region D_1 spacelike with respect to D_2 , then the fluctuation vanishes:

$$\omega_\beta(\delta N_0(x)) = 0 \text{ if } x \in D_2, \text{ Supp}\varphi_0 = D_2, \text{ } D_1 \times D_2. \tag{5.3.10}$$

Here Supp denotes the support of the function φ_0 (it is zero in the complement of the support).

Thus we easily recover the prediction of causality for $\theta_{\mu\nu} = 0$ [120].

5.3.0.2 Timelike Disturbances

In this case, the point of observation x is causally linked to the spacetime region D_2 . Hence $[N_0(x), N_0(y)]$ need not vanish if $x \in D_1$.

We can model the analysis of this case to the one in Section 2 if H_0 is the time translation generator of the Poincaré group for $\varphi_0 = 0$. We assume that to be the case.

Following section 2, we now introduce the correlator

$$W_0^\beta(x, y) = \omega_\beta(N_0(x)N_0(y)). \quad (5.3.11)$$

By relativistic invariance, W_0^β depends only on $(\vec{x} - \vec{y})^2$. Since $\theta(x_0 - y_0)$ is Lorentz invariant when $x - y$ is timelike, it can also depend on $\theta(x_0 - y_0)$. Thus W_0^β depends on $(\vec{x} - \vec{y})^2$ and $x_0 - y_0$ and we can rewrite (5.3.11) as

$$W_0^\beta((\vec{x} - \vec{y})^2, x_0 - y_0) = \omega_\beta(N_0(x)N_0(y)). \quad (5.3.12)$$

We can thus focus on

$$\overline{W}_0^\beta(\vec{x}^2, x_0) = \omega_\beta(N_0(x)N_0(y)). \quad (5.3.13)$$

It is important that *it is even in \vec{x}* . We cannot say that about x_0 because of the potential presence of $\theta(x_0)$.

Now

$$\overline{W}_0^\beta(\vec{x}^2, x_0) = \omega_\beta(N_0(0)N_0(x)) = W_0^\beta(\vec{x}^2, -x_0). \quad (5.3.14)$$

The presence of \vec{x} thus does not affect the symmetry properties in x_0 . That is the case also with regard to the KMS condition. We write all these conditions explicitly now: write

$$W_0^\beta(\vec{x}^2, x_0) = S_0^\beta(\vec{x}^2, x_0) + iA_0^\beta(\vec{x}^2, x_0), \quad (5.3.15)$$

where

$$\begin{aligned} S_0^\beta(\vec{x}^2, x_0) &= \frac{1}{2}\omega_\beta(N_0(x)N_0(0) + N_0(0)N_0(x)), \\ A_0^\beta(\vec{x}^2, x_0) &= -\frac{i}{2}\omega_\beta([N_0(x), N_0(0)]). \end{aligned} \quad (5.3.16)$$

Then

$$\chi_\beta(\vec{x}^2, x_0) = 2e\theta(x_0)A_0^\beta(\vec{x}^2, x_0), \quad (5.3.17)$$

where we have written the susceptibility as a function of \vec{x}^2 and x_0 . Then as before

1. S_0^β and A_0^β are real functions:

$$\overline{S_0^\beta(\vec{x}^2, x_0)} = S_0^\beta(\vec{x}^2, x_0), \quad \overline{A_0^\beta(\vec{x}^2, x_0)} = A_0^\beta(\vec{x}^2, x_0). \quad (5.3.18)$$

2. S_0^β is even in x_0 and A_0^β is odd in x_0 :

$$S_0^\beta(\vec{x}^2, -x_0) = S_0^\beta(\vec{x}^2, x_0), \quad A_0^\beta(\vec{x}^2, -x_0) = -A_0^\beta(\vec{x}^2, x_0). \quad (5.3.19)$$

3. We have the KMS condition

$$W_0^\beta(\vec{x}^2, -x_0 - i\beta) = W_0^\beta(\vec{x}^2, x_0), \quad (5.3.20)$$

where we have set the speed of light c equal to 1.

[We will rewrite $\chi_\beta, \tilde{\chi}_\beta$ as $\chi_0^\beta, \tilde{\chi}_0^\beta$ to emphasize that they correspond to $\theta_{\mu\nu} = 0$.]
Thus from the Fourier transforms distinguished by tildes, as in

$$\widetilde{W}_0^\beta(\vec{x}^2, \omega) = \int dx_0 e^{i\omega x_0} W_0^\beta(\vec{x}^2, x_0), \quad (5.3.21)$$

we get

$$\widetilde{W}_0^\beta(\vec{x}^2, \omega) = e^{\beta\omega} \widetilde{W}_0^\beta(\vec{x}^2, -\omega), \quad (5.3.22)$$

$$\text{Im} \widetilde{\chi}_0^\beta(\vec{x}^2, \omega) = -\frac{e}{2} (1 - e^{\beta\omega}) \widetilde{W}_0^\beta(\vec{x}^2, -\omega), \quad (5.3.23)$$

$$e \widetilde{S}_0^\beta(\vec{x}^2, \omega) = -\coth \frac{\beta\omega}{2} \text{Im} \widetilde{\chi}_0^\beta(\vec{x}^2, \omega) \quad (5.3.24)$$

Now following an argument analogous to the one that yielded (5.2.20), we are able to write

$$\text{Im} \widetilde{\chi}_0^\beta(\vec{x}^2, \omega) = -\frac{i}{2} \widetilde{\chi}'_0^\beta(\vec{x}^2, \omega), \quad (5.3.25)$$

where

$$\begin{aligned} \chi'_0{}^\beta(\vec{x}^2, x_0) &:= \text{sgn}(x_0, \vec{x}) \chi_0^\beta(\vec{x}^2, |x_0|), \\ \text{sgn}(x_0, \vec{x}) &= \theta(x_0 - |\vec{x}|) - \theta(-x_0 - |\vec{x}|). \end{aligned} \quad (5.3.26)$$

Therefore, (5.3.24) becomes

$$e \widetilde{S}_0^\beta(\vec{x}^2, \omega) = -\coth \frac{\beta\omega}{2} \text{Im} \widetilde{\chi}_0^\beta(\vec{x}^2, \omega) = \frac{i}{2} \coth \frac{\beta\omega}{2} \widetilde{\chi}'_0^\beta(\vec{x}^2, \omega). \quad (5.3.27)$$

The Fourier transform of (5.3.27) gives

$$e S_0^\beta(\vec{x}^2, x_0) = \frac{1}{2\beta} P \int dx'_0 \text{sgn}(x'_0 - x_0, \vec{x}) \chi_0^\beta(\vec{x}^2, |x'_0 - x_0|) \coth \frac{\pi x'_0}{\beta}. \quad (5.3.28)$$

The expression for the mean square equilibrium fluctuation $\omega_\beta(\Delta N_0^2)(\vec{x}^2, 0)$ follows as before:

$$\begin{aligned}
\frac{1}{2}\omega_\beta(\Delta N_0^2)((\vec{x} - \vec{y})^2, 0) &= \frac{1}{2}\omega_\beta((N_0(\vec{x}, x_0 + \Delta x_0) - N_0(\vec{y}, x_0))^2) \\
&= e(S_0^\beta(\vec{0}^2, 0) - S_0^\beta((\vec{x} - \vec{y})^2, \Delta x_0)) = \frac{1}{2\beta} \left\{ 2 \int_{|\vec{0}|}^\infty dx'_0 \chi_0^\beta(\vec{0}^2, |x'_0|) \coth \frac{\pi x'_0}{\beta} \right. \\
&\quad \left. - \int_{|\vec{x}-\vec{y}|}^\infty dx'_0 \chi_0^\beta((\vec{x} - \vec{y})^2, |x'_0|) \left(\coth \frac{\pi(x'_0 + \Delta x_0)}{\beta} + \coth \frac{\pi(x'_0 - \Delta x_0)}{\beta} \right) \right\}
\end{aligned} \tag{5.3.29}$$

So nothing much has changed until this point except for the additional dependence of correlations on \vec{x}^2 .

An ansatz like (5.2.24) for susceptibility is no longer appropriate now. That is because if

$$x_0^2 < \vec{x}^2, \tag{5.3.30}$$

then as we saw $\chi_0^\beta(\vec{x}^2, x_0)$ is zero by causality.

Thus the relaxation time τ in units of c has the lower bound $|\vec{x}|$:

$$\tau \geq |\vec{x}|. \tag{5.3.31}$$

τ is a function of \vec{x}^2 , and we write $\tau(\vec{x}^2)$. Then the generalization of the ansatz (5.2.24) is

$$\chi_0^\beta(\vec{x}^2, x_0) = \mu \left[1 - e^{-\frac{x_0 - |\vec{x}|}{\tau(\vec{x}^2)}} \right] \theta(x_0 - |\vec{x}|) \xrightarrow{x_0 - |\vec{x}| \gg \tau} \mu \theta(x_0 - |\vec{x}| - \tau(\vec{x}^2)). \tag{5.3.32}$$

This lets us evaluate the mean square fluctuation of number density

$$\frac{1}{2}\omega_\beta(\Delta N_0^2)((\vec{x} - \vec{y})^2, 0) = \frac{\mu\hbar}{\pi} \ln \frac{[\sinh \Omega |\Delta x_0 - \tau((\vec{x} - \vec{y})^2)| \sinh \Omega |\Delta x_0 + \tau((\vec{x} - \vec{y})^2)]^{\frac{1}{2}}}{\sinh \Omega \tau(0)}, \tag{5.3.33}$$

where $\Omega = \frac{\pi}{\hbar\beta}$.

Following Sinha and Sorkin [118], we assume that

$$\Delta x_0 \gg \tau(\vec{x}^2) \geq |\vec{x}|. \quad (5.3.34)$$

There are thus four time scales:

$$\beta\hbar, \quad |\vec{x}|, \quad \tau(\vec{x}^2), \quad \Delta x_0, \quad (5.3.35)$$

where we have restored \hbar . With the assumption (5.3.34), we have four possibilities to consider:

1. $\beta\hbar \ll |\vec{x}| \ll \tau(\vec{x}^2) \ll \Delta x_0$,
2. $|\vec{x}| \ll \beta\hbar \ll \tau(\vec{x}^2) \ll \Delta x_0$,
3. $|\vec{x}| \ll \tau(\vec{x}^2) \ll \beta\hbar \ll \Delta x_0$,
4. $|\vec{x}| \ll \tau(\vec{x}^2) \ll \Delta x_0 \ll \beta\hbar$.

Case 1: *The classical Regime*

Case 1 is the "classical" limit. We get back Einstein's result in this case:

$$\begin{aligned} & \frac{1}{2}\omega_\beta(\Delta N_0^2)((\vec{x} - \vec{y})^2, 0) \\ &= \frac{\mu}{\beta}(\Delta x_0 - \tau(0)) = \mu kT(\Delta x_0 - \tau(0)). \end{aligned} \quad (5.3.36)$$

Cases 2 and 3 interpolate the classical regime and the extreme quantum regime of case 4. So let us first consider Case 4.

Case 4: *The Extreme Quantum Regime*

This is the new regime where Sinha and Sorkin [118] found a logarithmic dependence on time Δt of mean square fluctuations. It is now changed significantly.

$$\frac{1}{2}\omega_\beta(\Delta N_0^2)((\vec{x} - \vec{y})^2, 0) = \frac{\mu\hbar}{\pi} \ln\left(\frac{\Delta x_0}{\tau(0)} \left[1 - \left(\frac{\tau((\vec{x} - \vec{y})^2)}{\Delta x_0}\right)^2\right]^{\frac{1}{2}}\right). \quad (5.3.37)$$

As for the cases 2 and 3, our results are as follows:

Case 2: The same as *Case 1*.

$$\frac{1}{2}\omega_\beta(\Delta N_0^2)((\vec{x} - \vec{y})^2, 0) = \frac{\mu}{\beta}(\Delta x_0 - \tau(0)). \quad (5.3.38)$$

Case 3:

$$\frac{1}{2}\omega_\beta(\Delta N_0^2)((\vec{x} - \vec{y})^2, 0) = \frac{\mu}{\beta}\Delta x_0 + \frac{\mu\hbar}{\pi} \ln \frac{\hbar\beta}{2\pi\tau(0)}. \quad (5.3.39)$$

5.4 Quantum Fields on the Moyal Plane

For the Moyal plane, we must use the twisted fields and interactions as explained in the Introduction. That leads to the following expression for δN_θ :

$$\delta N_\theta(x) = -i \int_{-\infty}^{\infty} dx'_0 \theta(x_0 - x'_0) \omega_\beta([N_\theta(x), H_I^\theta(x'_0)]), \quad (5.4.1)$$

where

$$N_\theta = N_0 e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}, \quad H_I(x_0) = e \int d^3x \mathcal{H}_I^0(x) e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}, \quad (5.4.2)$$

\mathcal{H}_I^0 being the interaction Hamiltonian density in the interaction representation.

Note that $e^{\frac{1}{2}\overleftarrow{\partial} \wedge P}$ reduces to $e^{\frac{1}{2}\overleftarrow{\partial}_0 \theta^{0i} P_i}$ on integration over d^3x . But we will not use this simplification yet.

We shall first discuss the dependence on θ of two-point correlators.

Let us first examine the twisted Wightman function:

$$\begin{aligned} W_\theta^\beta(x, y) &= \omega_\beta(N_\theta(x)N_\theta(y)) \\ &= e^{-\frac{i}{2}\frac{\partial}{\partial x^\mu}\theta^{\mu\nu}\frac{\partial}{\partial y^\nu}} \omega_\beta(N_0(x)N_0(y)e^{-\frac{i}{2}(\frac{\overleftarrow{\partial}}{\partial x^\mu} + \frac{\overleftarrow{\partial}}{\partial y^\mu})\theta^{\mu\nu}P_\nu}). \end{aligned} \quad (5.4.3)$$

We can write this as an integral (and sum) over states with total momentum p such as

$$\langle p, \dots | e^{-\beta P_0} N_0(x) N_0(y) e^{-\frac{i}{2} (\overleftarrow{\frac{\partial}{\partial x^\mu}} + \overleftarrow{\frac{\partial}{\partial y^\mu}}) \theta^{\mu\nu} P_\nu} | p, \dots \rangle, \quad (5.4.4)$$

where the dots indicate that there will in general be many states contributing to a state of given total momentum p . We can write (5.4.4) as

$$\langle p, \dots | e^{-\beta P_0} N_0(x) N_0(y) e^{-\frac{i}{2} \overleftarrow{ad} P_\mu \theta^{\mu\nu} P_\nu} | p, \dots \rangle, \quad (5.4.5)$$

where $ad P_\mu A = [P_\mu, A]$. for any operator A . But

$$\langle p, \dots | [P_\mu, A] | p, \dots \rangle = 0 \quad (5.4.6)$$

for any A . Consequently (5.4.4) is

$$W_\theta^\beta(x, y) = e^{-\frac{i}{2} \overleftarrow{\frac{\partial}{\partial x^\mu}} \theta^{\mu\nu} \overleftarrow{\frac{\partial}{\partial y^\nu}}} W_0^\beta(x, y). \quad (5.4.7)$$

But now we can write $W_0^\beta(x, y)$ as we wrote it earlier:

$$W_0^\beta(x, y) \rightarrow W_0^\beta((\vec{x} - \vec{y})^2, x_0 - y_0). \quad (5.4.8)$$

It depends on $x - y$. Hence in the exponential,

$$\frac{\partial}{\partial x^\mu} \theta^{\mu\nu} \frac{\partial}{\partial y^\nu} = -\frac{\partial}{\partial x^\mu} \theta^{\mu\nu} \frac{\partial}{\partial x^\nu} = 0. \quad (5.4.9)$$

Similarly,

$$\begin{aligned} S_\theta^\beta(x, y) &= \frac{1}{2} \omega_\theta(N_\theta(x) N_\theta(y) + N_\theta(y) N_\theta(x)) = S_0^\beta((\vec{x} - \vec{y})^2, x_0 - y_0), \\ A_\theta^\beta(x, y) &= -\frac{i}{2} \omega_\theta([N_\theta(x), N_\theta(y)]) = A_0^\beta((\vec{x} - \vec{y})^2, x_0 - y_0) \end{aligned} \quad (5.4.10)$$

and they have the properties listed earlier.

But we cannot conclude that δN_θ is independent of $\theta^{\mu\nu}$ as well. Specializing to

$$\mathcal{H}_I^0 = N_0 \varphi_0, \quad (5.4.11)$$

we find

$$\delta N_\theta(x) = \delta N_\theta^1(x) - \delta N_\theta^2(x), \quad (5.4.12)$$

$$\delta N_\theta^1(x) = -i \int d^4 x' \theta(x_0 - x'_0) e^{-\frac{i}{2} \frac{\partial}{\partial x^\mu} \theta^{\mu\nu} \frac{\partial}{\partial x'^\nu}} \omega_\beta(N_0(x) \mathcal{H}_I^0(x')) e^{-\frac{i}{2} (\frac{\partial}{\partial x^\mu} + \frac{\partial}{\partial x'^\mu}) \theta^{\mu\nu} P_\nu} \quad (5.4.13)$$

with a similar expression for $\delta N_\theta^2(x)$. The last exponential can be replaced by 1 as before. Also, integration over \vec{x}' reduces $e^{-\frac{i}{2} \frac{\partial}{\partial x^\mu} \theta^{\mu\nu} \frac{\partial}{\partial x'^\nu}}$ to $e^{-\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x'^0}}$,

$$e^{-\frac{i}{2} \frac{\partial}{\partial x^\mu} \theta^{\mu\nu} \frac{\partial}{\partial x'^\nu}} \rightarrow e^{-\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x'^0}}. \quad (5.4.14)$$

Thus

$$\delta N_\theta^1 = -i \int d^4 x' \theta(x_0 - x'_0) e^{-\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x'^0}} \omega_\beta(N_0(x) N_0(x')) \varphi_0(x') \quad (5.4.15)$$

and similarly

$$\delta N_\theta^2 = -i \int d^4 x' \theta(x_0 - x'_0) e^{\frac{i}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x'^0}} \omega_\beta(N_0(x') N_0(x)) \varphi_0(x'). \quad (5.4.16)$$

We now discuss the two cases where x is space- and time-like with respect to $\text{supp } \varphi_0$.

x spacelike with respect to Supp φ_0 :

This is the case where we anticipate qualitatively new results.

While calculating $\delta N_\theta^1(x') - \delta N_\theta^2(x')$, we cannot set

$$N_0(x) N_0(x') = N_0(x') N_0(x) \quad (\text{from causality}) \quad (5.4.17)$$

because the exponentials in the integrand translate the arguments x and x' , and can bring them to timelike separations. With this in mind, we can write

$$\begin{aligned}\delta N_\theta(x) &= -i \int d^4x' \theta(x_0 - x'_0) \cos\left[\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}\right] \omega_\beta([N_0(x), N_0(x')]) \varphi_0(x') \\ &\quad - \int d^4x' \theta(x_0 - x'_0) \sin\left[\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}\right] \omega_\beta(N_0(x)N_0(x') + N_0(x')N_0(x)) \varphi_0(x').\end{aligned}\tag{5.4.18}$$

We can replace $\cos(\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}})$ by $\cos(\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}) - 1 = 2 \sin^2(\frac{1}{4} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}})$ as the extra term contributes 0 by causality. This shows that this term is $O((\theta^{i0})^2)$. Finally,

$$\begin{aligned}\delta N_\theta(x) &= - \int d^4x' \theta(x_0 - x'_0) \sin\left[\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}\right] \omega_\beta(N_0(x)N_0(x') + N_0(x')N_0(x)) \varphi_0(x') \\ &\quad + 2i \int d^4x' \theta(x_0 - x'_0) \sin^2\left[\frac{1}{4} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}\right] \omega_\beta([N_0(x), N_0(x')]) \varphi_0(x').\end{aligned}\tag{5.4.19}$$

This shows clearly that there is an acausal fluctuation in $\delta N_\theta(x)$ when φ_0 (the ‘‘chemical potential’’) is fluctuated in a region D_2 spacelike with respect to x .

But it occurs only when time-space noncommutativity (θ^{0i}) is non-zero.

We will come back to this term after also briefly looking at the case where x is not spacelike with respect to D_2 .

x is not spacelike with respect to $Supp \varphi_0$

The only change as compared to the spacelike case is that we must restore the extra term, which contributed 0 in the spacelike case, but does not do that now.

We can simplify notation by defining $\Delta N_\theta(x)$ for any x as follows:

$$\begin{aligned}\Delta N_\theta(x) &= - \int d^4x' \theta(x_0 - x'_0) \sin\left[\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}\right] \omega_\beta(N_0(x)N_0(x') + N_0(x')N_0(x)) \varphi_0(x') \\ &\quad + 2i \int d^4x' \theta(x_0 - x'_0) \sin^2\left[\frac{1}{4} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x^{0i}}\right] \omega_\beta([N_0(x), N_0(x')]) \varphi_0(x').\end{aligned}\tag{5.4.20}$$

Then

a) If $x \times \text{Supp } \varphi_0$,

$$\delta N_\theta(x) = \Delta N_\theta(x). \quad (5.4.21)$$

b) If x is not spacelike with respect to $\text{Supp } \varphi_0$,

$$\delta N_\theta(x) = i \int d^4 x' \theta(x_0 - x'_0) \omega_\beta([N_0(x), N_0(x')]) \varphi_0(x') + \Delta N_\theta(x). \quad (5.4.22)$$

5.4.1 An exact expression for susceptibility

We want to write

$$\delta N_\theta(x) = \int d^4 x' \chi_\theta(x, x') \varphi_0(x'), \quad (5.4.23)$$

where χ_θ is the deformed susceptibility.

We will succeed in doing that by deriving an exact expression for the Fourier transform

$$\tilde{\chi}_\theta(k) = \int d^4 x e^{ikx} \chi_\theta(x), \quad kx = k_0 x_0 - \vec{k} \cdot \vec{x}, \quad (5.4.24)$$

in terms of $\tilde{\chi}_0(k)$. The corrections to $\tilde{\chi}_0(k)$ have remarkable zeros and direction dependence which we will soon point out.

We can write

$$\delta N_\theta(x) = \delta N_0(x) + \Delta N_\theta(x), \quad (5.4.25)$$

where

$$\delta N_0(x) = \int d^4 x' \chi_0(x - x') \varphi_0(x') \quad (5.4.26)$$

and

$$\begin{aligned}\Delta N_\theta(x) &= \Delta N_\theta^1(x) - \Delta N_\theta^2(x), \\ \Delta N_\theta^{(1)}(x) &= -2 \int d^4 x' \theta(x_0 - x'_0) \sin\left(\frac{1}{2} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x'_0}\right) S_0^\beta(x - x') \varphi_0(x') \\ &:= \int d^4 x' \chi_\theta^{(1)}(x - x') \varphi_0(x'),\end{aligned}\tag{5.4.27}$$

$$\begin{aligned}\Delta N_\theta^{(2)}(x) &= -4 \int d^4 x' \theta(x_0 - x'_0) \sin^2\left(\frac{1}{4} \frac{\partial}{\partial x^i} \theta^{i0} \frac{\partial}{\partial x'_0}\right) A_0^\beta(x - x') \varphi_0(x') \\ &:= \int d^4 x' \chi_\theta^{(2)}(x - x') \varphi_0(x').\end{aligned}\tag{5.4.28}$$

$$\tag{5.4.29}$$

In (5.4.27) and (5.4.28), $\frac{\partial}{\partial x'_0} = (\frac{\partial}{\partial x'_0})_1 + (\frac{\partial}{\partial x'_0})_2$, where the first differentiates just S_0^β and the second differentiates just φ_0 .

On partially integrating the second derivative, it cancels the first derivative acting on S_0^β leaving a derivative $\frac{\partial}{\partial x'_0}$ acting on $\theta(x_0 - x'_0)$. So finally

$$\chi_\theta^{(1)}(x) = 2S_0^\beta(x) \sin\left(\frac{1}{2} \overleftarrow{\frac{\partial}{\partial x^i}} \theta^{i0} \overrightarrow{\frac{\partial}{\partial x_0}}\right) \theta(x_0)\tag{5.4.30}$$

and similarly,

$$\chi_\theta^{(2)}(x) = -4A_0^\beta(x) \sin^2\left(\frac{1}{4} \overleftarrow{\frac{\partial}{\partial x^i}} \theta^{i0} \overrightarrow{\frac{\partial}{\partial x_0}}\right) \theta(x_0).\tag{5.4.31}$$

Let us Fourier transform these expressions setting

$$\tilde{\chi}_\theta^{(1)}(k) = \int d^4 x e^{ikx} \chi_\theta^{(1)}(x),\tag{5.4.32}$$

$$\tilde{\chi}_\theta^{(2)}(k) = \int d^4 x e^{ikx} \chi_\theta^{(2)}(x)\tag{5.4.33}$$

and similarly for $\tilde{S}(\vec{k})$, $\tilde{A}(\vec{k})$. Then

$$\begin{aligned}\tilde{\chi}_\theta^{(1)}(\vec{k}) &= \frac{1}{\pi} \int dx_0 \theta(x_0) \left[\int dq_0 e^{i(k_0 - q_0)x_0} \sin \frac{k_i \theta^{i0}(k_0 - q_0)}{2} \tilde{S}(\vec{k}, q_0) \right], \\ \tilde{\chi}_\theta^{(2)}(\vec{k}) &= -\frac{2}{\pi} \int dx_0 \theta(x_0) \left[\int dq_0 e^{i(k_0 - q_0)x_0} \sin^2 \frac{k_i \theta^{i0}(k_0 - q_0)}{4} \tilde{A}(\vec{k}, q_0) \right].\end{aligned}\tag{5.4.34}$$

Here we can write \tilde{S} and \tilde{A} in terms of $\text{Im}\tilde{\chi}_0$:

$$\tilde{S}(\vec{k}, k_0) = -\coth \frac{\beta k_0}{2} \text{Im}\tilde{\chi}_0(\vec{k}, k_0),\tag{5.4.35}$$

$$\tilde{A}(\vec{k}, k_0) = i \text{Im}\tilde{\chi}_0(\vec{k}, k_0).\tag{5.4.36}$$

Finally for the twisted susceptibility χ'_θ ,

$$\chi_\theta = \chi_0 + \chi_\theta^{(1)} + \chi_\theta^{(2)},\tag{5.4.37}$$

where we have *exact* expressions for $\chi_\theta^{(j)}$ in terms of $\text{Im}\chi_0$.

5.4.2 Zeros and Oscillations in $\tilde{\chi}_\theta^{(j)}$

A generic $\text{Im}\tilde{\chi}_0$ is the *superposition* of terms with δ -function supports at frequencies ω , that is, of terms

$$\delta(k_0 - \omega) \text{Im}\tilde{\chi}_0^R(\vec{k}, \omega)\tag{5.4.38}$$

(R standing for “reduced”).

We now focus on a single frequency ω , that is, the case where $\text{Im}\tilde{\chi}_0(\vec{k}, k_0)$ equals (5.4.38). Then

$$\tilde{\chi}_\theta^1(\vec{k}) = -\frac{i}{\pi} \coth \frac{\beta \omega}{2} \frac{1}{k_0 - \omega} \sin \frac{k_i \theta^{i0}(k_0 - \omega)}{2} \text{Im}\tilde{\chi}_0^R(\vec{k}, \omega)\tag{5.4.39}$$

$$\tilde{\chi}_\theta^2(\vec{k}) = \frac{2}{\pi} \frac{1}{k_0 - \omega} \sin^2 \frac{k_i \theta^{i0}(k_0 - \omega)}{4} \text{Im}\tilde{\chi}_0^R(\vec{k}, \omega).\tag{5.4.40}$$

These corrections have striking zeros and oscillations which would be characteristic signals for noncommutativity. Thus,

a)

$$\tilde{\chi}_\theta^{(1)}(k) = \tilde{\chi}_\theta^{(2)}(k) = 0 \quad \text{if} \quad \frac{k_i \theta^{i0} (k_0 - \omega)}{2} = 2n\pi, \quad n \in \mathbb{Z}. \quad (5.4.41)$$

$\tilde{\chi}_\theta^{(1)}$ actually vanishes at all $n\pi$.

b) Regarding the oscillations, they are from the \sin and \sin^2 terms. The sine repeats if its argument is changed by

$$2n\pi \quad (5.4.42)$$

while the \sin^2 term does so if its argument is changed by

$$n\pi \quad (5.4.43)$$

($n \in \mathbb{Z}$). These are multiplying backgrounds with no particular oscillatory behavior.

Both *a*) and *b*) are characteristic features of the Moyal Plane and in principle accessible to experiments. We emphasize that that both these effects are direction-dependent.

These features may have applications to the homogeneity problem in cosmology [121].

5.5 Finite temperature Lehmann representation

The Lehmann representation in QFT expresses the two-point vacuum correlation functions of a fully interacting theory in terms of their free field values. It is exact and captures the properties emerging from the spectrum of P_μ and Poincaré invariance in a useful manner.

We have seen in Section 4 that all the two-point correlations at finite temperature for $\theta^{\mu\nu} \neq 0$ can be expressed in terms of the corresponding expressions for $\theta^{\mu\nu} = 0$. In this section, we treat the $\theta^{\mu\nu} = 0$ case in detail which then also covers the $\theta^{\mu\nu} \neq 0$ case.

First we state some notation. The single particle states are normalized according to

$$\langle k'|k\rangle = 2|k_0|\delta^3(k' - k), \quad k_0 = (\vec{k}^2 + m^2)^{\frac{1}{2}}, \quad (5.5.1)$$

where m is the particle mass. The scalar product of n -particle states such as $|k_1, \dots, k_n\rangle$ then follows, (with appropriate symmetrization factors which we will not display here or below). We will also not display degeneracy indices such as those from color: their treatment is easy. For a similar reason, we consider spin 0 fields.

For the normalization (5.5.1), the volume form dV_n for the n -particle state is a product of factors

$$\frac{d^3k_j}{2|k_{j0}|},$$

$$dV_n = \prod_{j=1}^n d\mu_j, \quad d\mu_j = \frac{d^3k_j}{2|k_{j0}|}, \quad |k_{j0}| = \sqrt{\vec{k}_j^2 + m_j^2}. \quad (5.5.2)$$

Now consider

$$W_0^\beta(x) = \omega_\beta(\varphi_0(x)\varphi_0(x')), \quad (5.5.3)$$

where φ_0 is a scalar field for $\theta^{\mu\nu} = 0$ and H is the total time-translation generator of the Poincaré group. Its spacetime translation invariance implies that

$$\omega_\beta(\varphi_0(x)\varphi_0(x')) = \omega_\beta(\varphi_0(x - x')\varphi_0(0)). \quad (5.5.4)$$

We assume as usual that

$$\langle 0|\varphi_0(x)|0\rangle = 0. \quad (5.5.5)$$

We can write

$$\begin{aligned} W_0^\beta(x) &= \frac{\langle 0|e^{-\beta H}\varphi_0(x)\varphi_0(0)|0\rangle + \omega_\beta(\varphi_0(x)|0)\langle 0|\varphi_0(0)\rangle}{Z(\beta)} + \widehat{W}_0^\beta(x), \\ Z(\beta) &:= \text{Tr}e^{-\beta H}. \end{aligned} \quad (5.5.6)$$

We shall see that the vacuum contributions are separated out in the first two terms and that vacuum intermediate states do not contribute to \widehat{W}_0^β .

We now consider the three terms separately.

$$1) \quad \frac{1}{Z(\beta)}\langle 0|e^{-\beta H}\varphi_0(x)\varphi_0(0)|0\rangle = \frac{1}{Z(\beta)}W_0^0(x) \equiv \frac{1}{Z(\beta)}W(x). \quad (5.5.7)$$

Here $W(x)$ is the zero-temperature Wightman function with its standard spectral representation:

$$W(x) = \int dM^2 \rho(M^2)\Delta_+(x, M^2), \quad \Delta_+(x, M^2) = \int d^4p \delta(p^2 - M^2)\theta(p_0)e^{ipx}. \quad (5.5.8)$$

$$2) \quad \omega_\beta(\varphi_0(x)|0)\langle 0|\varphi_0(0)\rangle = \frac{1}{Z(\beta)} \sum_{n \geq 1} \int dV_n \langle k_1, \dots, k_n | e^{-\beta H} \varphi_0(x) | 0 \rangle \langle 0 | \varphi_0(0) | k_1, \dots, k_n \rangle \quad (5.5.9)$$

where the $n = 0$ term has been omitted in the sum as it contributes 0 by (5.5.5).

Using

$$\varphi_0(x) = e^{iPx}\varphi_0(0)e^{-iPx}, \quad (5.5.10)$$

where P_μ generates translations ($P_0 = H$), we find

$$\omega_\beta(\varphi_0(x)|0\rangle\langle 0|\varphi_0(0)) = \frac{1}{Z(\beta)} \int d^4k \theta(k_0) e^{-\beta k_0 + ikx} \rho(k^2), \quad (5.5.11)$$

$$\rho(k^2) = \sum_n \int \prod_{j=1}^n \delta(k_j^2 - m_j^2) \theta(k_{j0}) \delta^4(\sum k_j - k) |\langle k_1, \dots, k_n | \varphi_0(0) | 0 \rangle|^2, \quad (5.5.12)$$

ρ being the zero-temperature spectral function.

Thus

$$\omega_\beta(\varphi_0(x)|0\rangle\langle 0|\varphi_0(0)) = \frac{1}{Z(\beta)} \int dM^2 \rho(M^2) \Delta_+(x, M^2; \beta), \quad (5.5.13)$$

$$\Delta_+(x, M^2; \beta) = \int d^4k \theta(k_0) \delta(k^2 - M^2) e^{-\beta k_0 + ikx}. \quad (5.5.14)$$

For $\beta = 0$, $\Delta_+(x, M^2; 0)$ is the free field zero-temperature Wightman function. It vanishes when $\beta \rightarrow \infty$.

$$3) \widehat{W}_0^\beta(x) = \frac{1}{Z(\beta)} \sum_{n, m \geq 1} \int dV_n dV_m \langle k_1, \dots, k_n | e^{-\beta H} \varphi_0(x) | q_1, \dots, q_m \rangle \langle q_1, \dots, q_m | \varphi_0(0) | k_1, \dots, k_m \rangle.$$

The vacuum contributions (n and /or $m = 0$) have already been considered and need not be included here.

Elementary manipulations like those above show that

$$\begin{aligned} \widehat{W}_0^\beta(x) &= \frac{1}{Z(\beta)} \int d^4K d^4Q \theta(K_0) \theta(Q_0) e^{-\beta K_0 + i(K-Q)x} \times \\ &\{ \sum_{n, m \geq 1} \int \prod_{j=1}^n d^4k \theta(k_{j0}) \delta(k_j^2 - m_j^2) \prod_{j=1}^m d^4q \theta(q_{j0}) \delta(q_j^2 - m_j^2) \times \\ &\delta^4(\sum k_j - K) \delta^4(\sum q_j - Q) |\langle k_1, \dots, k_n | \varphi_0(0) | q_1, \dots, q_m \rangle|^2 \}. \end{aligned} \quad (5.5.15)$$

The term in braces, by relativistic invariance, depends only on K^2 , Q^2 and $(K + Q)^2$. As K_μ , Q_μ are timelike with $K_0, Q_0 > 0$, we have, as in scattering theory,

$$(K + Q)^2 \geq (\sqrt{K^2} + \sqrt{Q^2})^2. \quad (5.5.16)$$

Call the terms in braces as $\rho(K^2, Q^2, (K + Q)^2)$. Then

$$\begin{aligned} \widehat{W}_0^\beta(x) = & \frac{1}{Z(\beta)} \int dM^2 dN^2 dR^2 \rho(M^2, N^2, R^2) \times \\ & \{ \int d^4K \theta(K_0) \delta(K^2 - M^2) \int d^4Q \theta(Q_0) \delta(Q^2 - N^2) \delta((K + M)^2 - R^2) e^{-\beta K_0 + i(K-Q)x} \}. \end{aligned} \quad (5.5.17)$$

The term in braces here is the elementary function appropriate for \widehat{W}_θ^β .

The full spectral representation for W_θ^β is obtained by adding those of its terms given above.

5.6 Conclusions: Chapter 5

A major result of this chapter is the derivation of acausal and noncommutative effects in finite temperature QFT's. They are new and are expected to have applications for instance in the homogeneity problem in cosmology.

We have also treated the finite temperature Lehmann representation on the commutative and Moyal planes in detail. This representation succinctly expresses the spectral and positivity properties of the underlying QFT's in a transparent manner and are thus expected to be useful.

Chapter 6

Conclusions

We have given a brief review of quantum theory as well as an introduction to quantum field theory in noncommutative spacetime. The concept of deformed Lorentz invariance in noncommutative spacetime led to the following effects which may be susceptible to experimental tests.

1. Deformed statistics of quantum fields whose consequences include
 - 1) modification of the statistical interparticle force and hence degeneracy pressure which determines the fate of galactic nuclei after fuel burning ceases,
 - 2) the possibility of observing Pauli forbidden transitions,
 - 3) observation of Lorentz, P, PT, CP, CPT and causality violations.
2. The presence of noncommutativity dependent temperature fluctuations in the CMB radiation, through a noncommutativity dependent post inflation power spectrum; giving an estimated upper bound for the noncommutativity parameter and a corresponding lower bound for the energy scale.

3. Encounter with noncommutativity-induced causality violation and a non-Gaussian probability distribution during cosmological inflation.
4. Noncommutativity induces noncausal, and potentially periodic, corrections to the susceptibility in linear response theory.

To summarize we have investigated, in the context of quantum field theory, the scope of applicability of a new concept of Lorentz invariance. This new concept is a deformation of the usual concept of Lorentz invariance motivated by the form of invariance in Moyal's treatment of quantum mechanics. The investigations were based on certain available theoretical models and experimental data. Results of these investigations can point to alternative and hopefully simpler solutions to both expected and observed physical phenomena whose experimental energies fall within the range of validity of the noncommutativity models.

Appendix A

Some physical concepts

A.1 Motion of an electron in constant magnetic field

When an electron moves in a constant magnetic field the coordinates of the center of its circular motion (ie. guiding center) become noncommutative when the system is quantized canonically. The Lagrangian and equations of motion

$$L = \frac{m\vec{v}^2}{2} - e\vec{v} \cdot \vec{A}, \quad \vec{A} = -\frac{1}{2} \vec{x} \times \vec{B},$$
$$m \frac{d\vec{v}}{dt} = e\vec{v} \times \vec{B}, \quad \vec{v} = \frac{d\vec{x}}{dt}$$

have the solution

$$\vec{x}(t) = \vec{x}_0 + \frac{m}{eB} \hat{B} \times \vec{v}_0 + \hat{B} (\hat{B} \cdot \vec{v}_0)(t - t_0)$$
$$- \hat{B} \times \vec{v}_0 \frac{\cos(\omega(t - t_0))}{\omega} - \hat{B} \times (\hat{B} \times \vec{v}_0) \frac{\sin(\omega(t - t_0))}{\omega},$$
$$\omega = \frac{e|\vec{B}|}{m} = \frac{eB}{m}.$$

The position of the center of circular motion is

$$\vec{x}_c(t) = \vec{x}_0 + \frac{m}{eB} \hat{B} \times \vec{v}_0 + \hat{B} (\hat{B} \cdot \vec{v}_0)(t - t_0).$$

and the canonical momentum is

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} - e\vec{A}.$$

One gets the canonical commutation relations

$$\begin{aligned} [x^i(t), x^j(t)] &= 0 = [p^i(t), p^j(t)] \quad \forall t, \\ [x^i(t), p^j(t)] &= [x^i(t), mv^j(t) - eA^j(x(t))] = [x^i(t), mv^j(t)] = -i\hbar\delta^{ij} \quad \forall t, \end{aligned}$$

from which one can verify that

$$\begin{aligned} [v^i(t), v^j(t)] &= i\frac{e\hbar B}{m^2} \varepsilon^{ikj} \hat{B}^k \quad \forall t, \\ \Rightarrow [x_c^i(t), x_c^j(t)] &= i\theta^{ij} = i\frac{\hbar}{eB} \varepsilon^{ikj} \hat{B}^k \quad \forall t. \end{aligned} \quad (\text{A.1.1})$$

Here $\theta^{ij} = \frac{\hbar}{eB} \varepsilon^{ikj} \hat{B}^k$ is not invertible as a 3×3 matrix as $\det_{3 \times 3} \theta = 0$. However if we arrange the system such that $\hat{B} \cdot \vec{v}_0 = 0$, say with $B^k = B\delta^{zk}$, $v_0^k = v_0^x \delta^{xk} + v_0^y \delta^{yk}$, then the motion stays in the $x - y$ plane and $\theta^{ij} = \frac{\hbar}{eB} \varepsilon^{i3j}$ is now invertible as a 2×2 matrix.

A.2 Symmetries and the least action principle

A.2.1 Use of symmetries

A major reason for the use of symmetries to analyze physical systems stems from the fact that the kinematics and/or dynamics of a physical system can be cast in terms of nonanalytic and/or analytic (differential or integral) constraints or

equations which may also be derivable from a least action principle. The symmetry group of the action or Lagrangian is a subgroup of the symmetry group of the equations. The key observation that the solution space of the equations is invariant under the symmetry group of the equations implies that the complete space of solutions can be generated from only a few simple solutions. Moreover, most of the physically relevant information about the solution space of the equations is contained in their symmetry group. In particular, one expects that each independent solution of the equations has a simple correspondence with an irreducible representation of the symmetry group. Therefore instead of trying to solve the equations directly, one could rather consider the problem of finding the irreducible representations of the symmetry group. The group theoretic analysis is most useful for interacting physical systems where the interactions lead to coupled nonlinear equations for which even the simplest solution can be difficult to find. One may postulate that whenever two separate systems couple, one or more of the variables involved should be modified or extended such that their individual symmetry groups become either 1) independent symmetry groups of the coupled system or 2) subgroups of a larger symmetry group of the coupled system or 3) identified; that is, merged together into a larger unifying symmetry group. Another major reason for the use of symmetries is that they identify physically observable quantities, such as interaction amplitudes or potentials, as those that can survive the symmetry transformation. Together with an action principle, the symmetries also provide conservation laws and conserved quantities (Noether's theorem) which help simplify the analysis of interactions.

A.2.2 Analogy and least action principle

Biological systems, their developments and the interactions among them may be characterized by the way they respond to the variety of (certain) natural changes in their supporting environments (“external” changes) and also to a variety of changes in their most basic or defining configurations (“internal” changes) in these environments. Similarly, mathematical structures, operations on them and the relations between them can be characterized by the way they respond to a variety of special maps or transformations among their supporting spaces which are the spaces on which they are defined or configured and also to a variety of special maps or transformations among the spaces consisting of the structures and classes of structures themselves. Many mathematical models for (elementary) physical systems (their configurations and interactions in space and time or simply spacetime) can be based on a least action principle for a composite or derived mathematical structure on spacetime called the action functional. The action functional is a configuration-dependent variable that is written as a sum total of a Lagrangian over the domain (the region of spacetime in which the system can be variously configured) of the physical system. The Lagrangian is a quantity written in terms of spacetime variables and spacetime-dependent configuration variables for the physical system. A classical physical system is then characterized by its symmetries; those transformations or changes in spacetime variables and/or configuration variables and Lagrangian that do not alter the outcome of (or the equations of motion resulting from) the least action principle. The least action principle asserts that within a given spacetime domain, supporting all possible configurations of the physical system, the actual configuration of the physical system is the one for which the action functional is minimum. The domain of the system in spacetime

may either be a collection of points (eg. the system is a set of "events"), a one dimensional path (eg. the system is a mechanical "particle") or a hypersurface in general (eg. the system is either an extended classical object or a quantum event). In quantum theory, it turns out that one needs to average quantities over the configuration space domain of the physical system with a probability density function given by the exponential of the classical action. The exponential form of the probability distribution is due to the correspondence between the additive nature of the classical action and the multiplicative nature of the joint probability distribution for a collection of noninteracting systems.

A.3 Renormalizability

NB: Here the term "classical" is synonymous to "low energies" meanwhile the term "quantum" is synonymous to "all possible energies".

In quantum theory, the probability amplitude for the evolution of a physical system from an initial quantum configuration (or a set of possible initial quantum configurations) to a final quantum configuration (or a set of possible final quantum configurations) may be defined or postulated in terms of certain functionals known as Green's functions. For a noninteracting theory these probability amplitudes are finite. The introduction of interactions leads to initial/final quantum configuration-dependent quantum corrections to the probability amplitudes. Some of these corrections contain purely divergent parts. The finite parts of the divergent corrections can be isolated with the help of a regularization procedure. In some cases the remaining purely divergent parts can be eliminated by simple redefinitions of the parameters in the classical action and hence a few additional parameters to be determined experimentally.

This observation therefore suggests that whenever there are interactions one has corresponding initial/final state dependent quantum corrections to the experimentally measured values of the parameters found in the classical action as well. The elimination of the purely divergent parts of the corrections is known as renormalization and theories in which the simple parameter redefinitions are sufficient to eliminate all possible divergences are said to be renormalizable. Nonrenormalizable theories are known as effective (as opposed to fundamental) theories since due to divergences they can be valid only for a restricted range of initial/final quantum configurations. Effective theories are expected to arise as consequences of fundamental theories. There are several possible regularization procedures resulting in different renormalized values for the same quantity. A theory may have more than one symmetry and when none of the possible regularization procedures can preserve all the symmetries then anomalies, which may be presented as a failure of the conservation law of Noether currents, arise rendering the theory nonrenormalizable in some cases. Anomalies signal a possibility of incompleteness of the theory that may be for example due to a failure to take into account extra degrees of freedom (ie. a missing piece of the configuration space of the system) posing as topological nontriviality of spacetime and/or configuration space, or considering too many degrees of freedom such as the case where a reducible space rather than an irreducible one is used.

A consequence of renormalization is that requiring nondependence of the Green's functions and the measurable or measured coupling constant and/or mass on the regularization parameter implies a first order differential relationship between the Green's functions and the measured coupling constant and/or mass. The solution to this differential relationship indicates a scaling behavior for the Greens's function as the measured coupling constant and/or mass is varied through a single real

parameter that may be thought of as a parameter for the group of all possible renormalization schemes. The fixed points of this variation may indicate possible phase transitions which are marked changes in the behavior of the Green's functions as the initial/final quantum configurations of the system are varied. This is because a change in renormalization scheme causes a change in the renormalized or measured coupling constant and/or mass (which in turn depend only on the initial/final quantum configurations of the system) and so may be regarded as being equivalent to a change in the initial/final quantum configurations of the system. The scaling behavior together with symmetry properties of the Greens function give a qualitative description of the Green's function, and hence of the quantum configurations of the system, especially near the critical or fixed points.

A.4 Rules for writing probability amplitudes of physical processes

A sample Lagrangian is that of QED

$$\mathcal{L} = \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + i\bar{\psi}\gamma^\mu(\partial_\mu - iqA_\mu)\psi. \quad (\text{A.4.1})$$

1. Sketch all possible *connected* Feynman diagram(s) of the process and indicate momenta.
2. Each external (initial/final) line (“half of a propagator”) represents the *normalized* Fourier coefficient of the classical field; which includes polarization/spin “vectors” or directions.
3. Each internal line represent a full time-ordered (ie. Feynman) propagator.

4. Each vertex represents one or more of the following: coupling constants, momentum vectors, spin matrices, representation matrices/tensors, etc, as they appear in the (Fourier transformed) Lagrangian. When written out, a vertex with external lines has the form of the Fourier transform of a current from the Lagrangian.
5. Conserve overall momentum, conserve momentum at each vertex. This may be done either directly or by including delta functions.
6. For each loop, integrate over the residual momentum that remains after momentum conservation has been applied to all vertices surrounding the loop.
7. Trace over γ -matrices in a *purely fermion* loop.
8. Divide the amplitude of each diagram by its “symmetry factor” which represents how many times the given diagram has been over counted as compared to those other diagrams that lack the symmetries of the given diagram.
9. Add together the contributions from each diagram to get the total amplitude of the process.

Appendix B

Quantization

B.1 Canonical quantization, deformation quantization and noncommutative geometry

The form of the classical action in the Lagrangian and Hamiltonian pictures is

$$\begin{aligned} S[q] &= \int d\lambda L(\lambda, q, \dot{q}) = \int p_i dq^i - \int d\lambda H(\lambda, q, p), \quad p = \frac{\partial L}{\partial \dot{q}}, \\ L(\lambda, q, \dot{q}) &= \sum_i \mathcal{L}_i(\lambda, q, \dot{q}), \quad H(\lambda, q, p) = \sum_i \mathcal{H}_i(\lambda, q, p). \end{aligned} \quad (\text{B.1.1})$$

One can identify the canonical 1-form

$$\begin{aligned} A_I(\lambda, x) dx^I &\equiv p_i dq^i + d\alpha(\lambda, q, p), \quad x^I = (x_i^0, x_i^1) = (q_i(\lambda), p_i(\lambda)), \\ A(\lambda, x) &= (A_i^0(x, \lambda), A_i^1(x, \lambda)) = \left(\frac{\partial \alpha}{\partial q^i}, p_i(\lambda) + \frac{\partial \alpha}{\partial p^i} \right) \end{aligned} \quad (\text{B.1.2})$$

from the Legendre transformation $H(\lambda, d\lambda, q, p) = p_i dq^i - L(\lambda, d\lambda, q, dq)|_{p^i = \frac{\partial L}{\partial \dot{q}^i}}$.

A canonical transformation is any symmetry of the Lagrangian L (L changes by at most a total derivative) which is also a symmetry of the Hamiltonian L (H changes by at most a total derivative) and since $H = A - L$ it means that a

canonical transformation is any symmetry of L for which A changes by at most a total differential ($\delta A = d\beta$ or equivalently $\delta\Omega = \delta dA = d\delta A = 0$).

Therefore a canonical transformation is a transformation on phase space $T^*(\mathcal{M}) \simeq \{q, p\}$, $\mathcal{M} \simeq \{q\}$ that preserves the exterior differential $\Omega = dA$, $A \in T^*(T^*(\mathcal{M}))/\mathcal{M}$. The relationship between the canonical 2-form $\Omega = dA = \partial_I A_J dx^I \wedge dx^J$ and $A = A_I dx^I$ is analogous to the relationship between the electromagnetic 2-form and its 1-form. The infinitesimal transformation of any 2-form Ω

$$\begin{aligned} \delta\Omega &= \delta(\Omega_{IJ} dx^I \wedge dx^J) = \delta\Omega_{IJ} dx^I \wedge dx^J + \Omega_{IJ} \delta dx^I \wedge dx^J + \Omega_{IJ} dx^I \wedge \delta dx^J \\ &= \delta x^K \partial_K \Omega_{IJ} dx^I \wedge dx^J + \Omega_{KJ} \partial_I \delta x^K dx^I \wedge dx^J + \Omega_{IK} dx^I \partial_J \delta x^K \wedge dx^J \\ &= (\partial_K \Omega_{IJ} + \partial_J \Omega_{KI} + \partial_I \Omega_{JK}) \delta x^K dx^I \wedge dx^J - [\partial_I (\Omega_{JK} \delta x^K) - \partial_J (\Omega_{IK} \delta x^K)] \times \\ &\quad dx^I \wedge dx^J \end{aligned}$$

can be written in the general form

$$\delta\Omega = \mathcal{L}_{\delta x} \Omega = i_{\delta x} d\Omega + d(i_{\delta x} \Omega). \quad (\text{B.1.3})$$

Similarly the infinitesimal transformation of any 1-form A is given by

$$\delta A = \mathcal{L}_{\delta x} A = i_{\delta x} dA + d(i_{\delta x} A) \rightarrow i_{\delta x} \Omega + d(i_{\delta x} A). \quad (\text{B.1.4})$$

Therefore a canonical transformation $\delta A = d\beta$ is given by

$$i_{\delta x} \Omega = -df \quad (\text{ie. } \delta x^I = \Omega^{IK} \partial_K f \equiv \{x^I, f\}), \quad (\text{B.1.5})$$

where $\Omega_{IK} \Omega^{KJ} = \delta_I^J$ and the Poisson bracket $\{f, g\} = \Omega^{IJ} \partial_I f \partial_J g$ can be inferred.

This produces the desired symmetry conditions

$$\begin{aligned} \delta A &= i_{\delta x} \Omega + d(i_{\delta x} A) = d(-f + i_{\delta x} A), \\ \delta\Omega &= i_{\delta x} d\Omega + d(i_{\delta x} \Omega) = 0. \end{aligned} \quad (\text{B.1.6})$$

The vector field

$$\xi_f = \xi_f^I \partial_I \equiv \delta x^I \partial_I = -\Omega^{IJ} \partial_I f \partial_J \quad (\text{B.1.7})$$

associated with the canonical transformations is known as a Hamiltonian vector field. The following relations hold following the Jacobi identity for the Poisson bracket:

$$[\mathcal{L}_{\xi_f}, \mathcal{L}_{\xi_g}] \psi = \mathcal{L}_{[\xi_f, \xi_g]} \psi = \mathcal{L}_{\xi_{\{f, g\}}} \psi, \quad (\text{B.1.8})$$

where $\psi \in \frac{\mathbb{C}^N}{T^*(\mathcal{M})} \equiv \mathcal{F}(T^*(\mathcal{M}))$.

Remarks:

- One has the Liouville measure $d\mu = \Omega^{\wedge D} = \Omega \wedge \Omega^{\wedge(D-1)} = \sqrt{\det \Omega} d^{2D}x$ on $T^*(\mathcal{M})$.
- Canonical transformations (or canonical invariants rather) provide a way to derive quantization conditions. If only canonical path deformations are allowed then

$$\begin{aligned} \delta \oint_C A = \oint_C \delta A = 0 &\quad \Rightarrow \quad \oint_C A = \text{const.} \equiv K \quad \forall C, \\ [\text{Also verify using } \oint_C A = \int_{C^0} dA = \int_{C^0} \Omega], &\quad (\text{B.1.9}) \end{aligned}$$

which reproduces the Bohr-Sommerfeld quantization condition (1.1.7) when K takes on integer values. However $K \in \mathbb{R}$ in general and therefore one can have continuous as well as discrete values for the spectra of quantum mechanical observables. A generalization of the canonical invariant $\oint_C A$ to a situation where A is noncommutative (eg. an A that contains the nonabelian gauge potential) is the path ordered loop integral (known as Wilson loop)

$$\text{Tr } P e^{\oint_C A} = e^K, \quad \frac{\partial K}{\partial C} = 0, \quad (\text{B.1.10})$$

which is a gauge¹ invariant and P denotes path ordering.

- One learns that a canonical transformation is generated by a function on $T^*(\mathcal{M}) \simeq \{(q, p)\}$, where $\mathcal{M} \simeq \{q\}$ & $T(\mathcal{M}) \simeq \{(q, dq)\}$, through a Poisson bracket constructed from Ω . Note that $A \in \frac{T^*(T^*(\mathcal{M}))}{\mathcal{M}}$, $\Omega \in \frac{T^*(T^*(\mathcal{M})) \wedge T^*(T^*(\mathcal{M}))}{\mathcal{M}}$. In particular, the generator or generating function associated to time translations $\delta x^I = \delta t \frac{dx^I}{dt}$ is the Hamiltonian H . Conversely, to every function is associated a canonical transformation whose generator is the function.
- Canonical quantization is a parallel or correspondence where canonical transformations are mapped to unitary linear operators (or unitary transformations) of the set of operators $O(\mathcal{H})$ on a Hilbert space \mathcal{H} ; the classical observables or the generating functions of the canonical transformations are mapped to hermitian or antihermitian linear operators which are generators of the unitary transformations on \mathcal{H} and the Poisson bracket $\{, \}$ is mapped to the commutator $[,]$ in $O(\mathcal{H})$. Thus canonical transformations are to the symplectic 2-form Ω as unitary transformations are to the inner product $\langle | \rangle$ of the Hilbert space.

One can construct a quantum Hilbert space $\mathcal{H}_S = (\mathcal{F}(\mathcal{M}), \langle | \rangle)$ from the pointwise product algebra $(\mathcal{F}(\mathcal{M}), \text{pt-wise})$ of the space of complex functions $\mathcal{F}(\mathcal{M})$ on $\mathcal{M} \simeq \{q\}$ with an inner product given by $\langle f | g \rangle = \int d\mu(q) \overline{f(q)} g(q)$. On \mathcal{H}_S the commutation relation $[\hat{q}, \hat{p}] = i\hbar$ implies that $\hat{q} = \mu_q$, $\hat{p} = -i\hbar\partial_q = -i\hbar\frac{\partial}{\partial q}$ which is known as Schrodinger representation where the position operators \hat{q} act as multiplication operators

$$\mu_q : \mathcal{H}_S \rightarrow \mathcal{H}_S, \mu_q \xi(q) = q\xi(q). \quad (\text{B.1.11})$$

¹Gauge transformations are examples of canonical transformations.

The quantum operators $Q(T^*(\mathcal{M})) \subset O(\mathcal{H}_S)$ are then given by

$$Q(T^*(\mathcal{M})) = \{Q(f), f \in \mathcal{F}(T^*(\mathcal{M}))\},$$

$$Q : \mathcal{F}(T^*(\mathcal{M})) \rightarrow O(\mathcal{H}_S), f(q, p) \mapsto Q(f)(q, p) = f(q, -i\hbar\partial_q).$$

Since the points $x^I = (q^i, p^i)$ of $T^*(\mathcal{M})$ on which the commutative algebra of (generating) functions $\mathcal{F}(T^*(\mathcal{M}))$ is defined act like linear functionals $\delta_x : f \rightarrow \delta_x(f) = \int_{T^*(\mathcal{M})} dy \delta(y - x) f(y) = f(x)$ on the space of functions $\mathcal{F}(T^*(\mathcal{M}))$, the role of these points may be played by linear functionals $\mathcal{O}^*(\mathcal{H}) = \{\chi : O(\mathcal{H}) \rightarrow \mathbb{C}, \chi(a + b) = \chi(a) + \chi(b)\}$ on the algebra of linear operators $O(\mathcal{H})$ on the Hilbert space \mathcal{H} .

- Deformation quantization is an alternative method of quantization that arises because the algebra of operators $O(\mathcal{H})$ on the quantum mechanical Hilbert space \mathcal{H} can be shown to be equivalent to a noncommutative $*$ -product function algebra $(\mathcal{F}(T^*(\mathcal{M})), *$ -wise), the commutator in which reduces to the Poisson bracket of the classical function algebra $(\mathcal{F}(T^*(\mathcal{M})), \text{pt}\cdot\text{wise})$ in a certain limit. That is, the Poisson algebra can be obtained from a noncommutative deformation $(\mathcal{F}(T^*(\mathcal{M})), *$ -wise) of the commutative function algebra $(\mathcal{F}(T^*(\mathcal{M})), \text{pt}\cdot\text{wise})$:

$$(\mathcal{F}(T^*(\mathcal{M})), \text{pt}\cdot\text{wise}) \rightarrow (\mathcal{F}(T^*(\mathcal{M})), *$$
-wise),
$$(fg)(q, p) = f(q, p)g(q, p) \mapsto (f * g)(q, p) = f(q, p)e^{\overleftarrow{\partial}_I \Omega^{IJ} \overrightarrow{\partial}_J} g(q, p).$$

(B.1.12)

Here one may again construct a quantum Hilbert space $\mathcal{H}_M = (\mathcal{F}(T^*(\mathcal{M})), \langle | \rangle)$ from the $*$ -product product algebra $(\mathcal{F}(T^*(\mathcal{M})), *$ -wise) of the space of complex functions $\mathcal{F}(T^*(\mathcal{M}))$ on $T^*(\mathcal{M}) \simeq \{(q, p)\}$ with an inner product given

by $\langle f|g\rangle = \int d\mu(q,p) \overline{f(q,p)} * g(q,p)$. On \mathcal{H}_M the commutation relation $[\hat{q}, \hat{p}] = i\hbar$ implies that both $\hat{q} = \mu_q$, $\hat{p} = \mu_p$ act (reducibly) as multiplication operators

$$\begin{aligned} \mu_q, \mu_p : \mathcal{H}_M &\rightarrow \mathcal{H}_M, \quad \mu_q \xi(q,p) = q * \xi(q,p) = (q + \frac{i}{2} \partial_p) \xi(q,p), \\ \mu_p \xi(q,p) &= p * \xi(q,p) = (p - \frac{i}{2} \partial_q) \xi(q,p), \\ \frac{df(q_t, p_t)}{dt} &= (H * f - f * H)(q_t, p_t), \end{aligned} \tag{B.1.13}$$

which is however only left multiplication but we however have both Left and right independent multiplication operators $\mu_q^{L,R}, \mu_p^{L,R}$. Due to the simple nature of the algebra $\mu^c = \frac{1}{2}(\mu_{(q,p)}^L + \mu_{(q,p)}^R)$ gives a commutative coordinate representation $\mu^c = \mu_{(q_c, p_c)}$ that is insensitive to the $*$ -product.

Deformation quantization provides an example of noncommutative geometry since any C^* -algebra can be realized as an algebra of operators $O(\mathcal{H})$ on a Hilbert space \mathcal{H} and noncommutative geometry involves the representation of an arbitrary $*$ -algebra \mathcal{A} as a noncommutative algebra of functions on its dual $\mathcal{A}^* = \{\chi : \mathcal{A} \rightarrow \mathbb{C}^N, \chi(a+b) = \chi(a) + \chi(b)\}$. That is

$$R : \mathcal{A} \rightarrow R(\mathcal{A}) = \{\tilde{a} : \mathcal{A}^* \rightarrow \mathbb{C}^N, \tilde{a}(\chi) = \chi(a), (\tilde{a} * \tilde{b})(\chi) = \chi(ab)\}. \tag{B.1.14}$$

- Thus quantizing a given classical system involves the representation theory of the algebra(s) and symmetry group(s) of the classical system.

B.1.1 Star products and regularization

The star product construction is a trick one may use, whenever convenient, to find characteristic representations

$\pi : \mathcal{A} \rightarrow \pi(\mathcal{A}) \simeq \mathcal{F}(X)|_{fg \rightarrow f * g} \simeq \{\tilde{a} : \{\pi\} \simeq X \rightarrow \mathbb{C}^N, \pi \mapsto \tilde{a}(\pi) = \pi(a)\}$ of a given algebra \mathcal{A} by modifying the product on the algebra of functions $\mathcal{F}(X) = \mathbb{C}^N/X$ on some topological space $X \simeq \{\pi\}$. The characteristic representation of the algebra product on the function space is known as a star product:

$$\pi(ab) = (\tilde{a} * \tilde{b})(\pi). \quad (\text{B.1.15})$$

An example is given by the group algebra G^* of a group G .

$$\begin{aligned} G^* &= \text{Span}\{\hat{f} = L(f) = \sum_{g \in G} f(g) g, \quad f \in \mathcal{F}(G)\}, \quad \mathcal{F}(G) = \{f : G \rightarrow \mathbb{C}\}, \\ L(f)L(h) &= L(f * h), \\ (f * h)(g) &= \sum_{u \in G} f(u) h(u^{-1}g) = \sum_{u \in G} f(gu^{-1}) h(u) \neq (h * f)(g). \end{aligned}$$

As another example let \mathcal{A} be the Moyal-Weyl algebra;

$$\begin{aligned} \mathcal{A} &= \{W(f) = \hat{f}\}, \\ \hat{f} &= \sum_p \tilde{f}_p e_p(\hat{x}) = \sum_x f(x) \sum_p e_p(\hat{x} - x) = \sum_x f(x) \hat{\delta}_x, \\ \hat{\delta}_x &= \sum_y \delta(x - y) \hat{\delta}_y \equiv W(\delta_x), \quad x, p \in \mathbb{R}^{d+1} \end{aligned} \quad (\text{B.1.16})$$

generated by the linear operators $\{\hat{x}^\mu\}$;

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \quad (\text{B.1.17})$$

The major point here is to be able to invert (in an unambiguous way) the series expansion

$$\hat{f} = \sum_x f(x) \hat{\delta}_x. \quad (\text{B.1.18})$$

This is possible if a unique linear functional $\phi \equiv \sum_{\hat{x}}$ (an analog of the integral) can be found such that $\phi(\hat{\delta}_x \hat{\delta}_y) \sim \delta_{xy}$. To find this functional, consider the generators

of real translations $\{\hat{\partial}^\mu\}$ on this algebra given by

$$\begin{aligned}
[\hat{x}^\mu, \hat{\partial}_\nu] &= -\delta^\mu_\nu, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0, \\
(\text{compare with } \hat{y}_\nu &= -i\theta_{\nu\alpha}\hat{x}^\alpha = -i\theta^{-1\nu\alpha}\hat{x}^\alpha, \quad [\hat{y}_\mu, \hat{y}_\nu] = i\theta_{\mu\nu} \\
[\hat{x}^\mu, \hat{y}_\nu] &= -\delta^\mu_\nu). \tag{B.1.19}
\end{aligned}$$

That such $\hat{\partial}_\mu$'s exist may be seen by representing the algebra as an algebra of differential operators on a function space $\mathcal{F}(\mathbb{R}^{d+1}) = \mathcal{F}(\{x\})$, $\hat{x}^\mu = x^\mu + \frac{i}{2}\theta^{\mu\nu}\partial_\nu$. More simply, $\hat{\partial}_\mu = -i\theta^{-1\mu\nu}\text{ad}_{\hat{x}^\nu} \equiv -i\theta_{\mu\nu}\text{ad}_{\hat{x}^\nu} = \text{ad}_{\hat{y}_\mu}$ where the algebra (B.1.17) implies that

$$[\text{ad}_{\hat{x}^\mu}, \hat{x}^\nu] \hat{f} = \text{ad}_{\hat{x}^\mu}(\hat{x}^\nu \hat{f}) - \hat{x}^\nu \text{ad}_{\hat{x}^\mu}(\hat{f}) = i\theta^{\mu\nu} \hat{f} \tag{B.1.20}$$

and one easily sees that $[\text{ad}_{\hat{x}^\mu}, \text{ad}_{\hat{x}^\nu}] = \text{ad}_{[\hat{x}^\mu, \hat{x}^\nu]} = \text{ad}_{i\theta^{\mu\nu}} = 0$.

Then

$$\begin{aligned}
\hat{\delta}_x &= \sum_p e_p(\hat{x} - x) = \sum_p e^{ip(\hat{x}-x)} = e^{-ix\hat{\delta}} \sum_p e_p(\hat{x}) e^{ix\hat{\delta}}, \\
[\hat{\partial}_\mu, \hat{\delta}_x] &= -\partial_\mu \hat{\delta}_x \Rightarrow \partial_\mu \text{Tr} \hat{\delta}_x = 0 \text{ as } \text{Tr}(AB) = \text{Tr}(BA) \tag{B.1.21}
\end{aligned}$$

which means that $\text{Tr} \hat{\delta}_x = c = \text{const.}$ and the normalization $\text{Tr} \hat{\delta}_x = 1$ gives

$$\text{Tr}(\hat{\delta}_x \hat{\delta}_y) = \delta(x - y) \tag{B.1.22}$$

since

$$\hat{\delta}_x \hat{\delta}_y = \sum_{pp'} e^{i(p+p')\hat{x}} e^{\frac{i}{2}p \wedge p'} e^{-ipx - ip'y} \tag{B.1.23}$$

and²

$$\begin{aligned}
e^{ia\hat{x}} &= \sum_{\alpha} \delta(a - \alpha) e^{i\alpha\hat{x}} = \sum_{\alpha} \sum_z e^{i(a-\alpha)z} e^{i\alpha\hat{x}} = \sum_z e^{iaz} \sum_{\alpha} e^{i\alpha\hat{x}-i\alpha z} \\
&= \sum_z \hat{\delta}_z e^{iaz} \Rightarrow \text{Tr} e^{ia\hat{x}} = \sum_z \text{Tr} \hat{\delta}_z e^{iaz} = \sum_z e^{iaz} = \delta(a).
\end{aligned} \tag{B.1.25}$$

Thus the trace $\text{Tr} \equiv \sum_{\hat{x}}$ provides a means to invert the series (B.1.18).

We can therefore define the linear functionals $A^* \simeq \{\delta_x, x \in \mathbb{R}^D\} \simeq \mathbb{R}^D$ by

$$\delta_x : \mathcal{A} \rightarrow \mathbb{C}, \quad \delta_x(\hat{f}) = \text{Tr}(\hat{f}\hat{\delta}_x) = (\text{Tr}\hat{\delta}_x \circ W)(f) = f(x). \tag{B.1.26}$$

²Moreover one can simplify further to obtain

$$\begin{aligned}
\hat{\delta}_x \hat{\delta}_y &= \sum_{pp'} e^{i(p+p')\hat{x}} e^{\frac{i}{2}p\wedge p'} e^{-ipx-ip'y} = \frac{e^{-ix(\frac{\theta}{2})^{-1}y}}{\det(\frac{\theta}{2})^{-1}} \sum_z \hat{\delta}_z e^{iz(\frac{\theta}{2})^{-1}(x-y)} \\
&= e^{-ix(\frac{\theta}{2})^{-1}y} \sum_k \hat{\delta}_{\frac{\theta}{2}k} e^{-ik(x-y)} \equiv \Gamma_{xy}{}^z \hat{\delta}_z, \quad \sum_z \Gamma_{xy}{}^z = \delta(x-y), \\
\text{Tr}(\hat{\delta}_x \hat{\delta}_y \hat{\delta}_z) &= \Gamma_{xy}{}^z = \frac{1}{\det(\frac{\theta}{2})^{-1}} e^{-ix(\frac{\theta}{2})^{-1}y - ix(\frac{\theta}{2})^{-1}z - iy(\frac{\theta}{2})^{-1}z} \\
&= \frac{1}{\det(\frac{\theta}{2})^{-1}} e^{-i(x-z)(\frac{\theta}{2})^{-1}(y+z)} \equiv \text{Cycl}_{xyz} \left(\frac{1}{\det(\frac{\theta}{2})^{-1}} e^{-i(x-z)(\frac{\theta}{2})^{-1}(y+z)} \right), \\
\text{Tr}(\hat{\delta}_x \hat{\delta}_y \hat{\delta}_z \hat{\delta}_w) &= \Gamma_{xy}{}^\alpha \Gamma_{\alpha z}{}^w = \Gamma_{xy}{}^w \delta(x+y-z-w) = \Gamma_{xy}{}^{-z} \delta(x+y-z-w), \\
\text{Tr}(\hat{\delta}_{x_1} \hat{\delta}_{x_2} \dots \hat{\delta}_{x_{n-1}} \hat{\delta}_{x_n}) &= \Gamma_{x_1 x_2}{}^{z_1} \Gamma_{z_1 x_3}{}^{z_2} \Gamma_{z_2 x_4}{}^{z_3} \dots \Gamma_{z_{k-1} x_{k+1}}{}^{z_k} \Gamma_{z_k x_{k+2}}{}^{z_{k+1}} \\
&\quad \dots \Gamma_{z_{n-5} x_{n-3}}{}^{z_{n-4}} \Gamma_{z_{n-4} x_{n-2}}{}^{z_{n-3}} \Gamma_{z_{n-3} x_{n-1}}{}^{z_{n-2}} \Gamma_{z_{n-2} x_n}{}^{z_{n-1}} \text{Tr}(\hat{\delta}_{z_{n-1}}) \\
&= \Gamma_{x_1 x_2}{}^{z_1} \Gamma_{z_1 x_3}{}^{z_2} \Gamma_{z_2 x_4}{}^{z_3} \dots \Gamma_{z_{k-1} x_{k+1}}{}^{z_k} \Gamma_{z_k x_{k+2}}{}^{z_{k+1}} \\
&\quad \dots \Gamma_{z_{n-5} x_{n-3}}{}^{z_{n-4}} \Gamma_{z_{n-4} x_{n-2}}{}^{z_{n-3}} \Gamma_{z_{n-3} x_{n-1}}{}^{x_n} \\
&= \text{Tr}(\hat{\delta}_{x_1} \hat{\delta}_{x_2} \dots \hat{\delta}_{x_{n-4}} \hat{\delta}_{x_{n-3}} \hat{\delta}_{x_n+x_{n-1}-x_{n-2}}) \Gamma_{(x_n+x_{n-1}-x_{n-2})}{}^{x_{n-2}}{}^{x_n}. \tag{B.1.24}
\end{aligned}$$

The appearance of delta functions indicates that *Moyal noncommutativity is not strong enough to be able to regularize all possible n -point correlations. $(2k+1)$ -point correlations are fully regularized but $2k$ -point correlations are only partially regularized.*

Since $W(f_1)W(f_2)\dots W(f_n) = W(f_1 * f_2 * \dots * f_n)$ one has

$$\begin{aligned} \delta_x(\hat{f}_1\hat{f}_2\dots\hat{f}_n) &= \text{Tr}(\hat{\delta}_x\hat{f}_1\hat{f}_2\dots\hat{f}_n) = \text{Tr}(\hat{\delta}_x W(f_1)W(f_2)\dots W(f_n)) \\ &= (\text{Tr}\hat{\delta}_x \circ W)(f_1 * f_2 * \dots * f_n) = (f_1 * f_2 * \dots * f_n)(x), \\ (f * g)(x) &= f(x)e^{\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu}g(x), \quad [\hat{\partial}_\mu, W(f)] = W(\partial_\mu f). \end{aligned} \quad (\text{B.1.27})$$

For noncommutativity of the form

$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}{}_\alpha \hat{x}^\alpha$, $\text{Cyclic}_{\mu\nu\alpha}(C^{\mu\nu}{}_\rho C^{\rho\alpha}{}_\lambda) = 0$, for the purpose of inverting the series (B.1.18) one may define a conjugate

$\hat{\delta}_x^B = \sum_p A(p) e^{iB(p)\hat{x}-ipx}$ to $\hat{\delta}_x = \sum_p e^{ip\hat{x}-ipx}$ such that $\text{Tr}(\hat{\delta}_x\hat{\delta}_y^B) = \delta(x-y)$. Again one assumes that $\hat{\partial}_\mu$ can be found such that

$$[\hat{x}^\mu, \hat{\partial}_\nu] = -\delta^\mu{}_\nu, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0. \quad (\text{B.1.28})$$

That such $\hat{\partial}_\mu$'s exist may be seen by representing the algebra as an algebra of differential operators on a function space $\mathcal{F}(\mathbb{R}^{d+1}) = \mathcal{F}(\{x\})$.

$$\begin{aligned} \hat{x}^\mu &\rightarrow x^\nu E^\mu{}_\nu(i\partial), \quad \partial = \frac{\partial}{\partial x}, \\ E^\alpha{}_\nu(i\partial) \partial^\nu E^\beta{}_\mu(i\partial) - E^\beta{}_\nu(i\partial) \partial^\nu E^\alpha{}_\mu(i\partial) &= C^{\alpha\beta}{}_\nu E^\nu{}_\mu(i\partial), \end{aligned} \quad (\text{B.1.29})$$

$$\begin{aligned} \text{eg. } E^\nu{}_\mu(i\partial) &= \left(\frac{F(i\partial)}{1-e^{-F(i\partial)}}\right)^\nu{}_\mu = \left(\int_0^1 dt e^{-tF(i\partial)}\right)^{-1\nu}{}_\mu, \quad F^\mu{}_\nu(i\partial) = C^{\mu\alpha}{}_\nu i\partial_\alpha, \\ \text{as well as } E^\nu{}_\mu(i\partial) &= F^\nu{}_\mu(i\partial) \quad (\text{due to the Jacobi id}), \end{aligned} \quad (\text{B.1.30})$$

where the interchange $i\partial \leftrightarrow x$ in any given representation $\hat{x} \rightarrow f(x, i\partial)$ produces another representation $f(i\partial, x)$. In this case $\text{ad}_{\hat{x}^\mu}$'s are derivations but they rotate the coordinates rather than translate them as was the case with $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$.

The $\hat{\partial}_\mu$'s may be represented ³ by operators $\hat{\partial}_\mu = Q_\mu(\partial)$ such that

$$dQ_\mu(\partial) = E^{-1\nu}{}_\mu(i\partial) d\partial_\nu \stackrel{\text{eg.}}{=} \left(\int_0^1 dt e^{-tF(i\partial)}\right)^\nu{}_\mu d\partial_\nu$$

³The action of generators can be seen by making an infinitesimal variation and using the

and their action is as follows:

$$\begin{aligned}
Q_\mu(\partial) T_J(x, \partial) &= \frac{\partial}{\partial u^\mu} T_J(x, \partial) = \frac{\partial X^K}{\partial u^\mu} \left\{ \frac{\partial}{\partial X^K} T_J(x, \partial) + \Gamma_{KJ}{}^L(\partial) T_L(x, \partial) \right\}, \\
u^\mu &= x^\nu E^\mu{}_\nu(i\partial), \quad X^I = (x^\mu, \partial_\mu), \\
\Gamma_{KJ}{}^L &= \frac{\partial u^\rho}{\partial X^K} \frac{\partial}{\partial X^J} \frac{\partial X^L}{\partial u^\rho} = \frac{\partial u^\rho}{\partial X^J} \frac{\partial}{\partial X^K} \frac{\partial X^L}{\partial u^\rho} = -\frac{\partial^2 u^\rho}{\partial X^K \partial X^J} \frac{\partial X^L}{\partial u^\rho},
\end{aligned}$$

where we only need its restriction on x -functions, $T_J(x, \partial) \rightarrow f_\mu(x)$.

If $e^{i\alpha\hat{x}} e^{i\beta\hat{y}} = e^{iK(\alpha,\beta)\hat{x}}$ then $\text{Tr}(\hat{\delta}_x \hat{\delta}_y^B) = \delta(x-y)$ requires the functions A, B to

Hausdorf-Campbell formula:

$$\begin{aligned}
f(\hat{x} + \delta\hat{x}) &\approx f(\hat{x}) + \delta\hat{x}^i \int_0^1 ds e^{s \overleftarrow{\text{ad}}_{\hat{x}^j} \overrightarrow{\partial}^j} \partial_i f(\hat{x}), \\
\partial_i f(s \overleftarrow{\text{ad}}_{\hat{x}} + \hat{x}) &= \frac{\partial f(y)}{\partial y^i} \Big|_{y^j = s \overleftarrow{\text{ad}}_{\hat{x}^j} + \hat{x}^j}, \\
\text{ad}_{\hat{x}} &= (\text{ad}_{\hat{x}^i}) = (\text{ad}_{\hat{x}^1}, \text{ad}_{\hat{x}^2}, \dots, \text{ad}_{\hat{x}^D}), \\
\text{ad}_a(b) &= [a, b] = -[b, a] = -(b) \overleftarrow{\text{ad}}_a, \\
\delta\hat{x}^j &= d\hat{x}^i \hat{J}_i{}^j = d\hat{x}^i J_i{}^j(\hat{x}), \quad d\hat{x}^i \rightarrow 0,
\end{aligned} \tag{B.1.31}$$

where it is assumed that $\hat{f} = f(\hat{x})$ can be expanded in the specific form

$$f(\hat{x}) = \sum_{p \in \mathbb{C}^D} f_1(p) f_2(p \cdot \hat{x}), \quad p \cdot \hat{x} = p_i \hat{x}^i. \tag{B.1.32}$$

Rotations and translations are isometries of $g_{ij} = \delta_{ij}$ and are given by

$$\begin{aligned}
\partial_i J_{mj}(x) + \partial_j J_{mi}(x) &= 0, \\
J_{ij}(x) &= \delta_{ij} \quad \text{for translations} \quad \& \\
J_{mi}(x) &= (c_{mij} - c_{mji}) x^j \quad \text{for rotations,}
\end{aligned}$$

where the c 's are constants.

satisfy ⁴

$$K(p, B(-p)) = 0, \quad A(p) = \det \partial_{p'} K(p, B(p'))|_{p'=-p}. \quad (\text{B.1.33})$$

$$\begin{aligned} (\text{ Therefore } \hat{\delta}_x^B &= \sum_p A(p) e^{iB(p)\hat{x}-ipx} = \sum_p e^{ip\hat{x}-iB^{-1}(p)x} \\ &= \sum_z \hat{\delta}_z \sum_p e^{ipz-iB^{-1}(p)x}). \end{aligned} \quad (\text{B.1.34})$$

That is to say that $p + p' = 0$ is a solution of $K(p, B(p')) = 0$ and the factor $\det \partial_{p'} K(p, B(p'))|_{p'=-p}$ coming from $\delta(K(p, B(p')))$ at $p + p' = 0$ needs to be canceled by the amplitude $A(p)$. The uniqueness of the inversion here depends upon the solutions $\hat{\partial}_\mu$ and B of the relations

$$[\hat{x}^\mu, \hat{\partial}_\nu] = -\delta^\mu_\nu, \quad [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0, \quad K(p, B(-p)) = 0. \quad (\text{B.1.35})$$

Finally, with $W(f_1)W(f_2)\dots W(f_n) = W(f_1 * f_2 * \dots * f_n)$ one defines

$$\begin{aligned} \delta_x(\hat{f}_1 \hat{f}_2 \dots \hat{f}_n) &= \text{Tr}(\hat{\delta}_x^B \hat{f}_1 \hat{f}_2 \dots \hat{f}_n) = \text{Tr}(\hat{\delta}_x^B W(f_1)W(f_2)\dots W(f_n)) \\ &= (\text{Tr} \hat{\delta}_x^B \circ W)(f_1 * f_2 * \dots * f_n) = (f_1 * f_2 * \dots * f_n)(x), \\ [\hat{\partial}_\mu, W(f)] &= W(\partial_\mu f). \end{aligned} \quad (\text{B.1.36})$$

⁴ Note that

$$\begin{aligned} \hat{\delta}_x \hat{\delta}_y^B &= \sum_{pp'} A(p) e^{ip\hat{x}} e^{iB(p')\hat{x}} e^{-ipx-ip'y} \\ &= \sum_{pp'} A(p) e^{iK(p, B(p'))\hat{x}} e^{-ipx-ip'y} = \sum_{pp'} \sum_z \hat{\delta}_z A(p) e^{iK(p, B(p'))z} e^{-ipx-ip'y} \\ &= \Gamma_{xy}{}^z(B) \hat{\delta}_z, \quad \Gamma_{xy}{}^z(B) = \sum_{pp'} A(p) e^{iK(p, B(p'))z} e^{-ipx-ip'y}, \end{aligned}$$

$\text{Tr} \hat{\delta}_x = 1$ requires the existence of $[\hat{x}^\mu, \hat{\partial}_\nu] = -\delta^\mu_\nu, [\hat{\partial}_\mu, \hat{\partial}_\nu] = 0$.

Lie algebra type noncommutativity may be strong enough to regularize all possible n -point correlations unlike Moyal noncommutativity.

Note that one now also has the equivalent mirror algebra

$$\begin{aligned} \mathcal{A}^B &= \{W^B(f) = \hat{f}^B = \sum_x f(x) \hat{\delta}_x^B\}, \\ \hat{f}^B &= \sum_x f(x) \hat{\delta}_x^B = \sum_p \tilde{f}(p) A(p) e^{iB(p)\hat{x}}. \end{aligned} \quad (\text{B.1.37})$$

The corresponding set of linear functionals is

$$A_B^* \simeq \{\delta_x^B = \text{Tr} \circ \mu_{\hat{\delta}_x}, x \in \mathbb{R}^D\} \simeq \mathbb{R}^D \quad \forall B. \quad (\text{B.1.38})$$

B.2 The quantum field

A point particle's instance-wise trajectory $\Gamma :]0, 1[\subset \mathbb{R} \rightarrow \mathbb{R}^{d+1}$, $\tau \mapsto \gamma^\mu(\tau)$ in spacetime \mathbb{R}^{d+1} may be regarded as a field or collection

$\{c_\tau = (\gamma(\tau), \rho_\tau(\mathbb{R}^{d+1}); \tau \in]0, 1[\}$ of point-like spacetime distributions, with each instance $\gamma(\tau)$ represented by its localization or density or support function $\rho_\tau(y) = \delta(y - \gamma(\tau))$ in spacetime \mathbb{R}^{d+1} . For the value of any property P (eg. position, velocity, energy, momentum, etc) of the point particle that depends only on instances $\gamma(\tau)$ of its trajectory Γ one then has the decomposition

$$P = P(\gamma(\tau)) = \sum_{y \in \mathbb{R}^{d+1}} P(y) \delta(y - \gamma(\tau)) \equiv \sum_y p_y(\gamma(\tau)) \quad (\text{B.2.1})$$

where $\delta(y - \gamma(\tau))$ represents the density or support of the particle at the instant $\gamma(\tau)$ meanwhile $p_y(\gamma(\tau)) = P(y)\delta(y - \gamma(\tau)) \equiv P(\gamma(\tau))\delta(y - \gamma(\tau))$ is the (probability of) presence/influence, at the instance τ , of the property P at/on a generic point $y \in \mathbb{R}^{d+1}$.

Now consider a wave packet (generically a field) $\psi : \mathcal{D} \subset \mathbb{R}^{d+1} \rightarrow \mathbb{C}$, $x \mapsto \psi(x)$ which describes the energy (presence or existence) distribution or concentration of a large collection of particles. Just as the instance-wise trajectory Γ of the point

particle was decomposed into point-like (or spacetime δ -) distributions according to its instances $\{\gamma(\tau); \tau \in]0, 1[\}$ one can also decompose the wave packet into spacetime modes, which are spacetime δ -distributions, (δ -modes) as

$$\psi(x) = \sum_y \psi(y) \delta(y - x) = \sum_y \psi(x) \delta(y - x) \equiv \sum_y \psi_y \delta_y(x) = \sum_y \psi_x \delta_y(x).$$

where ψ_x is the amplitude of the space-time mode that is δ -localized at x .

The δ decomposition is done in analogy (and should be interpreted similarly) to the plane-wave (ie. Fourier) or exponential (e) decomposition

$$\psi(x) = \sum_k \tilde{\psi}_k e_k(x) \equiv \sum_k \tilde{\psi}(k) e^{ikx} \quad (\text{B.2.2})$$

where $\tilde{\psi}_k$ is the amplitude of the energy-momentum mode that is e -localized at x .

In principle one has an arbitrarily large number of possible types of decompositions (or transforms). The basic idea is to describe the interaction of two systems (wave packet, point particles, fields, etc) in terms of the interactions/correlations of their individual modes.

Of course one also has an e -decomposition of the instance-wise property P of the point-like particle:

$$P(\gamma(\tau)) = \sum_k \tilde{P}_k e_k(\gamma(\tau)) = \sum_k \tilde{P}_k e^{ik\gamma(\tau)}. \quad (\text{B.2.3})$$

The quantum field $\Psi : \mathcal{D} \subset \mathbb{R}^{d+1} \rightarrow U(\mathbb{C})$, $x \mapsto \Phi(x)$ is an operator-valued wave packet

$$\Psi(x) = \sum_k \tilde{\Psi}_k e_k(x) = \sum_y \Psi_y \delta_y(x) = \dots \quad (\text{B.2.4})$$

In a noncommutative spacetime \mathbb{R}_θ^{d+1} with coordinates \hat{x}^μ , $[\hat{x}^\mu, \hat{x}^\nu] \neq 0$ these

decompositions may be written analogously as

$$\begin{aligned}\Psi(\hat{x}) &= \sum_k \tilde{\Psi}_k \otimes \hat{e}_k = \sum_y \Psi_y \otimes \hat{\delta}_y = \dots, \\ \hat{e}_k &= e^{ik\hat{x}}, \quad \hat{\delta}_y = \sum_k e^{ik(\hat{x}-y)}.\end{aligned}\tag{B.2.5}$$

When \hat{x} is commutative, we have the two-point correlation duality

$$\begin{aligned}\langle \hat{e}_{k_1} \dots \hat{e}_{k_m} \hat{e}_{k'_1}^* \dots \hat{e}_{k'_n}^* \rangle_{NC} &= \sum_{\hat{x}} \hat{e}_{k_1} \dots \hat{e}_{k_m} \hat{e}_{k'_1}^* \dots \hat{e}_{k'_n}^* = \delta(\sum k - \sum k'), \\ \langle \hat{\delta}_{y_1} \dots \hat{\delta}_{y_m} \hat{\delta}_{y'_1}^* \dots \hat{\delta}_{y'_n}^* \rangle_{NC} &= \sum_{\hat{x}} \hat{\delta}_{y_1} \dots \hat{\delta}_{y_m} \hat{\delta}_{y'_1}^* \dots \hat{\delta}_{y'_n}^* = \prod_{i=1}^{m-1} \delta(y_i - y_{i+1}) \prod_{j=1}^{n-1} \delta(y'_j - y'_{j+1})\end{aligned}$$

where the former is an expression of momentum conservation. The purpose of noncommutativity(NC) is to spread out all the delta functions in the latter expression, ie. to make the spacetime δ -distribution nonsingular (although this does not happen for $2n + 1$ -point functions in the case of Moyal noncommutativity $[[\hat{x}^\mu, \hat{x}^\nu], \hat{x}^\alpha] = 0$. Moyal NC also maintains translational invariance/momentum conservation expressed by the former correlation expression but breaks rotational invariance and hence any angular momentum conservation). In Lie algebra type NC $[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}{}_\alpha \hat{x}^\alpha$ full smearing may be achieved (Here both rotational and translational invariance, and hence angular momentum and momentum conservation, are broken).

B.3 The algebra of quantum fields

Consider the algebra of free causal/accausal real (or hermitian) quantum fields $\mathcal{A} = \{\phi\}$

$$\begin{aligned} [\phi(x), \phi(y)] &= i\Delta(x, y) = -i \oint_C \frac{d^4x}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2}, \\ [\phi_x, \phi_y] &= i\Theta_{xy}, \quad \Theta_{xy} = - \oint_C \frac{d^4x}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2}, \end{aligned} \quad (\text{B.3.1})$$

where the integral in k_0 is an integral along any closed contour C in the complex k_0 plane that encloses all two poles of the integrand that are located at $k_0 = \pm\sqrt{\vec{k}^2 + m^2}$. Regarding $\theta^{\mu\nu}$ as a 2-point correlation function in the directions of spacetime, then one can employ the star product technique to calculate correlation functions of quantum fields:

$$\begin{aligned} g(\mu_1, \mu_2, \dots) &= \text{Tr}(\hat{x}^{\mu_1} \hat{x}^{\mu_2} \dots \hat{\delta}_x) = x^{\mu_1} * x^{\mu_2} * \dots, \\ W(x_1, x_2, \dots) &= \text{Tr}(\phi_{x_1} \phi_{x_2} \dots \hat{\delta}_\varphi) \equiv \text{Tr}_\varphi(\phi_{x_1} \phi_{x_2} \dots) = \varphi_{x_1} *_\Delta \varphi_{x_2} *_\Delta \dots, \\ \hat{\delta}_\varphi &= \int D\Pi e^{i \sum_y \Pi_y (\phi_y - \varphi_y)}, \\ G(x_1, x_2, \dots) &= \text{Tr}(T(\phi_{x_1} \phi_{x_2} \dots) T \hat{\delta}_\varphi) = T(\text{Tr}(\phi_{x_1} \phi_{x_2} \dots \hat{\delta}_\varphi)) = T(\varphi_{x_1} *_\Delta \varphi_{x_2} *_\Delta \dots), \\ T \hat{\delta}_\varphi &= \int D\Pi T e^{i \sum_y \Pi_y (\phi_y - \varphi_y)}, \\ A(x_1, x_2, \dots) &= \text{Tr}(T(e^{iS_E[\phi]} \phi_{x_1} \phi_{x_2} \dots \hat{\delta}_\varphi)), \quad (\text{amplitude of a dynamical process}), \\ *_\Delta &\equiv e^{-\frac{i}{2} \sum_{xy} \frac{\overleftarrow{\delta}}{\delta\varphi_x} \Theta_{xy} \frac{\overrightarrow{\delta}}{\delta\varphi_y}}, \\ G(x, y) &= \varphi_x \varphi_y + \text{sign}(x_0 - y_0) \Theta_{xy} \\ &= \varphi_x \varphi_y + \text{sign}(x_0 - y_0) \theta((x_0 - y_0)^2 - (\vec{x} - \vec{y})^2) \Theta_{xy}, \\ T(\phi_x \phi_y) &= \frac{1}{2}(\phi_x \phi_y + \phi_y \phi_x) + \text{sign}(x_0 - y_0) [\phi_x, \phi_y], \\ \theta((x_0 - y_0)^2 - (\vec{x} - \vec{y})^2) \Theta_{xy} &= \Theta_{xy}, \end{aligned} \quad (\text{B.3.2})$$

where T denotes time ordering. With this analogy, $\theta^{\mu\nu}$ may be interpreted as the probability amplitude or potential that a straight line trajectory into the μ direction will spontaneously turn in the ν direction.

If the spacetime on which the quantum field is defined is also noncommutative as the Moyal plane then the algebra of the free quantum fields

$$[\phi_x, \phi_y] = 0, \quad \text{whenever } \Theta_{xy} = 0 \quad (\text{B.3.3})$$

becomes

$$\begin{aligned} \phi_x \phi_y &= e^{i\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial x^\nu}} \phi_y \phi_x, \quad \text{whenever } \Theta_{xy} = 0, \\ \phi_x &= \phi_x^0 e^{\frac{1}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} P_\nu}, \quad [\phi_x^0, \phi_y^0] = 0 \quad \text{whenever } \Theta_{xy} = 0. \end{aligned} \quad (\text{B.3.4})$$

Thus the $*$ to be used in the Green's functions and process amplitudes is a composition of two $*$'s

$$\begin{aligned} * &= *_\Delta \circ *_\theta = e^{-\frac{i}{2} \sum_{yz} \frac{\overleftarrow{\delta}}{\delta \varphi_y} \Theta_{yz} \frac{\overrightarrow{\delta}}{\delta \varphi_z}} \circ e^{-\frac{i}{2} \overleftarrow{\delta}_\mu \theta^{\mu\nu} \overrightarrow{\delta}_\nu} = e^{-\frac{i}{2} \sum_{yz} \frac{\overleftarrow{\delta}}{\delta \varphi_y} \Theta_{yz} \frac{\overrightarrow{\delta}}{\delta \varphi_z} - \frac{i}{2} \overleftarrow{\delta}_\mu \theta^{\mu\nu} \overrightarrow{\delta}_\nu}, \\ G(x_1, x_2, \dots) &= (\text{Tr}_\varphi \circ \text{Tr}_\theta)(T\phi_{\hat{x}_1} \phi_{\hat{x}_2} \dots) = \text{Tr}_\varphi(\text{Tr}_\theta(T\phi_{\hat{x}_1} \phi_{\hat{x}_2} \dots)) \\ &= T(\varphi_{x_1} * \varphi_{x_2} * \dots). \end{aligned} \quad (\text{B.3.5})$$

The two $*$'s commute (ie. $*_\Delta \circ *_\theta = *_\theta \circ *_\Delta$) since they act on different spaces (spacetime and the internal space of the quantum fields). Therefore to consider the \hat{x} 's dynamical one may simply add a suitable term $\Gamma[\mathcal{X}(\hat{x})]$ to the action $S[\phi]$

$(S[\phi] \rightarrow S[\phi] + \Gamma[\mathcal{K}])$ which describes the dynamics ⁵ of ϕ

⁵ Time evolution in terms of the Hamiltonian is given by

$$\begin{aligned}
i\partial_0\phi_x &= [H, \phi_x] \Rightarrow i\partial_0\phi'_x = [H' - H_0, \phi'_x], \quad \partial_0 H_0 = 0, \\
H' &= e^{-ix_0 H_0} H e^{ix_0 H_0}, \quad \phi'_x = e^{-ix_0 H_0} \phi_x e^{ix_0 H_0} \quad (\phi_x = e^{ix_0 H_0} \phi'_x e^{-ix_0 H_0}), \\
\Rightarrow \text{(solution)} \quad \phi'_x &= C_{\bar{x}} + \overline{T} e^{-i \int_{-\infty}^{x_0} dt (H' - H_0)} \hat{\phi}_{\bar{x}} T e^{i \int_{-\infty}^{x_0} (H' - H_0)} = C_{\bar{x}} + \delta\phi'_x, \\
\partial_0 C_{\bar{x}} &= 0, \quad [H' - H_0, C_{\bar{x}}] = 0, \quad \partial_0 \hat{\phi}_{\bar{x}} = 0, \quad [H' - H_0, \hat{\phi}_{\bar{x}}] \neq 0, \\
\Rightarrow \phi_x &= e^{iH_0 x_0} C_{\bar{x}} e^{-iH_0 x_0} + \overline{T} e^{-i \int_{-\infty}^{x_0} dt (H - H_0)} e^{iH_0 x_0} \hat{\phi}_{\bar{x}} e^{-iH_0 x_0} T e^{-i \int_{-\infty}^{x_0} (H - H_0)} \\
&= C_x + \overline{T} e^{-i \int_{-\infty}^{x_0} dt (H - H_0)} \hat{\phi}_x T e^{i \int_{-\infty}^{x_0} (H - H_0)} = C_x + \delta\phi_x \\
&= C_x + \mathcal{S}^\dagger T(\mathcal{S}\hat{\phi}_x), \tag{B.3.6}
\end{aligned}$$

$$\begin{aligned}
T(\delta\phi_x \delta\phi_y \dots) &= T(\overline{T} e^{-i \int_{-\infty}^{x_0} (H - H_0)} \hat{\phi}_x T e^{i \int_{-\infty}^{x_0} (H - H_0)} \overline{T} e^{-i \int_{-\infty}^{y_0} (H - H_0)} \hat{\phi}_y T e^{i \int_{-\infty}^{y_0} (H - H_0)} \dots) \\
&= \mathcal{S}^\dagger T(\mathcal{S}\hat{\phi}_x \hat{\phi}_y \dots), \quad H - H_0 = \int d^{D-1}x \left(\frac{\partial(\mathcal{L} - \mathcal{L}_0)}{\partial\partial_0\phi} \partial_0\phi - (\mathcal{L} - \mathcal{L}_0) \right). \tag{B.3.7}
\end{aligned}$$

Here

$$T e^{i \int_{t_1}^{t_2} H} = e^{idt_2 H(t_2)} \dots e^{idt_1 H(t_1)}, \quad (T e^{i \int_{t_1}^{t_2} H})^{-1} = e^{-idt_1 H(t_1)} \dots e^{-idt_2 H(t_2)} = \overline{T} e^{-i \int_{t_1}^{t_2} H}.$$

$$\begin{aligned}
\overline{T} e^{-i \int_{-\infty}^{x_0} H_I} &= e^{-idt_{-\infty} H_I(-\infty)} \dots e^{-idx_0 H_I(x_0)} \\
&= e^{-idt_{-\infty} H_I(-\infty)} \dots e^{-idx_0 H_I(x_0)} e^{-idx_0 H_I(x_0)} \dots e^{-idt_{\infty} H_I(\infty)} e^{idt_{\infty} H_I(\infty)} \dots e^{idx_0 H_I(x_0)} \\
&= \overline{T} e^{-i \int_{-\infty}^{\infty} dt H_I} T e^{i \int_{x_0}^{\infty} dt H_I} = \mathcal{S}^\dagger T e^{i \int_{x_0}^{\infty} dt H_I}. \tag{B.3.8}
\end{aligned}$$

If $C_x = C_{\bar{x}}$, ie. all commuting, and \mathcal{L}_I contains no time derivatives then

$$\begin{aligned}
H_I &= H - H_0 = - \int d^{D-1}x \mathcal{L}_I, \\
S_I[\phi] &= \int dt H_I = S_I[C + U^{-1}\hat{\phi}U] = S_I[C + \hat{\phi}] \quad \text{as } S_I[\phi] \text{ is local,} \\
\end{aligned} \tag{B.3.9}$$

where (if it exists) $\hat{\phi}$ may be chosen such that $[\hat{\phi}_x, \hat{\phi}_y]$ commutes with all quantum operators, a property that does not hold for ϕ in general.

For example one can consider

$$\begin{aligned}
[\hat{x}_\mu(u), \hat{x}_\nu(v)] &= i\theta_{\mu\nu}(u, v), \quad *_{\theta} = e^{-\frac{i}{2} \sum_{uv} \overleftarrow{\frac{\delta}{\delta x^\mu(u)}} \theta^{\mu\nu}(u, v) \overrightarrow{\frac{\delta}{\delta x^\nu(v)}}}, \\
\Gamma[\hat{x}] &= \sum_u \alpha_\mu(u) \gamma^\mu(u, \hat{x}(u), \partial_u \hat{x}(u), \dots),
\end{aligned}
\tag{B.3.10}$$

The Moyal coordinate \hat{x} is seen to have been evolved from a general dynamical noncommutative coordinate \mathcal{X} to the form

$[\hat{x}_\mu(u), \hat{x}_\nu(v)] = i\theta_{\mu\nu}(u, v)$ by $\Gamma[\mathcal{X}] = \Gamma[c + U^{-1}\hat{x}U]$ in the same way that ϕ is evolved into $\hat{\phi}$ by $S[\phi] = S[C + U^{-1}\hat{\phi}U]$. Therefore the dynamical quantum theory of an “interacting” membrane embedded in spacetime is a theory of noncommutative spacetime.

Here one may say that the field ϕ propagates in a dynamical (ie. curved) spacetime (whose metric is induced by the classical path $\hat{x}(u)$ defined by

$$\frac{\delta\Gamma[\hat{x}]}{\delta\hat{x}^\mu(u)} = 0)$$

$$g_{\mu\nu} = h_{ab} \frac{\partial u^a}{\partial x^\mu} \frac{\partial u^b}{\partial x^\nu} = (h^{ab} \frac{\partial x^\mu}{\partial u^a} \frac{\partial x^\nu}{\partial u^b})^{-1}, \tag{B.3.11}$$

where for the special case of two parameters (ie. “string”) one can set $h_{ab} = \eta_{ab}$ and one would then say that the field ϕ is propagating in a “stringy spacetime”.

$$S[\phi] = \int Dx \mathcal{L}[\hat{x}, \phi[\hat{x}], \partial_{\hat{x}}\phi[\hat{x}], \dots] = \int (\prod_u d^{d+1}x(u)) \mathcal{L}[\hat{x}, \phi[\hat{x}], \partial_{\hat{x}}\phi[\hat{x}], \dots]$$

where any sum \sum_μ has been replaced by $\sum_\mu \sum_u$. One may also combine the φ and x spaces thus combining the two products:

$$\begin{aligned}
\varphi_i &= (\varphi_y, x_\mu(u)), \quad \Theta_{ij} = \begin{pmatrix} \Theta_{yz} & 0 \\ 0 & \theta_{\mu\nu}(u, v) \end{pmatrix}. \\
\frac{\delta}{\delta\varphi_i} &= \frac{\delta\psi_j(\varphi)}{\delta\varphi_i} \left(\frac{\delta}{\delta\psi_j} + \Gamma_j \right).
\end{aligned}
\tag{B.3.12}$$

B.3.1 Operator product ordering and physical correlations

- Linear transforms, such as the Fourier transform of functions, enable information to be processed (encoded/stored/transported/decoded) deterministically. In general, functions of the noncommuting variable ϕ may be analyzed by defining Fourier transforms (now however depending on the order in the operator products) in analogy to commutative variables. In particular for the description of natural processes or phenomena one can define a time-ordered Fourier transform required by their transitive past-future time direction; recall that ϕ can be expressed as a time ordered function of $\hat{\psi} = C + \hat{\phi}$. *Any natural process or phenomenon may be regarded as a sequence of localized spacetime “events” that is well ordered in time (non-relativistic sense) or proper time (relativistic sense) or any other suitable parameter.* This natural time ordering is trivial in a commutative theory but nontrivial in a noncommutative theory; it provides a starting point for defining kinematic variables by eliminating the inherent operator ordering ambiguity in the non-

commutative theory.

$$\begin{aligned}
f_{\mathcal{O}}[\hat{\psi}] &= \int DJ \tilde{f}[J] \mathcal{O}\left(\prod_x e^{-iJ_x \hat{\psi}_x}\right). \\
f_T[\hat{\psi}] &= \int DJ \tilde{f}[J] T\left(\prod_x e^{-iJ_x \hat{\psi}_x}\right), \\
T(e^{-iJ_x \hat{\psi}_x} e^{-iJ_y \hat{\psi}_y})^2 &= \theta(x_0 - y_0) e^{-iJ_x \hat{\psi}_x} e^{-iJ_y \hat{\psi}_y} + \theta(y_0 - x_0) e^{-iJ_y \hat{\psi}_y} e^{-iJ_x \hat{\psi}_x} \\
&= e^{-iJ_x \hat{\psi}_x - iJ_y \hat{\psi}_y} e^{-\frac{1}{2} J_x \text{sign}(x_0 - y_0) i \Theta_{xy} J_y} = e^{-iJ_x \hat{\psi}_x} e^{-iJ_y \hat{\psi}_y} e^{\frac{1}{2} J_x (1 - \text{sign}(x_0 - y_0)) i \Theta_{xy} J_y} \\
&= e^{-iJ_x \hat{\psi}_x} e^{-iJ_y \hat{\psi}_y} e^{J_x \theta(y_0 - x_0) i \Theta_{xy} J_y} = e^{-iJ_y \hat{\psi}_y} e^{-iJ_x \hat{\psi}_x} e^{-J_x \theta(x_0 - y_0) i \Theta_{xy} J_y}, \\
T(A_1 A_2 \dots A_n) &= \prod_{i < j} T_{ij} (A_1 A_2 \dots A_n), \quad T(\alpha A) = \alpha T(A) \quad \forall \alpha \in \mathbb{C}. \\
\Rightarrow f_T[\hat{\psi}] &= \int DJ \tilde{f}[J] e^{-i \sum_x J_x \hat{\psi}_x} e^{-\frac{1}{2} \sum_{xy} J_x \frac{1}{2} \text{sign}(x_0 - y_0) i \Theta_{xy} J_y} \\
&= e^{\frac{1}{2} \sum_{xy} \frac{1}{2} \text{sign}(x_0 - y_0) i \Theta_{xy} \frac{\delta}{\delta \hat{\psi}_x} \frac{\delta}{\delta \hat{\psi}_y}} f_W[\hat{\psi}], \tag{B.3.13}
\end{aligned}$$

where \mathcal{O} denotes general ordering, T is time ordering and W is symmetric (Weyl) ordering.

B.3.2 From Weyl or symmetric ordering to normal or classical ordering

One can further write Weyl ordering in terms of Normal ordering by the decomposition

$$\begin{aligned}
\hat{\psi}_x &= \hat{\psi}_x^+ + \hat{\psi}_x^-, \quad [\hat{\psi}_x^+, \hat{\psi}_y^+] = 0, \quad [\hat{\psi}_x^-, \hat{\psi}_y^-] = 0, \\
f_W[\psi] &= \int DJ \tilde{f}[J] W\left(\prod_x e^{-iJ_x \hat{\psi}_x}\right) = \int DJ \tilde{f}[J] e^{-i \sum_x J_x \hat{\psi}_x} \\
&= \int DJ \tilde{f}[J] e^{-i \sum_x J_x (\hat{\psi}_x^+ + \hat{\psi}_x^-)} \\
&= \int DJ \tilde{f}[J] e^{-i \sum_x J_x \hat{\psi}_x^+ - i \sum_y J_y \hat{\psi}_y^-} \\
&= \int DJ \tilde{f}[J] e^{\frac{1}{2} \sum_{xy} J_x J_y [\hat{\psi}_x^+, \hat{\psi}_y^-]} e^{-i \sum_x J_x \hat{\psi}_x^-} e^{-i \sum_y J_y \hat{\psi}_y^+} \\
&= \int DJ \tilde{f}[J] e^{\frac{1}{2} \sum_{xy} J_x J_y [\hat{\psi}_x^+, \hat{\psi}_y^-]} N(e^{-i \sum_x J_x \hat{\psi}_x}) \\
&= e^{-\frac{1}{2} \sum_{xy} [\hat{\psi}_x^+, \hat{\psi}_y^-] \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} \int DJ \tilde{f}[J] N(e^{-i \sum_x J_x \hat{\psi}_x}) \\
&= e^{-\frac{1}{2} \sum_{xy} [\hat{\psi}_x^+, \hat{\psi}_y^-] \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_N[\hat{\psi}]. \\
f_T[\hat{\psi}] &= e^{\frac{1}{2} \sum_{xy} \frac{1}{2} \text{sign}(x_0 - y_0) i \Theta_{xy} \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_W[\hat{\psi}] \\
&= e^{\frac{1}{2} \sum_{xy} (\frac{1}{2} \text{sign}(x_0 - y_0) i \Theta_{xy} - [\hat{\psi}_x^+, \hat{\psi}_y^-]) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_N[\hat{\psi}] \\
&= e^{\frac{1}{2} \sum_{xy} (\frac{1}{2} \text{sign}(x_0 - y_0) [\hat{\psi}_x, \hat{\psi}_y] - [\hat{\psi}_x^+, \hat{\psi}_y^-]) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_N[\hat{\psi}] \\
&= e^{\frac{1}{2} \sum_{xy} \Delta_F(x, y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_N[\hat{\psi}]. \\
\Delta_F(x, y) &= \frac{1}{2} \text{sign}(x_0 - y_0) [\hat{\psi}_x, \hat{\psi}_y] - \frac{1}{2} ([\hat{\psi}_x^+, \hat{\psi}_y^-] + [\hat{\psi}_y^+, \hat{\psi}_x^-]) \\
&\equiv \frac{1}{2} \text{sign}(x_0 - y_0) ([\hat{\psi}_x^+, \hat{\psi}_y^-] - [\hat{\psi}_y^+, \hat{\psi}_x^-]) - \frac{1}{2} ([\hat{\psi}_x^+, \hat{\psi}_y^-] + [\hat{\psi}_y^+, \hat{\psi}_x^-]) \\
&= \theta(x_0 - y_0) [\hat{\psi}_x^-, \hat{\psi}_y^+] + \theta(y_0 - x_0) [\hat{\psi}_y^-, \hat{\psi}_x^+]. \\
T(f_T[\hat{\psi}] g_T[\hat{\psi}]) &= (fg)_T[\hat{\psi}]. \tag{B.3.14}
\end{aligned}$$

Therefore in the coherent (ie. classical) state defined by

$$\begin{aligned}
\hat{\psi}_x^+ |\psi\rangle &= \psi_x^+ |\psi\rangle = |\psi\rangle \psi_x^+, \\
\langle \psi | f_T[\hat{\psi}] | \psi \rangle &= \langle \psi | e^{\frac{1}{2} \sum_{xy} \Delta_F(x,y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_N[\hat{\psi}] | \psi \rangle \\
&= \langle \psi | \psi \rangle e^{\frac{1}{2} \sum_{xy} \Delta_F(x,y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f[\psi] \\
&= \langle \psi | \psi \rangle \int D\psi' f[\psi'] \frac{1}{\sqrt{\det \Delta_F}} e^{\frac{1}{2} \sum_{xy} (\psi_x - \psi'_x) \Delta_F^{-1}(x,y) (\psi_y - \psi'_y)} \\
&= \langle \psi | \psi \rangle \int D\psi' f[\psi + \psi'] \frac{1}{\sqrt{\det \Delta_F}} e^{\frac{1}{2} \sum_{xy} \psi'_x \Delta_F^{-1}(x,y) \psi'_y}, \quad (\text{B.3.15})
\end{aligned}$$

where $\psi = 0$ corresponds to a possible vacuum state (ground state or local minimum energy configuration:

$\frac{\delta E[\psi, \dot{\psi}]}{\delta \psi_x} = 0$, $\frac{\delta E[\psi, \dot{\psi}]}{\delta \dot{\psi}_x} = 0$, $E[\psi, \dot{\psi}] = \int d^{D-1}x \left(\frac{\partial \mathcal{L}}{\partial \partial_0 \psi} \partial_0 \psi - \mathcal{L} \right)$ and ψ may be identified with $\langle \phi \rangle = C + \langle \hat{\varphi} \rangle$. In the case of more than one independent vacua $\{\psi_i\}$ the vacuum amplitude has a matrix structure

$$\begin{aligned}
\langle \psi_i | f_T[\hat{\psi}] | \psi_j \rangle &= \langle \psi_i | e^{\frac{1}{2} \sum_{xy} \Delta_F(x,y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f_N[\hat{\psi}] | \psi_j \rangle \\
&= \langle \psi_i | \psi_j \rangle e^{\frac{1}{2} \sum_{xy} \Delta_F(x,y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} f[\psi] \Big|_{\psi = \psi_i^+ + \psi_j^-}. \quad (\text{B.3.16})
\end{aligned}$$

One notes that the eigenfunctionals (which describe stationary or “elementary” processes) of the operator $\sum_{xy} \Delta_F(x,y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}$ are individually “unaffected” by the quantization. One may also consider expanding any given process in terms of these stationary processes.

Since $S[C + U^{-1} \hat{\varphi} U] = S[\phi]$ describes the background (as opposed to “particles” or excitations) dynamics $\frac{\delta S[\phi]}{\delta \phi_x} = 0$ of the quantum field ϕ , one can define a corresponding action $\Gamma[\psi]$ that describes the background dynamics $\frac{\delta \Gamma[\psi]}{\delta \psi_x} = 0$ of ψ (ie. the same dynamics in the classical picture) by

$$e^{-i\Gamma[\psi]} = e^{\frac{1}{2} \sum_{xy} \Delta_F(x,y) \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} e^{-iS[\psi]}, \quad (\text{B.3.17})$$

which is given by the choice

$$f[\psi] = e^{-iS[\psi]}; \text{ that is } \hat{\psi}_{x_1} = \hat{\psi}_{x_2} = \dots = \hat{\psi}_{x_n} = \text{const.}$$

- Thus n-point scattering amplitudes correspond to the choice

$$\begin{aligned} f[\psi] &= e^{-iS_I[\psi]} \psi_{x_1} \psi_{x_2} \dots \psi_{x_n}, \\ G(x_1, \dots, x_n; \psi) &= \frac{f_{eff}[\psi]}{e^{\frac{1}{2} \sum_{xy} \Delta_{xy} \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} e^{-iS_I[\psi]}} \\ &= \frac{e^{\frac{1}{2} \sum_{xy} \Delta_{xy} \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} (e^{-iS_I[\psi]} \psi_{x_1} \psi_{x_2} \dots \psi_{x_n})}{e^{\frac{1}{2} \sum_{xy} \Delta_{xy} \frac{\delta}{\delta \psi_x} \frac{\delta}{\delta \psi_y}} e^{-iS_I[\psi]}}. \end{aligned}$$

In the presence of spacetime noncommutativity an appropriate choice would be

$$f[\psi] = e^{-iS_{*I}[\psi]} \psi_{x_1} * \psi_{x_2} * \dots * \psi_{x_n}. \quad (\text{B.3.18})$$

- In momentum space one can choose

$$f[\psi] = e^{-iS_I[\psi]} \tilde{\psi}_{k_1} \tilde{\psi}_{k_2} \dots \tilde{\psi}_{k_n} \delta\left(\sum_i k_i\right), \quad (\text{B.3.19})$$

where $\tilde{\psi}_k = \int d^4x \psi_x e^{ikx}$ and in the presence of Moyal noncommutativity one can choose

$$\begin{aligned} f[\psi] &= e^{-iS_{*I}[\psi]} \tilde{\psi}_{k_1} \tilde{\psi}_{k_2} \dots \tilde{\psi}_{k_n} e^{-\frac{i}{2} \sum_{i < j} k_i^\mu \theta_{\mu\nu} k_j^\nu} \delta\left(\sum_i k_i\right). \\ (\tilde{\psi}_x^n)_q &= \sum_{q_1, q_2, \dots, q_n} \delta(q_1 + \dots + q_n - q) \tilde{\psi}_{q_1} \dots \tilde{\psi}_{q_n}. \end{aligned} \quad (\text{B.3.20})$$

Remarks:

- The definition of the functional includes the definition of pointwise products and therefore the use of alternative ordering to break Weyl symmetric

ordering in functionals generalizes the implementation/or detection of non-commutativity.

- To involve fermions one may directly extend the field ϕ to become a superfield; ie. $\phi_x \rightarrow \Phi_s$, $s_M = (x_\mu, \vartheta_i) = (x_\mu, \theta_\sigma, \bar{\theta}_\sigma)$, $\vartheta_i \vartheta_j = -\vartheta_j \vartheta_i$.

$$\begin{aligned} \Phi_s &= \phi_x + \psi_x \theta + \bar{\psi}_x \bar{\theta} + \theta A \bar{\theta} + F \theta^2 + \bar{F} \bar{\theta}^2 + \chi_x \theta \bar{\theta}^2 + \bar{\chi}_x \bar{\theta} \theta^2 + D \theta^2 \bar{\theta}^2. \\ \mu &= 0, 1, \dots, D-1, \quad i = 1, 2, \dots, 2^{\frac{D}{2}}, \quad \sigma = 1, 2, \dots, \frac{1}{2} 2^{\frac{D}{2}} = 1, 2, \dots, 2^{\frac{D}{2}-1}. \end{aligned}$$

A superspace extension of $g_x^{\mu\nu} \nabla_\mu \nabla_\nu \phi_x = 0$ would be

$$g_s^{MN} \nabla_M \nabla_N \Phi_s = 0. \quad (\text{B.3.21})$$

- If the semiclassical quantization procedure is applied to Moyal spacetime

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} = -i\theta_{\nu\mu}, \quad (\text{B.3.22})$$

one would obtain

$$\begin{aligned} \langle x | f_{\mathcal{O}}(\hat{x}) | x \rangle &= \langle x | x \rangle e^{\frac{i}{2} \sum_{\mu\nu} \Delta_{\mu\nu} \partial_\mu \partial_\nu} f(x) = \int d^D x' f(x') \frac{e^{\frac{i}{2} (x_\mu - x'_\mu) \Delta_{\mu\nu}^{-1} (x_\nu - x'_\nu)}}{\sqrt{\det \Delta}}, \\ \Delta_{\mu\nu} &= i\mathcal{O}_{\mu\nu} \theta_{\mu\nu} + i\mathcal{N}_{\mu\nu}, \quad \mathcal{O}_{\mu\nu} = -\mathcal{O}_{\nu\mu}, \quad \mathcal{N}_{\mu\nu} = \mathcal{N}_{\nu\mu} = \mathcal{N}_{\mu\nu}(\theta), \end{aligned}$$

where \mathcal{O}, \mathcal{N} are yet to be determined physical constants. If Lorentz invariance is required then $\Delta_{\mu\nu} = \lambda \eta^{\mu\nu}$.

By analogy we may write $\hat{x}_\mu = \hat{x}_\mu^+ + \hat{x}_\mu^-$, $[\hat{x}_\mu^+, \hat{x}_\nu^+] = 0$, $[\hat{x}_\mu^-, \hat{x}_\nu^-] = 0$; then

$$\begin{aligned} i\theta_{\mu\nu} &= [\hat{x}_\mu, \hat{x}_\nu] = [\hat{x}_\mu^+, \hat{x}_\nu^-] + [\hat{x}_\mu^-, \hat{x}_\nu^+] = [\hat{x}_\mu^-, \hat{x}_\nu^+] - [\hat{x}_\nu^-, \hat{x}_\mu^+], \\ \mathcal{O}_{\mu\nu} &= \frac{1}{2} \text{sign}(\mu - \nu), \quad i\mathcal{N}_{\mu\nu} = \frac{1}{2} ([\hat{x}_\mu^-, \hat{x}_\nu^+] + [\hat{x}_\nu^-, \hat{x}_\mu^+]), \\ \Delta_{\mu\nu} &= \theta(\mu - \nu) [\hat{x}_\mu^-, \hat{x}_\nu^+] + \theta(\nu - \mu) [\hat{x}_\nu^-, \hat{x}_\mu^+]. \end{aligned} \quad (\text{B.3.23})$$

However, for a particle moving in spacetime (or equivalently a particle-like spacetime), the coordinates $\mathcal{X}_\mu(\tau) = c_\mu + U^{-1}\hat{x}_\mu(\tau)U$ already possess the natural ordering wrt the parameter τ ,

$$\begin{aligned} [\hat{x}_\mu(\tau), \hat{x}_\nu(\tau')] &= i\theta_{\mu\nu}(\tau, \tau'), \\ [\hat{x}_M, \hat{x}_N] &= i\theta_{MN}, \quad M = (\tau, \mu), \quad N = (\tau', \nu). \\ \mathcal{X}_M &= c_\mu + U^{-1}\hat{x}_M U. \\ d\tau^2 &= \eta_{\mu\nu}dx^\mu(\tau)dx^\nu(\tau). \end{aligned} \tag{B.3.24}$$

Recall that *Any natural process or phenomenon may be regarded as a sequence of localized spacetime “events” that is well ordered in time (non-relativistic sense) or proper time (relativistic sense) or any other suitable parameter.*

One can therefore write

$$\Delta_{MN} = \theta(\tau - \tau') [\hat{x}_M^-, \hat{x}_N^+] + \theta(\tau' - \tau) [\hat{x}_N^-, \hat{x}_M^+]. \tag{B.3.25}$$

Similarly for a string-like space-time

$$\begin{aligned} [\hat{x}_\mu(\tau, \sigma), \hat{x}_\nu(\tau', \sigma')] &= i\theta_{\mu\nu}(\tau, \sigma, \tau', \sigma'), \\ [\hat{x}_M, \hat{x}_N] &= i\theta_{MN}, \quad M = (\tau, \sigma, \mu), \quad N = (\tau', \sigma', \nu), \\ d\tau^2 &= \sum_{\sigma} \eta_{\mu\nu}dx_\sigma^\mu(\tau)dx_\sigma^\nu(\tau), \end{aligned} \tag{B.3.26}$$

one can write

$$\Delta_{MN} = \theta(\tau - \tau') [\hat{x}_M^-, \hat{x}_N^+] + \theta(\tau' - \tau) [\hat{x}_N^-, \hat{x}_M^+]. \tag{B.3.27}$$

and for a brane-like spacetime one has

$$\begin{aligned} [\hat{x}_M, \hat{x}_N] &= i\theta_{MN}, \quad M = (u_a, \mu), \quad N = (u'_a, \nu), \\ \Delta_{MN} &= \theta(u_0 - u'_0) [\hat{x}_M^-, \hat{x}_N^+] + \theta(u'_0 - u_0) [\hat{x}_N^-, \hat{x}_M^+]. \end{aligned} \tag{B.3.28}$$

B.4 Hamilton-Jacobi theory

Here is an example of how solutions of differential equations may be found using knowledge of their symmetries.

Consider the configurations or canonical “flow” parameter λ of (B.1.2) and replace q by $x \in \mathbb{R}^{d+1}$, then the symplectic potential and λ -flow (with generating function \mathcal{J}) equations are given by

$$\begin{aligned} A(x, p) &= p_\mu dx^\mu = p^0 dt - p_i dx^i, & \mathcal{J} &= \mathcal{J}(x, p) \\ \frac{dx^\mu}{d\lambda} &= \frac{\partial \mathcal{J}}{\partial p^\mu}, & \frac{dp^\mu}{d\lambda} &= -\frac{\partial \mathcal{J}}{\partial x^\mu}, & \frac{d\mathcal{J}}{d\lambda} &= 0, \end{aligned} \quad (\text{B.4.1})$$

where the Poisson bracket is given by $\{f, g\}_1 = \frac{\partial f}{\partial p^\mu} \frac{\partial g}{\partial x^\mu} - \frac{\partial g}{\partial p^\mu} \frac{\partial f}{\partial x^\mu}$.

Choosing the flow parameter λ to be timelike ($\mathcal{J} \rightarrow H$), ie. $\frac{dt}{d\lambda} = 1 = \frac{\partial H}{\partial p^0}$, implies that $\mathcal{J} = H = p^0 + h(t, x^i, p^i) = p^0(t) + h(t, x^i(t), p^i(t))$ and the equations of motion become

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p^i} = \frac{\partial h}{\partial p^i}, \quad \frac{dp^i}{dt} = -\frac{\partial H}{\partial x^i} = -\frac{\partial h}{\partial x^i}, \quad \frac{dp^0}{dt} = -\frac{\partial H}{\partial t} = -\frac{\partial h}{\partial t}.$$

Thus for any given $F = F(x, p) = F(t, x^i(t), p^i(t))$,

$$\begin{aligned} \frac{dF}{dt} &= \{H, F\}_1 = \frac{\partial H}{\partial p^\mu} \frac{\partial F}{\partial x^\mu} - \frac{\partial F}{\partial p^\mu} \frac{\partial H}{\partial x^\mu} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial p^i} \frac{\partial h}{\partial x^i} - \frac{\partial h}{\partial p^i} \frac{\partial F}{\partial x^i} \\ &= \frac{\partial F}{\partial t} + \{h, F\}, & \{f, g\} &= \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p^i} \frac{\partial f}{\partial x^i}. \end{aligned} \quad (\text{B.4.2})$$

Note that $H = p^0 + h$ defines a hypersurface since H is a constant of motion. Recall that a canonical transformation $(t, x^i, p^i, p^0) \rightarrow (\tilde{t}, \tilde{x}^i, \tilde{p}^i, \tilde{p}^0)$ is one where

$$\begin{aligned} \tilde{A}(\tilde{x}, \tilde{p}) &= \tilde{p}^0 d\tilde{t} - \tilde{p}_i d\tilde{x}^i \equiv A(x, p) + dW = p^0 dt - p_i dx^i + dW, \\ \tilde{H} &= \tilde{p}^0 + \tilde{h} = \tilde{p}^0(\tilde{t}) + \tilde{h}(\tilde{t}, \tilde{x}^i(\tilde{t}), \tilde{p}^i(\tilde{t})), \\ \frac{dF}{d\tilde{t}} &= \frac{\partial F}{\partial \tilde{t}} + \{\tilde{h}, F\}, & F &= F(\tilde{t}, \tilde{x}^i(\tilde{t}), \tilde{p}^i(\tilde{t})), \\ \frac{d\tilde{x}^i}{d\tilde{t}} &= \frac{\partial \tilde{h}}{\partial \tilde{p}^i}, & \frac{d\tilde{p}^i}{d\tilde{t}} &= -\frac{\partial \tilde{h}}{\partial \tilde{x}^i}, & \{f, g\} &= \frac{\partial f}{\partial \tilde{p}^i} \frac{\partial g}{\partial \tilde{x}^i} - \frac{\partial g}{\partial \tilde{p}^i} \frac{\partial f}{\partial \tilde{x}^i}. \end{aligned} \quad (\text{B.4.3})$$

That is, the transformation preserves the symplectic form (and hence the Poisson bracket) while changing the symplectic potential by a total differential.

It may be possible to choose the function W such that $\tilde{t} = t$ (synchronization), $\tilde{H} = 0$ ($\tilde{p}^0 = -\tilde{h}$), $\tilde{p}^0 - p^0 = -h$ in which case \tilde{x}^i, \tilde{p}^i become (arbitrary) constants of motion given by

$$\begin{aligned} \tilde{p}^i &= -\frac{\partial W}{\partial \tilde{x}^i}, \quad p^i = -\frac{\partial W}{\partial x^i}, \quad 0 = \frac{\partial W}{\partial \tilde{p}^i}, \\ \tilde{p}^0 - p^0 &= \frac{\partial W}{\partial t} = -h(t, x^i, \frac{\partial W}{\partial x^i}). \\ W &= W(t, x^i, \tilde{x}^i). \end{aligned} \tag{B.4.4}$$

If $\frac{\partial h}{\partial t} = 0$ (ie. $h(t, x, p) = h(x, p)$) then we may write

$$\begin{aligned} W(t, x^i, \tilde{x}^i) &= S(x^i, \tilde{x}^i, \tilde{p}^i) - \alpha(\tilde{x}^i, \tilde{p}^i)t, \quad \alpha(\tilde{x}^i, \tilde{p}^i) = h(x^i, \frac{\partial S}{\partial x^i}), \\ \tilde{p}^j &= -\frac{\partial S(x^i(t), \tilde{x}^i, \tilde{p}^i)}{\partial \tilde{x}^j} + \frac{\partial \alpha(\tilde{x}^i, \tilde{p}^i)}{\partial \tilde{x}^j}t, \end{aligned}$$

where the constants depend on initial conditions thus

$$\begin{aligned} \alpha &= \alpha(\tilde{x}^i, \tilde{p}^i) = h(x^i(0), p^i(0)) \equiv h_0, \\ \tilde{x}^j &= \tilde{x}^j(x^i(0), p^i(0)), \quad \tilde{p}^j = \tilde{p}^j(x^i(0), p^i(0)). \end{aligned} \tag{B.4.5}$$

The choice of how the \tilde{x}^i and \tilde{p}^i depend on $x^i(0)$ and $p^i(0)$ is a matter of convenience and one can for example choose $\tilde{x}^i = x^i(0)$, $\tilde{p}^i = p^i(0)$. In this case

$W(t, x^i, \tilde{x}^i) = S(x^i, x^i(0), p^i(0)) - h(x^i(0), p^i(0))t$ where we need to remember that

$$\begin{aligned} \frac{\partial W(t, x^i, \tilde{x}^i)}{\partial \tilde{p}^i} &= \frac{\partial S(x^i, \tilde{x}^i, \tilde{p}^i)}{\partial \tilde{p}^i} - \frac{\partial \alpha(\tilde{x}^i, \tilde{p}^i)}{\partial \tilde{p}^i}t = 0 \\ \Rightarrow \frac{\partial W(t, x^i, x^i(0))}{\partial p^i(0)} &= \frac{\partial S(x^i, x^i(0), p^i(0))}{\partial p^i(0)} - \frac{\partial h(x^i(0), p^i(0))}{\partial p^i(0)}t = 0. \end{aligned} \tag{B.4.6}$$

For the case $h = \vec{p}^2 + V(\vec{x})$, one has

$$\begin{aligned}\alpha(\tilde{x}^i, \tilde{p}^i) &= h(x^i(0), p^i(0)) = \left(\frac{\partial S}{\partial \vec{x}}\right)^2 + V(\vec{x}), \\ \vec{p}^2 &= \left(\frac{\partial S}{\partial \vec{x}}\right)^2 = \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \varphi}\right)^2 = p_r^2 + \frac{1}{r^2 \sin^2 \theta} p_\theta^2 + \frac{1}{r^2} p_\varphi^2\end{aligned}\tag{B.4.7}$$

so that

$$\begin{aligned}\vec{p} &= \frac{\partial S_u(x^i, \tilde{x}^i, \tilde{p}^i)}{\partial \vec{x}} = \vec{u}(x^i, \tilde{x}^i, \tilde{p}^i) \sqrt{\alpha(\tilde{x}^i, \tilde{p}^i) - V(\vec{x})}, \quad \vec{u}^2 = 1, \\ S_u(x^i, \tilde{x}^i, \tilde{p}^i) &= \int_{x^i(0)}^{x^i} d\vec{y} \cdot \vec{u}(y^i, \tilde{x}^i, \tilde{p}^i) \sqrt{\alpha(\tilde{x}^i, \tilde{p}^i) - V(\vec{y})} + S_u(x^i(0), \tilde{x}^i, \tilde{p}^i),\end{aligned}\tag{B.4.8}$$

where a convenient choice of integration path depends on the choice of \vec{u} whose set is as large as $SO(d)$ for $x^i \in \mathbb{R}^d$. Similarly for the case $h = \sqrt{\vec{p}^2 + m^2} + V(\vec{x})$ one obtains

$$\begin{aligned}S_u(x^i, \tilde{x}^i, \tilde{p}^i) &= \int_{x^i(0)}^{x^i} d\vec{y} \cdot \vec{u}(y^i, \tilde{x}^i, \tilde{p}^i) \sqrt{[\alpha(\tilde{x}^i, \tilde{p}^i) - V(\vec{y})]^2 - m^2} + S_u(x^i(0), \tilde{x}^i, \tilde{p}^i), \\ W_u(t, x^i, \tilde{x}^i) &= \int_{x^i(0)}^{x^i} d\vec{y} \cdot \vec{u}(y^i, \tilde{x}^i, \tilde{p}^i) \sqrt{[\alpha(\tilde{x}^i, \tilde{p}^i) - V(\vec{y})]^2 - m^2} + S_u(x^i(0), \tilde{x}^i, \tilde{p}^i) \\ &\quad - \alpha(\tilde{x}^i, \tilde{p}^i)t.\end{aligned}$$

If we set $\tilde{p} = t_0 = -\frac{\partial W}{\partial \alpha}$ then we obtain

$$\int_{x^i(0)}^{x^i} d\vec{y} \cdot \vec{u}(y^i, \tilde{x}^i, \tilde{p}^i) \frac{\alpha(\tilde{x}^i, \tilde{p}^i) - V(\vec{y})}{\sqrt{[\alpha(\tilde{x}^i, \tilde{p}^i) - V(\vec{y})]^2 - m^2}} = t - t_0.\tag{B.4.9}$$

For bounded motion in a central potential where

$$\begin{aligned}h &= \sqrt{\vec{p}^2 + m^2} + V(r) = \sqrt{p_r^2 + \frac{L^2}{r^2} + m^2} + V(r) = \varepsilon, \\ L^{ij} &= \frac{1}{\sqrt{2}}(x^i p^j - x^j p^i), \quad \frac{dL^{ij}}{dt} = \{h, L^{ij}\} = 0, \\ L^2 &= L_{ij}L^{ij} = \vec{x}^2 \vec{p}^2 - (\vec{x} \cdot \vec{p})^2 = r^2(\vec{p}^2 - p_r^2), \quad p_r = \frac{\vec{x}}{r} \cdot \vec{p},\end{aligned}\tag{B.4.10}$$

the solutions to

$p_r = -\frac{\partial S(r,\varepsilon,L)}{\partial r} = -\sqrt{[\varepsilon - V(r)]^2 - m^2 - \frac{L^2}{r^2}} = 0$ give the extreme bounds r_-, r_+, \dots of the orbit and thus with $t_0 = -\frac{\partial W}{\partial \varepsilon}$

$$\begin{aligned} \int_{r(t_0)}^{r(t)} dr' \frac{\varepsilon - V(r')}{\sqrt{[\varepsilon - V(r')]^2 - m^2 - \frac{L^2}{r'^2}}} &= t - t_0 \\ \Rightarrow \int_{r_-}^{r_+} dr \frac{\varepsilon - V(r)}{\sqrt{[\varepsilon - V(r)]^2 - m^2 - \frac{L^2}{r^2}}} &= T. \end{aligned} \quad (\text{B.4.11})$$

gives a measure of the period(s) T of the motion.

For the simple time dependent case $h = p^2 + xt$, one can write

$W(t, x) = a(t) + xb(t) + x^2c(t)$ and then solve for the t dependent coefficients using $\frac{\partial W}{\partial t} = -h(t, x^i, \frac{\partial W}{\partial x^i})$.

B.5 (Orbital) angular momentum and spherical functions

Consider the angular momentum operator given by $L_{ij} = \frac{1}{\sqrt{2}}(x_i \partial_j - x_j \partial_i)$, then

$$\begin{aligned} L^2 &= r^2 \vec{\partial}^2 - (D-2)\vec{x} \cdot \vec{\partial} - (\vec{x} \cdot \vec{\partial})^2 \\ &= r^2 \vec{\partial}^2 - (D-2)r \frac{\partial}{\partial r} - r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} = r^2 \vec{\partial}^2 - (D-1)r \frac{\partial}{\partial r} - r^2 \frac{\partial^2}{\partial r^2} \\ \Rightarrow \vec{\partial}^2 &= \frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} + \frac{L^2}{r^2} = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{L^2}{r^2}. \\ \vec{\partial}^2 \psi &= \left(\frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{L^2}{r^2} \right) \psi = \frac{1}{r^{\frac{D-1}{2}}} \left(\frac{\partial^2}{\partial r^2} + \frac{L^2 - (1-D)(3-D)/4}{r^2} \right) (r^{\frac{D-1}{2}} \psi), \\ \frac{1}{f} \frac{d}{dx} f \frac{d}{dx} \psi &= \frac{1}{f^{\frac{1}{2}}} \left\{ \frac{d^2}{dx^2} + f^{-\frac{1}{2}} (f(f^{-\frac{1}{2}})')' \right\} (f^{\frac{1}{2}} \psi) = \frac{1}{f^{\frac{1}{2}}} \left\{ \frac{d^2}{dx^2} + \frac{f'^2 - 2ff''}{4f^2} \right\} (f^{\frac{1}{2}} \psi) \\ &\equiv \frac{1}{f^{\frac{1}{2}}} \left\{ \frac{d^2}{dx^2} + \beta(f) \right\} (f^{\frac{1}{2}} \psi), \\ \frac{1}{g} \frac{d}{dx} g \frac{d}{dx} \psi &= \sqrt{\frac{f}{g}} \left\{ \frac{1}{f} \frac{d}{dx} f \frac{d}{dx} + \beta(g) - \beta(f) \right\} \left(\sqrt{\frac{g}{f}} \psi \right). \end{aligned} \quad (\text{B.5.1})$$

Therefore, a function satisfying

$$\vec{\partial}^2 G(r) = \delta^D(\vec{r}) \text{ is } G(r) = r^\alpha, \quad \alpha = 3-D, \quad L^2 = -\alpha(\alpha+D) = -3(3-D), \quad D \neq 2.$$

B.5.1 ∂^2 in a minimally coupled system?

In arbitrary orthogonal coordinates $\vec{x} = \vec{x}(u^a)$, the divergence operator is given by

$$\begin{aligned} \partial_i V^i &= \frac{1}{\sqrt{h}} \partial_a (\sqrt{h} V^a) = \frac{1}{\sqrt{h}} \partial_a \left(\frac{\sqrt{h}}{h_a^2} e_a^i V^i \right) \equiv \frac{1}{\sqrt{h}} \partial_a \left(\frac{\sqrt{h}}{h_a^2} V_a \right), \\ h &= \prod_a h_a, \quad h_a = |\vec{e}_a|, \quad \vec{e}_a = e_a^i = \frac{\partial x^i}{\partial u^a}. \end{aligned} \quad (\text{B.5.2})$$

For a $U(1)$ gauge covariant divergence $(\partial_i + A_i)V^i$ one may simply make the replacement

$\sqrt{h} \rightarrow U\sqrt{h}$ so that

$$(\partial_i + A_i)V^i = \frac{1}{U\sqrt{h}} \partial_a \left(\frac{U\sqrt{h}}{h_a^2} V_a \right), \quad (\text{B.5.3})$$

where $\partial_i U = A_i U$. Writing $P_i = \partial_i + A_i$, $J_{ij} = x_i P_j - x_j P_i$,

$$\begin{aligned} [P_i, x_j] &= \delta_{ij}, \quad [P_i, P_j] = \partial_i A_j - \partial_j A_i, \quad [x^2, J_{ij}] = 0, \quad [P^2, J_{ij}] \neq 0 \\ \vec{P}^2 &= \frac{D-2}{r^2} \vec{x} \cdot \vec{P} + \frac{1}{r^2} (\vec{x} \cdot \vec{P})^2 + \frac{J^2}{r^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{D-1+2rA_r}{r} \frac{\partial}{\partial r} + \frac{(D-2)rA_r + r\partial_r(rA_r) + r^2A_r^2 + J^2}{r^2} \\ &= \frac{1}{f(r)} \frac{\partial}{\partial r} f(r) \frac{\partial}{\partial r} + \frac{(D-2)rA_r + r\partial_r(rA_r) + r^2A_r^2 + J^2}{r^2}, \\ f(r) &= e^{\int^r dr' \frac{D-1+2r'A_{r'}}{r'}}, \quad rA_r = \vec{x} \cdot \vec{A}(\vec{x}), \quad \partial_r = \frac{\partial}{\partial r}. \end{aligned} \quad (\text{B.5.4})$$

The condition $[x^2, J_{ij}] = 0$ is essential for the independence of the angular part from the radial parts. Notice that for the case where $\vec{x} \cdot \vec{A} = \text{const}$, such as

$A^i = \alpha \frac{x^i}{r^2} + \frac{1}{2} B^{[ij]}(\vec{x}) x^j = -\alpha \partial_i \left(\frac{1}{r} \right) + \frac{1}{2} B^{[ij]}(\vec{x}) x^j$, the expression for P^2 simplifies

drastically and is almost equivalent to ∂^2 except that the angular momentum J_{ij} (and hence J^2) is slightly modified.

In the case $A_i = \frac{1}{2}B_{ij}x^j$, $B_{ij} = -B_{ji} = \text{const}$ one also has $\partial_i A^i = 0$ and so expanding P^2 implies that ∂^2 can be written as

$$\partial^2 = \frac{1}{r^{D-1}} \frac{d}{dr} r^{D-1} \frac{d}{dr} + \frac{J^2}{r^2} - \frac{1}{2} B_{ij} J^{ij} + \frac{1}{4} B_{ki} B_{kj} x_i x_j. \quad (\text{B.5.5})$$

B.5.2 Spherical eigenfunctions

In spherical coordinates (with $\theta_0 = 0$)

$$x_0 = y = r \cos \theta_0 \sin \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \dots \sin \theta_{D-1}, \quad (\text{B.5.6})$$

$$x_1 = x = r \cos \theta_1 \sin \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \dots \sin \theta_{D-1} \quad (\text{B.5.7})$$

$$x_2 = z = r \cos \theta_2 \sin \theta_3 \sin \theta_4 \sin \theta_5 \dots \sin \theta_{D-1} \quad (\text{B.5.8})$$

$$x_3 = r \cos \theta_3 \sin \theta_4 \sin \theta_5 \dots \sin \theta_{D-1} \quad (\text{B.5.9})$$

$$x_4 = r \cos \theta_4 \sin \theta_5 \dots \sin \theta_{D-1} \quad (\text{B.5.10})$$

$$\dots \quad (\text{B.5.11})$$

$$x_k = r \cos \theta_k \sin \theta_{k+1} \dots \sin \theta_{D-1} \quad (\text{B.5.12})$$

$$\dots \quad (\text{B.5.13})$$

$$x_{D-2} = r \cos \theta_{D-2} \sin \theta_{D-1} \quad (\text{B.5.14})$$

$$x_{D-1} = r \cos \theta_{D-1} \quad (\text{B.5.15})$$

the Laplacian is given by

$$\begin{aligned} \partial^2 &= \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \sum_{k=1}^{D-1} \frac{1}{\sin^2 \theta_{k+1} \sin^2 \theta_{k+2} \dots \sin^2 \theta_{D-1}} \frac{1}{\sin^{k-1} \theta_k} \frac{\partial}{\partial \theta_k} \sin^{k-1} \theta_k \frac{\partial}{\partial \theta_k} \\ &= \frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{1}{r^2} \left\{ \frac{1}{\sin^2 \theta_2 \dots \sin^2 \theta_{D-1}} \frac{\partial^2}{\partial \theta_1^2} + \frac{1}{\sin^2 \theta_3 \dots \sin^2 \theta_{D-1}} \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \sin \theta_2 \frac{\partial}{\partial \theta_2} \right. \\ &\quad + \frac{1}{\sin^2 \theta_4 \dots \sin^2 \theta_{D-1}} \frac{1}{\sin^2 \theta_3} \frac{\partial}{\partial \theta_3} \sin^2 \theta_3 \frac{\partial}{\partial \theta_3} + \frac{1}{\sin^2 \theta_5 \dots \sin^2 \theta_{D-1}} \frac{1}{\sin^3 \theta_4} \frac{\partial}{\partial \theta_4} \sin^3 \theta_4 \frac{\partial}{\partial \theta_4} \\ &\quad + \dots + \\ &\quad \left. \frac{1}{\sin^2 \theta_{D-1}} \frac{1}{\sin^{D-3} \theta_{D-2}} \frac{\partial}{\partial \theta_{D-2}} \sin^{D-3} \theta_{D-2} \frac{\partial}{\partial \theta_{D-2}} + \frac{1}{\sin^{D-2} \theta_{D-1}} \frac{\partial}{\partial \theta_{D-1}} \sin^{D-2} \theta_{D-1} \frac{\partial}{\partial \theta_{D-1}} \right\} \end{aligned} \quad (\text{B.5.16})$$

Now any operator Q that has a complete set of eigenfunctions $\mathcal{Q} = \{\psi, Q\psi = \lambda\psi, \lambda \in \mathbb{C}\}$ on a space X (ie. every element in $\mathcal{F}(X)$ can be expressed as a linear combination of the elements of \mathcal{Q} , $\mathcal{F}(X) = \text{Span}\mathcal{Q}$) can be defined in X and/or $\mathcal{F}(X)$ only up to similarity transformations since the spectrum of PQP^{-1} is isomorphic to the spectrum of Q . The Laplacian ∂^2 is such an operator. One can make the identification

$$L^2 = \partial_\Omega^2 = \sum_{k=1}^{D-1} \frac{1}{\sin^2 \theta_{k+1} \sin^2 \theta_{k+2} \dots \sin^2 \theta_{D-1}} \frac{1}{\sin^{k-1} \theta_k} \frac{\partial}{\partial \theta_k} \sin^{k-1} \theta_k \frac{\partial}{\partial \theta_k} \quad (\text{B.5.17})$$

The solution in a spherically symmetric potential $f(r)$ may be separated as follows

$$\begin{aligned} (\partial^2 + f(r))\psi(\vec{x}) &= 0, \\ \psi(\vec{x}) &= \sum_{m_1 \dots m_{D-2} \tilde{L}} \alpha_{m_1 \dots m_{D-2} \tilde{L}} \Theta_1^{m_1}(\theta_1) \Theta_2^{m_1 m_2}(\theta_2) \dots \Theta_k^{m_{k-1} m_k}(\theta_k) \dots \Theta_{D-1}^{m_{D-2} \tilde{L}}(\theta_{D-1}) \times \\ &R^{\tilde{L}}(r) \equiv \sum_{m_1 \dots m_{D-2} \tilde{L}} \alpha_{m_1 \dots m_{D-2} \tilde{L}} Y^{m_1 \dots m_{D-2} \tilde{L}}(\theta_1, \dots, \theta_{D-1}) R^{\tilde{L}}(r), \\ \frac{\partial^2}{\partial \theta_1^2} \Theta_1^{m_1}(\theta_1) &= m_1^2 \Theta_1^{m_1}(\theta_1), \\ \left(\frac{m_1^2}{\sin^2 \theta_2} + \frac{1}{\sin \theta_2} \frac{\partial}{\partial \theta_2} \sin \theta_2 \frac{\partial}{\partial \theta_2} \right) \Theta_2^{m_1 m_2}(\theta_2) &= m_2^2 \Theta_2^{m_1 m_2}(\theta_2), \\ \dots & \\ \left(\frac{m_{k-1}^2}{\sin^2 \theta_k} + \frac{1}{\sin^{k-1} \theta_k} \frac{\partial}{\partial \theta_k} \sin^{k-1} \theta_k \frac{\partial}{\partial \theta_k} \right) \Theta_k^{m_{k-1} m_k}(\theta_k) &= m_k^2 \Theta_k^{m_{k-1} m_k}(\theta_k), \\ \dots & \\ \left(\frac{m_{D-2}^2}{\sin^2 \theta_{D-1}} + \frac{1}{\sin^{D-2} \theta_{D-1}} \frac{\partial}{\partial \theta_{D-1}} \sin^{D-2} \theta_{D-1} \frac{\partial}{\partial \theta_{D-1}} \right) \Theta_{D-1}^{m_{D-2} \tilde{L}}(\theta_{D-1}) & \\ &= \tilde{L}^2 \Theta_{D-1}^{m_{D-2} \tilde{L}}(\theta_{D-1}), \\ \left(\frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{\tilde{L}^2}{r^2} + f(r) \right) R^{\tilde{L}}(r) &= 0. \end{aligned} \quad (\text{B.5.18})$$

Writing $s_k = \cos \theta_k$ the k th equation is

$$\begin{aligned}
& \left\{ \frac{1}{(1-s_k^2)^{\frac{k}{2}}} \frac{d}{ds_k} (1-s_k^2)^{\frac{k}{2}} \frac{d}{ds_k} + \frac{m_{k-1}^2 - (1-s_k^2)m_k^2}{(1-s_k^2)^2} \right\} \Theta_k^{m_{k-1}m_k}(s_k) = 0, \\
& \frac{1}{(1-s_k^2)^{\frac{k}{4}}} \left\{ \frac{d^2}{ds_k^2} + \frac{4k - k^2 + m_{k-1}^2 - (1-s_k^2)(2k - k^2 + m_k^2)}{(1-s_k^2)^2} \right\} \times \\
& [(1-s_k^2)^{\frac{k}{4}} \Theta_k^{m_{k-1}m_k}] = 0. \tag{B.5.19}
\end{aligned}$$

The Legendre equation

$$\begin{aligned}
& ((1-x^2)y')' + (l(l+1) - \frac{m^2}{1-x^2})y = 0, \\
& \frac{1}{(1-x^2)^{\frac{1}{2}}} \left(\frac{d^2}{dx^2} + \frac{4 - m^2 + l(l+1)(1-x^2)}{(1-x^2)^2} \right) ((1-x^2)^{\frac{1}{2}}y) = 0
\end{aligned} \tag{B.5.20}$$

has solutions

$$y_{ml}(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l, \quad -l \leq m \leq l. \tag{B.5.21}$$

To get the solution to (B.5.19) one should replace m^2 with $(2-k)^2 - m_{k-1}^2$ and replace $l(l+1)$ with $-(2k - k^2 + m_k^2)$.

The eigenvalue equation with $f(r) = \frac{\beta}{r}$ may be written as

$$\begin{aligned}
(\vec{\partial}^2 + \frac{\beta}{r} - \lambda)R &= \left(\frac{1}{r^{D-1}} \frac{\partial}{\partial r} r^{D-1} \frac{\partial}{\partial r} + \frac{L^2}{r^2} + \frac{\beta}{r} - \lambda \right) R \\
&= \frac{1}{r^{\frac{D-1}{2}}} \left(\frac{\partial^2}{\partial r^2} + \frac{L^2 - (1-D)(3-D)/4}{r^2} + \frac{\beta}{r} - \lambda \right) (r^{\frac{D-1}{2}} R) \\
&= \frac{1}{r^{\frac{D-1}{2}}} \left(\frac{\partial^2}{\partial r^2} + \frac{\tilde{L}^2}{r^2} + \frac{\beta}{r} - \lambda \right) (r^{\frac{D-1}{2}} R) = 0,
\end{aligned} \tag{B.5.22}$$

As $r \rightarrow 0$ the term in \tilde{L}^2 dominates and as $r \rightarrow \infty$ the term in λ dominates.

These suggest the change of variables

$F(r) = r^{\frac{D-1}{2}} R(r) = r^{\tilde{l}+1} e^{-r\sqrt{\lambda}} u(r)$, $-\tilde{L}^2 = \tilde{l}(\tilde{l}+1)$ that may simplify the equation.

$$\begin{aligned}
(r^{\tilde{l}+1} e^{-r\sqrt{\lambda}} u(r))'' &= [(-\sqrt{\lambda} + \frac{\tilde{l}+1}{r} + \frac{u'}{u})F]' \\
&= [(-\sqrt{\lambda} + \frac{\tilde{l}+1}{r} + \frac{u'}{u})^2 - \frac{\tilde{l}+1}{r^2} + \frac{u''}{u} - \frac{u'^2}{u^2}]F \\
&= [\lambda + \frac{\tilde{l}(\tilde{l}+1)}{r^2} - \frac{2(\tilde{l}+1)\sqrt{\lambda}}{r} + (-2\sqrt{\lambda} + \frac{2(\tilde{l}+1)}{r})\frac{u'}{u} + \frac{u''}{u}]F \\
\{ \frac{d^2}{dr^2} + (-2\sqrt{\lambda} + \frac{2(\tilde{l}+1)}{r})\frac{d}{dr} + \frac{\beta - 2(\tilde{l}+1)\sqrt{\lambda}}{r} \} u &= 0 \\
\{ r \frac{d^2}{dr^2} + (-2r\sqrt{\lambda} + 2(\tilde{l}+1))\frac{d}{dr} + \beta - 2(\tilde{l}+1)\sqrt{\lambda} \} u &= 0
\end{aligned} \tag{B.5.23}$$

Let the solution be in the form $u = \sum_{k=N}^{\infty} \alpha_k r^k$, then

$$\begin{aligned}
&\sum_{k=N}^{\infty} k(k-1)\alpha_k r^{k-1} - 2\sqrt{\lambda} \sum_{k=N}^{\infty} k\alpha_k r^k \\
&\quad + 2(\tilde{l}+1) \sum_{k=N}^{\infty} k\alpha_k r^{k-1} + (\beta - 2(\tilde{l}+1)\sqrt{\lambda}) \sum_{k=N}^{\infty} \alpha_k r^k = 0 \\
&\sum_{k=N}^{\infty} k(k-1)\alpha_k r^{k-1} - 2\sqrt{\lambda} \sum_{k=N}^{\infty} k\alpha_k r^k \\
&\quad + 2(\tilde{l}+1) \sum_{k=N}^{\infty} k\alpha_k r^{k-1} + (\beta - 2(\tilde{l}+1)\sqrt{\lambda}) \sum_{k=N}^{\infty} \alpha_k r^k = 0 \\
&\sum_{k=N} (k(k-1) + 2k(\tilde{l}+1))\alpha_k r^{k-1} + \sum_{k=N} (\beta - 2\sqrt{\lambda}(k + \tilde{l}+1))\alpha_k r^k = 0, \\
&(N(N-1) + 2N(\tilde{l}+1))\alpha_N r^{N-1} \\
&\quad + \sum_{k=N-1} (k+1)(k+2(\tilde{l}+1))\alpha_{k+1} r^k + \sum_{k=N} (\beta - 2\sqrt{\lambda}(k + \tilde{l}+1))\alpha_k r^k = 0,
\end{aligned} \tag{B.5.24}$$

With $N = 0$,

$$\alpha_{k+1} = -\frac{\beta - 2\sqrt{\lambda}(k + \tilde{l} + 1)}{(k + 1)(k + 2(\tilde{l} + 1))} \alpha_k \equiv [k]_{\beta\lambda\tilde{l}} \alpha_k = [k]_{\beta\lambda\tilde{l}}! \alpha_0, \quad (\text{B.5.25})$$

and the series terminates (eg. when bound states, $|R(r)| < \infty \quad \forall r$, $r = |\vec{x}_1 - \vec{x}_2|$, are desired) at

$$\begin{aligned} \lambda \rightarrow \lambda_{kl} &= \frac{\beta^2}{4(k + \tilde{l} + 1)^2}, & E_{12}^{kl} &= -\frac{2m_1 m_2}{\hbar^2(m_1 + m_2)} \left(\frac{q_1 q_2}{4\pi\epsilon_0}\right)^2 \frac{1}{4(k_{12} + \tilde{l}_{12} + 1)^2}. \\ \lambda &= -\frac{2m}{\hbar^2} E, & \beta &= \frac{2m}{\hbar^2} \frac{qq}{4\pi\epsilon_0}. \\ R(r) &= \alpha_0 r^{\frac{1-D}{2}} r^{\tilde{l}+1} e^{-r\sqrt{\lambda}} \sum_{k=0}^{\infty} [k]_{\beta\lambda\tilde{l}}! r^k \\ &\xrightarrow{D \rightarrow \infty} r^{\frac{D}{2}} e^{-r\sqrt{\lambda}} e^{r\sqrt{\lambda}} = r^{\frac{D}{2}}, \\ [k]_{\beta\lambda\tilde{l}} &= \frac{2\sqrt{\lambda}(k + \tilde{l} + 1) - \beta}{(k + 1)(k + 2(\tilde{l} + 1))}. \end{aligned} \quad (\text{B.5.26})$$

where

$$\tilde{L}^2 = L^2 - (1 - D)(3 - D)/4 \Rightarrow \tilde{l}(\tilde{l} + 1) = l(l + 1) - \frac{(D - 1)(D - 3)}{4}.$$

Thus (B.5.26) implies that radial and angular motion decouple and motion becomes free from any interactions. Therefore systems that can travel in extra dimensions can avoid interactions with those confined to fewer dimensions.

In spectroscopic notation

$$E_n = -\frac{2m}{\hbar^2} \left(\frac{qq}{4\pi\epsilon_0}\right)^2 \frac{1}{4n^2}, \quad n = k + \tilde{l} + 1 \Rightarrow 0 \leq \tilde{l} \leq n - 1. \quad (\text{B.5.27})$$

If a constant weak B_{ij} -field, $B^2 \ll B$, is included, then $\lambda \rightarrow \lambda + \frac{e}{\hbar} j^B B$ where j^B is the angular momentum quantum number satisfying

$$B_{ij} J_{ij} \psi_{j^B} = j^B \psi_{j^B}. \quad (\text{B.5.28})$$

Therefore

$$E_{nj^B} = \frac{\hbar^2}{2m} \frac{eB}{2\hbar} j^B - \frac{2m}{\hbar^2} \left(\frac{qq}{4\pi\epsilon_0} \right)^2 \frac{1}{4n^2} = \frac{e\hbar B}{4m} j^B - \frac{2m}{\hbar^2} \left(\frac{qq}{4\pi\epsilon_0} \right)^2 \frac{1}{4n^2}$$

$$n = k + \tilde{j} + 1 \Rightarrow 0 \leq \tilde{j} \leq n - 1. \quad (\text{B.5.29})$$

In spectroscopic notation, quantum orbits (known as orbitals) are label are given as

$$n^{2s+1}l_j, \quad |l - s| \leq j \leq l + s, \quad n = k + j + 1.$$

$$l = 0, 1, 2, 3, 4, 5, 6 \dots$$

$$\equiv S \text{ (sharp), } P \text{ (principal), } D \text{ (diffused), } F \text{ (fundamental), } G, H, \dots \quad (\text{B.5.30})$$

where the possible total angular momentum quantum numbers

$\vec{J}^2 = (\vec{L} + \vec{S})^2 = j(j + 1)$ follow from the vector inequality

$$(|\vec{L}| - |\vec{S}|)^2 \leq (\vec{L} + \vec{S})^2 \leq (|\vec{L}| + |\vec{S}|)^2 \quad (\text{B.5.31})$$

and

$$\vec{L}^2 = l(l + 1), \quad \vec{S}^2 = s(s + 1). \quad (\text{B.5.32})$$

One observes that to label representations

$$R : SO(n) \rightarrow GL(N) \rightarrow \mathcal{O}(\mathcal{F}(\mathbb{R}^n)), \quad g \triangleright |m_1, \dots, m_{n-2}, l \rangle$$

$$= \sum_{m'_1, \dots, m'_{n-2}, l'} |m'_1, \dots, m'_{n-2}, l \rangle \langle m'_1, \dots, m'_{n-2}, l' | g |m_1, \dots, m_{n-2}, l \rangle$$

$$\mapsto R(g) \triangleright Y^{m_1 \dots m_{n-2} \tilde{L}} \quad (\text{B.5.33})$$

for $SO(n)$, one can consider the sequence $SO(n) \subset SO(n-1) \dots \subset SO(3) \subset SO(2)$ with their respective quadratic Casimirs $L_{(n)}^2, L_{(n-1)}^2 \dots L_{(3)}^2, L_{(2)}^2$ all commuting and therefore can serve as labels. These Casimirs may be identified with the numbers $m_1, m_2, \dots, m_{n-2}, \tilde{L}$ given in the spherically symmetric equation above.

Since $L_{(k)}^2 \leq L_{(k+1)}^2$, numbers may be assigned such that

$-|l_{k+1}| \leq l_k \leq |l_{k+1}|$, $k = 2, 3, \dots, n$, where $L_{(k)}^2 = L_{(k)}^2(l_k) = l_k(l_k + 1)$. States may be labeled as $|l_2 l_3 \dots l_{n-1} l_n\rangle \equiv |m_1, \dots, m_{n-2}, l\rangle$.

Appendix C

Variation principle and classic symmetries

C.1 Division of spaces

If A, B are two spaces, then their quotient is given by the set of isomorphic maps

$$A/B = \{f; f : B \rightarrow A, f(a) = f(b) \Leftrightarrow a = b\}.$$

If one wishes the maps to be parallel then the following condition may be included:

$$f(a) = g(a) \Leftrightarrow f = g \quad \forall f, g \in A/B.$$

If $A \simeq F \times B$ then one says that A is a bundle of fibers $\{F\}$ (or fiber bundle $\pi : A \rightarrow B$) over B and A/B is the space of sections of A by B . The partial equality \simeq means “similar to” and its actual meaning depends on the context.

C.1.1 Spectrum of a group algebra

In the case of groups, if $A = G$ a group and subgroup $F = H \subset G$, with an action $\rho : G \times H \rightarrow H$, $(g, h) \rightarrow \rho(g, h) = gh$; eg. $G = SO(n+1)$, $H = SO(n)$, then $B = G/H \simeq \{gH, g \in G\} \simeq \{Hg, g \in G\}$. Thus one has a fiber bundle structure $G \simeq H \times G/H$.

If a subgroup H is not normal; ie. $gH = Hg$ does not hold for at least one $g \in G$, then a normal subgroup H_G may be constructed from it as

$$H_G = \{ghg^{-1}, G \in G, h \in H\} \quad (\text{C.1.1})$$

since $\tilde{g} gHg^{-1} = \tilde{g}gH(\tilde{g}g)^{-1} \tilde{g}$. One can also define a commuting element $S_0(s)$ for any element $s \in G$, $S_0(s)$ being an element of the group algebra

$$\begin{aligned} \mathcal{A}_G &\simeq \{a = a(\alpha) = \int_{g \in G} d\mu(g) \alpha(g) g, \alpha : G \rightarrow \mathbb{C}^N\}, \\ a(\alpha)a(\beta) &= a(\alpha * \beta), \\ (\alpha * \beta)(g) &= \int_{x \in G} d\mu(x) \alpha(gx^{-1})\beta(x) = \int_{x \in G} d\mu(x) \alpha(x)\beta(x^{-1}g), \end{aligned}$$

where μ is the left-translation invariant measure¹ on G .

$$S_0(s) = \{h_0 = \int_{g \in G} d\mu(g) gsg^{-1}, s \in H\}, \quad \int_{g \in G} d\mu(g'g) = \int_{g \in G} d\mu(g) \quad \forall g' \in G. \quad (\text{C.1.2})$$

The number of unique such elements is equal to the number of conjugacy classes of G since $S_0(g) = S_0(hgh^{-1})$. That is, the center $Z(\mathcal{A}_G)$ of \mathcal{A}_G is as large as the set of conjugacy classes $\{[g], g \in G\}$.

$$\begin{aligned} Z(\mathcal{A}_G) &= \{S_0([g]), g \in G\} \simeq \{[g], g \in G\}. \\ Z(G) &= G \cap Z(\mathcal{A}_G) \end{aligned} \quad (\text{C.1.3})$$

¹Section E.15.1.

and the irreducible representations of G or of \mathcal{A}_G are parametrized by the spectrum² $\sigma(Z(\mathcal{A}_G))$.

C.2 Gauge symmetry and Noether's theorem

A $U(1)$ gauge transformation is a continuous local transformation of the electromagnetic potential $A(x) \rightarrow A(x) - \frac{1}{g(x)}dg(x)$ that preserves the Maxwell Lagrangian for electromagnetism. Gauge symmetry may also be defined for an interacting theory, in which case, it may be associated to the conservation of electric charge by Noether's theorem which associates, along the classical path $\delta S = 0$, a “complete” set of conservation laws and hence a “complete” set of conserved charges to any continuous global symmetry of a classical theory. A continuous symmetry of a classical theory is a transformation that changes the differential action or Lagrangian only by an exact form $\delta_\xi \mathcal{L} = d\mathcal{K}$ (a canonical transformation) and hence does not change the equations of motion $\delta S = \delta \int \mathcal{L} = 0$. The Noether charges $\{Q^a\}$ for a given symmetry give a canonical representation for the generators of the symmetry group. The characteristic values or spectra, which may be referred to as possible physical realizations of the Noether charges $\{Q^a\}$, correspond to the irreducible representations of the symmetry group.

C.3 Symmetry breaking/violation

Certain internal (non spacetime) symmetries are broken by the observation that “elementary” particles come with different masses. A natural way to characterize this symmetry breaking is through a procedure known as dynamical or “sponta-

²Section F.4

neous” (implicit in general) symmetry breaking. In this procedure the physical system around its ground configuration (lowest energy configuration) is seen to have evolved from a more symmetric system at high energy/temperature configurations (ie. high kinetic energy, referring to a situation where the kinetic terms are dominating in the Lagrangian). As the system evolves to lower energy configurations (a situation where the interaction or potential energy terms are dominating) it has more than one local minimum energy configuration to randomly/spontaneously choose from. The space of all configurations with a given local minimum of energy is known as a vacuum or a vacuum manifold. The local extrema may be obtained by solving

$$\begin{aligned} \frac{\partial H}{\partial\{\partial_0\varphi\}} = 0, \quad \frac{\partial H}{\partial\{\varphi\}} = 0, \\ H = \int d^3x \mathcal{H} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial\{\partial_0\varphi\}} \{\partial_0\varphi\} - \mathcal{L} \right), \end{aligned} \quad (\text{C.3.1})$$

where $\{\varphi\}$ is the collection of all fields involved.

In one case all fields take zero values in the vacuum and in this case the field theory around this vacuum retains the original symmetry and the vacuum is said to be invariant under the symmetry. In the other case, one or more of the fields assume non-zero values in the vacuum and consequently the field theory around the vacuum cannot retain all of the original symmetry and the symmetry is said to have been spontaneously broken by this supposedly spontaneous or random choice of the vacuum.

There is also empirical (explicit in general) symmetry breaking which involves the introduction of noninvariant terms into the Lagrangian in order that theoretical results (eg. calculated interaction amplitudes) agree with experimental results of certain processes observed to violate the symmetry.

C.4 Action/on-shell symmetries

At the level of the action, any two theories with the same number of degrees of freedom (dofs) are equivalent in that they can be related by an invertible transformation

$$\{\varphi_1\} \rightarrow \{\varphi_2\} = g_{12}(\{\varphi_1\}), \quad S_1[\{\varphi_1\}] \rightarrow S_2[\{\varphi_2\}] = f_{12}[\varphi_1, S_1[\varphi_1], \delta_{\varphi_1} S_1[\varphi_1], \dots].$$

$$\begin{aligned} S_1[\{\varphi_1\}] &= \int d\mu(x) \mathcal{L}_1(x, \{\varphi_1\}, \partial\{\varphi_1\}, \dots), \\ S_2[\{\varphi_2\}] &= \int d\mu(x) \mathcal{L}_2(x, \{\varphi_2\}, \partial\{\varphi_2\}, \dots). \end{aligned} \quad (\text{C.4.1})$$

However the equations of motion

$$\frac{\delta S_1[\{\varphi_1\}]}{\delta\{\varphi_1\}(x)} = 0, \quad \frac{\delta S_2[\{\varphi_2\}]}{\delta\{\varphi_2\}(x)} = 0 \quad (\text{C.4.2})$$

may not be invariant under the transformation (that is the two solution spaces are not isomorphic). Thus the space of all action equivalent theories $T = \{S_i[\{\varphi_i\}]\}$ for a given number of degrees of freedom has an action intertheory symmetry group \mathcal{G} that interconnects the different theories in T . The usual symmetry groups of physics are “fixed points” of T .

$$\begin{aligned} \{\varphi_1\} \rightarrow \{\varphi_2\} &= g_{12}(\{\varphi_1\}), \\ S_1[\{\varphi_1\}] \rightarrow S_2[\{\varphi_2\}] &= \alpha S_1[a\{\varphi_1\} + b] + \beta \\ \Rightarrow \frac{\delta S_2[\{\varphi_2\}]}{\delta\{\varphi_2\}(x)} &= \alpha \frac{\delta\{\varphi_1\}(y)}{\delta g_{12}(\{\varphi_1\}(x))} \frac{\delta S_1[a\{\varphi_1\} + b]}{\delta\{\varphi_1\}(y)} = 0. \end{aligned} \quad (\text{C.4.3})$$

where α, β, a, b are constants. That is, they map an action to one that is similar to itself and for these special cases, the equations of motion are interrelated even though the symmetry group of the equations of motion can be larger than that of the action. The duality symmetries of string theory arise as special cases of $\alpha, a \neq 1$ and $\beta, b = 0$.

C.5 Noether's theorem and Ward-Takahashi identities

For simplicity we consider an action with at most first derivatives but the discussion can be extended to the case with any number of higher derivatives.

Consider a physical system described by a particular configuration (trajectory or path)

$\phi \in \{f : \mathcal{D} \subseteq \mathbb{R}^{d+1} \rightarrow \mathcal{E} \subseteq \mathcal{M} = \mathbb{C}^N\}$. Assume that the dynamics of the system is determined by a least action principle with an action $S[f] = \int_{\mathcal{D}} d\mu \mathcal{L}(x, f(x), \partial f(x))$. That is, among all the possible configurations f marked by any given *bound* $\partial\mathcal{E} = f(\partial\mathcal{D})$ (ie. $\delta f|_{f \in \partial\mathcal{E}} \equiv \delta f(x)|_{x \in \partial\mathcal{D}} = 0$), the classical physical configuration(s) is (are) the one(s) for which the action is extremized $\delta S[f]|_{f=\phi} = 0$. Therefore physically the Euler-Lagrange equations describe the only classically possible dependence(s) of ϕ on $x \in \mathcal{D}$ for any given boundary $\partial\mathcal{E} = \phi(\partial\mathcal{D})$.

$$\begin{aligned} \delta S[f] &= \int_{\mathcal{D}} d^n x \left\{ \partial_\mu (\delta f \frac{\partial \mathcal{L}}{\partial \partial_\mu f}) + \delta f \left(\frac{\partial \mathcal{L}}{\partial f} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu f} \right) \right\} \\ &= \int_{x \in \partial\mathcal{D}} dS_\mu(u) \delta f(x(u)) \frac{\partial \mathcal{L}}{\partial \partial_\mu f}(x(u)) + \int_{x \in \mathcal{D}} d^n x \delta f(x) \left(\frac{\partial \mathcal{L}}{\partial f}(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu f}(x) \right), \\ dS_\mu(u) &= \varepsilon_{\mu\nu_1 \dots \nu_{D-1}} \frac{\partial(x^{\nu_1}, \dots, x^{\nu_{D-1}})}{\partial(u^1, \dots, u^{D-1})} d^{D-1}u. \end{aligned} \tag{C.5.1}$$

Therefore $\delta S[f]|_{f=\phi} = 0$, $\delta f(x)|_{x \in \partial\mathcal{D}} = 0$ implies that

$$\frac{\partial \mathcal{L}}{\partial \phi}(x) - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}(x) = 0 \quad \forall x \in \mathcal{D} \tag{C.5.2}$$

or simply

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0. \tag{C.5.3}$$

Now different physical observers describe the behavior of the system with different points of view, which range from the use of different coordinates (labels or parameters) $x \mapsto y$ and/or different integration domains $\mathcal{D} \rightarrow \mathcal{D}'$ to reordering and/or

rescaling/translating of the components of the field variable ϕ ($\phi(x) \mapsto \phi'(x')$) and of the Lagrangian \mathcal{L} . According to the **theory (or principle) of relativity**, these observers should still use the same least action principle and hence the same equations of motion, among other things, to describe the behavior of ϕ . That is, $\delta S'[f]_{f=\phi'} = 0$ implies that

$$\begin{aligned} \frac{\partial \mathcal{L}'}{\partial \phi'}(x', \dots) - \partial'_\mu \frac{\partial \mathcal{L}'}{\partial \partial'_\mu \phi'}(x', \dots) &= 0 \quad \forall x' \in \mathcal{D}', \\ (\partial_\mu f)'(x') &= \frac{\partial}{\partial x^{\mu'}} f'(x') \equiv \partial'_\mu f'(x'). \end{aligned} \tag{C.5.4}$$

One can check that for smooth transformations (ie. smoothly related observers), the difference between \mathcal{D} and \mathcal{D}' can be fully specified through a change

in the integration measure of the action.

$$\begin{aligned}
S[\phi, \mathcal{D}] &= \int_{\mathcal{D}} d^n x \mathcal{L}(x, \phi(x), \partial\phi(x)), \\
x \rightarrow x' &= x + \delta x, \\
\phi(x) \rightarrow \phi'(x') &\equiv M(x, \phi(x), \partial\phi(x), \dots) = \phi'(x') - \phi(x') + \phi(x') \equiv \bar{\delta}\phi(x') + \phi(x') \\
&\Rightarrow \quad \bar{\delta}\phi(x) = \phi'(x) - \phi(x) = M(x - \delta x, \phi(x - \delta x), \partial\phi(x - \delta x), \dots) - \phi(x), \\
d^n x \rightarrow \det \frac{\partial(x + \delta x)}{\partial x} d^n x &= \det(\mathbb{I} + \partial\delta x) d^n x \approx (1 + \text{Tr}(\partial\delta x)) d^n x = (1 + \partial_\mu \delta x^\mu) d^n x, \\
\delta\mathcal{L} &= \bar{\delta}\mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \bar{\delta}\phi \frac{\partial\mathcal{L}}{\partial\phi} + \bar{\delta}\partial_\mu \phi \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = \bar{\delta}\mathcal{L} + \delta x^\mu \partial_\mu \mathcal{L} + \bar{\delta}\phi \frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu \bar{\delta}\phi \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}, \\
&= \bar{\delta}\mathcal{L} - \partial_\mu \delta x^\mu \mathcal{L} + \partial_\mu (\delta x^\mu \mathcal{L} + \bar{\delta}\phi \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}) + \bar{\delta}\phi (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}), \\
\delta S[\phi, \mathcal{D}] &:= S'[\phi', \mathcal{D}'] - S[\phi, \mathcal{D}] \\
&= \int_{\mathcal{D}'} d^n x' \mathcal{L}'(x', \phi'(x'), \partial'\phi'(x')) - \int_{\mathcal{D}} d^n x \mathcal{L}(x, \phi(x), \partial\phi(x)) \\
&= \int_{\mathcal{D}} (\delta d^n x \mathcal{L} + d^n x \delta\mathcal{L}) \approx \int_{\mathcal{D}} \{ (1 + \partial_\mu \delta x^\mu) d^n x \mathcal{L} + d^n x \delta\mathcal{L} \} \\
&= \int_{\mathcal{D}} d^n x \{ \bar{\delta}\mathcal{L} + \partial_\mu (\delta x^\mu \mathcal{L} + \bar{\delta}\phi \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}) + \bar{\delta}\phi (\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi}) \} \\
&= \int_{\mathcal{D}} d^n x \{ \bar{\delta}\mathcal{L} + \partial_\mu J^\mu + \bar{\delta}\phi E \} = \int_{\mathcal{D}} d^n x (\bar{\delta}\mathcal{L} + \partial_\mu \alpha^\mu) = 0 \quad \forall \mathcal{D}, \\
\alpha^\mu &= \delta x^\mu \mathcal{L} + \bar{\delta}\phi \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi} + f_1^\mu + f_2^\mu, \quad \partial_\mu f_1^\mu = 0, \quad \int_{\partial\mathcal{D}} dS_\mu f_2^\mu = 0.
\end{aligned} \tag{C.5.5}$$

In particular, for domains that can be continuously shrunk to a point, one has that $\bar{\delta}\mathcal{L} + \partial_\mu \alpha^\mu = 0$ at every point. Here, $\bar{\delta}$ is the functional variation $\bar{\delta}F(u) = (F' - F)(u) = F'(u) - F(u)$; ie. it is the part of the variation that is not due to the “visible” arguments for the function involved.

$$\begin{aligned}
\bar{\delta}\phi(x) &= \phi'(x) - \phi(x), \\
\bar{\delta}\mathcal{L}(x, \phi(x), \partial\phi(x)) &= \mathcal{L}'(x, \phi(x), \partial\phi(x)) - \mathcal{L}(x, \phi(x), \partial\phi(x)). \tag{C.5.6}
\end{aligned}$$

For any system of observers whose functional forms of the Lagrangian \mathcal{L} can differ

only by a total divergence $\partial_\mu \beta^\mu$ there is a conserved current $J^\mu = \alpha^\mu + \beta^\mu$;

$$\bar{\delta} \mathcal{L} = \partial_\mu \beta^\mu \Rightarrow \delta S = \int \partial_\mu J^\mu = \int \partial_\mu (\alpha^\mu + \beta^\mu) = 0. \quad (\text{C.5.7})$$

In the case that involves a system with a dynamic domain \mathcal{D} such as a smoothly expanding universe, the dynamics of \mathcal{D} can be accounted for by introducing a dynamical metric field $g_{\mu\nu}$ whose dynamics is also determined by the Euler-Lagrange equations. In a more convenient form for other purposes, one may express the general variation of the action as

$$\delta S = \int d^D x \{ \tilde{\delta} \mathcal{L} + \partial_\mu (\delta x^\mu \mathcal{L}) \} = 0 \Rightarrow \tilde{\delta} \mathcal{L} + \partial_\mu (\delta x^\mu \mathcal{L}) = 0, \quad (\text{C.5.8})$$

where $\tilde{\delta} x = x \delta$, $\tilde{\delta} \partial = \partial \delta$.

In $D = 1 + 0$ dimensions for example,

$$\begin{aligned} \int d^D x \mathcal{L}(x, \phi(x), \partial \phi(x)) &\rightarrow \int d\lambda L(\lambda, q(\lambda), \dot{q}(\lambda)), \quad \partial_\mu \rightarrow \frac{d}{d\lambda}, \\ \alpha &= \delta \lambda L + \bar{\delta} q^i(\lambda) \frac{\partial L}{\partial \dot{q}^i}, \quad \dot{q} = \frac{dq}{d\lambda}. \\ \bar{\delta} L &= \frac{d}{d\lambda} \beta \Rightarrow \frac{d}{d\lambda} (\alpha + \beta) = 0. \end{aligned} \quad (\text{C.5.9})$$

Analogously in quantum field theory where we have the quantum measure $d\mu_\phi$ involving a sum over all possible ϕ configurations (or "paths") in \mathcal{D} , an amplitude G for a physical process is given by the expectation value $\langle F(\{\phi\}) \rangle$, wrt the quantum measure, of a homogeneous polynomial $F(\{\phi\})$ of the fields. The invariance of G

may be expressed as follows:

$$\begin{aligned}
G &= \langle F(\{\phi\}) \rangle = \int_{\varphi \in \mathcal{M}/\mathcal{D}} d\mu_\varphi F(\{\varphi\}) = \int_{\varphi \in \mathcal{M}/\mathcal{D}} D\varphi F(\{\varphi\}) e^{\frac{i}{\hbar} S[\varphi, \mathcal{D}]} \\
&\rightarrow \int_{\varphi \in \mathcal{M}/\mathcal{D}} D'\varphi' F'(\{\varphi'\}) e^{\frac{i}{\hbar} S'[\varphi', \mathcal{D}']} \\
&= \int_{\varphi \in \mathcal{M}/\mathcal{D}} D\varphi \det\left(\frac{D'\varphi'}{D\varphi}\right) (F(\{\varphi\}) + \delta F(\{\varphi\})) e^{\frac{i}{\hbar} (S[\varphi, \mathcal{D}] + \delta S[\varphi, \mathcal{D}])} \\
&\approx \int_{\varphi \in \mathcal{M}/\mathcal{D}} D\varphi e^{\text{tr} \frac{\tilde{\delta}(\tilde{\delta}\varphi)}{\tilde{\delta}(\varphi)}} (F(\{\varphi\}) + \delta F(\{\varphi\})) e^{\frac{i}{\hbar} (S[\varphi, \mathcal{D}] + \delta S[\varphi, \mathcal{D}])}, \\
\delta G &\approx \int_{\varphi \in \mathcal{M}/\mathcal{D}} D\varphi \left[\delta F(\{\varphi\}) + F(\{\varphi\}) \left(\frac{i}{\hbar} \delta S[\varphi, \mathcal{D}] + \text{tr} \frac{\tilde{\delta}(\tilde{\delta}\varphi)}{\tilde{\delta}(\varphi)} \right) \right] e^{\frac{i}{\hbar} S[\varphi, \mathcal{D}]} = 0 \\
\langle \delta F(\{\phi\}) \rangle + \langle F(\{\phi\}) \left(\frac{i}{\hbar} \int_{\mathcal{D}} \partial_\mu J^\mu + \text{tr} \frac{\tilde{\delta}(\tilde{\delta}\phi)}{\tilde{\delta}(\phi)} \right) \rangle &= 0. \tag{C.5.10}
\end{aligned}$$

The relation (C.5.10) is the quantum analog of the classical Noether's theorem and is known as Ward-Takahashi identity.

An expression for the trace tr is

$$\begin{aligned}
\text{tr} \frac{\tilde{\delta}(\tilde{\delta}\phi)}{\tilde{\delta}(\phi)} &= \sum_{xy} \sum_{\xi} \xi^*(x) \frac{\tilde{\delta}\tilde{\delta}\phi(y)}{\tilde{\delta}\phi(x)} \xi(y), \quad \frac{\tilde{\delta}\phi(y)}{\tilde{\delta}\phi(x)} = \delta^n(x-y). \\
\sum_{\xi} \xi^*(x) \xi(y) &= \delta^n(x-y), \quad \sum_x \xi'^*(x) \xi(x) = \delta_{\xi'\xi}. \\
\phi(x) &= \sum_{\xi} \tilde{\phi}_\xi \xi(x). \tag{C.5.11}
\end{aligned}$$

For the case of global spacetime translations where $\tilde{\delta}\phi = -b^\mu \partial_\mu \phi$,

$$\text{tr} \frac{\tilde{\delta}(\tilde{\delta}\phi)}{\tilde{\delta}(\phi)} = - \sum_{\xi} \int_{\mathcal{D}} d^n x b^\mu \xi^*(x) \partial_\mu \xi(x), \quad \xi|_{\partial\mathcal{D}} = 0. \tag{C.5.12}$$

This vanishes if the domain \mathcal{D} is symmetric as we have an odd integrand.

In general,

$$\begin{aligned} \text{tr} \frac{\tilde{\delta}(\tilde{\delta}\phi)}{\tilde{\delta}(\phi)} &= \text{Tr}(P \cdot \frac{\tilde{\delta}(\tilde{\delta}\phi)}{\tilde{\delta}(\phi)}) = \sum_{xy} P(x, y) \frac{\tilde{\delta}(\tilde{\delta}\phi(x))}{\tilde{\delta}(\phi(y))} \\ P(x, y) &= \sum_{\xi\xi'} \xi^*(x) g^{\xi\xi'} \xi'(y), \quad g^{\xi\xi'} = g_{\xi\xi'}^{-1}, \quad g_{\xi\xi'} = \sum_x \xi^*(x) \xi'(x). \end{aligned}$$

where P is a projection that may be constructed from a complete set of functions which can span the solution space of the classical trajectory given by $\frac{\tilde{\delta}S[\phi]}{\tilde{\delta}\phi} = 0$.

In order to obtain an analogous situation to the classical case, we need to define

$$Z[\mathcal{C}] = \int_{\varphi \in \mathcal{C}(\mathcal{D})} D\varphi e^{\frac{i}{\hbar} S[\varphi, \mathcal{D}]}, \quad (\text{C.5.13})$$

where $\mathcal{C}(\mathcal{D}) = \{f; f : \mathcal{D} \subseteq \mathbb{R}^{d+1} \rightarrow \mathcal{M}, f_1(\partial\mathcal{D}) = f_2(\partial\mathcal{D}) \forall f_1, f_2\} \subset \mathcal{M}/\mathcal{D}$ is the configurations space.

C.5.1 Dynamics using differential forms

In terms of differential forms the terms of matter and fermion actions are

$$\begin{aligned} i \int_{\mathcal{D}} d^D x \bar{\psi} \gamma^\mu (\partial_\mu - ieA_\mu) \psi &= i \int_{\mathcal{D}} \bar{\psi} \gamma * (d\psi - ieA\psi), \quad \gamma = \gamma_\mu dx^\mu, \\ \frac{1}{2} \int_{\mathcal{D}} d^D x \partial_\mu \phi^\dagger \partial^\mu \phi &= \frac{1}{2} \int_{\mathcal{D}} d\phi^\dagger * d\phi, \\ \frac{1}{4} \int_{\mathcal{D}} F_{\mu\nu} F^{\mu\nu} &= \frac{1}{4} \int_{\mathcal{D}} F * F, \quad F = dA, \quad A = A_\mu dx^\mu. \end{aligned} \quad (\text{C.5.14})$$

Infinitesimal transformations are given by

$$\delta f = i_{\delta x} df + d(i_{\delta x} f) \quad \forall f = f_{\mu_1 \dots \mu_p} dx^{\mu_1 \dots \mu_p},$$

where

$$i_{\delta x} f = f_{\mu_1 \dots \mu_p} \delta x^{[\mu_1} dx^{\mu_2 \dots \mu_p]}$$

and the $*$ and d operations are

$$\begin{aligned} *f &= f_{\mu_1 \dots \mu_p} \varepsilon^{\mu_1 \dots \mu_p}_{\mu_{p+1} \dots \mu_D} dx^{\mu_{p+1} \dots \mu_D} \equiv f_{\mu_{p+1} \dots \mu_D}^* dx^{\mu_{p+1} \dots \mu_D}, \\ df &= \partial_{[\mu} f_{\mu_1 \dots \mu_p]} dx^{\mu \mu_1 \dots \mu_p}, \quad f_{\mu_1 \dots \mu_p} = f_{\mu_1 \dots \mu_p}(x). \end{aligned}$$

Example of transformation:

$$\begin{aligned} \delta(F * F) &= 2(\delta F) * F = 2 d(i_{\delta x} F) * F \\ &= 2 d((i_{\delta x} F) * F) - 2 (i_{\delta x} F)(d * F). \end{aligned} \tag{C.5.15}$$

The Lagrangian in general is given by

$$\mathcal{L} = \mathcal{L}(f, *f, df, *df, d * f, *d * f, \dots). \tag{C.5.16}$$

C.6 Faddeev-Popov gauge gixing method

The definition of gauge fields in the classical or low energy action involves irrelevant degrees of freedom (in the form of invariance under gauge transformations) that must be eliminated (through gauge fixing: ie. by imposing any constraint that breaks the gauge symmetry completely) when attempting to obtain physical solutions to the equations of motion resulting from the least action principle. Similarly this elimination has to be done when attempting to quantize (ie. extend to all possible energies) the classical gauge theory since quantization involves summing over contributions from relevant degrees of freedom only. The Faddeev-Popov gauge fixing method is one method of implementing gauge fixing in quantum theory.

The convenient (i.e. Euclidean) measure $d\mu(\mathcal{A})$ and action $S[A]$, in the parti-

tion function

$$Z_1[J] = \int DA e^{-S[A]+JA} \equiv \int d\mu(\mathcal{A}) e^{-S[A]+JA},$$

$$JA = \int d^Dx \operatorname{Tr}(J_\mu(x)A^\mu(x)) = \int d^Dx J_\mu^a(x)A^{a\mu}(x),$$

are invariant under the gauge transformation

$$A \rightarrow A^g = g^{-1}Ag + g^{-1}dg = A + g^{-1}Dg, \quad Dg = dg + [A, g]. \quad (\text{C.6.1})$$

Here \mathcal{A} is the \mathcal{G} -bundle $\mathcal{A} = \{A\} \simeq \mathcal{A}/\mathcal{G} \times \mathcal{G}$ of all gauge equivalent potentials $[A]$. This means that

$$d\mu(\mathcal{A}) = d\mu(\mathcal{A}/\mathcal{G}) d\mu(\mathcal{G}),$$

$$\mathcal{A}/\mathcal{G} = \{[A], A \in \mathcal{A}\}, \quad [A] = \{g^{-1}Ag + g^{-1}dg, g \in \mathcal{G}\}.$$

Therefore there is over counting in the partition function (C.6.1) as it includes integration over the group \mathcal{G} under which the integrand is invariant at $J = 0$ which is the most important point in the definition and applications of the partition function to averaging of quantities

$\langle Q(A) \rangle = \frac{1}{Z[0]} Q(\frac{\delta Z[J]}{\delta J}|_{J=0})$ as well as in evaluating effective actions. One simply needs to divide $Z[0]$ by the volume of the group $\int d\mu(\mathcal{G})$ in order to remove the redundant factor and so the corrected partition function is

$$Z[0] = \int d\mu(\mathcal{A}/\mathcal{G}) e^{-S[A]} \quad (\text{C.6.2})$$

now having the less convenient measure $d\mu(\mathcal{A}/\mathcal{G})$. The Faddeev-Popov method involves rewriting $Z[0]$ in terms of the more convenient measure $d\mu(\mathcal{A})$ by choosing a path (a section or gauge fixing condition $G[A] - h = 0$, $\frac{\partial h}{\partial A} = 0$) other than $g = \text{const}$ through the bundle $\{[A]\} \times \mathcal{G} \simeq \{([A], g)\}$. The path should cut through

any given fiber $[A]$ only once : ie $G[A^{g'}] \neq G[A^g]$ unless $g = g'$. We insert the identity

$$1 = \int_{g \in \mathcal{G}} DG[A^g] \delta(G[A^g] - h) = \int_{g \in \mathcal{G}} d\mu(\mathcal{G}) \left| \det \frac{\delta G[A^g]}{\delta g} \right| \delta(G[A^g] - h)$$

into the integral expression for $Z[0]$.

$$\begin{aligned} Z[0] &= \int d\mu(\mathcal{A}/\mathcal{G}) e^{-S[A]} \\ &= \int d\mu(\mathcal{A}/\mathcal{G}) DG[A^g] \delta(G[A^g] - h) e^{-S[A]} \\ &= \int d\mu(\mathcal{A}/\mathcal{G}) d\mu(\mathcal{G}) \left| \det \frac{\delta G[A^g]}{\delta g} \right| \delta(G[A^g] - h) e^{-S[A]} \\ &= \int d\mu(\mathcal{A}) \left| \det \frac{\delta G[A^g]}{\delta g} \right| \delta(G[A^g] - h) e^{-S[A]} \\ &= \int d\mu(\mathcal{A}) \left| \det \frac{\delta G[A^g]}{\delta g} \right| \delta(G[A^g] - h) e^{-S[A^g]} \\ &= \int d\mu(\mathcal{A}) \left| \det \frac{\delta G[A^g]}{\delta g} \right|_{g=1} \delta(G[A] - h) e^{-S[A]} \\ &= \int d\mu(\mathcal{A}) \left| \det \frac{\delta(G[A^g] - h)}{\delta g} \right|_{g=1} \delta(G[A] - h) e^{-S[A]} \\ &= \int d\mu(\mathcal{A}) \left| \det \frac{\delta(G[A + g^{-1}Dg] - h)}{\delta g} \right|_{g=1} \delta(G[A] - h) e^{-S[A]}. \end{aligned}$$

But

$$\begin{aligned} G[A + g^{-1}Dg] - h &= G[A] - h + \frac{\delta(g^{-1}D_\mu g)}{\delta g} \frac{\delta G[A]}{\delta A_\mu} \\ &+ \frac{1}{2!} \frac{\delta(g^{-1}D_\mu g g^{-1}D_\nu g)}{\delta g} \frac{\delta^2 G[A]}{\delta A_\mu \delta A_\nu} + \dots \\ &= G[A] - h + \frac{\delta(g^{-1}D_\mu g)}{\delta g} \frac{\delta G[A]}{\delta A_\mu} + g^{-1}D_\mu g \frac{\delta(g^{-1}D_\nu g)}{\delta g} \frac{\delta^2 G[A]}{\delta A_\mu \delta A_\nu} + \dots \end{aligned} \tag{C.6.3}$$

and so only the first derivative term in the expansion can survive in the presence

of $\delta(G[A] - h)$ and upon setting $g = 1$.

$$\begin{aligned}
Z[0] &= \int d\mu(\mathcal{A}) \left| \det \frac{\delta(g^{-1}D_\mu g)}{\delta g} \frac{\delta G[A]}{\delta A_\mu} \right|_{g=1} \delta(G[A] - h) e^{-S[A]} \\
&= \int d\mu(\mathcal{A}) \left| \det D_\mu \frac{\delta G[A]}{\delta A_\mu} \right| \delta(G[A] - h) e^{-S[A]} \\
&= \int d\mu(\mathcal{A}) DcD\bar{c} \delta(G[A] - h) e^{-S[A] - \text{Tr}(\bar{c}D_\mu(\frac{\delta G[A]}{\delta A_\mu})c)} \\
&= \int d\mu(\mathcal{A}) DcD\bar{c} \delta(G[A] - h) e^{-S[A] - \bar{c}^\alpha D_\mu(\frac{\delta G^\alpha[A]}{\delta A_\mu^b})c^b} \quad \forall h, \\
c^a c^b &= -c^b c^a, \quad c^a \bar{c}^b = -\bar{c}^b c^a, \quad \bar{c}^a \bar{c}^b = -\bar{c}^b \bar{c}^a
\end{aligned} \tag{C.6.4}$$

where integration over spacetime is understood.

Since h is arbitrary we can use equivalently

$$Z[0] = \int d\mu(\mathcal{A}) DcD\bar{c} Dh F[h] \delta(G[A] - h) e^{-S[A] - \bar{c}^\alpha D_\mu(\frac{\delta G^\alpha[A]}{\delta A_\mu^b})c^b}. \tag{C.6.5}$$

In particular $F[h] = e^{-\frac{1}{2\alpha}\text{Tr}h^2} = e^{-\frac{1}{2\alpha}h^a h^a}$ gives

$$\begin{aligned}
Z[0] &= \int d\mu(\mathcal{A}) DcD\bar{c} e^{-S[A] - \frac{1}{2\alpha}G^\alpha[A]G^\alpha[A] - \bar{c}^\alpha D_\mu(\frac{\delta G^\alpha[A]}{\delta A_\mu^b})c^b}, \\
Z[J] &= \int d\mu(\mathcal{A}) DcD\bar{c} e^{-S[A] - \frac{1}{2\alpha}G^\alpha[A]G^\alpha[A] - \bar{c}^\alpha D_\mu(\frac{\delta G^\alpha[A]}{\delta A_\mu^b})c^b + AJ}.
\end{aligned} \tag{C.6.6}$$

Appendix D

Geometry and Symmetries

D.1 Manifold structure

A real D -dimensional manifold \mathcal{M}_D is a collection of differentiable invertible maps from an arbitrarily given space \mathcal{M} onto \mathbb{R}^D . That is $\mathcal{M}_D \simeq \frac{\mathcal{M}}{\mathbb{R}^D} \subset \{\varphi : \mathbb{R}^D \rightarrow \mathcal{M}\}$. One may ignore the dimension label D when it is understood and write simply $\mathcal{M} \simeq \frac{\mathcal{M}}{\mathbb{R}^D}$. The function space over \mathcal{M} is $\mathcal{F}(\mathcal{M}) = \mathbb{C}^N/\mathcal{M}$ and the tangent vector bundle $T(\mathcal{M})$ and dual tangent vector bundle $T^*(\mathcal{M})$ are given by

$$\begin{aligned} T(\mathcal{M}) &= \{t : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{M}), \quad t \circ (fg) = t \circ f \, g + f \, t \circ g, \\ &\quad t \circ (f + g) = t \circ f + t \circ g\}, \\ T^*(\mathcal{M}) &= \{t^* : T(\mathcal{M}) \rightarrow \mathbb{C}^N, \quad t^* \circ (t_1 + t_2) = t^* \circ t_1 + t^* \circ t_2\}. \end{aligned} \tag{D.1.1}$$

and their fields (or sections) are $T(\mathcal{M})/\mathcal{M}$ and $T^*(\mathcal{M})/\mathcal{M}$ respectively. One can equally construct tensor fields and dual tensor fields which are sections

$\mathcal{T}(\mathcal{M})/\mathcal{M}$, $\mathcal{T}^*(\mathcal{M})/\mathcal{M}$ of the tensor and dual tensor algebras

$$\begin{aligned}\mathcal{T}(\mathcal{M}) &= \mathbb{C}^N \oplus \bigoplus_{k=1}^{\infty} T(\mathcal{M})^{\otimes k} = \mathbb{C}^N \oplus T(\mathcal{M}) \otimes \bigoplus_{k=0}^{\infty} T(\mathcal{M})^{\otimes k}, \\ \mathcal{T}^*(\mathcal{M}) &= \mathbb{C}^N \oplus \bigoplus_{k=1}^{\infty} T^*(\mathcal{M})^{\otimes k} = \mathbb{C}^N \oplus T^*(\mathcal{M}) \otimes \bigoplus_{k=0}^{\infty} T^*(\mathcal{M})^{\otimes k}.\end{aligned}\tag{D.1.2}$$

D.2 Relativity or Observer Symmetry

According to a universal observer, the dynamics of a physical system may be described by a “path”¹ $\Gamma : X \rightarrow E$; ie. $\Gamma \in E/X = \{\psi : X \rightarrow E\}$, in the universal (experimental, operational or investigational) space $E = \{f : \mathcal{A}/X \rightarrow \mathcal{A}/X\}$ of space and time (spacetime) $X \simeq \mathbb{R}^D$, where \mathcal{A} is any suitable algebra. However a local or limited observer (sees the path as $\psi : \mathcal{D} \subseteq X \rightarrow E$, $\psi(\mathcal{D}) = \Gamma(X)$) can only access an observer domain \mathcal{D} of spacetime that serves as a parameter space and is different for different local observers although the physical system (ie. its “path” in the universal space), and of course the universal space, look the same according to the different observers. We assume that the local observers are careful observers, where a careful observer is one that is aware that he needs to make several observations using as many different frames of reference \mathcal{D} as possible before attempting to make any general conclusions about the behavior of the physical system.

According to the universal observer, it is therefore natural to regard each local observer $O \in G$ as merely a member of the set of structure preserving transformations $G = \{g : E/X \rightarrow E/X, (g \circ \Gamma)(X) = \Gamma(X)\} \subset E$ on spacetime based

¹A “path in the universal space E ” is a “configuration in spacetime X ”.

systems or paths, where the structure to be preserved is the “path” Γ of the physical system in the universal space E of X .

Therefore $\Gamma \in \mathcal{C} = \{c = G \circ f \equiv [f], \quad f \in E\} \subset E/G$ since

$g \circ G := \{gh; \quad h \in G\} = G \quad \forall g \in G$, where \mathcal{C} is the space of all symmetric path configurations.

For simplicity, we will make the restriction $E \simeq \mathbb{C}^{N_1} \times \mathbb{C}^{N_2} \times \dots \times \mathbb{C}^{N_n}$. Now two local observers O, O' define the path Γ as $\psi : \mathcal{D} \subseteq X \rightarrow E$, $\psi(\mathcal{D}) = \Gamma(X)$ and $\psi' : \mathcal{D}' \subseteq X \rightarrow E$, $\psi'(\mathcal{D}') = \Gamma(X)$ respectively.

Which means that $\psi'(\mathcal{D}') = \psi(\mathcal{D})$. From experimenting with local observer re-labeling properties of a function, components of a vector field, components of a spinor field, components of a tensor field (infinitesimal polygons), ... in ordinary spaces one finds that respectively,

$$\begin{aligned}
\alpha'(u') &= \alpha(u), \quad \alpha : X \rightarrow \mathbb{R}, \\
\alpha'(u') &= e^{i\theta(u, u', \frac{\partial u'}{\partial u}, \dots)} \alpha(u), \quad \alpha : X \rightarrow \mathbb{C}, \quad \theta(u, u', \frac{\partial u'}{\partial u}, \dots) \in \mathbb{R}, \\
\partial'_\mu &= \frac{\partial u^\nu}{\partial u'^\mu} \partial_\nu, \\
\alpha'^\mu(u') &= \frac{\partial u'^\mu}{\partial u^\nu} \alpha^\nu(u), \\
\alpha'^a(u') &= S^a_b(u, u', \frac{\partial u'}{\partial u}) \alpha^b(u), \\
&\dots
\end{aligned} \tag{D.2.1}$$

Therefore “pointwise”, $\psi'(\mathcal{D}') = \psi(\mathcal{D})$ may be written as

$$\begin{aligned}
\psi'_{a_1 a_2 \dots a_n}(u') &= R_{a_1}^{b_1}(u, u', \frac{\partial u'}{\partial u}, \dots) R_{a_2}^{b_2}(u, u', \frac{\partial u'}{\partial u}, \dots) \dots R_{a_n}^{b_n}(u, u', \frac{\partial u'}{\partial u}, \dots) \psi_{b_1 b_2 \dots b_n}(u) \\
\forall u' \in \mathcal{D}', \quad u \in \mathcal{D},
\end{aligned} \tag{D.2.2}$$

a more general form of which being

$$\begin{aligned} \psi'_{a_1 a_2 \dots a_n}(u') &= R_{a_1 a_2 \dots a_n}{}^{b_1 b_2 \dots b_n}(u, u', \frac{\partial u'}{\partial u}, \dots) \psi_{b_1 b_2 \dots b_n}(u) + b_{a_1 a_2 \dots a_n}(u, u', \frac{\partial u'}{\partial u}, \dots). \\ \forall u' \in \mathcal{D}', u \in \mathcal{D}. \end{aligned} \tag{D.2.3}$$

and yet a more general form being

$$\begin{aligned} \psi'_{a_1 a_2 \dots a_n}(u') &= R_{a_1 a_2 \dots a_n}(u, u', \partial u', \dots, \psi(u), \partial \psi(u), \dots). \\ \forall u' \in \mathcal{D}', u \in \mathcal{D}. \\ \psi'(u') &= R(u, u', \partial u', \dots, \psi(u), \partial \psi(u), \dots). \end{aligned} \tag{D.2.4}$$

In general ψ may be expanded as a sum of products of elementary functions $\{e\}$:

$$\psi(u) = \sum_k \tilde{\psi}_{i_1 i_2 \dots i_k} e^{i_1}(u) e^{i_2}(u) \dots e^{i_k}(u). \tag{D.2.5}$$

Internal symmetries are those for which $u = u' \quad \forall u \in \mathcal{D}, u' \in \mathcal{D}'$. One notes here that the observer domains $\mathcal{D}, \mathcal{D}'$ may be specified through differentiable-invertible maps $\varphi : U \subset \mathcal{M} \rightarrow X, m \mapsto u$ and $\varphi' : U' \subset \mathcal{M} \rightarrow X, m' \mapsto u'$ (with $\varphi : U \cap U' \rightarrow \mathcal{D} \subseteq X, \varphi' : U \cap U' \rightarrow \mathcal{D}' \subseteq X$) so that u' and u correspond to the same point $m = \varphi^{-1}(u) = \varphi'^{-1}(u')$ in the intersection $U \cap U'$ on some abstract space \mathcal{M} , in which case any given complete collection $\{U_a\}, \bigcup_a U_a \supseteq \mathcal{M}$ of pre-observer domains is said to define a differentiable manifold \mathcal{M} over X meanwhile any corresponding appropriate choices $\{E(\mathcal{M}) = \{f : \mathcal{A}/\mathcal{M} \rightarrow \mathcal{A}/\mathcal{M}\}\}$ for the universal space E are fiber bundles $\{\pi : E \rightarrow \mathcal{M}\}$ over \mathcal{M} . The

(internal) transition relation among the various observers may be expressed thus

$$\begin{aligned}
u_a &= \varphi_a(m) = \varphi_a \circ \varphi_b^{-1} \circ \varphi_b(m) = \varphi_a \circ \varphi_b^{-1}(u_b) \\
&= \varphi_{ab}(u_b) = \varphi_{ab}(\varphi_{bc}(u_c)) = \varphi_{ab} \circ \varphi_{bc}(u_c) = \varphi_{ac}(u_c). \\
\varphi_{ac} \circ \varphi_{cb} &= \varphi_{ab}. \tag{D.2.6}
\end{aligned}$$

$$\begin{aligned}
\psi_a(\mathcal{D}_a) = \psi_b(\mathcal{D}_b) &\iff \psi_a \circ \varphi_a(U_a \cap U_b) = \psi_b \circ \varphi_b(U_a \cap U_b) \\
&\iff \psi_a = \psi_b \circ \varphi_{ba}, \quad \varphi_{ab} = \varphi_a \circ \varphi_b^{-1}. \tag{D.2.7}
\end{aligned}$$

The form of the representation function R in (D.2.4) is determined by consistency with observational facts. For example, in quantum theory E is a noncommutative space [as decided by observations] meanwhile X can also be noncommutative [as decided by observations]; then for the noncommutativity to be physical or observable, the underlying algebraic (eg. commutation) relations, or their functional form equivalently, need to be the same (just as the path $\Gamma(X)$ is) for each local observer and consequently the (local) observer relabeling or reparametrization tensor R needs to take on a form that can support this preservation of the algebraic relations on relabeling. The following section D.3 is an attempt at such transformations.

If the path Γ is defined by a least action principle

$$\begin{aligned}
S[\psi, \mathcal{D}] &= \int_{\mathcal{D}} d\mu(\mathcal{D}) \mathcal{L}(u, \psi(u), \partial\psi(u), \dots), \\
\frac{\delta S[\psi, \mathcal{D}]}{\delta\psi(u)} &= 0, \quad \delta\psi(u)|_{u \in \partial\mathcal{D}} = 0, \tag{D.2.8}
\end{aligned}$$

then $\psi'(\mathcal{D}') = \psi(\mathcal{D})$ requires that

$$\begin{aligned}
S'[\psi', \mathcal{D}'] &= \int_{\mathcal{D}'} d\mu(\mathcal{D}') \mathcal{L}'(u', \psi'(u'), \partial\psi'(u'), \dots), \\
\frac{\delta S'[\psi', \mathcal{D}']}{\delta\psi'(u')} &= 0, \quad \delta\psi'(u')|_{u' \in \partial\mathcal{D}'} = 0 \tag{D.2.9}
\end{aligned}$$

as well.

In the case where X is an algebra $\hat{X} = \{\hat{x}\}$ specified by commutation relations, one may first determine the spectra

$\sigma(\hat{x}^\mu) = \{\lambda^\mu \in \mathbb{C}; (\hat{x}^\mu - \lambda^\mu 1)^{-1} \nexists\} \quad \forall \mu$. Then the “functional” form of the relativistic path $\hat{\psi} : \hat{\mathcal{D}} \subseteq \hat{X} \rightarrow \hat{E}$, $\hat{\psi}'(\hat{\mathcal{D}}') = \hat{\psi}(\hat{\mathcal{D}})$ may be expressed as

$$\begin{aligned} \hat{\psi}(\hat{u}) &= \sum_k \tilde{\psi}_{i_1 \dots i_k} e^{i_1}(\hat{u}) e^{i_2}(\hat{u}) \dots e^{i_k}(\hat{u}), \\ e^i(\hat{u}) &= (e^i \circ \varphi)(\hat{x}) = \oint_{D(\sigma(\hat{x}))} (e^i \circ \varphi)(z) \prod_\mu \frac{dz^\mu}{z^\mu - \hat{x}^\mu}, \end{aligned} \quad (\text{D.2.10})$$

where one now has (index) reordering or permutation or braiding symmetry due to the noncommutativity and again this reordering must be consistent with the relation $\hat{\psi}'(\hat{\mathcal{D}}') = \hat{\psi}(\hat{\mathcal{D}})$.

D.3 Hopf symmetry transformations

The permutation group \mathcal{S}_n and braid group \mathcal{B}_n may be defined as follows:

$$\begin{aligned}
T_n &= \{1, t_{12}, t_{23}, \dots, t_{n-1 n}\} \equiv \{1, t_1, t_2, \dots, t_{n-1}\}, \\
t_k &= t_{k k+1} : V^{\otimes N} \rightarrow V^{\otimes N}, \quad v_1 \otimes \dots \otimes v_N \mapsto v_1 \otimes \dots \otimes t(v_k \otimes v_{k+1}) \otimes \dots \otimes v_N, \\
t &: V \otimes W \rightarrow W \otimes V, \\
B_n &= \{b_i \in T_n ; \quad b_i b_j = \delta_{ij} b_i^2 + b_j b_i b_j^{-1} \delta_{i \pm 1 j} + b_j b_i (1 - \delta_{ij} - \delta_{i \pm 1 j})\} \\
&= \{b_i \in T_n ; \quad [b_i, b_j] = b_j b_i (b_j b_i^{-1} - 1) \delta_{i \pm 1 j} = b_i b_j (1 - b_i b_j^{-1}) \delta_{j \pm 1 i}\} \\
\mathcal{B}_n &= \{g_i = u_1^{\circ n_1} \circ u_2^{\circ n_2} \circ \dots \circ u_n^{\circ n_i}, \quad n_1 + \dots + n_i = i, \quad n_r \in \mathbb{Z}; \\
&\quad (u_1, \dots, u_n) \in B_n^n \equiv B_n \times B_n^{n-1}\}. \\
S_n &= \{s_i \in T_n ; \quad s_i s_j = \delta_{ij} + (s_j s_i)^2 \delta_{i \pm 1 j} + s_j s_i (1 - \delta_{ij} - \delta_{i \pm 1 j})\} \\
\mathcal{S}_n &= \{g_i = u_1^{\circ n_1} \circ u_2^{\circ n_2} \circ \dots \circ u_n^{\circ n_i}, \quad n_1 + \dots + n_i = i, \quad n_r \in \{0, 1\}; \\
&\quad (u_1, \dots, u_n) \in S_n^n \equiv S_n \times S_n^{n-1}\}.
\end{aligned} \tag{D.3.1}$$

One may summarize the defining properties of a Hopf algebra $H = (A = \{a, b, \dots\}, F, \mu, \Delta, \eta, \varepsilon, S, \tau) \equiv (\text{Vector space, Field, product, coproduct, unit, counit, antipode, braiding})$ as follows.

- Unit:

$$\begin{aligned}
\eta_a &: F \rightarrow A, \quad \lambda \mapsto \lambda a. \quad \forall a \in A. \\
\eta &:= \eta_{1_A} : F \rightarrow A, \quad \lambda \mapsto \lambda 1_A.
\end{aligned} \tag{D.3.2}$$

- Product:

$$\mu : A \otimes A \rightarrow A.$$

(D.3.3)

- Coproduct

$$\Delta = \Delta^2 : A \rightarrow A \otimes A, \quad a \mapsto \Delta(a) = \sum_{\alpha\beta} C_{\alpha\beta} \alpha(a) \otimes \beta(a) \equiv a_{(1)} \otimes a_{(2)}$$

$$\equiv a_\alpha \otimes a^\alpha,$$

$$\Delta(a \otimes b) = a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)} \quad (\text{an alternative}).$$

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta), \quad \Delta(ab) = \Delta(a)\Delta(b).$$

$$\Delta^1 = id.$$

$$\Delta^3 := (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta,$$

$$\Delta^3(g) = g_{(1)} \otimes g_{(2)(1)} \otimes g_{(2)(2)} = g_{(1)(1)} \otimes g_{(1)(2)} \otimes g_{(2)} \equiv g_{(1)} \otimes g_{(2)} \otimes g_{(3)}.$$

$$\Delta^k : A \rightarrow A^{\otimes k} = A \otimes A^{\otimes(k-1)},$$

$$\Delta^{k+1} = ((id \otimes)^{i-1} \Delta (\otimes id)^{k-i}) \circ \Delta^k, \quad i = 1, 2, \dots, k.$$

$$\Delta^k(g) = g_{(1)} \otimes \dots \otimes g_{(i-1)} \otimes g_{(i)(1)} \otimes g_{(i+1)(2)} \otimes g_{(i+2)} \otimes \dots \otimes g_{(k-1)}$$

$$\equiv g_{(1)} \otimes g_{(2)} \otimes \dots \otimes g_{(k)}.$$

(D.3.4)

- Counit:

$$\begin{aligned}
\varepsilon : A &\rightarrow F, \quad a \mapsto \varepsilon(a), \quad \varepsilon(1_A) = 1_F, \\
\varepsilon \circ \mu_A &= \mu_F \circ (\varepsilon \otimes \varepsilon), \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b). \\
(id \otimes \varepsilon) \circ \Delta &= (\varepsilon \otimes id) \circ \Delta = id, \\
g_{(1)}\varepsilon(g_{(2)}) &= \varepsilon(g_{(1)})g_{(2)} = g. \\
((id \otimes)^{i-1} \varepsilon(\otimes id)^{k-i}) \circ \Delta^k &= \Delta^{k-1}, \quad i = 1, 2, \dots, k. \\
\varepsilon(g_{(i)}) g_{(1)} \otimes \dots \otimes g_{(i-1)} \otimes g_{(i+1)} \otimes \dots \otimes g_{(k)} &= g_{(1)} \otimes \dots \otimes g_{(k-1)}.
\end{aligned} \tag{D.3.5}$$

- Antipode:

$$\begin{aligned}
S : A &\rightarrow A, \\
\mu \circ (id \otimes S) \circ \Delta &= \mu \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon, \\
g_{(1)}S(g_{(2)}) &= S(g_{(1)})g_{(2)} = \eta(\varepsilon(g)) = \varepsilon(g)1_G, \\
\Rightarrow S(gh) &= S(h)S(g), \quad S(1_G) = 1_G, \quad (S \otimes S) \circ \Delta = \tau \circ \Delta \circ S, \quad \varepsilon \circ S = \varepsilon. \\
((id \otimes)^{i-1} S(\otimes id)^{k-i}) \circ \Delta^k &\stackrel{?}{=} \eta \circ \varepsilon \Delta^{k-1}, \\
g_{(1)} \otimes \dots \otimes g_{(i-1)} \otimes S(g_{(i)})g_{(i+1)} \otimes g_{(i+2)} \otimes \dots \otimes g_{(k)} \\
&\stackrel{?}{=} \varepsilon(g) 1_G \otimes g_{(1)} \otimes g_{(2)} \otimes \dots \otimes g_{(k-2)}.
\end{aligned}$$

- Boundary:

$$\begin{aligned}
\mu^k : A^{\otimes k} &\rightarrow A, \quad a_1 \otimes a_2 \otimes \dots \otimes a_k \mapsto a_1 a_2 \dots a_k. \\
\partial_i &\equiv \mu_{i \bmod_{k+1}(i+1)} = (id \otimes)^{i-1} \mu^2 (\otimes id)^{k-i} : A^{\otimes k} \rightarrow A^{\otimes(k-1)}, \quad i = 1, 2, \dots, k. \\
\text{mod}_N(n) &= \begin{cases} n, & n < N \\ \min(\{n\}), & n = N \\ \text{mod}_N(n - N), & n > N \end{cases} \\
&= n \theta(N - n) + \min(\{n\}) \delta_{nN} + \text{mod}_N(n - N) \theta(n - N), \\
\partial &= \sum_{i=1}^k (-1)^{i-1} \partial_i, \quad \partial^2 = 0. \tag{D.3.6}
\end{aligned}$$

D.3.0.1 Example

$H = (\mathcal{A} = \{a, b, c, \dots\}, \mu = \mu_{\mathcal{A}}, \Delta, \tau, \varepsilon, \eta, S)$, $\mu, \Delta, \tau, \varepsilon, \eta, S$ linear. $Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$. “First” define Δ such that (ie. check that) $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$. For example, if $\pi : H \rightarrow O(\mathcal{B})$, $\mathcal{B} = (B, \mu_B)$ then Δ will be defined such that $\pi(a) \circ \mu_B = \mu_B \circ \Delta(\pi(a)) \equiv \mu_B \circ \Delta(\pi) \circ \Delta(a)$.

$$\begin{aligned}
\Delta(a) &= a_\alpha \otimes a^\alpha, \quad \Delta(ab) = (ab)_\alpha \otimes (ab)^\alpha, \\
\Delta(a)\Delta(b) &= a_\alpha b_\beta \otimes a^\alpha b^\beta, \\
(ab)_\alpha &= a_\gamma b_\rho \lambda^{\gamma\rho}_\alpha, \quad (ab)^\alpha = \lambda^{\gamma\rho}_\alpha a^\gamma b^\rho, \quad \lambda^{\gamma\rho}_\alpha \lambda^{\alpha\gamma'}_{\rho'} = \delta^{\gamma\gamma'} \delta^{\rho\rho'} \\
&\Rightarrow \Delta(ab) = \Delta(a)\Delta(b). \\
S(a_\alpha) &= \eta(\varepsilon(a)) (a_\rho a^\rho)^{-1} a_\alpha, \quad S(a^\alpha) = a^\alpha (a_\rho a^\rho)^{-1} \eta(\varepsilon(a)), \\
\varepsilon(a_\alpha) &= \delta_{aa_\alpha}, \quad \varepsilon(a^\alpha) = \delta_{aa^\alpha}, \\
\varepsilon(ab) &= \varepsilon(a)\varepsilon(b), \quad S(ab) = S(b)S(a). \tag{D.3.7}
\end{aligned}$$

D.3.1 Quasi-triangular Hopf algebras and R-matrix

If $\tau \circ \Delta = Q \circ \Delta$, then one may write $\mathcal{T} = Q \circ \tau$, where $\mathcal{T} \circ \Delta = \Delta$.

$$\begin{aligned} \mathcal{T}_i \mathcal{T}_{i+1} \mathcal{T}_i &= \mathcal{T}_{i+1} \mathcal{T}_i \mathcal{T}_{i+1} \\ \Rightarrow Q_{i+1} Q_i Q_{i+1} &= Q_{i+1} Q_i Q_{i+1}. \end{aligned} \quad (\text{D.3.8})$$

For example, if $Q = \text{ad}_R$; ie. $\tau \circ \Delta(h) = R \circ \Delta(h) \circ R^{-1}$ then we also have

$$R_{i+1} R_i R_{i+1} = R_{i+1} R_i R_{i+1}. \quad (\text{D.3.9})$$

D.3.2 Action

$H = \{h, g, \dots\}$ acts on a product algebra $A = A_1 \otimes A_2 \otimes \dots \otimes A_k = \{a, b, c, \dots\}$ through an action ρ .

$$\rho : H \otimes A \rightarrow A, \quad (h, a) \mapsto \rho(h, a) = \rho(h)a \equiv \rho_h a \equiv h \triangleright a,$$

$$\rho_h a = \Delta(\rho_h)(a_1 \otimes a_2 \otimes \dots \otimes a_k) = \rho_{h_{(1)}} a_1 \otimes \rho_{h_{(2)}} a_2 \otimes \dots \otimes \rho_{h_{(k)}} a_k.$$

$$\rho_h(abc\dots) = \rho_{h_{(1)}} a \rho_{h_{(2)}} b \rho_{h_{(3)}} c \dots, \quad a, b, c, \dots \in A.$$

$$\text{e.g. left action (left "translation")} \quad \rho_h a = L_h a = \Delta(h)(a_1 \otimes a_2 \otimes \dots \otimes a_k)$$

$$= h_{(1)} a_1 \otimes h_{(2)} a_2 \otimes \dots \otimes h_{(k)} a_k.$$

$$\text{adjoint action } \rho_h a = \text{ad}_h a := h_{(1)} a S(h_{(2)}) = \text{ad}_{h_{(1)}} a_1 \otimes \text{ad}_{h_{(2)}} a_2 \otimes \dots \otimes \text{ad}_{h_{(k)}} a_k$$

$$\Rightarrow \text{ad}_h(ab) = h_{(1)} ab S(h_{(2)}) = h_{(1)} a S(h_{(2)}) h_{(3)} b S(h_{(4)}) = \text{ad}_{h_{(1)}} a \text{ad}_{h_{(2)}} b.$$

(D.3.10)

More simply for the adjoint action, one can write

$$\begin{aligned}
ga &= g_{(1)}aS(g_{(2)})g_{(2)} = ad_{g_{(1)}}a g_{(2)}, \\
gab &= ad_{g_{(1)}}a g_{(2)}b = ad_{g_{(1)}}a ad_{g_{(2)(1)}}b g_{(2)(2)} = ad_{g_{(1)}}a ad_{g_{(2)}}b g_{(3)} \\
&= ad_{g_{(1)}}(ab) g_{(2)}. \\
g\psi_1\psi_2\dots\psi_k &= ad_{g_{(1)}}(\psi_1\psi_2\dots\psi_k) g_{(2)} = ad_{g_{(1)}}\psi_1 ad_{g_{(2)}}\psi_2 \dots ad_{g_{(k)}}\psi_k g_{(k)},
\end{aligned} \tag{D.3.11}$$

where each of ψ_i 's may be a tensor product as well; ie.

$$\psi_i \in \mathcal{T}(A) = F \oplus A \oplus A^{\otimes 2} \oplus A^{\otimes 3} \oplus \dots \oplus A^{\otimes n} \quad \forall i. \tag{D.3.12}$$

D.3.3 Duality and integration

The set of linear functionals $H^* \equiv A^* = \{f, f : H \rightarrow F, a \rightarrow f(a) \equiv (f, a)\}$ is the dual of H . That is, $(,) : H^* \otimes H \rightarrow F$. For purposes of (co)homology indicated by the maps $\partial_i = \mu_i \text{ mod}_{k+1}(i+1) : A^{\otimes k} \rightarrow A^{\otimes(k-1)}$, $\Delta_i : A^{\otimes k} \rightarrow A^{\otimes(k+1)}$, μ and Δ are dual to each other. Similarly, ε and η are duals.

$$\begin{aligned}
(ff', a) &:= (\mu(f \otimes f'), a) = (f \otimes f', \Delta(a)). \\
(\Delta(f), a \otimes b) &:= (f, \mu(a \otimes b)) = (f, ab). \\
(f, \eta(\lambda)) = (\varepsilon(f), \lambda) &\iff (f, 1_A) = \varepsilon(f), \quad \lambda \neq 0. \\
(\eta(\alpha), a) = (\alpha, \varepsilon(a)) &\iff (1_{A^*}, a) = \varepsilon(a), \quad \alpha \neq 0. \\
(S(f), a) = (f, S(a)), &\quad S \text{ is self dual.}
\end{aligned} \tag{D.3.13}$$

A left integral $\int \phi \in H^*$ of an element $\phi \in H^*$ is a left-invariant linear

functional $\int \phi : H \rightarrow F$, $\int L_h^* \phi = \varepsilon(h) \int \phi \ \forall h \in H$, where

$$(L_h a, \phi) = (a, L_h^* \phi),$$

$$\begin{aligned} \text{ie. } L_h^* \phi(a) &= \phi(L_h a) = \phi(ha) = \phi(\mu(h \otimes a)) = \mu_F \circ \Delta(\phi)(h \otimes a) \\ &= \phi_{(1)}(h) \phi_{(2)}(a). \end{aligned}$$

$$\phi \circ \mu_H = \mu_F \circ \Delta(\phi). \tag{D.3.14}$$

Therefore a left integral \int on H is given by

$$\int L_h^*(\phi) = \varepsilon(h) \int \phi \quad \forall (h \in H, \phi \in H^*) \tag{D.3.15}$$

and similarly, a left integral in H is any $\mathcal{I} \in H$ such that

$$L_h \mathcal{I} \equiv h \mathcal{I} = \varepsilon(h) \mathcal{I} \quad \forall h \in H. \tag{D.3.16}$$

Appendix E

Some math concepts

E.1 Groups, Rings (Algebras), Fields, Vector spaces, Modules

A group G is a set S with an identity e , closed under an associative binary operation $S \times S \rightarrow S$, $(a, b) \mapsto ab$ and in which every element has an inverse. That is, $\forall a, b, c \in S \quad \exists e, a^{-1} \in S$ such that

$$\begin{aligned} ae = a, \quad a(bc) = (ab)c, \quad a^{-1}a = e. \\ G = (S, S \times S \rightarrow S). \end{aligned} \tag{E.1.1}$$

G is an Abelian group G_0 if $ab = ba \quad \forall a, b \in G$. The binary operation of the Abelian group is written as $+$ and the identity is written as 0 and the inverse a^{-1} of $a \in G_0$ is written as $-a$. That is $G_0 = (S, + : S \times S \rightarrow S)$.

A ring (or an algebra) R is an Abelian group G_0 that is closed under an additional associative binary operation $\cdot : G_0 \times G_0 \rightarrow G_0$, $(a, b) \mapsto ab$ that is

distributive over $+$. That is

$$a(b+c) = ab+ac, \quad (b+c)a = ba+ca, \quad a(bc) = (ab)c \quad \forall a, b, c \in G_0.$$

$$R = (G_0, \cdot : G_0 \times G_0 \rightarrow G_0) = (S, +, \cdot : S \times S \rightarrow S) \equiv (G_0, \cdot) \equiv (S, +, \cdot).$$

A field F is a ring $(S_0, +, \cdot)$ such that $(S_0 \setminus \{0\}, \cdot)$ is an Abelian group. That is $F = (S_0, +, \cdot)|_{(S_0 \setminus \{0\}, \cdot) \in \mathcal{G}}$ where $\mathcal{G} = \{G\}$ is the family of groups. A field F is ordered iff there is $P \subseteq F$ such that

$$+, \cdot : P \times P \rightarrow P, \quad P \cap -P = \{\}, \quad P \cup \{0\} \cup -P = F, \quad (\text{E.1.2})$$

where $A^{-1} = \{a^{-1}, a \in A\}$, $AB = \{ab, a \in A, b \in B\}$, $A^2 = AA$.

If $I \subset R_0$ is an ideal (ie. $I + I = I$, $R_0 I = I R_0 = I$) of an Abelian ring $R_0 = (S_0, +, \cdot)$ then $F_I = R_0 \setminus I = \{a + I, a \in R_0\}$ is a field.

With notation understood, one defines a vector space V_F over a field F and a module M_R over a ring R as $V_F = (V, +, F \times V \rightarrow V)$,

$M_R^{\text{left}} = (M, +, R \times M \rightarrow M)$, $M_R^{\text{right}} = (M, +, M \times R \rightarrow M)$ respectively.

E.2 Set commutant algebra

Let the commutator of two subsets A, B of an algebra $\mathcal{A} := (\mathcal{A}, +, \star)$ be

$$[A, B] = \{[a, b], a \in A, b \in B\}. \quad (\text{E.2.1})$$

Let $S \subseteq \mathcal{A}$, then the commutant $S' \subseteq \mathcal{A}$ of S in \mathcal{A} is defined to be the subset of \mathcal{A} , with the highest possible number of elements that each commute with every element of S .

$$S' = \max_{[U, S] = \{0\}} U, \quad U \subseteq \mathcal{A}, \quad 0 + g = g \quad \forall g \in \mathcal{A}. \quad (\text{E.2.2})$$

If $A, B \subseteq \mathcal{A}$ and $A \subseteq B$, then

$$B' \subseteq A' \tag{E.2.3}$$

since some of the elements of A' may fail to commute with $B \setminus A \equiv B \setminus A \cap B$ and hence fail to commute with B .

Also, since both S and S'' (the commutant of S') commute with S' and S'' is supposed to be the maximum of all sets that commute with S' , it follows that

$$S \subseteq S'' \tag{E.2.4}$$

One can then deduce using (E.2.3) and (E.2.4) that

$$\begin{aligned} S \subseteq S'' = S'''' = \dots = S^{(2n)}, \quad n > 2, \\ S' = S''' = \dots = S^{(2n-1)}, \quad n \geq 1. \end{aligned} \tag{E.2.5}$$

As a check $S \subseteq S'' \Rightarrow (S'')' \subseteq S'$. Also $S' \subseteq (S')''$ and so $S' = S'''$ follows by the identification $(S'')' = (S')'' \equiv S'''$. Therefore

$$\mathcal{A} = S' \cup S'' \tag{E.2.6}$$

Furthermore

$$(A \cup B)' \subseteq A' \cap B' \tag{E.2.7}$$

since $A \subseteq A \cup B$, $B \subseteq A \cup B \Rightarrow (A \cup B)' \subseteq A'$, $(A \cup B)' \subseteq B'$. Similarly

$$\begin{aligned} A \cap B \subseteq A, \quad A \cap B \subseteq B \Rightarrow A' \subseteq (A \cap B)', \quad B' \subseteq (A \cap B)'. \\ \text{Hence } A' \cup B' \subseteq (A \cap B)'. \end{aligned} \tag{E.2.8}$$

If A, B are Von Neumann algebras $A = A'', B = B''$ then it follows from (E.2.7) and (E.2.8) that

$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'. \quad (\text{E.2.9})$$

Also one observes that the center $Z(S) := S \cap S'$ of S is always a commuting set and so S is a commuting set iff $S \subseteq S'$.

Remarks:

- Although S is merely an arbitrary subset, the derived sequence of subsets $S^{(i)}$, $i = 1, 2, \dots, n$ is a sequence of subalgebras (that is, these subsets are closed under $+$ & $*$) if S is self adjoint; ie. both $a, a^* \in S$.
- Consequently S'' is seen as the closure of S since it is the smallest closed set that contains S in this sense of closure. Thus S is closed (ie. a subalgebra) iff $S'' \subseteq S$ and hence iff $S'' = S$ since we also know that $S \subseteq S''$. S is open iff its complement $S^c = \mathcal{A} \setminus S$ is closed (ie. a subalgebra).
- In particular if S is a single element set with element a then a' is the symmetry algebra of a and $a' \cap a''$ is the largest commutative set that contains a . $\sigma(a) \subseteq \sigma(a' \cap a'')$.
- A representation $R : S \rightarrow B(\mathcal{H})$, where $B(\mathcal{H})$ is the set of bounded linear operators on a Hilbert space \mathcal{H} , of a subalgebra S is irreducible iff $R(S)' = \mathbb{C} 1_{B(\mathcal{H})}$ meaning that the commutant $R(S)'$ of $R(S)$ is proportional to the identity (ie. trivial) in $B(\mathcal{H})$.

E.3 Projector algebra

Let $p \in \mathcal{A}$, $p^2 = p$, $D_p = pDp \equiv \{pap; a \in D\}$, $D \subseteq \mathcal{A}$. Then in

$$\mathbb{N}_p(D) := \{D_p^n; n \in \mathbb{N}\},$$

one has that

- $D_p^m D_p^n = D_p^{m+n}$.
- $\bigcup_{n \in \mathbb{N}} D_p^n \subseteq p' = \{c \in \mathcal{A}; pc = cp\}$.
- If $p \in D$ then $m \leq n \Rightarrow D_p^m \subseteq D_p^n$.
- $p \in D$, $DD = D \Rightarrow p \in D_p \subseteq D$, $D_p D_p = D_p$. Therefore for $D = \mathcal{A}$ one sees that projectors correspond to “closed” subspaces of \mathcal{A} .

An Abelian group $\mathbb{Z}_p(D) = \{g^n; n \in \mathbb{Z}\}$ may also be defined with elements

$$g^n = \{(D_p^m, D_p^{m+n}); m \in \mathbb{N}\},$$

$$g^{-n} = \{(D_p^{n+m}, D_p^m); m \in \mathbb{N}\}, \quad (x, y)(z, w) := (xz, yw).$$

E.4 Matrix-valued functions and BCH formula

E.4.1 Limits

For complex numbers α, β, \dots and matrices

$$A = e^a, B = e^b, \dots,$$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^{\lim_{n \rightarrow \infty} n \ln\left(1 + \frac{\alpha}{n}\right)} = e^{\lim_{n \rightarrow \infty} \frac{\ln\left(1 + \frac{\alpha}{n}\right)}{\frac{1}{n}}} = e^\alpha \equiv \lim_{n \rightarrow \infty} \left(e^{\frac{\alpha}{n}}\right)^n.$$

$$\text{ie } \lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n = \lim_{n \rightarrow \infty} \left(e^{\frac{\alpha}{n}}\right)^n = e^\alpha.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\left(1 + \frac{\alpha}{n}\right)^n \left(1 + \frac{\beta}{n}\right)^n \right) &= \lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 + \frac{\beta}{n}\right)^n = \lim_{n \rightarrow \infty} \left(\left(e^{\frac{\alpha}{n}}\right)^n \left(e^{\frac{\beta}{n}}\right)^n \right) \\ &= \lim_{n \rightarrow \infty} \left(e^{\frac{\alpha}{n}}\right)^n \lim_{n \rightarrow \infty} \left(e^{\frac{\beta}{n}}\right)^n = e^\alpha e^\beta = e^{\alpha+\beta}. \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\left(1 + \frac{\alpha}{n}\right) \left(1 + \frac{\beta}{n}\right) \right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{\alpha + \beta}{n} + \frac{\alpha\beta}{n^2} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{\alpha + \beta}{n} \right)^n \\ &= \lim_{n \rightarrow \infty} \left(e^{\frac{\alpha+\beta}{n}} \right)^n = e^{\alpha+\beta} = \lim_{n \rightarrow \infty} \left(e^{\frac{\alpha}{n}} e^{\frac{\beta}{n}} \right)^n, \quad \Rightarrow \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left(\left(1 + \frac{\alpha}{n}\right)^n \left(1 + \frac{\beta}{n}\right)^n \right) = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{\alpha}{n}\right) \left(1 + \frac{\beta}{n}\right) \right)^n. \quad (\text{E.4.1})$$

Similarly,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n = \lim_{n \rightarrow \infty} \left(e^{\frac{a}{n}}\right)^n = e^a.$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\left(1 + \frac{a}{n}\right)^n \left(1 + \frac{b}{n}\right)^n \right) &= \lim_{n \rightarrow \infty} \left(1 + \frac{a}{n}\right)^n \lim_{n \rightarrow \infty} \left(1 + \frac{b}{n}\right)^n = \lim_{n \rightarrow \infty} \left(e^{\frac{a}{n}}\right)^n \lim_{n \rightarrow \infty} \left(e^{\frac{b}{n}}\right)^n \\ &= \lim_{n \rightarrow \infty} \left(\left(e^{\frac{a}{n}}\right)^n \left(e^{\frac{b}{n}}\right)^n \right) = e^a e^b. \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\left(1 + \frac{a}{n}\right) \left(1 + \frac{b}{n}\right) \right)^n &= \lim_{n \rightarrow \infty} \left(1 + \frac{a+b}{n} + \frac{ab}{n^2} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{a+b}{n} \right)^n = e^{a+b} \\ &= \lim_{n \rightarrow \infty} \left(e^{\frac{a}{n}} e^{\frac{b}{n}} \right)^n, \quad \Rightarrow \end{aligned}$$

$$\lim_{n \rightarrow \infty} \left(\left(1 + \frac{a}{n}\right)^n \left(1 + \frac{b}{n}\right)^n \right) \neq \lim_{n \rightarrow \infty} \left(\left(1 + \frac{a}{n}\right) \left(1 + \frac{b}{n}\right) \right)^n. \quad (\text{E.4.2})$$

E.4.2 Matrix functions

Let $t = \text{real parameter}$.

$$\begin{aligned}
 \frac{d}{dt} e^{a+bt} &= \frac{d}{dt} \lim_{n \rightarrow \infty} (e^{\frac{a}{n}} e^{\frac{bt}{n}})^n = \lim_{n \rightarrow \infty} \frac{d}{dt} (e^{\frac{a}{n}} e^{\frac{bt}{n}})^n \\
 &= \lim_{n \rightarrow \infty} \sum_{k=1}^n (e^{\frac{a}{n}} e^{\frac{bt}{n}})^{k-1} \frac{d}{dt} (e^{\frac{a}{n}} e^{\frac{bt}{n}}) (e^{\frac{a}{n}} e^{\frac{bt}{n}})^{n-k} \\
 &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (e^{\frac{a}{n}} e^{\frac{bt}{n}})^{n-k-1} \frac{d}{dt} (e^{\frac{a}{n}} e^{\frac{bt}{n}}) (e^{\frac{a}{n}} e^{\frac{bt}{n}})^k \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (e^{\frac{a}{n}} e^{\frac{bt}{n}})^k b (e^{\frac{a}{n}} e^{\frac{bt}{n}})^{n-k} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} (e^{\frac{a}{n}} e^{\frac{bt}{n}})^{n-k} b (e^{\frac{a}{n}} e^{\frac{bt}{n}})^k
 \end{aligned} \tag{E.4.3}$$

$$\begin{aligned}
 \sum_{k=1}^n \alpha^k &= \alpha \sum_{k=1}^n \alpha^{k-1} = \alpha \sum_{k=0}^{n-1} \alpha^k = \alpha (\sum_{k=1}^n \alpha^k + 1 - \alpha^n), \Rightarrow \\
 \sum_{k=1}^n \alpha^k &= \frac{\alpha(1 - \alpha^n)}{1 - \alpha} \\
 \sum_{k=0}^{n-1} \alpha^k &= \sum_{k=1}^n \alpha^k + 1 - \alpha^n = \frac{\alpha(1 - \alpha^n)}{1 - \alpha} + 1 - \alpha^n = \frac{(1 - \alpha^n)}{1 - \alpha}
 \end{aligned} \tag{E.4.4}$$

Therefore,

$$\begin{aligned}
 \frac{d}{dt} e^{a+bt} \Big|_{t=0} &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n e^{a \frac{k}{n}} b e^{a(1-\frac{k}{n})} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n (e^{[\frac{a}{n}, \cdot]})^k b e^a \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \frac{e^{[\frac{a}{n}, \cdot]} (I - e^{[a, \cdot]})}{I - e^{[\frac{a}{n}, \cdot]}} b e^a = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{I - e^{[\frac{a}{n}, \cdot]}} (I - e^{[a, \cdot]}) b e^a \\
 &= -\frac{1}{[a, \cdot]} (I - e^{[a, \cdot]}) b e^a = \frac{e^{[a, \cdot]} - I}{[a, \cdot]} b e^a, \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{a(1-\frac{k}{n})} b e^{a \frac{k}{n}} = e^a \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-a \frac{k}{n}} b e^{a \frac{k}{n}} = e^a \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} e^{-[\frac{a}{n}, \cdot]} b \\
 &= e^a \lim_{n \rightarrow \infty} \frac{1}{n} \frac{I - e^{-[a, \cdot]}}{I - e^{-[\frac{a}{n}, \cdot]}} b = e^a \frac{I - e^{-[a, \cdot]}}{[a, \cdot]} b
 \end{aligned} \tag{E.4.5}$$

For a general matrix function $f(t) = \sum_{r=0}^{\infty} \frac{f^{(r)}(a)}{r!} (t-a)^r$,

$$\begin{aligned} \frac{d}{dt} e^{f(t)} &= \frac{d}{dt} \lim_{n \rightarrow \infty} \left(\prod_{r=0}^{\infty} e^{\frac{f^{(r)}(a)}{r!} \frac{(t-a)^r}{n}} \right)^n = \lim_{n \rightarrow \infty} \frac{d}{dt} \left(\prod_{r=0}^{\infty} e^{\frac{f^{(r)}(a)}{r!} \frac{(t-a)^r}{n}} \right)^n \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \left(\prod_{r=0}^{\infty} e^{\frac{f^{(r)}(a)}{r!} \frac{(t-a)^r}{n}} \right)^{n-k-1} \frac{d}{dt} \left(\prod_{r=0}^{\infty} e^{\frac{f^{(r)}(a)}{r!} \frac{(t-a)^r}{n}} \right) \left(\prod_{r=0}^{\infty} e^{\frac{f^{(r)}(a)}{r!} \frac{(t-a)^r}{n}} \right)^k \end{aligned} \quad (\text{E.4.6})$$

Therefore,

$$\begin{aligned} \frac{d}{da} e^{f(a)} &:= \frac{d}{dt} e^{f(t)} \Big|_{t=a} = e^{f(a)} \frac{I - e^{-[f(a), \cdot]}}{[f(a), \cdot]} \frac{df(a)}{da}. \\ f(1) &= f(0) + \int_0^1 dt \frac{adf(t)}{I - e^{-adf(t)}} \left(e^{-f(t)} \frac{d}{dt} e^{f(t)} \right) \end{aligned}$$

That is,

$$\begin{aligned} de^f &= e^f \frac{I - e^{-[f, \cdot]}}{[f, \cdot]} df = \frac{e^{[f, \cdot]} - I}{[f, \cdot]} df e^f, \quad \Rightarrow \\ df &= \frac{[f, \cdot]}{I - e^{-[f, \cdot]}} (e^{-f} de^f) = \frac{\ln e^{[f, \cdot]}}{I - e^{-[f, \cdot]}} (e^{-f} de^f), \quad \text{in other words} \\ e^{-adf}(d) &= e^{-f} de^f = \frac{I - e^{-adf}}{\ln e^{adf}} (df) = \int_0^1 d\alpha e^{-\alpha adf} df = \int_0^1 d\alpha e^{-\alpha f} df e^{\alpha f} \\ &= Df = \bar{d}f - d \left(\int_0^1 d\alpha e^{-\alpha adf} \right) f = [d - d \left(\int_0^1 d\alpha e^{-\alpha adf} \right)] f \\ de^f &= \int_0^1 d\alpha e^{(1-\alpha)f} df e^{\alpha f} = \int_0^1 d\alpha e^f e^{-\alpha adf} df = \int_0^1 d\alpha e^{f-\alpha adf} df. \end{aligned} \quad (\text{E.4.7})$$

Similarly, the commutator of any operator, a , with the exponential, e^b , of another operator b can be written as

$$\begin{aligned}
[a, e^b] &= [a, \lim_{n \rightarrow \infty} (e^{\frac{b}{n}})^n] = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} (e^{\frac{b}{n}})^{n-k-1} [a, e^{\frac{b}{n}}] (e^{\frac{b}{n}})^k \\
&= e^b \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} e^{-\frac{k}{n}b} e^{-\frac{b}{n}} [a, e^{\frac{b}{n}}] e^{\frac{k}{n}b} \\
&= e^b \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} e^{-\frac{k}{n}adb} e^{-\frac{b}{n}} [a, e^{\frac{b}{n}}] \\
&= e^b \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} e^{-\frac{k}{n}adb} e^{-\frac{b}{n}} [a, I + \frac{b}{n}] \\
&= e^b \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{n} e^{-\frac{k}{n}adb} e^{-\frac{b}{n}} [a, b] \\
[a, e^b] &= e^b \frac{I - e^{-adb}}{adb} [a, b] \\
\text{ie. } \quad ade^b &= e^b \frac{I - e^{-adb}}{adb} adb = e^b(I - e^{-adb}) \tag{E.4.8}
\end{aligned}$$

More generally,

$$\begin{aligned}
[a, f(b)] &= \int_0^1 dt f'(b - t \mathbf{adb}) [a, b] = \partial_b f(b) \int_0^1 dt e^{-\overleftarrow{\partial}_{bt} \mathbf{adb}} [a, b] \\
a^{-1} f(b) &= f(a^{-1}ba) a^{-1} = [a^{-1}, f(b)] + f(b) a^{-1}
\end{aligned} \tag{E.4.9}$$

$$[f(a), g(b)] = \int_0^1 d\alpha \int_0^1 d\beta g'(b - \alpha \mathbf{adb}) f'(a - \beta \mathbf{ada}) [a, b] \tag{E.4.10}$$

E.4.3 Symmetric ordered extension

With the help of the Fourier transform, a general function of a matrix might be written as

$$g(f) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f^n = \sum_{n=0}^{\infty} \alpha_n f^n = \int d\mu(k) \tilde{g}(k) e^{-ikf},$$

$$f = \text{matrix}, \quad \alpha_n \in \mathbb{C}, \quad \tilde{g} : \mathbb{C} \rightarrow \mathbb{C}, \quad (\text{E.4.11})$$

has differential

$$dg(f) = \frac{dg(f)}{df} \int_0^1 d\alpha e^{-\alpha \frac{\partial}{\partial f} \text{ad}f} df = \sum_{n=0}^{\infty} \frac{d^{n+1}g(f)}{df^{n+1}} \frac{(-1)^n}{(n+1)!} (\text{ad}f)^n(df)$$

$$dg(f) \stackrel{?}{=} \int_0^1 d\alpha \frac{\partial g}{\partial f}(f - \alpha \text{ad}f) df \quad , \quad ? = (\text{if } [f, \text{ad}f] = [\frac{\partial}{\partial f}, \text{ad}f] = 0).$$

(E.4.12)

Note that given any φ ,

$$\frac{\partial}{\partial f_i} f_j = I\delta_i^j,$$

$$[\frac{\partial}{\partial f_i}, \text{ad}f_j]\varphi = \frac{\partial}{\partial f_i}(\text{ad}f_j(\varphi)) - \text{ad}f_j(\frac{\partial}{\partial f_i}(\varphi)) = \frac{\partial}{\partial f_i}[f_j, \varphi] - [f_j, \frac{\partial}{\partial f_i}\varphi]$$

$$= [f_j, \frac{\partial}{\partial f_i}\varphi] - [f_j, \frac{\partial}{\partial f_i}\varphi] = 0,$$

$$[f_i, \text{ad}f_j]\varphi = f_i \text{ad}f_j(\varphi) - \text{ad}f_j(f_i\varphi) = f_i[f_j, \varphi] - [f_j, f_i\varphi] = [f_i, f_j]\varphi.$$

That is,

$$[\frac{\partial}{\partial f_i}, \text{ad}f_j] = 0, \quad [f_i, \text{ad}f_j] = [f_i, f_j]. \quad (\text{E.4.13})$$

Therefore, $[\frac{\partial}{\partial f}, \text{ad}f] = 0$ always. However, if we have only one variable f , then $[f, f] = 0$, but with more than one f 's, $[f_i, f_j] \neq 0$. Therefore one needs to write the general case with care:

$$dg(f) = \int_0^1 d\alpha \frac{\partial g}{\partial f}(f - \alpha \text{ad}f|_f) df \quad (\text{E.4.14})$$

where $\text{adf}|_f$ (a "partial" adjoint, just like the partial derivative, whose target independent variables are f and df) is the adjoint action that leaves the f , which is in the same function argument as itself, "constant". In the case where one defines $g(f) := \int d\mu(k) \tilde{g}(k) e^{-ik_i f^i}$, then because of the complete contraction, the chain rule formula,

$$dg(f) = \int_0^1 d\alpha \frac{\partial g}{\partial f_i} (f - \alpha \text{adf}) df_i, \quad f = (f_i) = (f_1, f_2, \dots, f_n)$$

holds without any restriction such as $\text{adf}|_f$ since

$$[k_i f_i, \text{ad}(k_j f_j)] = [k_i f_i, k_j f_j] = 0.$$

E.4.4 Baker-Campbell-Hausdorff (BCH) formula

If one defines $f(t)$ by $e^{f(t)} = e^A e^{Bt}$ ($e^{-f(t)} = (e^{f(t)})^{-1} = (e^A e^{Bt})^{-1} = e^{-Bt} e^{-A}$), then

$$\begin{aligned} f(1) &= \ln(e^A e^B) \\ &= A + \int_0^1 dt \left(\frac{I - (e^{\text{ad}A} e^{\text{ad}Bt})^{-1}}{\ln(e^{\text{ad}A} e^{\text{ad}Bt})} \right)^{-1} (B) = A + \int_0^1 dt \frac{1}{\int_0^1 d\alpha e^{-\alpha t} \text{ad}B e^{-\alpha \text{ad}A}} (B). \\ &= A + \int_0^1 dt \frac{\ln(e^{\text{ad}A} e^{\text{ad}Bt})}{I - (e^{\text{ad}A} e^{\text{ad}Bt})^{-1}} (B) \\ &= A + \int_0^1 dt e^{\text{ad}A} e^{\text{ad}Bt} \frac{\ln(e^{\text{ad}A} e^{\text{ad}Bt})}{e^{\text{ad}A} e^{\text{ad}Bt} - I} (B) \\ &= A + \int_0^1 dt e^{\text{ad}A} e^{\text{ad}Bt} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (e^{\text{ad}A} e^{\text{ad}Bt} - I)^{n-1} (B) \\ &= A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] - \frac{1}{48}[B, [A, [A, B]]] \\ &\quad - \frac{1}{48}[A, [B, [A, B]]] + \dots \\ &= A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] - \frac{1}{12}[B, [A, B]] - \frac{1}{24}[B, [A, [A, B]]] + \dots \end{aligned} \tag{E.4.15}$$

Similarly,

$$\begin{aligned}
\ln(e^A e^B e^C) &= \ln(e^A e^B) + \int_0^1 dt \frac{\ln(e^{\text{ad}A} e^{\text{ad}B} e^{\text{ad}Ct})}{I - e^{-\text{ad}Ct} e^{-\text{ad}B} e^{-\text{ad}A}}(C) \\
&= A + \int_0^1 dt \frac{\ln(e^{\text{ad}A} e^{\text{ad}Bt})}{I - e^{-\text{ad}Bt} e^{-\text{ad}A}}(B) + \int_0^1 dt \frac{\ln(e^{\text{ad}A} e^{\text{ad}B} e^{\text{ad}Ct})}{I - e^{-\text{ad}Ct} e^{-\text{ad}B} e^{-\text{ad}A}}(C), \\
\ln(fe^A) &= \ln f + \int_0^1 dt \frac{\ln(e^{\text{ad}(\ln f)} e^{\text{ad}At})}{I - e^{-\text{ad}At} e^{-\text{ad}(\ln f)}}(A), \\
\ln(AB) &= \ln A + \int_0^1 dt \frac{\ln(e^{\text{ad}(\ln A)} e^{\text{ad}(\ln B)t})}{I - e^{-\text{ad}(\ln B)t} e^{-\text{ad}(\ln A)}}(\ln B)
\end{aligned}
\tag{E.4.16}$$

E.5 Complex analytic transforms

Given a complex function

$$f(z, z^*) = f_1(x_1, x_2) + i f_2(x_1, x_2), \quad z = x_1 + i x_2, \quad z^* = x_1 - i x_2 \text{ and a closed contour}$$

$C \subset \mathbb{C} \setminus \infty$, $\infty = \lim_{r \rightarrow +\infty} \{r e^{i\theta}, 0 \leq \theta \leq 2\pi\}$, Stokes's theorem implies

$$\begin{aligned}
\oint_{\partial D} dz f(z, z^*) &= \oint_C dx_1 f_1 - dx_2 f_2 + i(dx_1 f_2 + dx_2 f_1) \\
&= \oint_{\partial D} (dx_1 f_1 + dx_2(-f_2) + i(dx_1 f_2 + dx_2 f_1)) \\
&= \oint_{\partial D} (dx_1 f_1 + dx_2(-f_2) + i(dx_1 f_2 + dx_2 f_1)) \\
&= \int_D d^2 x \{(\partial_1(-f_2) - \partial_2 f_1 + i(\partial_1 f_1 - \partial_2 f_2))\} \\
&= \int_D d^2 x \{-(\partial_1 f_2 + \partial_2 f_1) + i(\partial_1 f_1 - \partial_2 f_2)\} \\
&= 2i \int_D d^2 x \frac{\partial f(z, z^*)}{\partial z^*} \\
&= - \int_D dz \wedge dz^* \frac{\partial f(z, z^*)}{\partial z^*} \\
\frac{\partial f(z, z^*)}{\partial z^*} &= \frac{1}{2} \{\partial_1 f_1 - \partial_2 f_2 + i(\partial_1 f_2 + \partial_2 f_1)\}. \\
\oint_{\partial D} dz f(z, z^*) + \int_D dz \wedge dz^* \frac{\partial f(z, z^*)}{\partial z^*} &= 0 \tag{E.5.1}
\end{aligned}$$

if f has no singularities in D .

Therefore if f is nonsingular (has no singularities) inside $C = \partial D$ then $\partial f(z, z^*) = 0$ iff

$$\oint_C dz f(z) = 0 \quad \text{or} \quad \int_{\Gamma=g(C)} dz \frac{dg^{-1}(z)}{dz} f \circ g^{-1}(z) = 0 \tag{E.5.2}$$

for any invertible analytic function g .

However in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \simeq S^2$, the formula must also hold for the “exterior” of the closed contour C for any continuation (which can of course be singular) of the function f into the exterior of C . Therefore it may be more correct to say: if a closed contour contains either 1) none or 2) all of the singularities of f in $\overline{\mathbb{C}}$ then $\partial f(z, z^*) = 0$ iff

$$\oint_C dz f(z) = 0. \tag{E.5.3}$$

We also have that

$$\int_0^{2\pi} e^{in\theta} d\theta = 0, \quad \forall n \in \mathbb{Z}. \quad (\text{E.5.4})$$

Therefore if $f = f(z)$ is analytic (ie. can be expanded as a power series in z) then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} d\theta f(re^{i\theta} + z) \quad \forall r = \text{const}. \quad (\text{E.5.5})$$

Thus if ω is a point on a circle of constant radius C_r centered at z ; ie. $\omega - z = re^{i\theta}$ then

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_0^{2\pi} d\theta f(re^{i\theta} + z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{d(re^{i\theta})}{re^{i\theta}} f(re^{i\theta} + z) \\ &= \frac{1}{2\pi i} \oint_{C_r} \frac{d(\omega - z)}{\omega - z} f(\omega - z + z) = \frac{1}{2\pi i} \oint_{C_r} \frac{d\omega}{\omega - z} f(\omega) \\ &= \frac{1}{2\pi i} \oint_{C_r} d\omega \frac{f(\omega)}{\omega - z}. \end{aligned} \quad (\text{E.5.6})$$

Let $f(\omega)$ be nonsingular inside C and be analytic about $\omega = z$ then $g(\omega) = \frac{1}{2\pi i} \frac{f(\omega)}{\omega - z}$ is nonsingular in the region between C and some circle C_r lying in C and centered at z . We will write $C(z)$ to mean that the point z lies inside the closed contour C . Therefore

$$\oint_{C(z)-C_r(z)} dz g(z) = 0 = \frac{1}{2\pi i} \oint_{C(z)} d\omega \frac{f(\omega)}{\omega - z} - \frac{1}{2\pi i} \oint_{C_r(z)} d\omega \frac{f(\omega)}{\omega - z}. \quad (\text{E.5.7})$$

That is, if a complex function $f = f(z)$ has none of its poles inside any given closed contour C then

$$f(z) = \frac{1}{2\pi i} \oint_{C(z)} d\omega \frac{f(\omega)}{\omega - z}. \quad (\text{E.5.8})$$

This easily extends to a nonsingular function in D as

$$\begin{aligned} f(z, z^*) &= \frac{1}{2\pi i} \oint_{\partial D(z)} d\omega \frac{f(\omega, \omega^*)}{\omega - z} + \frac{1}{2\pi i} \int_{D(z)} d\omega \wedge d\omega^* \frac{1}{\omega - z} \frac{\partial f(\omega, \omega^*)}{\partial \omega^*} \\ &= \frac{1}{2\pi i} \oint_{\partial D(z)} d\omega \frac{f(\omega, \omega^*)}{\omega - z} + \frac{1}{2\pi i} \int_{D(z)} d\omega \wedge df(\omega, \omega^*) \frac{1}{\omega - z} \end{aligned} \quad (\text{E.5.9})$$

E.5.1 Laurent series

If f is known to be singular at $a \in \mathbb{C}$ then for any two inner/outer curves C_1, C_2 each containing a , f in the region between C_1, C_2 that excludes a is given by

$$\begin{aligned}
2\pi i f(z) &= - \oint_{C_1(a)} d\omega_1 \frac{f(\omega_1)}{\omega_1 - z} + \oint_{C_2(a,z)} d\omega_2 \frac{f(\omega_2)}{\omega_2 - z} \\
&= - \oint_{C_1(a)} d\omega_1 f(\omega_1) \frac{z - a}{\omega_1 - a - (z - a)} \frac{1}{z - a} \\
&\quad + \oint_{C_2(a,z)} d\omega_2 f(\omega_2) \frac{\omega_2 - a}{\omega_2 - a - (z - a)} \frac{1}{\omega_2 - a} \\
&= - \oint_{C_1(a)} d\omega_1 f(\omega_1) \frac{1}{\frac{\omega_1 - a}{z - a} - 1} \frac{1}{z - a} + \oint_{C_2(a,z)} d\omega_2 f(\omega_2) \frac{1}{1 - \frac{z - a}{\omega_2 - a}} \frac{1}{\omega_2 - a} \\
&= \oint_{C_1(a)} d\omega_1 f(\omega_1) \sum_{n=0}^{\infty} \left(\frac{\omega_1 - a}{z - a} \right)^n \frac{1}{z - a} + \oint_{C_2(a,z)} d\omega_2 f(\omega_2) \sum_{n=0}^{\infty} \left(\frac{z - a}{\omega_2 - a} \right)^n \frac{1}{\omega_2 - a} \\
&\quad \left\{ \left| \frac{\omega_1 - a}{z - a} \right| < 1 \quad \forall \omega_1, \quad \left| \frac{z - a}{\omega_2 - a} \right| < 1 \quad \forall \omega_2 \right\} \\
&= \oint_{C_1(a)} d\omega_1 f(\omega_1) \sum_{n=0}^{\infty} \frac{(\omega_1 - a)^n}{(z - a)^{n+1}} + \oint_{C_2(a,z)} d\omega_2 f(\omega_2) \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\omega_2 - a)^{n+1}}, \\
&= \oint_{C_1(a)} d\omega_1 f(\omega_1) \sum_{n=1}^{\infty} \frac{(\omega_1 - a)^{n-1}}{(z - a)^n} + \oint_{C_2(a,z)} d\omega_2 f(\omega_2) \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\omega_2 - a)^{n+1}}, \\
&= \oint_{C_1(a)} d\omega_1 f(\omega_1) \sum_{n=-\infty}^{-1} \frac{(z - a)^n}{(\omega_1 - a)^{n+1}} + \oint_{C_2(a,z)} d\omega_2 f(\omega_2) \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\omega_2 - a)^{n+1}}, \\
f(z) &= \sum_{n=-\infty}^{\infty} \alpha_n^f(a) (z - a)^n, \tag{E.5.10}
\end{aligned}$$

$$\begin{aligned}
\alpha_n^f(a) &= \left\{ \begin{array}{l} \frac{1}{2\pi i} \oint_{C_1(a)} d\omega \frac{f(\omega)}{(\omega - a)^{n+1}}, \quad n \leq -1 \\ \frac{1}{2\pi i} \oint_{C_2(a,z)} d\omega \frac{f(\omega)}{(\omega - a)^{n+1}}, \quad n \geq 0 \end{array} \right\} \\
&= \frac{1}{2\pi i} \oint_{C_1(a)\theta(-1-n) + C_2(a,z)\theta(n)} d\omega \frac{f(\omega)}{(\omega - a)^{n+1}} \\
&\equiv \frac{1}{2\pi i} \oint_{\Gamma(a)} d\omega \frac{f(\omega)}{(\omega - a)^{n+1}}, \quad a \in C_1^0 \subset \Gamma^0 \subset C_2^0, \quad n \in \mathbb{Z},
\end{aligned}$$

ie. $0 < |\omega_1 - a| \leq |\omega - a| \leq |\omega_2 - a| \quad \forall \omega \in \Gamma$.

$$0 < |\omega_1 - a| < |z - a| < |\omega_2 - a| \quad \forall \omega_1 \in C_1, \omega_2 \in C_2. \tag{E.5.11}$$

The condition $|\omega_1 - a| < |z - a| < |\omega_2 - a| \quad \forall \omega_1 \in C_1, \omega_2 \in C_2$ is satisfied for the case where C_1, C_2 are circular so that the domain D of convergence of the series is any strip

$$D = D(r_1, r_2) = \{z, \quad 0 < r_1 < |z - a| < r_2\} \quad (\text{E.5.12})$$

where r_1 is the radius of C_1 about a and r_2 is the radius of C_2 about a .

If all k poles of f lie in a region of finite size L and f has no poles at ∞ then C_2 may be taken to ∞ and C_1 can be chosen to consist of a chain of “small” circles, each of radius $r_1 \rightarrow 0$ and encircling one pole, covering all poles $\{a_i, i = 1, \dots, k\}$ of f .

For the nonholomorphic case

$$f(z, z^*) = \sum_{n=-\infty}^{\infty} \alpha_n^f(a, a^*) (z - a)^n.$$

$$\alpha_n^f(a, a^*) = \frac{1}{2\pi i} \oint_{\Gamma(a)} d\omega \frac{f(\omega, \omega^*)}{(\omega - a)^{n+1}} + \frac{1}{2\pi i} \int_{\Gamma^0(a)} d\omega \wedge d\omega^* \frac{1}{(\omega - a)^{n+1}} \frac{\partial f(\omega, \omega^*)}{\partial \omega^*}.$$

E.5.2 Fourier series and other derived transforms

The Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \alpha_n^f(a) (z - a)^n$ may also be rewritten as

$$f(a + q^z) = \sum_{n=-\infty}^{\infty} \alpha_n^f(a) q^{nz}, \quad \forall q \in \mathbb{C} \quad (\text{E.5.13})$$

since it is true in general that $f(g(z)) = \sum_{n=-\infty}^{\infty} \alpha_n^f(a) (g(z) - a)^n$ for any function $g : \mathbb{C} \rightarrow \mathcal{A}(\mathbb{C})$ and $g(z) = a + q^z$ is an example. On the other hand, letting $f \rightarrow f \circ g^{-1}$, we have

$$f(z) = \sum_{n=-\infty}^{\infty} \alpha_n^{f \circ g^{-1}}(a) (g(z) - a)^n,$$

$$\alpha_n^{f \circ g^{-1}}(a) = \frac{1}{2\pi i} \oint_{\Gamma(a)} d\omega \frac{f \circ g^{-1}(\omega)}{(\omega - a)^{n+1}}, \quad a \in C_1^0 \subset \Gamma^0 \subset C_2^0, \quad n \in \mathbb{Z},$$

ie. $0 < |\omega_1 - a| \leq |\omega - a| \leq |\omega_2 - a| \quad \forall \omega \in \Gamma,$

for any invertible $g : \mathbb{C} \rightarrow \mathbb{C}$, where we must now restrict the function f to a domain where g^{-1} is single-valued. For example, in the case $g(z) = a + e^z$, $g^{-1}(z) \in \{g_k^{-1}(z) = 2\pi ki + \ln(z - a), k \in \mathbb{Z}\}$ we must choose only one from the following infinite sequence of regions

$$D_k = \{z = x + iy, x \in \mathbb{R}, 2\pi k \leq y < 2\pi(k + 1)\}, \quad k \in \mathbb{Z}. \quad (\text{E.5.14})$$

The same trick applied to Cauchy's integral formula implies that

$$\begin{aligned} f(z) &\equiv f \circ g^{-1}(g(z)) = \frac{1}{2\pi i} \oint_{\Gamma(z)} d\omega \frac{f \circ g^{-1}(\omega)}{\omega - g(z)} \\ &= \frac{1}{2\pi i} \int_{\Gamma' = g^{-1}(\Gamma(z))} du \frac{dg(u)}{du} \frac{f(u)}{g(u) - g(z)} \quad \forall g, \end{aligned}$$

whenever $f \circ g^{-1}$ has no singularities in $D = \Gamma^0 \cup \Gamma$ and g^{-1} is single-valued on $\partial D = \Gamma$ (ie. $g(u_1) = g(u_2) \Rightarrow u_1 = u_2 \quad \forall u_1, u_2 \in \Gamma' = g^{-1}(\Gamma)$).

Thus if $f \circ g^{-1}$ is singular (ie. undetermined) at 0 (ie. f is singular at $g^{-1}(0)$) [and g^{-1} is single-valued on $\partial D = \Gamma$] then $\Gamma^0 \cup \Gamma$ must be chosen to avoid this singularity and thus in a strip about $g^{-1}(0)$ f will have the Laurent expansion

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} \alpha_n^{f \circ g^{-1}} (g(z))^n \equiv \sum_{n=-\infty}^{\infty} \tilde{f}_g(n) (g(z))^n, \\ \alpha_n^{f \circ g^{-1}} &= \frac{1}{2\pi i} \oint_{\Gamma(0)} d\omega \frac{f \circ g^{-1}(\omega)}{\omega^{n+1}} = \frac{1}{2\pi i} \int_{\Gamma' = g^{-1}(\Gamma(0))} du \frac{dg(u)}{du} \frac{f(u)}{(g(u))^{n+1}} \equiv \tilde{f}_g(n), \\ 0 &\in C_1^0 \subset \Gamma^0 \subset C_2^0, \quad n \in \mathbb{Z}, \quad \omega = g(u), \end{aligned}$$

$$\text{ie. } 0 < |\omega_1| \leq |\omega| \leq |\omega_2| \quad \forall \omega \in \Gamma. \quad 0 < |\omega_1| < |g(z)| < |\omega_2| \quad \forall \omega_1 \in C_1, \omega_2 \in C_2,$$

(E.5.15)

where $\Gamma = \Gamma(0)$ means that Γ is a closed curve in a strip S about 0 [note that the expansion of f is about $g^{-1}(0)$ and the corresponding image curve is $\Gamma' = g^{-1}(\Gamma) \simeq \Gamma'(g^{-1}(0))$, a curve in or on $g^{-1}(S)$ that may approach but may not reach $g^{-1}(0)$]. One notes that $\ln 0 = \infty$ (ie. in the case of

$g(z) = e^z = e^x e^{iy} = e^x \cos(y) + i e^x \sin(y)$, $g^{-1}(z) = \ln z = \ln |z| + i \text{Arg}(z)$. If $\Gamma(0) = \{\omega\}$ is chosen to be any circle of radius r , $\varepsilon = |\omega_1| \leq r = |\omega| \leq \rho = |\omega_2|$ centered at 0, then the resulting series is

$$f(z) = \sum_{n=-\infty}^{\infty} \tilde{f}_n e^{nz}, \quad \tilde{f}_n = \frac{1}{2\pi i} \int_{\ln r - \pi i}^{\ln r + \pi i} du f(u) e^{-nu} \equiv \frac{1}{2\pi i} \int_{\ln r}^{\ln r + 2\pi i} du f(u) e^{-nu},$$

$$\varepsilon = |\omega_1| \leq |\omega| = |e^u| \leq \rho = |\omega_2|, \quad \varepsilon = |\omega_1| < |e^z| = e^{\text{Re}(z)} = e^x \leq \rho = |\omega_2|,$$

$$\Rightarrow \ln \varepsilon \leq x = \text{Re}(z) \leq \ln \rho \quad (\text{convergence requirement}). \quad (\text{E.5.16})$$

Therefore if $\varepsilon \rightarrow 0$, $\rho \rightarrow \infty$ then $-\infty < x = \text{Re}(z) < \infty$ and so

$$f(z) \equiv f_\gamma(z) = \sum_{n=-\infty}^{\infty} \tilde{f}_n e^{nz},$$

$$\tilde{f}_n = \frac{1}{2\pi i} \int_{\gamma - \pi i}^{\gamma + \pi i} du f(u) e^{-nu} \equiv \frac{1}{2\pi i} \int_\gamma^{\gamma + 2\pi i} du f(u) e^{-nu},$$

$$-\infty < \gamma < \infty, \quad -\infty < x = \text{Re}(z) < \infty,$$

ie. $\forall z \in \mathbb{C}$ & $\forall f$ st. f is may be undetermined only at ∞ .

The integral $\frac{1}{2\pi i} \int_{-\pi i}^{\pi i} du e^{nu} e^{-mu} = \frac{\sin(n-m)\pi}{(n-m)\pi} = \delta_{nm}$ (the analog of $\oint_{\Gamma(a)} dz \frac{(z-a)^{n-m}}{z-a} = \delta_{nm}$) is useful for motivating the series from an alternative point of view where $\{e_n(z) = e^{nz}, n \in \mathbb{Z}\}$ may be regarded as a complete set of orthonormal functions in terms of which $f(z)$ can be expanded. One can similarly define a continuous series with the help of the function:

$$\lim_{a \rightarrow \infty} \frac{1}{2a} \int_{-a}^a dq e^{uq} e^{-vq} = \lim_{a \rightarrow \infty} \frac{\sin(u-v)a}{(u-v)a} = \delta_{uv}.$$

$$\lim_{a \rightarrow \infty} \frac{1}{2} \int_{-a}^a dq e^{uq} e^{-vq} = \lim_{a \rightarrow \infty} \frac{\sin(u-v)a}{(u-v)} = \delta(u-v). \quad (\text{E.5.17})$$

In this case, if one considers only periodic functions of the form $y \equiv y + 2\pi$ then the choice of $\Gamma(0)$ is no longer restricted to the region where $g^{-1}(z) = \ln z$ is single-valued but is only restricted by the singular/non-singular requirement for f as usual.

Notice that in the integral formula with transformed contour C

$$f(z) = \frac{1}{2\pi i} \int_{C=g^{-1}(\Gamma(z))} du \frac{dg(u)}{du} \frac{f(u)}{g(u) - g(z)}, \quad (\text{E.5.18})$$

setting $\frac{1}{f(z)} = \frac{dg(z)}{dz}$ implies that

$$\frac{1}{\frac{dg(z)}{dz}} = \frac{1}{2\pi i} \int_{C=g^{-1}(\Gamma(z))} \frac{du}{g(u) - g(z)}, \quad (\text{E.5.19})$$

where $g' \circ g^{-1}$ has no singularities in $D = \Gamma^0 \cup \Gamma$ and g^{-1} is single-valued on $\partial D = \Gamma$.

In the case $g(z) = \frac{az+b}{cz+d}$, $g^{-1}(z) = -\frac{zd-b}{zc-a}$ for example one has

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C=g^{-1}(\Gamma(z))} du \frac{dg(u)}{du} \frac{f(u)}{g(u) - g(z)} \\ &= \frac{1}{2\pi i} \int_{C=-\frac{\Gamma(z)d-b}{\Gamma(z)c-a}} du \frac{cz+d}{cu+d} \frac{f(u)}{u-z}, \end{aligned}$$

where $f \circ g^{-1}(\omega) = f(-\frac{\omega d-b}{\omega c-a})$ has no singularities in $D(z) = \Gamma^0(z) \cup \Gamma(z)$ and $g^{-1}(\omega) = -\frac{\omega d-b}{\omega c-a}$ is single-valued on $\partial D(z) = \Gamma(z)$.

E.5.3 Groups of invertible functions and related transforms

To summarize the properties of the contour integral, let Γ be a closed contour with interior Γ^0 and

$$\delta_{\Gamma}(z) = \begin{cases} 1, & z \in D = \Gamma^0 \cup \Gamma \\ 0, & z \notin D = \Gamma^0 \cup \Gamma \end{cases}, \quad (\text{E.5.20})$$

then

$$\begin{aligned} \oint_{\Gamma} \frac{d\omega}{\omega - g(z)} &= \delta_{\Gamma}(g(z)), \quad \text{ie.} \quad \oint_{\Gamma(g(z))} \frac{d\omega}{\omega - g(z)} = 1 \quad \forall g. \\ \oint_{\Gamma(g(z))} \frac{d\omega}{\omega - g(z)} &= \int_{C(z)=g^{-1}(\Gamma(g(z)))} \frac{dg(u)}{du} \frac{du}{g(u) - g(z)} = 1. \end{aligned} \quad (\text{E.5.21})$$

If g^{-1} exists and $f \circ g^{-1}$ is non-singular in $D = \Gamma^0 \cup \Gamma$, $\Gamma^0 \ni g(z)$; ie. $\Gamma = \Gamma(g(z))$, then

$$f(z) = f \circ g^{-1}(g(z)) = \oint_{\Gamma(g(z))} d\omega \frac{f \circ g^{-1}(\omega)}{\omega - g(z)} = \int_{C(z)=g^{-1}(\Gamma(g(z)))} du \frac{dg(u)}{du} \frac{f(u)}{g(u) - g(z)}. \quad (\text{E.5.22})$$

That is $\forall f, g, C$ such that $g(C) = \Gamma$ is a closed contour, $f \circ g^{-1}$ is non-singular in $D = \Gamma \cup \Gamma^0$ and g^{-1} is single-valued in $D = \Gamma \cup \Gamma^0$ we have

$$f(z) = f \circ g^{-1}(g(z)) = \int_{C(z)=g^{-1}(\Gamma(g(z)))} du \frac{dg(u)}{du} \frac{f(u)}{g(u) - g(z)}. \quad (\text{E.5.23})$$

It may also be possible to restrict f and/or g to a class of functions where C would also be a closed contour.

If $G = \{g \in \mathcal{F}(\mathbb{C}), g^{-1} \exists\}$ is a group of complex invertible functions (maps in general) with function composition as the group product, then any given function f has a G -representation f_G for all possible G 's and may be decomposed, for each G , through the insertion of an identity as follows

$$\begin{aligned} f_G(z) &= \frac{1}{|G|} \sum_{g \in G} f \circ g^{-1}(g(z)) \equiv \int_{g \in G} d\mu(g) f \circ g^{-1}(g(z)) \\ &= \int_{g \in G} d\mu(g) \sum_{n=-\infty}^{\infty} \alpha_n^{f \circ g^{-1}}(0) (g(z))^n = \sum_{\mathcal{J}(n,g) \in \mathbb{Z} \times G} d\mu(g) \tilde{f}_n^g (g(z))^n, \\ \tilde{f}_n^g &= \alpha_n^{f \circ g^{-1}}(0), \quad \int_{g \in G} d\mu(g) 1(g) = 1, \quad \sum_{g \in G} 1(g) = |G|. \end{aligned} \quad (\text{E.5.24})$$

Such decompositions may be used to represent solutions, of differential equations, which typically determine \tilde{f} . Boundary/initial conditions can then be used to determine the actual form or “shape” of G . Note that G may also be chosen to contain the space of inverses of g if one wishes to extend to domains where g^{-1} is not unique.

If one takes the example $G = \{g : z \mapsto g(z) = e^{\omega z}, \quad d\mu(g) = \frac{d\omega}{\omega},$
 $\omega \in \Gamma = \Gamma(0) = \partial D, \quad 0 \in D \subset \mathbb{C}\}$, as $\oint_{\Gamma(a)} \frac{d\omega}{\omega - a} = 1$, then

$$\begin{aligned}
f_{\Gamma(a)}(z) &= \oint_{\Gamma(a)} \frac{d\omega}{\omega - a} \sum_{n=-\infty}^{\infty} \alpha_n^{f \circ g^{-1}}(0) e^{n\omega z} = \oint_{\Gamma(a)} \frac{d\omega}{\omega - a} \sum_{n=-\infty}^{\infty} \alpha_n^{f \circ g_n^{-1}}(0) e^{\omega z} \\
&= \oint_{\Gamma(a)} d\omega \tilde{f}_a(\omega) e^{\omega z}, \\
f \circ g^{-1}(z) &= f\left(\frac{1}{\omega} \ln z\right), \quad f \circ g_n^{-1}(z) = f\left(\frac{n}{\omega} \ln z\right). \\
\alpha_n^{f \circ g_n^{-1}}(0) &= \oint_{C(0)} dv \frac{f \circ g_n^{-1}(v)}{v^{n+1}} = \oint_{C(0)} dv \frac{f\left(\frac{n}{\omega} \ln(v)\right)}{v^{n+1}} \\
&= \int_{C' = g_n^{-1}(C(0))} du \frac{dg_n(u)}{du} \frac{f(u)}{(g_n(u))^{n+1}} = \int_{C' = \frac{n}{\omega} \ln(C(0))} du \frac{\omega}{n} e^{-\frac{\omega u}{n}} f(u). \\
\tilde{f}_a(\omega) &= \frac{1}{\omega - a} \sum_{n=-\infty}^{\infty} \alpha_n^{f \circ g_n^{-1}}(0) = \frac{1}{\omega - a} \oint_{C(0)} dv \sum_{n=-\infty}^{\infty} \frac{f\left(\frac{n}{\omega} \ln(v)\right)}{v^{n+1}} \\
&= \frac{\omega}{\omega - a} \sum_{n=-\infty}^{\infty} \int_{C' = \frac{n}{\omega} \ln(C(0))} du f(u) \frac{e^{-\frac{\omega u}{n}}}{n} \\
&= \frac{\omega}{\omega - a} \int_{C' = \frac{1}{\omega} \ln(C(0))} du e^{-\omega u} \sum_{n=-\infty}^{\infty} f(nu). \\
|v_1| \leq |v| \leq |v_2|, \quad 0 < |v_1| < |e^{\omega z}| < |v_2|.
\end{aligned} \tag{E.5.17}$$

This Fourier-like transform verifies the existence of the Fourier transform.

E.5.4 Several variables

We may also consider n complex variables $Z = (z_1, \dots, z_n)$ for which case the integral formula applied to each argument, of the holomorphic function, separately becomes

$$\begin{aligned}
f(Z) &= \frac{1}{(2\pi i)^n} \oint_{S(Z)} d^n \Omega \frac{f(\Omega)}{\prod_{i=1}^n (z_i - \omega_i)}, \quad \Omega = (\omega_1, \dots, \omega_n), \\
\oint_{S(Z)} d^n \Omega &\equiv \oint_{C_1(z_1)} d\omega_1 \oint_{C_2(z_2)} d\omega_2 \dots \oint_{C_n(z_n)} d\omega_n.
\end{aligned} \tag{E.5.17}$$

One may write $Z_1 = (z_1, 0, \dots, 0)$, $Z_2 = (0, z_2, 0, \dots, 0), \dots, Z_n = (0, \dots, 0, z_n)$, then for each i contour $C_i(z_i)$ can be replaced by a $(2n - 1)$ -dimensional (hollow) cylinder-like hypersurface $\mathcal{C}_i(Z_i)$ in \mathbb{C}^n and thus $S(Z) = \bigcap_i \mathcal{C}_i(Z_i)$ is the $(2n - 1)$ -dimensional hypersurface in \mathbb{C}^n formed by the intersection $\bigcap_i \mathcal{C}_i(Z_i)$ of the $(2n - 1)$ -dimensional (hollow) cylinder-like hypersurfaces.

Similarly one can define a Fourier-like transform

$$f_{S(A)}(Z) = \oint_{S(A)} d^n \Omega \tilde{f}_A(\Omega) e^{\Omega Z} = \oint_{\bigcap_i \mathcal{C}_i(A_i)} d^n \Omega \tilde{f}_A(\Omega) e^{\Omega Z},$$

$$S(A) = \bigcap_i \mathcal{C}_i(A_i), \quad A_1 = (a_1, 0, \dots, 0), \quad A_2 = (0, a_2, 0, \dots, 0), \dots, \quad A_n = (0, \dots, 0, a_n).$$

E.6 Some inequalities

E.6.1 Young's inequality

Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $\varphi(0) = 0$, $\lim_{x \rightarrow \infty} \varphi(x) = +\infty$ be increasing (ie. $\frac{d\varphi(x)}{dx} \equiv \varphi'(x) \geq 0 \quad \forall x \geq 0$). Then φ^{-1} is also increasing as $\varphi^{-1}(\varphi(x)) = x \Rightarrow \varphi^{-1'}(\varphi(x)) = \frac{1}{\varphi'(x)} \geq 0$. We also have

$$f(c) = \int_0^c dx \varphi(x) + \int_0^{\varphi(c)} dx \varphi^{-1}(x) = c\varphi(c) \quad \forall c \geq 0 \quad (\text{E.6.1})$$

since $f'(c) = \varphi(c) + \varphi'(c) \varphi^{-1}(\varphi(c)) = \varphi(c) + c\varphi'(c) = (c\varphi(c))'$.

Therefore the *continuous* function

$$g : \mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad a \mapsto g(a) = \frac{ab}{\int_0^a dx \varphi(x) + \int_0^b dx \varphi^{-1}(x)} \equiv h(a, b), \quad b \in \mathbb{R}^+$$

is stationary at $a = \varphi^{-1}(b)$ (by $g'(a) \equiv \partial_a h(a, b) = 0$).

Furthermore one can check that

$$g(\varphi^{-1}(b)) = 1, \quad \lim_{a \rightarrow 0} g(a) = 0 = \lim_{a \rightarrow +\infty} g(a), \quad (\text{E.6.1})$$

hence $g(a) \leq 1 \quad \forall a$ since g is continuous. That is, we have the inequality

$$ab \leq \int_0^a dx \varphi(x) + \int_0^b dx \quad \forall a, b \quad (\text{E.6.2})$$

where equality holds when $b = \varphi(a)$.

Setting $\varphi(x) = x^{p-1}$, $p \in \mathbb{R}^+$, the conditions $\varphi(0) = 0$, $\lim_{x \rightarrow \infty} \varphi(x) = \infty$ are satisfied if $p > 1$ and one obtains

$$ab \leq \frac{a^p}{p} + \frac{b^{\frac{p}{p-1}}}{\frac{p}{p-1}} \equiv \frac{a^p}{p} + \frac{b^{p'}}{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (\text{E.6.2})$$

Equality holds iff $a^p = b^{p'}$.

E.6.2 Holder's inequality

With $p > 1$ define $\|f\|_p = (\int d\mu(x) |f(x)|^p)^{\frac{1}{p}} \equiv (\int d\mu |f|^p)^{\frac{1}{p}}$ and set

$a = \frac{|f(x)|}{\|f\|_p}$, $b = \frac{|g(x)|}{\|g\|_{p'}}$. Then

$$\begin{aligned} \frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_{p'}} &\leq \frac{1}{p} \frac{|f(x)|^p}{(\|f\|_p)^p} + \frac{1}{p'} \frac{|g(x)|^{p'}}{(\|g\|_{p'})^{p'}} \\ \frac{|f(x)g(x)|}{\|f\|_p \|g\|_{p'}} &\leq \frac{1}{p} \frac{|f(x)|^p}{(\|f\|_p)^p} + \frac{1}{p'} \frac{|g(x)|^{p'}}{(\|g\|_{p'})^{p'}} \\ \frac{1}{\|f\|_p \|g\|_{p'}} \int d\mu |fg| &\leq \frac{1}{p} \frac{\int d\mu |f|^p}{(\|f\|_p)^p} + \frac{1}{p'} \frac{\int d\mu |g|^{p'}}{(\|g\|_{p'})^{p'}} = \frac{1}{p} + \frac{1}{p'} = 1, \\ \int d\mu |fg| &\leq \|f\|_p \|g\|_{p'}. \end{aligned} \quad (\text{E.6.0})$$

Equality holds iff $|f(x)|^p = \alpha |g(x)|^{p'}$, $\alpha \in \mathbb{R}^+$, $\alpha \neq 0$.

For $0 < p < 1$, $q = \frac{1}{p} > 1$, writing $f = u^{-p} = u^{-\frac{1}{q}}$, $g = u^{\frac{1}{q}} v^{\frac{1}{q}} = (uv)^{\frac{1}{q}}$, $u(x) \geq 0$, $v(x) \geq 0 \quad \forall x$ one obtains

$$\int d\mu uv \geq \left(\int d\mu v^p \right)^{\frac{1}{p}} \left(\int d\mu u^{p'} \right)^{\frac{1}{p'}}. \quad (\text{E.6.1})$$

E.6.3 Minkowski's inequality

For $p > 1$

$$\begin{aligned}
(\|f + g\|_p)^p &= \int d\mu |f + g|^p = \int d\mu |f + g|^{p-1} |f + g| \leq \int d\mu |f + g|^{p-1} (|f| + |g|) \\
&= \int d\mu |f + g|^{p-1} |f| + \int d\mu |f + g|^{p-1} |g| \\
&\leq \|f\|_p \int (d\mu |f + g|^{(p-1)p'})^{\frac{1}{p'}} + \|g\|_p \int d\mu |f + g|^{(p-1)p'} \\
&= (\|f\|_p + \|g\|_p) \int (d\mu |f + g|^p)^{\frac{1}{p}} = (\|f\|_p + \|g\|_p) (\|f + g\|_p)^{\frac{p}{p'}}, \\
(\|f + g\|_p)^{p - \frac{p}{p'}} &= \|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{E.6.-2}
\end{aligned}$$

For $0 < p < 1$ the same argument and Holder's inequality for $0 < p < 1$ gives

$$\|f + g\|_p \geq \|f\|_p + \|g\|_p. \tag{E.6.-1}$$

E.7 Map continuity

A map $f : \mathcal{A} \rightarrow \mathcal{B}$ between two linear metric spaces $\mathcal{A} = (\mathcal{A}, \|\cdot\|)$, $\mathcal{B} = (\mathcal{B}, \|\cdot\|)$ is continuous iff any of the following is true

- (1) (uniformly) $x \rightarrow y \Rightarrow f(x) \rightarrow f(y)$.
- (2) (uniformly) $|x - y| \rightarrow 0 \Rightarrow |f(x) - f(y)| \rightarrow 0$.
- (3) (uniformly) $|x - y| < \varepsilon \rightarrow 0^+$
 $\Rightarrow \exists \delta = \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} 0^+ \text{ st } |f(x) - f(y)| < \delta(\varepsilon)$.
- (4) $\forall B_\varepsilon(x), \varepsilon \rightarrow 0^+, \exists B_{\delta(\varepsilon)}(f(x))$
 $\text{st } f(B_\varepsilon(x)) \subseteq B_{\delta(\varepsilon)}(f(x)), \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} 0^+$
- where $B_\varepsilon(x) = \{y, |x - y| < \varepsilon\}$.
- (5) [if $f^{-1} \exists$] $\forall B_\varepsilon(x), \varepsilon \rightarrow 0^+, \exists B_{\delta(\varepsilon)}(f(x))$
 $\text{st } B_\varepsilon(x) \subseteq f^{-1}(B_{\delta(\varepsilon)}(f(x))), \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} 0^+$.

(E.7.-8)

It follows that a composition $f \circ g$ of two continuous maps f, g is continuous since

$$\begin{aligned}
 |x - y| < \varepsilon &\Rightarrow |g(x) - g(y)| < \delta(\varepsilon) \equiv \varepsilon_g \\
 &\Rightarrow |f \circ g(x) - f \circ g(y)| < \delta(\varepsilon_g).
 \end{aligned}$$

The same is true for sums and products of continuous maps by the triangle inequality:

$$\begin{aligned}
|x - y| < \varepsilon &\Rightarrow \\
|f(x) + g(x) - (f(y) + g(y))| &\leq |f(x) - f(y)| + |g(x) - g(y)| \\
&< \delta_f(\varepsilon) + \delta_g(\varepsilon) \equiv \delta(\varepsilon), \\
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\
&\leq |f(x) - f(y)||g(x)| + |f(y)||g(x) - g(y)| \\
&< \delta_f(\varepsilon)|g(x)| + \delta_g(\varepsilon)|f(y)| \equiv \delta(\varepsilon).
\end{aligned}
\tag{E.7.-15}$$

A set S is open iff for any $x \in S$ one can find $B_\varepsilon(x) \subseteq S$. Notice that in the definition of map continuity if \mathcal{B} is open then $B_{\delta(\varepsilon)}(f(x)) \subseteq \mathcal{B}$ for sufficiently small ε . But if $f^{-1} \exists$ then $B_{\delta(\varepsilon)}(f(x)) \subseteq \mathcal{B} \Rightarrow f^{-1}(B_{\delta(\varepsilon)}(f(x))) \subseteq f^{-1}(\mathcal{B}) = \mathcal{A}$ and hence $B_\varepsilon(x) \subseteq f^{-1}(B_{\delta(\varepsilon)}(f(x)))$ guarantees that \mathcal{A} must also be open since one has $B_\varepsilon(x) \subseteq f^{-1}(B_{\delta(\varepsilon)}(f(x))) \subseteq \mathcal{A}$ and this is true for any $x \in \mathcal{A}$. Therefore the map continuity condition implies that the inverse image of any open set is open. For the converse, if the inverse image of every open set is open under f then for any $x \in \mathcal{A}$ $f^{-1}(B_\delta(f(x)))$ is open for all $\delta > 0$ since $B_\delta(f(x))$ is open. Now since $x \in f^{-1}(B_\delta(f(x)))$ one can find $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq f^{-1}(B_\delta(f(x)))$ and in particular, since $\delta > 0$ was arbitrary, one can choose $\delta = \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0^+} 0^+$, which is the condition for continuity. Hence a map f is continuous iff the inverse image of every open set is open. One observes here that map continuity can also be stated as: for any nbd $B_\delta(f(x))$ one can find $\varepsilon = \varepsilon(\delta)$ such that $f(B_\varepsilon(x)) \subseteq B_\delta(f(x))$ OR for any nbd $B_\delta(u)$ of $u \in \mathcal{B}$ one can find $\varepsilon = \varepsilon(\delta)$ such that $f(B_\varepsilon(f^{-1}(u))) \subseteq B_\delta(u)$

A map $f : D \subseteq \mathcal{A} \rightarrow \mathcal{B}$ is said to be (uniformly) bounded if

$$\forall x, y \in D, \quad |f(x) - f(y)| \leq M, \quad 0 \leq M < \infty. \quad (\text{E.7.-14})$$

A map $f : D \subseteq \mathcal{A} \rightarrow \mathcal{B}$ is (uniformly) differentially bounded iff

$$\forall x, y \in D, \quad |f(x) - f(y)| \leq M|x - y|, \quad 0 \leq M < \infty. \quad (\text{E.7.-13})$$

It is clear that a differentially bounded map is continuous as one may simply set $\delta(\varepsilon) = M\varepsilon$. The composition or sum of two differentially bounded maps is differentially bounded.

In a general metric space (\mathcal{S}, d) rather than a linear metric space $(\mathcal{H}, | \cdot |)$ one needs to replace $|a - b|$ by $d(a, b)$.

E.7.1 Uniform continuity in terms of sets

Uniform continuity means

$$B_\varepsilon(x) \cap B_\varepsilon(y) \neq \{\} \Rightarrow B_{\delta(\varepsilon)}(f(x)) \cap B_{\delta(\varepsilon)}(f(y)) \neq \{\}. \quad (\text{E.7.-12})$$

On the other hand continuity requires

$$f(B_\varepsilon(x)) \subseteq B_{\delta(\varepsilon)}(f(x)), \quad f(B_\varepsilon(y)) \subseteq B_{\delta(\varepsilon)}(f(y)) \quad (\text{E.7.-11})$$

which implies

$$f(B_\varepsilon(x)) \cap f(B_\varepsilon(y)) \subseteq B_{\delta(\varepsilon)}(f(x)) \cap B_{\delta(\varepsilon)}(f(y)). \quad (\text{E.7.-10})$$

Since $f(B_\varepsilon(x) \cap B_\varepsilon(y)) \subseteq f(B_\varepsilon(x)) \cap f(B_\varepsilon(y))$ we identify the condition for uniform continuity as

$$f(B_\varepsilon(x) \cap B_\varepsilon(y)) \subseteq B_{\delta(\varepsilon)}(f(x)) \cap B_{\delta(\varepsilon)}(f(y)) \quad (\text{E.7.-9})$$

which reflects the fact that a uniformly continuous maps is continuous but the converse may not be true. Let a set A be **uniformly open** iff $\forall x, y \in A$ one can find $\varepsilon > 0$ such that $B_\varepsilon(x) \cap B_\varepsilon(y) \subseteq A$. Then a uniformly open set is open but the converse may not be true. Uniform continuity/openness and continuity/openness are equivalent in a separable space (one in which every pair $(x, y; x \neq y)$ of distinct points have disjoint neighborhoods $B_{\varepsilon_1}(x), B_{\varepsilon_2}(y), B_{\varepsilon_1}(x) \cap B_{\varepsilon_2}(y) = \{\}$) since an open set would be automatically uniformly open if one decides that $\{\} \subseteq A$ for any set A , but not necessarily in a nonseparable space.

One can check as in the case of continuity that a map f is uniformly continuous iff the inverse image $f^{-1}(\check{O})$ of every uniformly open set \check{O} is uniformly open.

The fact that the intersection $A \cap B$ of two open sets is open follows because for $a \in A \cap B$, $\exists \varepsilon_1, \varepsilon_2 > 0$ such that

$$B_{\varepsilon_1}(a) \subseteq A, \quad B_{\varepsilon_2}(a) \subseteq B \quad (\text{E.7.-8})$$

which implies that

$$B_{\varepsilon_1}(a) \cap B_{\varepsilon_2}(a) \subseteq A \cap B. \quad (\text{E.7.-7})$$

But this means that $B_{\min(\varepsilon_1, \varepsilon_2)}(a) \subseteq B_{\varepsilon_1}(a) \cap B_{\varepsilon_2}(a) \subseteq A \cap B$ and hence $A \cap B$ is open. One can similarly check that $B_{\max(\varepsilon_1, \varepsilon_2)}(a) \subseteq B_{\varepsilon_1}(a) \cup B_{\varepsilon_2}(a) \subseteq A \cup B$ and hence $A \cup B$ is open. With the same steps one can show that the unions and intersections of uniformly open sets are uniformly open.

E.8 Sequences and series

A sequence s in a set \mathcal{S} is an ordered selection of objects in \mathcal{S} ; ie. a map from the natural numbers \mathbb{N} to a set of objects \mathcal{S} .

$$s : \mathbb{N} \rightarrow \mathcal{S}, \quad n \mapsto s_n. \quad (\text{E.8.1})$$

The sequence s is bounded iff one can find $M \in \mathbb{R}$ such that

$$d(s_n, s_m) \leq M \quad \forall n, m \in \mathbb{N}. \quad (\text{E.8.2})$$

Define the ε neighborhood $\mathcal{N}_\varepsilon(A)$ of a set $A \subseteq \mathcal{S}$ by

$$\mathcal{N}_\varepsilon(A) = \{y \in \mathcal{S}, \quad d(a, y) < \varepsilon, \quad \forall a \in A\} \equiv \bigcup_{a \in A} \mathcal{N}_\varepsilon(a). \quad (\text{E.8.3})$$

A sequence in a metric space \mathcal{S} is convergent (ie. converges to a point $L \in \mathcal{S}$) iff

- 1) $\forall \varepsilon > 0, \exists N = N(\varepsilon) < \infty$ s.t. $d(s_n, L) < \varepsilon \quad \forall n > N(\varepsilon)$.
- 2) $\forall \mathcal{N}_\varepsilon(L) \exists N(\varepsilon) < \infty$ s.t. $s_n \in \mathcal{N}_\varepsilon(L) \quad \forall n > N(\varepsilon)$.
- (3) $\forall N < \infty, \exists 0 < \varepsilon = \varepsilon(N) \xrightarrow{N \rightarrow \infty} 0$ s.t. $\forall n > N, \quad d(s_n, L) < \varepsilon(N)$.
- (3) $\forall N < \infty, \exists 0 < \varepsilon = \varepsilon(N) \xrightarrow{N \rightarrow \infty} 0$ s.t. $n > N \Rightarrow d(s_n, L) < \varepsilon(N)$.
- (4) $\forall N < \infty, \exists 0 < \varepsilon = \varepsilon(N) \xrightarrow{N \rightarrow \infty} 0$ s.t. $n > N \Rightarrow s_n \in \mathcal{N}_{\varepsilon(N)}(L)$.

A sequence in a metric space is (uniformly) converging or Cauchy iff

- 1) $\forall \varepsilon > 0, \exists N = N(\varepsilon) < \infty$ s.t. $d(s_m, s_n) < \varepsilon \quad \forall m, n > N(\varepsilon)$.
- 2) $\forall \varepsilon > 0 \exists N = N(\varepsilon)$ s.t. $\mathcal{N}_\varepsilon(s_m) \cap \mathcal{N}_\varepsilon(s_n) \neq \{\}$ $\forall m, n > N(\varepsilon)$.
- (3) $\forall N < \infty, \exists 0 < \varepsilon = \varepsilon(N) \xrightarrow{N \rightarrow \infty} 0$ s.t. $\forall n, m > N, \quad d(s_n, s_m) < \varepsilon(N)$.
- (3) $\forall N < \infty, \exists 0 < \varepsilon = \varepsilon(N) \xrightarrow{N \rightarrow \infty} 0$ s.t. $n, m > N \Rightarrow d(s_n, s_m) < \varepsilon(N)$.
- (4) $\forall N < \infty, \exists 0 < \varepsilon = \varepsilon(N) \xrightarrow{N \rightarrow \infty} 0$ s.t.
$$m, n > N \Rightarrow \mathcal{N}_{\varepsilon(N)}(s_m) \cap \mathcal{N}_{\varepsilon(N)}(s_n) \neq \{\}.$$

Every convergent sequence, $\lim_{n \rightarrow \infty} s_n = L$, is (uniformly) converging since

$$d(s_m, s_n) \leq d(s_m, L) + d(L, s_n) < \varepsilon + \varepsilon = 2\varepsilon \quad \forall n, m > N(\varepsilon). \quad (\text{E.8.-7})$$

Every Cauchy sequence is bounded; one simply needs to set $M = \max_{N \in \mathbb{N}} \varepsilon(N)$. One can also check that sums and products of Cauchy sequences are Cauchy sequences.

A metric space \mathcal{S} is complete if every Cauchy sequence in \mathcal{S} converges to a point in \mathcal{S} . The Cauchy completion of a space \mathcal{S} is the union of the space \mathcal{S} and the set consisting of the limit points of all Cauchy sequences in \mathcal{S} . That is, a space \mathcal{S} is complete iff any (uniformly) converging sequence in \mathcal{S} converges to a point in \mathcal{S} .

A series $S = S(s)$ is the sum of the terms of a sequence s ,

$$S(s) = \sum_{k=1}^{\infty} s_k. \quad (\text{E.8.-6})$$

A series $S(s)$ is convergent iff the sequence of partial sums

$$S_n = S_n(s) = \sum_{k=1}^n s_k \text{ is convergent.}$$

One can check that ***a set S is open iff only a finite number of points of any sequence that converges to a point $L \in S$ can lie outside of S .***

A Cauchy sequence in a closed set C must converge to a point in C for if it converges to a point in the complement \tilde{C} which is open (ie. CP is closed) then that sequence lies in \tilde{C} instead as it would then have only a finite number of points in C . Thus ***a closed set is complete.*** Also the complement \widetilde{CP} of a complete set CP is open for if \widetilde{CP} were not open then one can find a point $b \in \widetilde{CP}$ such that $N_\varepsilon(b) \cap CP \neq \{\}$ $\forall \varepsilon > 0$ meaning that one can construct a Cauchy sequence in CP that converges to $b \notin CP$ in contradiction to the completeness of CP . Thus ***a complete set is closed*** and hence ***a set is closed iff it is complete.***

The ***closure*** \bar{A} of a set A is its Cauchy ***completion.***

E.9 Connectedness and convexity

A space \mathcal{S} is *connected* iff for any two points $x, y \in \mathcal{S}$ one can find a continuous path $\Gamma : [0, 1] \rightarrow \mathcal{S}$, $t \mapsto \Gamma(t)$ such that $\Gamma(0) = x$, $\Gamma(1) = y$. The points x, y are said to be connected by the path Γ . The space \mathcal{S} is topologically trivial iff its power set $\mathcal{P}(\mathcal{S}) = \{A; A \subseteq \mathcal{S}\}$ is connected.

A metric space (\mathcal{S}, d) is *convex* iff any two points $x, y \in \mathcal{S}$ can be connected by a *unique* continuous path

$$\Gamma_0 \in [x, y] = \{\gamma : [0, 1] \rightarrow \mathcal{S}, t \mapsto \gamma(t), \gamma(0) = x, \gamma(1) = y\} \quad (\text{E.9.1})$$

such that

$$\min_{\gamma \in [x, y]} l[\gamma] = l[\Gamma_0] = d(x, y), \quad l[\gamma] = \int_0^1 d(\gamma(t), \gamma(t + dt)). \quad (\text{E.9.2})$$

E.10 Some topology

A **set** is a collection of objects where each object individually satisfies a certain basic condition.

The inverse or complement \tilde{A} of a set A is given by

$$B = A \cap B \cup \tilde{A} \cap B \quad \forall B. \quad (\text{E.10.1})$$

A set O is (uniformly) **open** (or [uniformly] **continuous**) iff its complement \tilde{O} is (uniformly) **complete**. *The union or intersection of an arbitrary number of open sets is also an open set.*

A set A is said to be **closed**, $A \in \mathcal{C} = \{C\}$, iff its is **complete** (or iff its inverse is open, $\tilde{A} \in \mathcal{O}$). *The union or intersection of any finite number of closed sets is also a closed set.*

A **neighborhood** (nbd) $\mathcal{N}(A)$ of a set A is any open superset of A . That is

$$A \subseteq \mathcal{N}(A) \in \mathcal{O}. \quad (\text{E.10.2})$$

Equality is possible only when A is open. The **closure** \bar{A} of a set A is the intersection of all closed supersets of A and is thus the smallest closed superset of A ,

$$\bar{A} = \bigcap_{A \subseteq C \in \mathcal{C}} C = \min_{A \subseteq C \in \mathcal{C}} C \quad (\text{E.10.3})$$

and the **interior** A^0 of A is the union of all open subsets of A and is thus the largest open subset of A ,

$$A^0 = \bigcup_{A \supseteq O \in \mathcal{O}} O = \max_{A \supseteq O \in \mathcal{O}} O. \quad (\text{E.10.4})$$

The **boundary** ∂A of A is given by

$$\partial A = \widetilde{A^0} \cap \bar{A}. \quad (\text{E.10.5})$$

A point $x \in A$ is a **limit point** of A iff $\mathcal{N}(x) \cap A \neq \{x\} \quad \forall \mathcal{N}(x)$. A set A is closed iff $\bar{A} = A$ iff A contains all its limit points. A set A is open iff $A = A^0$.

A collection $\Sigma = \{\sigma\}$ of (open) sets such that

$$A \subseteq \bigcup_{\sigma \in \Sigma} \sigma$$

is called a **cover** $\Sigma(A)$ of A .

A set A is **compact**, $A \in \mathcal{K} = \{K\}$, if every cover $\Sigma(A)$ contains a finite subcover $\mathcal{O}_n(A)$,

$$\Sigma(A) \supseteq \mathcal{O}_n(A) = \{O_k \in \mathcal{O}, k = 1, \dots, n, A \subseteq \bigcup_{k=1}^n O_k\}. \quad (\text{E.10.6})$$

Since any cover $\Sigma(A \cup B)$ for $A \cup B$ is also a cover of A and of B , it follows that ***the union of a finite number of compact sets is also a compact set.***

A ***space*** is a structured collection of one or more sets whose elements are known as points; the elements or points of the space are obtained through well defined interactions between the elements of the defining sets. A topology $\mathcal{T}(\mathcal{S})$ for a space \mathcal{S} is any subfamily (ie. is closed under union and intersection) of the family of open subsets of \mathcal{S} that covers \mathcal{S} and which contains both \mathcal{S} and $\{\}$. ie. $\mathcal{T}(\mathcal{S}) \subseteq \mathcal{O}(\mathcal{S}) = \{O \subseteq \mathcal{S}; O \in \mathcal{O}\}$, $\cup, \cap : \mathcal{T}(\mathcal{S}) \times \mathcal{T}(\mathcal{S}) \rightarrow \mathcal{T}(\mathcal{S})$,

$\mathcal{S} \subseteq \bigcup \mathcal{T}(\mathcal{S}) = \bigcup_{O \in \mathcal{T}(\mathcal{S})} O$, $\mathcal{S}, \{\} \in \mathcal{T}(\mathcal{S})$. A ***topological space*** is any given pair $X = (\mathcal{S}, \mathcal{T}(\mathcal{S}))$. $\{\}, \mathcal{S}$ are both open and closed as they are members of $\mathcal{T}(\mathcal{S})$ and $\mathcal{S} = \widetilde{\{\}}, \{\} = \widetilde{\mathcal{S}}$.

A topological space $X = (\mathcal{S}, \mathcal{T}(\mathcal{S}))$ is ***separable*** (or ***Hausdorff***) iff for any $A, B \subseteq X$ such that $A \cap B = \{\}$ one can find nbds $\mathcal{N}_1(A), \mathcal{N}_2(B)$ such that

$$\mathcal{N}_1(A) \cap \mathcal{N}_2(B) = \{\}. \quad (\text{E.10.7})$$

A space \mathcal{S} is ***uniformly open*** iff for any $A, B \subseteq \mathcal{S}$ one can find nbds $\mathcal{N}_1(A), \mathcal{N}_2(B)$ whose intersection lies in \mathcal{S} ,

$$\mathcal{N}_1(A) \cap \mathcal{N}_2(B) \subseteq \mathcal{S}. \quad (\text{E.10.8})$$

A space \mathcal{S} is ***locally compact*** iff every point $x \in \mathcal{S}$ has a nbd $\mathcal{N}(x)$ whose closure $\overline{\mathcal{N}(x)}$ is compact.

A subset $D \subseteq \mathcal{S}$ is ***dense*** in \mathcal{S} iff $\overline{D} = \mathcal{S}$. A set \mathcal{S} is ***countable*** iff its is isomorphic to \mathbb{N} ; ie. $\exists i : \mathcal{S} \rightarrow \mathbb{N}$.

A ***map*** m between two topological spaces $m : S \rightarrow T$ is ***continuous*** iff the inverse image $m^{-1}(O)$ of every open set $O \subseteq T$ is open, ie. $m^{-1}(O)$ belongs to $\mathcal{O}(S)$.

A **sequence** $s : \mathbb{N} \rightarrow S$, $n \mapsto s_n$ in a topological space S **converges** to a point L iff

$$\forall \mathcal{N}(L) \exists N = N(\mathcal{N}(L)) < \infty \text{ s.t. } s_n \in \mathcal{N}(L) \quad \forall n > N. \quad (\text{E.10.9})$$

That is, every nbd of L contains an infinite number of points of the sequence since there is an infinite number of terms between ∞ and any $N < \infty$.

A **sequence** on a topological space S is **Cauchy** iff

$$\forall O \in \mathcal{O}(S) \exists N = N(O) < \infty \text{ s.t. } \mathcal{N}_O(s_m) \cap \mathcal{N}_O(s_n) \neq \{\} \quad \forall m, n > N(O)$$

where $\mathcal{N} : \mathcal{O}(S) \rightarrow \mathcal{O}(S)$, $O \mapsto \mathcal{N}_O$ is a map that assigns O as a nbd \mathcal{N}_O of a point or set. That is, each point s_n of the sequence becomes increasingly nonseparable from its neighbors as n increases. If the set of all nbds of $A \subseteq S$ is $\mathcal{N}[A]$ then

$$\mathcal{N}[A] \ni \mathcal{N}_O(A) = \begin{cases} \{\}, & A \not\subseteq O \\ O, & A \subseteq O. \end{cases} \quad (\text{E.10.9})$$

Every convergent sequence, $\lim_{n \rightarrow \infty} s_n = L$, is a Cauchy sequence since

$$\mathcal{N}_O(s_m) \cap \mathcal{N}_O(s_n) \supseteq \mathcal{N}_O(s_m) \cap \mathcal{N}_O(L) \cup \mathcal{N}_O(L) \cap \mathcal{N}_O(s_n) \neq \{\}. \quad (\text{E.10.10})$$

A topological space S is **complete** iff every Cauchy sequence in S converges to a point in S . The **Cauchy completion** of a space S is the union of the space S and the set consisting of the limit points of all Cauchy sequences in S .

A map m is a **P -map** iff the image $m(A)$ has the property P whenever A has the property P ; ie. m preserves the property P . For example one has singular/nonsingular maps, open/closed maps, measurable/nonmeasurable maps, bounded/unbounded maps, compact/noncompact maps, connected/nonconnected, convex/nonconvex, etc.

Since the identity map (or linear map in general) is both invertible and open it follows that continuity of a space and its open topology are equivalent concepts. That is, a continuous or topological space is one that has an open topology and continuity of a map m measures how much of the continuity or topology of a space is preserved by the inverse map m^{-1} . Thus reassigning continuity to sets means that a map m is said to be continuous iff m^{-1} is a continuous map.

E.11 More on compactness and separability

We work in a Hausdorff space where any two disjoint sets have disjoint nbds. For simplicity we will denote $A \cap B$ as AB and $A \cup B$ as $A + B$.

- Let K be compact and C be closed. Then \tilde{C} is open. If $\Sigma(KC)$ is any cover for KC ; ie.

$$KC \subseteq \bigcup_{\sigma \in \Sigma(KC)} \sigma$$

then

$$K = KC + K\tilde{C} \subseteq \bigcup_{\sigma \in \Sigma(KC)} \sigma + K\tilde{C} \subseteq \bigcup_{\sigma \in \Sigma(KC)} \sigma + \tilde{C}$$

and so $\{\Sigma(KC), \tilde{C}\}$ is a cover for K and therefore has a finite subcover $\mathcal{O}_n(K) = \{O_1, \dots, O_n, \tilde{C}\}$ as K is compact. That is $K \subseteq \bigcup_{k=1}^n O_k + \tilde{C}$, which implies that $KC \subseteq \bigcup_{k=1}^n O_k$ which means that $\{O_1, \dots, O_n\}$ is a finite subcover for KC and hence KC is also compact. That is, if K is compact and C is closed then KC is compact. It follows that ***every closed subset of a compact set is also compact.***

- Let K be compact and O, O' be open and $K \subseteq O + O'$. Then $K \subseteq O + O' \Rightarrow \tilde{K} \supseteq \tilde{O}\tilde{O}' \Rightarrow K\tilde{K} \supseteq K\tilde{O} K\tilde{O}' \Rightarrow \{\} \supseteq K\tilde{O} K\tilde{O}'$

$\Rightarrow K\widetilde{O} \cap K\widetilde{O}' = \{\}$. Therefore $K\widetilde{O}$ and $K\widetilde{O}'$ are disjoint compact sets, since $\widetilde{O}, \widetilde{O}'$ are closed and K is compact, and since we are in a Hausdorff space we can find disjoint open sets O_1, O_2 such that $K\widetilde{O} \subseteq O_1$ and $K\widetilde{O}' \subseteq O_2$.

$$\begin{aligned} K\widetilde{O} \subseteq O_1 &\Rightarrow \widetilde{O}_1 \subseteq \widetilde{K} + O \Rightarrow K_1 = K\widetilde{O}_1 \subseteq O, \\ K\widetilde{O}' \subseteq O_2 &\Rightarrow \widetilde{O}_2 \subseteq \widetilde{K} + O' \Rightarrow K_2 = K\widetilde{O}_2 \subseteq O'. \end{aligned} \tag{E.11.-1}$$

Therefore we have found compact sets K_1, K_2 such that $K_1 \subseteq O$, $K_2 \subseteq O'$ and $K_1 + K_2 = K\widetilde{O}_1 + K\widetilde{O}_2 = K(\widetilde{O}_1 + \widetilde{O}_2) = K\widetilde{O}_1\widetilde{O}_2 = K\widetilde{\{\}} = K$.

- Let $A \subseteq B$ in a Hausdorff space. Since $A \subseteq B \Rightarrow A\widetilde{B} = \{\}$, one can find disjoint open sets O_1, O_2 such that $A \subseteq O_1$, $\widetilde{B} \subseteq O_2$ where $\widetilde{B} \subseteq O_2 \Rightarrow \widetilde{O}_2 \subseteq B$.

But $O_1 \cap O_2 = \{\} \Rightarrow O_1 \subseteq \widetilde{O}_2$ and therefore one has the sandwich relations

$$A \subseteq O_1 \subseteq \widetilde{O}_2 \subseteq B \tag{E.11.0}$$

which can also be iterated to obtain sequences of inclusions. Thus given any two sets A, B in a Hausdorff space one can connect them with sequences through $A \cap B$ and/or $A \cup B$ since

$$A \cap B \subseteq A \subseteq A \cup B, \quad A \cap B \subseteq B \subseteq A \cup B. \tag{E.11.1}$$

E.12 On the realization of compact spaces

Here "cover" will mean "open cover".

Let a *minimal or essential cover* for a set \mathcal{S} be one that contains no proper subcovers. That is $\Sigma_0(\mathcal{S})$ is minimal iff

$$\Sigma(\mathcal{S}) \subseteq \Sigma_0(\mathcal{S}) \Rightarrow \Sigma(\mathcal{S}) = \Sigma_0(\mathcal{S}). \quad (\text{E.12.1})$$

It follows that any cover of \mathcal{S} can be generated from one or more minimal covers. That is the set of all minimal covers of \mathcal{S} is basic and generates the rest of the nonminimal covers.

Then compactness of \mathcal{S} implies that every minimal cover of \mathcal{S} is a finite cover since every cover contains a finite subcover and this also implies that any subcover of the finite cover must in turn contain a finite subcover. Conversely, suppose that every minimal cover of a space \mathcal{S} is finite. Then \mathcal{S} must be compact since (by the generating property of the set of minimal covers) every cover of \mathcal{S} contains at least one minimal cover, which is finite (a finite subcover) by the supposition. This means that ***a set \mathcal{S} is compact iff every minimal cover of \mathcal{S} is finite.*** In other words a compact set is one that is essentially finite in the topological sense.

A nonempty open set $O \neq \{\}$ in a (separable) metric space cannot be compact since

$\lim_{\varepsilon \rightarrow 0} \{\mathcal{N}_\varepsilon(a), a \in O\}$ is a minimal cover of O that is not finite. Partially open sets cannot be compact either since an open set, which is not compact, can be obtained through the union of a finite number of partially open sets. Also unbounded sets are isomorphs of open and partially open sets and so cannot be compact. Hence ***a compact set in a metric space must be closed and bounded.***

One also notes that the image $m(\mathcal{S})$ of a compact space \mathcal{S} under an isomorphic map $m : \mathcal{S} \rightarrow m(\mathcal{S})$ is also compact. It follows therefore that a (free) compact space is actually an equivalence class of all such spaces under all possible isomorphisms. In particular ***a space is compact iff it is compact as a subspace.*** To see

this, one notes that a compact space \mathcal{S} is a compact subspace of itself. Conversely if \mathcal{S} is a compact subspace of some space then the equivalence class $[\mathcal{S}] = \{i(\mathcal{S}), i : \mathcal{S} \rightarrow i(\mathcal{S}), i \in I\}$ of its images under all possible isomorphisms $I = \{i\}$ generates the (free) compact space.

[To verify that the image of a compact set \mathcal{S} under an isomorphism i is compact, one notes that in general

$f(A \cup B) \subseteq f(A) \cup f(B)$, $f(A \cap B) \subseteq f(A) \cap f(B)$ since either of $A \cup B$ and $A \cap B$ has less points to transform than has A and B separately. But under an isomorphism ($i(a) = i(b)$ iff $a = b$) one has

$$i(A \cup B) = i(A) \cup i(B), \quad i(A \cap B) = i(A) \cap i(B) \quad \forall A, B \quad (\text{E.12.2})$$

and so all the structures and/or statements that characterize compactness are preserved implying that if \mathcal{S} is compact then so is $i(\mathcal{S})$. It may also be worth recalling that $A \subseteq B$ iff $A \cap B = A$ iff $A \cup B = B$. Also ***the direct product of a finite number of compact spaces is also a compact space.***]

Thus whenever possible one can check noncompactness of a space \mathcal{S} by embedding it into a (separable) metric space (\mathcal{H}, d) , $i : \mathcal{S} \rightarrow S \subseteq (\mathcal{H}, d)$ and using the fact that a compact subspace S of a (separable) metric space (\mathcal{H}, d) must be closed (\tilde{S} is open in $(\mathcal{H}, d : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}^+)$) and bounded ($\max_{x,y \in S} d(x,y) < \infty$).

The arguments concerning compactness may be adapted to other properties such as openness (or continuity), closedness (or completeness), connectedness, convexity, measurability and so on.

E.13 Metric topology of \mathbb{R}

In \mathbb{R} the finite interval $I_0(a, b) = \{c, a < c < b\}$ is the basic open subset and every open set can be written as a union and/or intersection of finite open intervals. The finite interval $I(a, b) = \{c, a \leq c \leq b\}$ is the basic closed subset which is also the closure $\overline{I_0(a, b)}$. The finite closed interval is compact since the only possible noncompact sets are open and half open intervals and their isomorphs. \mathbb{R} is open in that every point has a finite open interval as a neighborhood, and since the closure of any finite open interval is the compact interval it means that \mathbb{R} is a locally compact space. The direct product space $\mathbb{R}^n = \{x = (x^1, x^2, \dots, x^n), x^1, x^2, \dots, x^n \in \mathbb{R}\}$, $n \in \mathbb{N}^+$ inherits the topological properties of \mathbb{R} alongside additional ones. One has as possible metrics

$$d_1(x, y) = \max_{i \in \mathbb{N}_n} |x^i - y^i|, \quad d_2(x, y) = \sqrt{\sum_{i=1}^n (x^i - y^i)^2}. \quad (\text{E.13.1})$$

A subset of \mathbb{R}^n is compact iff it is closed (complete) and bounded.

Consider the real maps $\mathcal{F}(\mathbb{R}, \mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{R}\}$. Then a subspace S of \mathbb{R}^n may be specified through implicit relations imposed pointwise (ie. simultaneously) on a sequence of functions

$$S = \{x \in \mathbb{R}^n, f_i(x) \sim 0, f_i \in \mathcal{F}(\mathbb{R}, \mathbb{R}^n), i \in \mathbb{N}\}. \quad (\text{E.13.2})$$

where \sim includes relations such as $=, <, \leq, >, \geq$, etc.

E.14 On Measures I

A *content* λ is a finite, positive, subadditive, additive, and monotone function on the set of compact sets $\mathcal{K} = \{K\}$.

$$\lambda : \mathcal{K} \rightarrow \mathbb{R}^+ \setminus \{\infty\},$$

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2).$$

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2) \quad (\text{subadditivity}).$$

$$K_1 \cap K_2 = \{\} \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2) \quad (\text{additivity}).$$

Additivity implies that $\lambda(\{\}) = 0$.

An *inner content* λ_* induced by λ ;

$$\lambda_*(A) = \sup_{K \subseteq A} \lambda(K), \quad \lambda_*(\{\}) = \lambda(\{\}) = 0, \quad (\text{E.14.-3})$$

is the *content of the biggest compact subset* of A .

If $\mathcal{O} = \{O\}$ is the set of open sets, the *outer measure* μ_o ;

$$\mu_o(A) = \inf_{A \subseteq O} \lambda_*(O), \quad \mu_o(\{\}) = \lambda_*(\{\}) = 0, \quad (\text{E.14.-2})$$

is the *inner content of the smallest open superset* of A .

Remarks

- The content (measure) of a set is unique if the inner and outer contents (measures) coincide.
- Let $A \subseteq B$ then

$$\begin{aligned} \lambda_*(A) &= \sup_{K \subseteq A} \lambda(K), \quad \lambda_*(B) = \sup_{K \subseteq B} \lambda(K) \geq \sup_{K \subseteq A} \lambda(K) = \lambda_*(A) \\ \Rightarrow \lambda_*(A) &\leq \lambda_*(B). \end{aligned} \quad (\text{E.14.-2})$$

Similarly,

$$\begin{aligned}
\mu_o(A) &= \inf_{A \subseteq O} \lambda_*(O), \\
\mu_o(B) &= \inf_{B \subseteq O} \lambda_*(O) \geq \inf_{A \subseteq O} \lambda_*(O) = \mu_o(A) \\
\Rightarrow \mu_o(A) &\leq \mu_o(B).
\end{aligned} \tag{E.14.-3}$$

- From these inequalities one sees that

$$\lambda_*(K) = \sup_{K' \subseteq K} \lambda(K') \leq \lambda(K) \quad \forall K \in \mathcal{K} \tag{E.14.-2}$$

and

$$\mu_o(A) = \inf_{A \subseteq O} \lambda_*(O) \geq \lambda_*(A) \quad \forall A. \tag{E.14.-1}$$

In particular

$$\lambda(K) \leq \lambda_*(K) \leq \mu_o(K) \quad \forall K \in \mathcal{K}. \tag{E.14.0}$$

- Also

$$\mu_o(O) = \inf_{O \subseteq O'} \lambda_*(O') \leq \lambda_*(O) \quad \forall O \in \mathcal{O} \tag{E.14.1}$$

since $O \subseteq O$. But from (E.14.-1) $\mu_o(O) \geq \lambda_*(O)$. Therefore

$$\mu_o(O) = \lambda_*(O) \quad \forall O \in \mathcal{O}. \tag{E.14.2}$$

Similarly

$$\lambda_*(K) = \sup_{K' \subseteq K} \lambda(K') \geq \lambda(K) \quad \forall K \in \mathcal{K} \tag{E.14.3}$$

since $K \subseteq K$. But from (E.14.-2) $\lambda_*(K) \leq \lambda(K)$. Therefore

$$\lambda_*(K) = \lambda(K) \quad \forall K \in \mathcal{K}. \tag{E.14.4}$$

- One also deduces that

$$\mu_o(\text{int}K) = \lambda_*(\text{int}K) \leq \lambda_*(K) = \lambda(K) \leq \mu_o(K). \quad (\text{E.14.5})$$

- For open sets O_1, O_2 , $\lambda_*(O_1 + O_2) \leq \lambda_*(O_1) + \lambda_*(O_2)$. This follows because for any compact $K \subseteq O_1 + O_2$ one can find compacts $K_1 \subseteq O_1$, $K_2 \subseteq O_2$ such that $K \subseteq K_1 + K_2$. Therefore

$$\begin{aligned} K \subseteq K_1 + K_2 &\Rightarrow \\ \lambda(K) &\leq \lambda(K_1 + K_2) \leq \lambda(K_1) + \lambda(K_2) \\ &\Rightarrow \sup_{K \subseteq O_1 + O_2} \lambda(K) \leq \sup_{K_1 \subseteq O_1} \lambda(K_1) + \sup_{K_2 \subseteq O_2} \lambda(K_2) \\ &\Rightarrow \lambda_*(O_1 + O_2) \leq \lambda_*(O_1) + \lambda_*(O_2). \end{aligned} \quad (\text{E.14.3})$$

Furthermore if $O_1 O_2 = \{\}$ then since by construction $K_1 = K\tilde{O}_1$, $K_2 = K\tilde{O}_2$ one sees that

$$\begin{aligned} K_1 K_2 &= K\tilde{O}_1 K\tilde{O}_2 = K \tilde{O}_1 \tilde{O}_2 = K(\widetilde{O_1 + O_2}) = \{\}, \\ K_1 + K_2 &= K\tilde{O}_1 + K\tilde{O}_2 = K(\widetilde{O_1 O_2}) = K\{\}^c = K. \end{aligned} \quad (\text{E.14.3})$$

Therefore

$$\begin{aligned} K &= K_1 + K_2 \Rightarrow \\ \lambda(K) &= \lambda(K_1 + K_2) = \lambda(K_1) + \lambda(K_2) \\ &\Rightarrow \sup_{K \subseteq O_1 + O_2} \lambda(K) = \sup_{K_1 \subseteq O_1} \lambda(K_1) + \sup_{K_2 \subseteq O_2} \lambda(K_2), \\ &\Rightarrow \lambda_*(O_1 + O_2) = \lambda_*(O_1) + \lambda_*(O_2). \end{aligned} \quad (\text{E.14.1})$$

That is, $O_1 O_2 = \{\} \Rightarrow \lambda_*(O_1 + O_2) = \lambda_*(O_1) + \lambda_*(O_2)$.

These results are automatically valid for μ_o since $\mu_o(O) = \lambda_*(O)$.

Given any A, B then for any open supersets $O_1 \supseteq A$, $O_2 \supseteq B$ one can find $\varepsilon > 0$ such that

$$\begin{aligned} \mu_o(O_1) &\leq \mu_o(A) + \frac{\varepsilon_1}{2}, \quad \mu_o(O_2) \leq \mu_o(B) + \frac{\varepsilon_2}{2} \\ \Rightarrow \mu_o(A + B) &\leq \mu_o(O_1 + O_2) \leq \mu_o(O_1) + \mu_o(O_2) \\ &\leq \mu_o(A) + \mu_o(B) + \frac{\varepsilon_1 + \varepsilon_2}{2}. \end{aligned}$$

Due to separability one can continue to generate smaller and smaller intermediate subsets $O_1 \supseteq \tilde{O}_{1i} \supseteq O_{1i'} \supseteq A$, $O_2 \supseteq \tilde{O}_{2i} \supseteq O_{2i'} \supseteq B$ until $\varepsilon_1, \varepsilon_2 \rightarrow 0$. Thus $\mu_o(A + B) \leq \mu_o(A) + \mu_o(B) \quad \forall A, B$.

These results can be iterated and verified through induction.

E.14.1 Measurability

The set A is μ_o -measurable iff

$$\mu_o(B) = \mu_o(AB) + \mu_o(\tilde{A}B) \quad (\text{E.14.-1})$$

for any set B . ie. measurability is defined by requiring that the additivity property holds for μ_o as A^c is defined by

$$B = BA + B\tilde{A} \quad \forall B, \quad A\tilde{A} = \{\}. \quad (\text{E.14.0})$$

E.15 On Measures II

A *measure* μ on sets is defined as

$$\begin{aligned} \mu(A) &\geq 0, \quad A \subseteq B \Rightarrow \mu(A) \leq \mu(B), \\ \mu(A + B) &\leq \mu(A) + \mu(B), \\ AB = \{\} &\Rightarrow \mu(A + B) = \mu(A) + \mu(B). \end{aligned} \quad (\text{E.15.-1})$$

Upon making the replacements $A \rightarrow AB$, $B \rightarrow \tilde{A}B$ in $\mu(A + B) \leq \mu(A) + \mu(B)$ one obtains

$$\mu(B) \leq \mu(AB) + \mu(\tilde{A}B) \quad (\text{E.15.0})$$

using the assumptions that

$$B = B\{\tilde{\cdot}\} = B(\widetilde{A\tilde{A}}) = B(\tilde{A} + A) = AB + \tilde{A}B.$$

A set A is μ -*measurable* iff equality holds in (E.15.0) for all sets B . That is, A is μ -measurable iff

$$\mu(B) = \mu(AB) + \mu(\tilde{A}B) \quad \forall B. \quad (\text{E.15.0})$$

Any collection $\Sigma_0 = \{\sigma\}$ of nonintersecting sets $\sigma_1\sigma_2 = \{\cdot\} \quad \forall \sigma_1, \sigma_2 \in \Sigma_0$ is a partition and the partition Σ_0 is a μ -*measuring scale* (or simply μ -*measurable*) iff

$$\mu(B) = \sum_{\sigma \in \Sigma_0} \mu(\sigma B) \quad \forall B. \quad (\text{E.15.1})$$

Thus a set A is μ -measurable iff the partition $\{A, \tilde{A}\}$ is μ -measurable.

Measurability can also be expressed entirely in terms of open sets: a set M is μ_o -measurable iff

$$\mu_o(O) \geq \mu_o(OM) + \mu_o(O\tilde{M}) \quad \forall O \in \mathcal{O}. \quad (\text{E.15.2})$$

This is because one has that

$$\begin{aligned} \mu_o(O) \geq \mu_o(OM) + \mu_o(O\tilde{M}) &\Rightarrow \\ \mu_o(A) = \inf_{A \subseteq O} \lambda_*(O) = \inf_{A \subseteq O} \mu_o(O) &\geq \inf_{A \subseteq O} \{\mu_o(OM) + \mu_o(O\tilde{M})\} \\ &\geq \mu_o(AM) + \mu_o(A\tilde{M}), \end{aligned} \quad (\text{E.15.1})$$

and $\mu_o(A) \leq \mu_o(AM) + \mu_o(A\widetilde{M})$ by subadditivity and so

$\mu_o(A) = \mu_o(AM) + \mu_o(A\widetilde{M}) \quad \forall A$. Conversely, if M is μ_o -measurable, ie.

$\mu_o(A) = \mu_o(AM) + \mu_o(A\widetilde{M}) \quad \forall A$ then for any open set O in particular we have

$\mu_o(O) = \mu_o(OM) + \mu_o(O\widetilde{M})$ which satisfies $\mu_o(O) \geq \mu_o(OM) + \mu_o(O\widetilde{M})$.

The product $\Sigma_A \Sigma_B = \{A_i B_j; A_i \in \Sigma_A, B_j \in \Sigma_B\}$ of two μ -measurable partitions $\sigma_A = \{A_i\}$, $\Sigma_B = \{B_j\}$ is μ -measurable:

$$\mu(M) = \sum_i \mu(A_i M) = \sum_i \sum_j \mu(B_j A_i M) = \sum_{ij} \mu(B_j A_i M) \quad \forall B. \quad (\text{E.15.1})$$

If σ_A is a measurable partition and $\Sigma_A \leq \Sigma_B$ (meaning that each $A_i \in \Sigma_A$ is a subset of some $B_j \in \Sigma_B$) then Σ_B is also measurable:

$$\begin{aligned} \sum_i \mu(B_i M) &= \sum_i \sum_j \mu(A_j B_i M) = \sum_i \sum_j \mu(A_j B_i M) = \sum_j \sum_{i, A_j \subseteq B_i} \mu(A_j M) \\ &= \sum_j \mu(A_j M) = \mu(M). \end{aligned} \quad (\text{E.15.1})$$

A partition $\Sigma = \{A_i; \forall i\}$ is measurable iff each of A_i is measurable: A_i is measurable $\forall i$ iff $\Sigma_i = \{A_i, \widetilde{A}_i\}$ is measurable $\forall i$ and so is their product;

$$\begin{aligned} \prod_i \{A_i, \widetilde{A}_i\} &= (A_1, A_2, \dots, A_n, \prod_{i=1}^n \widetilde{A}_i) \text{ measurable} \Rightarrow \\ \mu(M) &= \sum_{i=1}^n \mu(A_i M) + \mu\left(\prod_{i=1}^n \widetilde{A}_i M\right) \geq \sum_{i=1}^n \mu(A_i M). \end{aligned} \quad (\text{E.15.1})$$

But we also have $\mu(M) \leq \sum_{i=1}^n \mu(A_i M)$ and so $\mu(M) = \sum_{i=1}^n \mu(A_i M)$ and thus Σ is measurable. Conversely if Σ is measurable then $\Sigma_i = \{A_i, \widetilde{A}_i\}$ is also measurable for each i since for any given i , $\Sigma \leq \Sigma_i$.

If A, B are measurable then so are $\widetilde{A}, AB, A + B$ since

$$\begin{aligned} A &= \widetilde{\widetilde{A}}, \quad AB \in \{A, \widetilde{A}\}\{B, \widetilde{B}\} = \{AB, A\widetilde{B}, \widetilde{A}B, \widetilde{A}\widetilde{B}\}, \\ A + B &= \widetilde{\widetilde{A+B}}. \end{aligned} \quad (\text{E.15.1})$$

If the sets $A_1, A_2, \dots, A_n, \forall n \in \mathbb{N}$ are measurable, then so is $A = \sum_{i=1}^n A_i \forall n \in \mathbb{N}$ by induction. One notes that one can write A in terms of disjoint sets:

$$\begin{aligned} A &= \sum_{i=1}^n A_i = A_1 + \tilde{A}_1 A_2 + \tilde{A}_1 \tilde{A}_2 A_3 + \dots + \tilde{A}_1 \tilde{A}_2 \dots \tilde{A}_{n-1} A_n \\ &= \sum_{i=1}^n A_i \prod_{1 \leq j \leq i-1} \tilde{A}_j. \end{aligned} \quad (\text{E.15.1})$$

For any two sets A, B where one of them, say A , is measurable (ie. $\mu(M) = \mu(MA) + \mu(M\tilde{A}) \forall M$), then we have

$$\begin{aligned} \mu(B) &= \mu(BA) + \mu(B\tilde{A}), \\ \mu(A+B) &= \mu((A+B)A) + \mu((A+B)\tilde{A}) = \mu(A+BA) + \mu(B\tilde{A}) \\ &= \mu(A) + \mu(B\tilde{A}) = \mu(A) + \mu(A) - \mu(BA). \\ \mu(A+B) &= \mu(A) + \mu(A) - \mu(BA). \end{aligned} \quad (\text{E.15.-1})$$

One notes that the second line could have simply been expressed as

$$\mu(A+B) = \mu(A + \tilde{A}B) = \mu(A) + \mu(\tilde{A}B) \quad (\text{E.15.0})$$

without using measurability of A .

Every open set is μ_o -measurable: Given $O_1, O_2 \in \mathcal{O}$ consider $K_1, K_2 \in \mathcal{K}$ such that $K_1 \subseteq O_1 O_2$ (ie. $\tilde{O}_1 + \tilde{O}_2 \subseteq \tilde{K}_1$ or $\tilde{O}_1 O_2 \subseteq \tilde{K}_1 O_2$),

$K_2 \subseteq \tilde{K}_1 O_2$ (ie. $K_2 \subseteq \tilde{K}_1 O_2$) then $K_1 K_2 = \{\}$ (as $K_2 \subseteq \tilde{K}_1$) and

$$\begin{aligned} K_1, K_2 \subseteq O_2 &\Rightarrow K_1 + K_2 \subseteq O_2, \\ \mu_o(O_2) = \lambda_*(O_2) &= \sup_{K \subseteq O_2} \lambda(K) \geq \sup_{K=K_1+K_2 \subseteq O_2} \lambda(K_1 + K_2) \\ &= \sup_{K_1 \subseteq O_2} \lambda(K_1) + \sup_{K_2 \subseteq O_2} \lambda(K_2) \geq \lambda_*(O_1 O_2) + \lambda_*(\tilde{K}_1 O_2) \\ &= \mu_o(O_1 O_2) + \mu_o(\tilde{K}_1 O_2) \geq \mu_o(O_1 O_2) + \mu_o(\tilde{O}_1 O_2). \\ \Rightarrow \mu_o(O_2) &\geq \mu_o(O_1 O_2) + \mu_o(\tilde{O}_1 O_2) \quad \forall O_1, O_2 \in \mathcal{O}. \end{aligned} \quad (\text{E.15.-3})$$

E.15.1 Haar measure: existence and uniqueness

Let \mathcal{S} be a measurable space and λ_1, λ_2 be two contents defined on compact subsets $\mathcal{K}(\mathcal{S})$ of \mathcal{S} . If $h : \mathcal{S} \rightarrow \mathcal{S}$ is a self homeomorphism of \mathcal{S} such that $\lambda_2 = \lambda_1 \circ h$ then the induced measures μ_1, μ_2 of λ_1, λ_2 are also related as $\mu_2 = \mu_1 \circ h$, where \circ denotes map composition.

Let G be a locally compact topological group (Topological in that $() \cdot () : G \times G \rightarrow G$, or equivalently $L_u : G \rightarrow G, g \mapsto ug$, $R_u : G \rightarrow G, g \mapsto gu \ \forall u \in G$, and $()^{-1} : G \rightarrow G, g \mapsto g^{-1}$ are continuous maps). Thus in G the existence of a left-invariant content λ will imply the existence of a left-invariant measure μ since left translation $L_u : G \rightarrow G, g \mapsto L_u(g) = ug$ by $u \in G$ is a homeomorphism. One simply needs to set

$\lambda_1 = \lambda, h = L_u, \lambda_2 \equiv \lambda_u = \lambda \circ L_u$; then $\lambda_2 = \lambda_1 \circ L_u \Rightarrow \mu_2 = \mu_1 \circ L_u$. Thus $\lambda_1 = \lambda_2 \Rightarrow \mu_1 = \mu_2$ or, equivalently, that $\lambda = \lambda \circ L_u \Rightarrow \mu = \mu \circ L_u$.

Let $K \in \mathcal{K}(G)$ be a compact subset of G and $o \in \mathcal{O}(G), o \neq \{\}$ be a *small* nonempty open subset of G . It is possible to form a cover $\Sigma_n^o(K) = \{g_i o; i = 1, 2, \dots, n\}$ for K (this is possible for any $A \subseteq G$) made up of a number of translated copies $g_i o$ of o by some elements $\{g_i \in G; i = 1, 2, \dots, n\}$. That is

$$K \subseteq \bigcup_{i=1}^n g_i o \equiv \bigcup \Sigma_n^o(K). \quad (\text{E.15.-2})$$

Consider the map

$$n_o : \mathcal{K}(G) \rightarrow \mathbb{R}_+, \quad A \mapsto n_o(K) = \min_{K \subseteq \bigcup \Sigma_n^o(K)} n. \quad (\text{E.15.-1})$$

That is $n_o(K)$ *is the smallest number of copies of o needed to just cover K* . Since only the number n of copies of o , and not the elements $\{g_i\}$, is important

we may simply write the inclusion (E.15.-2) as

$$K \subseteq n_o(K) \circ \quad (\text{E.15.0})$$

and use heuristics to deduce the following properties. Let $A \subseteq G$, $A^0 \neq \{\}$. Then

$$\begin{aligned} K \subseteq n_A(K) \circ A, \quad A \subseteq n_o(A) \circ \Rightarrow K \subseteq n_A(K) \circ A \subseteq n_A(K) n_o(A) \circ \\ \Rightarrow n_o(K) \leq n_A(K) n_o(A) \Rightarrow \frac{n_o(K)}{n_o(A)} \leq n_A(K). \end{aligned} \quad (\text{E.15.0})$$

$$K \subset K_1 \Rightarrow n_o(K) \leq n_o(K_1).$$

$$K_1 + K_2 \subseteq n_o(K_1 + K_2) \circ, \quad K_1 \subseteq n_o(K_1) \circ, \quad K_2 \subseteq n_o(K_2) \circ$$

$$\Rightarrow K_1 + K_2 \subseteq (n_o(K_1) + n_o(K_2)) \circ \Rightarrow n_o(K_1 + K_2) \leq n_o(K_1) + n_o(K_2).$$

$$\begin{aligned} (n_o \circ L_u)(K) = n_o(u \cdot K) = \min_{u \cdot K \subseteq \bigcup \Sigma_n^o(u \cdot K)} n = \min_{K \subseteq \bigcup \Sigma_n^o(K)} n \\ = n_o(K) \quad \forall u \in G. \end{aligned} \quad (\text{E.15.-3})$$

If $K_1 K_2 = \{\}$ then it is possible to find nbds $\mathcal{N}(K_1), \mathcal{N}(K_2)$ such that $\mathcal{N}(K_1) \mathcal{N}(K_2) = \{\}$. Consequently, \circ can be chosen arbitrarily small so that

$$K_1 K_2 = \{\} \Rightarrow n_o(K_1 + K_2) = n_o(K_1) + n_o(K_2). \quad (\text{E.15.-2})$$

Thus we have additivity. However the number $n_o(K)$ can clearly diverge and so we now define a regularized version (see E.15.0)

$$\lambda_o(K) = \frac{n_o(K)}{n_o(A)} \leq n_A(K). \quad (\text{E.15.-1})$$

Then λ_o clearly inherits all the essential properties of n_o and is bounded by $n_A(K) \forall o$. One can define the desired content λ as

$$\lambda(K) = \min_{\{\} \neq o \in \mathcal{O}(G)} \lambda_o(K). \quad (\text{E.15.0})$$

To check uniqueness, let μ, ν be two left invariant measures on G and consider two continuous functions $\alpha, \beta : K \in \mathcal{K}(G) \rightarrow \mathbb{C}$. Then

$$\begin{aligned}
& \int_K d\mu(x) \alpha(x) \int_K d\nu(y) \beta(y) = \int_K d\mu(x) d\nu(y) \alpha(x) \beta(y) \\
& = \int_K d\mu(x) d\nu(y) \alpha(y^{-1}x) \beta(y) \\
& = \int_K d\mu(x) d\nu(y) \alpha((x^{-1}y)^{-1}) \beta(x(x^{-1}y)) = \int_K d\mu(x) d\nu(y) \alpha(y^{-1}) \beta(xy) \\
& = \int_K d\nu(y) \alpha(y^{-1}) \int_K d\mu(x) \beta(xy).
\end{aligned}$$

One can left translate β to obtain

$$\int_K d\mu(x) \alpha(x) \int_K d\nu(y) \beta(yg) = \int_K d\nu(y) \alpha(y^{-1}) \int_K d\mu(x) \beta(xyg).$$

Now integrate over g to obtain

$$\begin{aligned}
& \int_K d\mu(x) \alpha(x) \int_K d\nu(y) \int_K d\rho(g) \beta(yg) \\
& = \int_K d\nu(y) \alpha(y^{-1}) \int_K d\mu(x) \int_K d\rho(g) \beta(xyg)
\end{aligned}$$

where ρ can be either μ or ν . Thus

$$\begin{aligned}
& \int_K d\mu(x) \alpha(x) \int_K d\nu(y) \int_K d\rho(g) \beta(g) \\
& = \int_K d\nu(y) \alpha(y^{-1}) \int_K d\mu(x) \int_K d\rho(g) \beta(g) \\
& \Rightarrow |K|_\nu \int_K d\mu(x) \alpha(x) = |K|_\mu \int_K d\nu(y) \alpha(y^{-1}), \quad |K|_\mu = \int_K d\mu(x).
\end{aligned}$$

In particular for any function α such that $\alpha(y^{-1}) = \alpha(y)$, eg

$\alpha(y) = \gamma(y) + \gamma(y^{-1})$ or $\alpha(y) = \gamma(y)\gamma(y^{-1})$, one has that

$$\begin{aligned}
& \int_K d\mu(g) \alpha(g) = \frac{|K|_\mu}{|K|_\nu} \int_K d\nu(g) \alpha(g) \quad \forall \alpha : K \rightarrow \mathbb{C}, \quad \alpha(g) = \alpha(g^{-1}). \\
& \mu = c \nu, \quad c = \frac{|K|_\mu}{|K|_\nu}. \tag{E.15.-10}
\end{aligned}$$

Also since

$$\begin{aligned}\int_K d\mu(x) \alpha(xa) &= \int_K d\mu(xaa^{-1}) \alpha(xa) = \int_K d\mu(xa^{-1}) \alpha(x), \\ \int_K d\mu(x) \alpha(uxa) &= \int_K d\mu(x) \alpha(xa) = \int_K d\mu(xaa^{-1}) \alpha(xa) = \int_K d\mu(xa^{-1}) \alpha(x),\end{aligned}$$

one sees that if μ is a left invariant measure, ie $\mu \circ L_G = \mu$ then

$\mu_a := \mu \circ R_a \quad \forall a \in G$, where R_a denotes right translation, is another left invariant measure. This means that μ and μ_a , by uniqueness, differ only by a constant.

$$\mu_a = \mu \circ R_a = c(a) \mu \quad \forall a \in G.$$

$$\mu_{ab} = c(b)\mu_a = c(b) c(a) \mu = c(ab) \mu \quad \Rightarrow \quad c(ab) = c(a)c(b).$$

E.15.2 Invariant linear maps

Let \mathcal{A}, \mathcal{B} be two algebras and

$L(\mathcal{B}/\mathcal{A}) \subset \mathcal{A}/\mathcal{B} = \{\pi : \mathcal{A} \rightarrow \mathcal{B}, \quad \pi(a_1 + a_2) = \pi(a_1) + \pi(a_2)\}$ be the set of all linear maps from \mathcal{A} to \mathcal{B} .

Also let $J = J(L(\mathcal{B}/\mathcal{A})) : \mathcal{A} \rightarrow \mathcal{B}$ be the addition/composition $(+, \circ)$ algebra of these linear maps, which is a \mathcal{B} -module as $\mathcal{B}J, JB \subset J$.

One notes that each $\pi : \mathcal{A} \rightarrow \mathcal{B}$ is equivalent to a bilinear pairing

$$P_\pi : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}.$$

The induced relative central set $\mathcal{A}_\mathcal{B}^*$ of \mathcal{A} is the “kernel” of J given by

$$\mathcal{A}_\mathcal{B}^* = \{s \in J, \quad s : \mathcal{A} \rightarrow Z(\mathcal{B}) = \mathcal{B} \cap \mathcal{B}'\} \quad (\text{E.15.-13})$$

where \mathcal{B}' is the commutant of \mathcal{B} . That is, $\pi \in \mathcal{A}_\mathcal{B}^*$ if $\pi(a) \in Z(\mathcal{B}) = \mathcal{B} \cap \mathcal{B}' \quad \forall a \in \mathcal{A}$.

Let another algebra $\mathcal{C} = \{c\}$ act on \mathcal{A} through a linear representation ρ , ie.

$$\begin{aligned}\rho : \mathcal{C} &\rightarrow O(\mathcal{A}), \quad c \rightarrow \rho_c : \mathcal{A} \rightarrow \mathcal{A}, \\ \rho_c \rho_{c_1} &= \rho_{cc_1}, \quad \rho_c(a + a_1) = \rho_c(a) + \rho_c(a_1).\end{aligned} \quad (\text{E.15.-13})$$

Then a linear map \int ,

$$\int : \mathcal{A}_{\mathcal{B}}^* \rightarrow \mathcal{A}_{\mathcal{B}}^*, \quad s \rightarrow \int s, \quad \int (s + s_1) = \int s + \int s' \quad (\text{E.15.-12})$$

is a ρ -integral if there is some $s_0 \in \mathcal{A}_{\mathcal{B}}^*$ such that

$$\int s \circ \rho_c = s_0(c) \int s \quad \forall (c \in \mathcal{C}, s \in \mathcal{A}_{\mathcal{B}}^*). \quad (\text{E.15.-11})$$

That is,

$$\int s \circ \rho_c(a) = \int s(\rho_c(a)) = s_0(c) \int s(a) \quad \forall (a \in \mathcal{A}, c \in \mathcal{C}, s \in \mathcal{A}_{\mathcal{B}}^*).$$

More generally, $\int : L(\mathcal{B}/\mathcal{A}) \rightarrow L(\mathcal{B}/\mathcal{A})$ is a ρ -integral if there is some $\pi_0 \in \mathcal{A}_{\mathcal{B}}^*$ such that

$$\int \pi \circ \rho_c = \pi_0(c) \int \pi \quad \forall (c \in \mathcal{C}, \pi \in L(\mathcal{B}/\mathcal{A})). \quad (\text{E.15.-11})$$

On the other hand, $a \in \mathcal{A}$ is a ρ -integral element under the map π if there is some $s_0 \in \mathcal{A}_{\mathcal{B}}^*$ such that

$$\pi \circ \rho_c(a) = \pi(\rho_c(a)) = s_0(c) \pi(a) \quad \forall c \in \mathcal{C}. \quad (\text{E.15.-10})$$

Even more generally, if there is an equivalence relation \sim among the elements of $L(\mathcal{B}/\mathcal{A})$ which separate into equivalence classes $\{[\pi]\}$ then

$\int : L(\mathcal{B}/\mathcal{A}) \rightarrow L(\mathcal{B}/\mathcal{A})$ is a ρ -integral if there is some $\pi_0 \in \mathcal{A}_{\mathcal{B}}^*$ such that

$$[\int \pi \circ \rho_c] = [\int \pi] \quad \forall (c \in \mathcal{C}, \pi \in L(\mathcal{B}/\mathcal{A})). \quad (\text{E.15.-9})$$

Appendix F

C^* -algebras

F.1 Cauchy-Schwarz inequality

Let us define a $*$ -algebra \mathcal{A} to be an associative algebra over the field of complex numbers \mathbb{C} that is closed under an operation $*$ (that is, $a^* \in \mathcal{A} \ \forall a \in \mathcal{A}$) with the following properties.

$$\begin{aligned} a^{**} &= a, & (ab)^* &= b^*a^*, & (a+b)^* &= a^*+b^*, \\ \alpha^* &= \bar{\alpha} \quad \forall a, b \in \mathcal{A}, \quad \alpha \in \mathbb{C}, \end{aligned} \tag{F.1.0}$$

where $\bar{\alpha}$ denotes the complex conjugate of α .

Let \mathcal{A} be a $*$ -algebra. Consider any $A \in \mathcal{A}$, a collection $\{B_i \in \mathcal{A}, \ i = 1, 2, \dots, p\}$ and $\phi \in \mathcal{A}_+^*$, the set of positive linear functionals on \mathcal{A} .

$$\begin{aligned} \phi(a+b) &= \phi(a) + \phi(b), & \phi(a^*) &= \overline{\phi(a)}, \\ \phi(\lambda a) &= \lambda\phi(a), & \phi(a^*a) &\geq 0 \quad \forall a, b \in \mathcal{A}, \lambda \in \mathbb{C}. \end{aligned} \tag{F.1.0}$$

Also define the function $f : \mathbb{C}^p \rightarrow \mathbb{R}^+$, $\lambda \mapsto f(\lambda, \bar{\lambda})$,

$$\begin{aligned}
f(\lambda, \bar{\lambda}) &= \phi((A + \lambda_i B_i)^*(A + \lambda_i B_i)) \\
&= \phi(A^* A) + \lambda_i \phi(A^* B_i) + \bar{\lambda}_i \phi(B_i^* A) + \bar{\lambda}_i \lambda_j \phi(B_i^* B_j) \\
&\equiv \phi(A^* A) + \lambda_i \bar{N}_i + \bar{\lambda}_i N_i + \bar{\lambda}_i \lambda_j M_{ij} \geq 0, \tag{F.1.-1} \\
N_i &= \phi(B_i^* A), \quad M_{ij} = \phi(B_i^* B_j), \quad \bar{M}_{ij} = M_{ji}. \tag{F.1.0}
\end{aligned}$$

The value of f at its extreme point gives the Cauchy-Schwarz inequality. That is,

$$\frac{\partial}{\partial \bar{\lambda}_i} f(\lambda', \bar{\lambda}') = 0 \iff \lambda'_i = -M_{ij}^{-1} N_j, \quad \det \bar{M}_{ij} > 0, \tag{F.1.1}$$

$$\Rightarrow f(\lambda', \bar{\lambda}') = \phi(A^* A) - \bar{N}_i \bar{M}_{ij}^{-1} N_j \geq 0. \tag{F.1.2}$$

If we write $B_I = (A, B_i) \equiv (B_0, B_i)$, $M_{IJ} = \phi(B_I^* B_J)$, $I, J = 0, 1, \dots, p$ then the inequality (F.1.2) becomes

$$f(\lambda', \bar{\lambda}') = \frac{\det \bar{M}_{IJ}}{\det \bar{M}_{ij}} \geq 0 \Rightarrow \det \bar{M}_{IJ} \geq 0. \tag{F.1.3}$$

In the case $p = 1$ one has

$$\phi(A^* B) \phi(B^* A) = \phi(A^* B) \overline{\phi(A^* B)} \leq \phi(A^* A) \phi(B^* B). \tag{F.1.4}$$

F.2 Hilbert space and operator norm

Define the operator norm $\|a\|$ for a bounded operator $a \in B(\mathcal{H}) = \{b : \mathcal{H} \rightarrow \mathcal{H}\}$ on a Hilbert space \mathcal{H} (inner product vector space $(V, \langle | \rangle) : V \times V \rightarrow \mathbb{C}$) which is complete with respect to the inner product norm $\| \cdot \| : \xi \mapsto \|\xi\| = \sqrt{\langle \xi | \xi \rangle}$) as

$$\|a\| = \sup_{\xi \in \mathcal{H}} \frac{\|a\xi\|}{\|\xi\|}, \quad \|\xi\| = \sqrt{\langle \xi | \xi \rangle}. \tag{F.2.1}$$

It follows from the definition (F.2.1) that

$$\|a\xi\| \leq \|a\|\|\xi\| \quad \forall \xi \in \mathcal{H} \quad (\text{F.2.2})$$

and since $b\mathcal{H} = \mathcal{H} \quad \forall b \in \mathcal{H}$ we then have that

$$\|ab\xi\| \leq \|a\|\|b\xi\| \leq \|a\|\|b\|\|\xi\| \quad (\text{F.2.3})$$

and therefore

$$\begin{aligned} \|ab\| &= \sup_{\xi \in \mathcal{H}} \frac{\|ab\xi\|}{\|\xi\|} \leq \sup_{\xi \in \mathcal{H}} \frac{\|a\|\|b\|\|\xi\|}{\|\xi\|} = \sup_{\xi \in \mathcal{H}} \|a\|\|b\| = \|a\|\|b\|. \\ \text{ie. } \|ab\| &\leq \|a\|\|b\|. \end{aligned} \quad (\text{F.2.3})$$

This in turn implies that one could also define the norm

$$\|a\|_{B(\mathcal{H})} = \sup_{b \in B(\mathcal{H})} \frac{\|ab\|}{\|b\|} \leq \|a\|. \quad (\text{F.2.4})$$

If one defines a^* by $\langle a^*\eta|\xi \rangle = \langle \eta|a\xi \rangle$ then

$$\begin{aligned} (ab)^* &= b^*a^*, \quad a^{**} = a, \quad (\lambda a)^* = \bar{\lambda}a^* \quad \forall a, b \in B(\mathcal{H}) \text{ \& } \lambda \in \mathbb{C}, \\ \|a^*\| &= \sup_{\xi \in \mathcal{H}} \frac{\|a^*\xi\|}{\|\xi\|} = \sup_{\xi \in \mathcal{H}} \frac{\sqrt{|\langle a^*\xi|a^*\xi \rangle|}}{\|\xi\|} = \sup_{\xi \in \mathcal{H}} \frac{\sqrt{|\langle \xi|aa^*\xi \rangle|}}{\|\xi\|}, \\ \|a\| &= \sup_{\xi \in \mathcal{H}} \frac{\|a\xi\|}{\|\xi\|} = \sup_{\xi \in \mathcal{H}} \frac{\sqrt{|\langle a\xi|a\xi \rangle|}}{\|\xi\|} = \sup_{\xi \in \mathcal{H}} \frac{\sqrt{|\langle \xi|a^*a\xi \rangle|}}{\|\xi\|}. \end{aligned} \quad (\text{F.2.3})$$

Thus the mean-center inequality (F.2.2, F.2.3) and the norm inequality (F.2.3) give

$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\|, \quad \Rightarrow \quad \|a\| \leq \|a^*\|. \quad (\text{F.2.4})$$

And $a^{**} = a$ therefore gives

$$\|a\| \leq \|a^*\| \leq \|a^{**}\| = \|a\| \quad \Rightarrow \quad \|a^*\| = \|a\|, \quad \|a^*a\| = \|a\|^2. \quad (\text{F.2.5})$$

Remarks

- Each vector $\xi \in \mathcal{H}$ corresponds to an operator $\xi_o = \frac{1}{\sqrt{\langle \xi | \xi \rangle}} |\xi\rangle \langle \xi|$ whose norm with respect to a positive linear functional ϕ_ξ , defined below in terms of the operator $\rho_\xi \equiv \frac{1}{\langle \xi | \xi \rangle} |\xi\rangle \langle \xi| \in B(\mathcal{H})$, gives the norm of ξ . That is

$$\begin{aligned}\phi_\xi(a) &= \text{Tr}(\rho_\xi a) = \frac{1}{\langle \xi | \xi \rangle} \text{Tr}(|\xi\rangle \langle \xi| a) = \frac{\langle \xi | a | \xi \rangle}{\langle \xi | \xi \rangle}, \\ \|a\|_\xi &= \sqrt{\phi_\xi(a^*a)}, \quad \|a\| = \sup_{\xi \in \mathcal{H}} \|a\|_\xi, \\ \|\xi_o\|_\xi &= \sqrt{\phi_\xi(\xi_o^* \xi_o)} = \sqrt{\phi_\xi(|\xi\rangle \langle \xi|)} = \sqrt{\text{Tr} \rho_\xi |\xi\rangle \langle \xi|} \\ &= \sqrt{\text{Tr} |\xi\rangle \langle \xi|} = \sqrt{\langle \xi | \xi \rangle} = \text{Tr} \xi_o = \|\xi\|.\end{aligned}$$

One has the Cauchy-Schwarz inequality

$$\phi_\xi(a^*b) \overline{\phi_\xi(a^*b)} = |\phi_\xi(a^*b)|^2 \leq \phi_\xi(a^*a) \phi_\xi(b^*b). \quad (\text{F.2.2})$$

Setting $b = 1$ in (F.2.2) gives

$$\begin{aligned}\phi_\xi(a^*) \phi_\xi(a) (2 - \phi(1)) &\leq \phi_\xi(a^*a), \\ \text{or } \phi_\xi((a - \phi_\xi(a))^* (a - \phi_\xi(a))) &\geq 0,\end{aligned} \quad (\text{F.2.2})$$

which in turn gives

$$\phi_\xi(a^*b) \overline{\phi_\xi(a^*b)} (2 - \phi(1)) \leq \phi_\xi((a^*b)^* a^*b) = \phi_\xi(b^* a a^* b), \quad (\text{F.2.3})$$

as well as

$$\phi_\xi(a^*b) \overline{\phi_\xi(a^*b)} \leq \phi_\xi(a^*a) \phi_\xi(b^*b) \leq \frac{1}{(2 - \phi(1))} \phi_\xi(a^* a b^* b). \quad (\text{F.2.4})$$

- The triangular inequality also follows thus:

$$\begin{aligned}
\phi_\xi((a+b)^*(a+b)) &= \phi_\xi(a^*a) + \phi_\xi(a^*b) + \phi_\xi(b^*a) + \phi_\xi(b^*b) \\
&= \phi_\xi(a^*a) + 2\Re\phi_\xi(a^*b) + \phi_\xi(b^*a^*) \leq \phi_\xi(a^*a) + 2\sqrt{\phi_\xi(a^*b)\overline{\phi_\xi(a^*b)}} + \phi_\xi(b^*b) \\
&\leq \phi_\xi(a^*a) + 2\sqrt{\phi_\xi(a^*a)\phi_\xi(b^*b)} + \phi_\xi(b^*b) = (\phi_\xi(a^*a) + \phi_\xi(b^*b))^2, \\
\|a+b\|_\xi &\leq \|a\|_\xi + \|b\|_\xi \quad \forall \xi \in \mathcal{H}, \\
\Rightarrow \quad \|a+b\| &\leq \|a\| + \|b\|. \tag{F.2.1}
\end{aligned}$$

Using $\|a\|_\xi \leq \sup_{\xi \in \mathcal{H}} \|a\|_\xi = \|a\|$ one can also check that

$$\|a\|_\xi \|b\|_\xi \leq \sup_{\xi \in \mathcal{H}} (\|a\|_\xi \|b\|_\xi) \leq \sup_{\xi \in \mathcal{H}} (\|a\|_\xi \|b\|) = \|a\| \|b\|. \tag{F.2.2}$$

- Since $a\mathcal{H} = \mathcal{H}$, $\|a\|$ is the same for all elements in the conjugacy class $[a] = \{b, \exists c \in \mathcal{H}, b = c^*ac\}$.
- For the finite dimensional case, a^*a may be diagonalized: ie. $a^*a = P\Lambda P^{-1}$ and if one chooses a basis $\{|i\rangle\}$ for \mathcal{H} then

$$\begin{aligned}
\xi &= |i\rangle\xi_i, \quad a^* = (a^*a)_{ij}|i\rangle\langle j|, \quad \Lambda = \lambda_i |i\rangle\langle i|, \quad \langle i|j\rangle = \delta_{ij}, \\
\|a\| &= \sup_{\xi \in \mathcal{H}} \frac{\sqrt{\sum_i \lambda_i |\xi_i|^2}}{\sqrt{\sum_j |\xi_j|^2}} = \sup_{\xi \in \mathcal{H}} \|a\|_\xi. \\
\frac{\partial}{\partial |\xi_i|} \|a\|_\xi &= |\xi_i| \frac{1}{\|a\|_\xi} \{\lambda_i - \|a\|_\xi^2\} = 0 \quad \forall i. \tag{F.2.1}
\end{aligned}$$

(F.2.1) may have several different solutions but one should take the one on which $\|a\|$ is biggest. In particular one can choose ξ to be in the direction i_{\max} where $\lambda_{i_{\max}} = \max_i \lambda_i$. That is

$$\xi_i = |\xi| \delta_{ii_{\max}} \Rightarrow \|a\| = \sqrt{\max_i \lambda_i}. \tag{F.2.2}$$

The eigenvalue character may be defined more generally as

$$\lambda_\xi(a) = \text{Extr}_{\eta \in \mathcal{H}} \frac{\langle \eta | a \xi \rangle}{\langle \eta | \xi \rangle}, \quad (\text{F.2.3})$$

where $\text{Extr}_{\eta \in \mathcal{H}}$ refers to extremization in \mathcal{H} .

- A C^* -algebra given abstractly as

$$\mathcal{A} = (B = \{a, b, c, \dots\}, B \xrightarrow{*, \mathbb{C}} B, B \xrightarrow{\|\cdot\|} \mathbb{R}^+) \quad (\text{F.2.4})$$

has the following defining properties

$$(ab)^* = b^* a^*, (a + b)^* = a^* + b^*, (\lambda a)^* = \bar{\lambda} a^*, a^{**} = a, \quad (\text{F.2.5})$$

(involutive algebra),

$$\|a\| \geq 0, \quad \|ab\| \leq \|a\| \|b\|, \quad (\text{F.2.5})$$

also with $\|a^*\| = \|a\|$ (normed involutive algebra),

also $\|\cdot\|$ – **complete** (normed involutive Banach algebra),

$$\|a^* a\| = \|a\|^2 \quad (C^*\text{-algebra}) \quad (\text{F.2.4})$$

(F.2.5) and (F.2.4) give

$$\|a^* a\| = \|a\|^2 \leq \|a^*\| \|a\| \Rightarrow \|a\| \leq \|a^*\|. \quad (\text{F.2.5})$$

(F.2.5) and (F.2.5) then give

$$\|a^*\| \leq \|a^{**}\| = \|a\|. \quad \text{ie } \|a^*\| \leq \|a\| \quad (\text{F.2.6})$$

and therefore $\|a^*\| = \|a\|$. Therefore the algebra of bounded operators $B(\mathcal{H})$ is a C^* -algebra with the operator norm. In certain cases it may also be possible that one can obtain the norm inequality (F.2.5) when given only the C^* -condition (F.2.4) and the Cauchy-Schwarz inequality.

- Examples of C^* -algebras are given by pointwise product (denoted \star) algebras $B(\mathcal{H}) = \mathcal{F}_\mu(X) = \{\mu_f : \mathcal{F}(X) \rightarrow \mathcal{F}(X), g \rightarrow (f \star g)(x), f \in \mathcal{F}(X)\}$ of complex functions $\mathcal{H} = \mathcal{F}(X) = \{\xi : X \rightarrow \mathbb{C}\}$ over a topological space X under a suitably defined operator norm.

$$\begin{aligned} \|\xi\| &= \sqrt{\sum_{x \in X} |\xi(x)|^2}, & \|\mu_f\| &= \sup_{\xi \in \mathcal{H}} \frac{\|\mu_f \xi\|}{\|\xi\|}, \\ \Rightarrow \|\mu_f \xi\| &\leq \|\mu_f\| \|\xi\| \quad \forall \xi \in \mathcal{H}. \end{aligned} \quad (\text{F.2.6})$$

For case of a separable (ie. local) pointwise product any $f \in \mathcal{H} = \mathcal{F}(X)$ corresponds to an operator $\mu_f \in \mathcal{F}_\mu(X)$ that acts on $\mathcal{H} = \mathcal{F}(X)$ linearly as

$$\begin{aligned} \mu_f \xi(x) &= (f \star \xi)(x) = f(x)\xi(x), \\ \|\mu_f\| &= \sqrt{\max_{x \in X} |f(x)|^2} = \max_{x \in X} |f(x)| \end{aligned} \quad (\text{F.2.6})$$

where in the norm we have used the fact that for each $x \in X$, $f(x)$ is regarded as an eigenvalue of μ_f . Also the last step is due to the fact that $\frac{\partial |f(x)|^2}{\partial x} = 2|f(x)| \frac{\partial |f(x)|}{\partial x}$.

One also has the pointwise convolution product algebra $\mathcal{F}_\nu(X)$

$$\begin{aligned} \nu_f \xi(x) &= \sum_{y \in X} f(x-y)\xi(y) \equiv \sum_{k \in X} e^{ikx} \tilde{f}(k) \tilde{\xi}(k), \\ \|\xi\| &= \sqrt{\sum_{x \in X} |\xi(x)|^2}, \\ \|a\| &= \sup_{\tilde{\xi} \in \mathcal{H}} \frac{\sqrt{\sum_{k \in X} |\tilde{f}(k)|^2 |\tilde{\xi}_k|^2}}{\sqrt{\sum_{k' \in X} |\tilde{\xi}_{k'}|^2}} = \max_{k \in X} |\tilde{f}(k)|, \end{aligned} \quad (\text{F.2.5})$$

where \tilde{f} is the Fourier transform of f .

One notes that if the product is noncommutative, then there are two possible and independent representations μ^L , μ^R of the product corresponding to left and right multiplication respectively

$$\begin{aligned}\mu_f^L \xi(x) &= (f \star \xi)(x), & \mu_f^R \xi(x) &= (\xi \star f)(x), \\ \mu_f^L \mu_g^L &= \mu_{f \star g}^L, & \mu_f^R \mu_g^R &= \mu_{g \star f}^R, & \mu_f^L \mu_g^R &= \mu_g^R \mu_f^L.\end{aligned}\quad (\text{F.2.5})$$

Therefore the derived multiplication $\mu^\lambda = \alpha \mu^L + \beta \mu^R$, where $\lambda = (\alpha, \beta)$ are commuting or central numbers, has the commutation relation

$$[\mu_f^\lambda, \mu_g^\lambda] = \alpha^2 \mu_{f \star g - g \star f}^L + \beta^2 \mu_{-f \star g + g \star f}^R. \quad (\text{F.2.6})$$

For a subset of elements $A = \{a\} \subseteq \mathcal{F}(X)$ where $a \star b - b \star a$ is central for all $a, b \in A$ one has $\mu_{a \star b - b \star a}^L = -\mu_{-f \star g + g \star f}^R$ and so μ^λ will give a commutative representation $\mu_{a_c} = \mu_a^\lambda \quad \forall a \in \mathcal{A}$ of A on $\mathcal{H} = \mathcal{F}(X)$ with $\alpha^2 = \beta^2$.

- For a self adjoint operator $a^* = a$, the C^* condition $\|a^*a\| = \|a\|^2$ becomes $\|a^2\| = \|a\|^2$. Thus if one defines \sqrt{a} then

$$\begin{aligned}\|a\| &= \|\sqrt{a^2}\| = \|\sqrt{a}\|^2 = \left(\sup_{\xi \in \mathcal{H}} \frac{\sqrt{|\langle \xi | \sqrt{a^*} \sqrt{a} \xi \rangle|}}{\|\xi\|} \right)^2 = \left(\sup_{\xi \in \mathcal{H}} \frac{\sqrt{|\langle \xi | a \xi \rangle|}}{\|\xi\|} \right)^2 \\ &= \sup_{\xi \in \mathcal{H}} \frac{\langle \xi | a \xi \rangle}{\langle \xi | \xi \rangle}.\end{aligned}\quad (\text{F.2.6})$$

- The following names are used:

$$a^*a = 1 \quad (a \text{ is an isometry}),$$

$$(a^*a)^2 = a^*a \quad (a \text{ is a partial isometry}),$$

$$a^*a = aa^* \quad (a \text{ is a normal element}),$$

$$a^*a = aa^* = 1 \quad (a \text{ is a normal isometry or a unitary element}).$$

F.3 Convex subspaces

In a Hilbert space one has the basic expansion identity

$$\|\xi + \eta\|^2 = \|\xi\|^2 + \langle \xi | \eta \rangle + \langle \eta | \xi \rangle + \|\eta\|^2. \quad (\text{F.3.1})$$

$C \subset \mathcal{H}$ is convex if $\forall c, c' \in C, \alpha c + (1 - \alpha)c' \in C, \forall 0 < \alpha < 1$. Alternatively $C \subset \mathcal{H}$ is convex if

$$d(\xi, C) = \min_{c \in C} \|\xi - c\| = \|\xi - \xi_C\| \quad (\text{F.3.2})$$

is unique for any $\xi \in \mathcal{H}$. Consider the collection $\partial C = \{\xi_C, \xi \in \mathcal{H}\}$ of extreme points of C . Then one deduces from the primary definition that any point $\eta \in C$ can be expanded as

$$\eta = \sum_{b \in \partial C} \eta_b b, \quad \sum_{b \in \partial C} \eta_b = 1, \quad 0 \leq \eta_b \leq 1, \quad (\text{F.3.3})$$

The points of $\partial C = \{b\}$ are pure in that any element b defines a unique equivalence class of elements of \mathcal{H} given by $[b] = \{\xi \in \mathcal{H}, d(\xi, C) = d(\xi, b) > 0\}$. The impure elements of C are those that do not belong to ∂C .

From the definition (F.3.2) it follows that

$$\|\xi - \xi_C\| \leq \|\xi - c\| \quad \forall c \in C. \quad (\text{F.3.4})$$

In particular $\xi_C + \frac{\langle \xi - \xi_C | c \rangle}{\|c\|^2} c \equiv c_p \in C$ because of the fact that both ξ_C & $\frac{\langle \xi - \xi_C | c \rangle}{\|c\|^2} c \in C$ assuming that C is a closed linear subspace (Here $\frac{\langle \xi - \xi_C | c \rangle}{\|c\|^2} c$ is the projection of $\xi - \xi_C$ in the direction of an arbitrary $c \in C, c \neq 0$).

Therefore

$$\begin{aligned}
\|\xi - \xi_C\|^2 &\leq \|\xi - c_p\|^2 \\
&= \left\| \xi - \left(\xi_C + \frac{\langle \xi - \xi_C | c \rangle}{\|c\|^2} c \right) \right\|^2 \\
&= \left\| \xi - \xi_C - \frac{\langle \xi - \xi_C | c \rangle}{\|c\|^2} c \right\|^2 \\
&= \|\xi - \xi_C\|^2 - \frac{|\langle \xi - \xi_C | c \rangle|^2}{\|c\|^2} \\
&\Rightarrow \langle \xi - \xi_C | c \rangle = 0 \quad \forall c \in C, \quad c \neq 0.
\end{aligned} \tag{F.3.1}$$

That is $d(\xi, C) = \|\xi - \xi_C\|$ implies that $\xi - \xi_C$ is orthogonal or normal to C and therefore if $P_C \in B(\mathcal{H})$ is the orthogonal projection unto C then

$$\begin{aligned}
d(\xi, C) &= \|\xi - \xi_C\| = \|\xi - P_C \xi\| = d(\xi, P_C \xi), \\
P_C \mathcal{H} &= C, \quad P_C^* = P_C^2 = P_C, \quad \|P_C\| = 1.
\end{aligned} \tag{F.3.2}$$

In particular

$$\langle \xi - \xi_C | \xi_C \rangle = 0, \quad \Rightarrow \quad \langle \xi | \xi_C \rangle = \langle \xi_C | \xi_C \rangle = \|\xi_C\|^2 \tag{F.3.2}$$

and therefore

$$\begin{aligned}
\|\xi\|^2 &= \|\xi - \xi_C\|^2 + \langle \xi | \xi_C \rangle + \langle \xi_C | \xi \rangle - \|\xi_C\|^2. \\
\|\xi\|^2 &= \|\xi_C\|^2 + \|\xi - \xi_C\|^2.
\end{aligned} \tag{F.3.2}$$

F.3.1 States of a *-algebra

A linear functional $\phi : \mathcal{A} \rightarrow \mathbb{C}$, $\phi \in \mathcal{A}_+^*$ on an algebra $\mathcal{A} = \{a\}$ is said to be positive if it maps positive elements $P(\mathcal{A}) = \{p = a^*a, \quad a \in \mathcal{A}\}$ to positive numbers. Consider the possibility of introducing a basis $\{\varphi_i\}$ for the set of positive linear

functionals (plfs). Then a general plf may be expanded as

$$\phi = \sum_i \lambda_i \varphi_i. \quad (\text{F.3.3})$$

Then $\phi(p) = \sum_i \lambda_i \varphi_i(p) \geq 0 \quad \forall p \in P(\mathcal{A})$ implies that $\lambda_i \geq 0 \quad \forall i$. Therefore for the set $S(\mathcal{A}) = \{\rho \in \mathcal{A}_+^*, \rho(1) = 1\}$ of normalized plfs one has $\rho = \sum_i \lambda_i \varphi_i$, $\lambda_i \geq 0$ and $\rho(1) = 1$ gives $\sum_i \lambda_i = 1$, $0 \leq \lambda_i \leq 1$ if $\varphi_i(1) = 1 \quad \forall i$. Therefore $S(\mathcal{A})$ is a convex set generated by the basis elements $\{\varphi_i\}$ which are known as pure states due to their role as the extreme points in the convexity of $S(\mathcal{A})$.

F.4 Spectral theory

Spectrum $\sigma(a)$ /spectral radius $\rho(a)$

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in C, (\lambda - a)^{-1} = \sum_{n=0}^{\infty} \lambda^{-(n+1)} a^n \notin \text{in } \mathcal{A}\} \quad (\text{F.4.0})$$

The spectral radius or radius of convergence ¹ of $\sum_{n=0}^{\infty} \lambda^{-(n+1)} a^n$ is

$$\begin{aligned} \rho(a) &= \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = e^{\lim_{n \rightarrow \infty} \frac{\ln \|a^n\|}{n}} \leq e^{\lim_{n \rightarrow \infty} \frac{\ln \|a\|^n}{n}} \\ &= e^{\lim_{n \rightarrow \infty} \ln \|a\|} = \|a\|. \end{aligned} \quad (\text{F.4.1})$$

¹ This utilizes L'Hospital's rule: if f, g are differentiable functions and $\lim_{x \rightarrow a} f(x) = f(a) = 0 = \lim_{x \rightarrow a} g(x) = g(a)$ then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \lim_{b \rightarrow x} \frac{\frac{f(b)-f(a)}{b-a}}{\frac{g(b)-g(a)}{b-a}} = \lim_{x \rightarrow a} \frac{\lim_{b \rightarrow x} \frac{f(b)-f(a)}{b-a}}{\lim_{c \rightarrow x} \frac{g(c)-g(a)}{c-a}} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}, \quad (\text{F.4.1})$$

and similarly for $\lim_{x \rightarrow a} f(x) = \infty = \lim_{x \rightarrow a} g(x)$.

The series is convergent (ie. $\lambda - a$ is invertible or $\lambda \notin \sigma_{\mathcal{A}}(a)$) if

$$\lim_{n \rightarrow \infty} \|\lambda^{-(n+1)} a^n\|^{\frac{1}{n}} = |\lambda|^{-1} \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = |\lambda|^{-1} \rho(a) < 1, \quad (\text{F.4.2})$$

and similarly the series is not convergent (ie. $\lambda \in \sigma_{\mathcal{A}}(a)$) if

$$\rho(a) < |\lambda|. \quad (\text{F.4.3})$$

Therefore as $\rho_{\mathcal{A}}(a)$ is finite, $\sigma_{\mathcal{A}}(a)$ cannot be empty in \mathbb{C} .

That is $\forall a \in \mathcal{A}, \sigma_{\mathcal{A}}(a) \neq \{\}$.

Corresponding to any single variable function f , one can define an \mathcal{A} -valued function $f : \mathcal{A} \rightarrow \mathcal{A}, a \mapsto f(a)$. In particular one can make use of holomorphic functions

$$f(a) = \frac{1}{2\pi i} \oint_{\Gamma(\sigma_{\mathcal{A}}(a))} dz \frac{f(z)}{z - a}, \quad (\text{F.4.4})$$

where $\Gamma(\sigma_{\mathcal{A}}(a))$ is any closed curve in C that encloses $\sigma_{\mathcal{A}}(a)$.

- $aa^* = a^*a \Rightarrow \rho(a^*a) = \|a\|^2$ since

$$\begin{aligned} \rho(a^*a) &= \lim_{n \rightarrow \infty} \|(a^*a)^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|(a^n)^* a^n\|^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \|a^n\|^{\frac{2}{n}} = \|a\|^2. \end{aligned} \quad (\text{F.4.4})$$

- $a = a^* \Rightarrow \rho(a) = \|a\|$ since by the C^* condition (F.2.4)

$$\rho(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|. \quad (\text{F.4.5})$$

- One may also verify that $\sigma(ab) = \sigma(ba) \forall a, b \in \mathcal{A}$ due to the following identity:

$$\begin{aligned} (1 - ab)^{-1} &= 1 + ab + (ab)^2 + (ab)^3 + \dots \\ &= 1 + a(1 + ba + (ba)^2 + (ba)^3 + \dots)b \\ &= 1 + a(1 - ba)^{-1}b. \end{aligned} \quad (\text{F.4.4})$$

F.4.1 Gelfand-Mazur theorem

If \mathcal{A} has unit then $\lambda - a \in \mathcal{A} \quad \forall \lambda \in \mathbb{C}, \forall a \in \mathcal{A}$. Therefore if every element $a \in \mathcal{A}$ is invertible except when $a = 0$ then so does $(a - \lambda)^{-1} \exists$ except when $a - \lambda = 0$.

But $\sigma(a) = \{\lambda, (a - \lambda)^{-1} \nexists\} \neq \{\}$ and therefore for each $a \in \mathcal{A}, \exists \lambda \in \mathbb{C}$ such that $a - \lambda = 0$. That is if \mathcal{A} has a unit and if every element $a \in \mathcal{A}$ is invertible except when $a = 0$ then $\mathcal{A} \simeq \mathbb{C}$.

It follows that if \mathcal{A} has unit and $I \subset \mathcal{A}$ is a maximal (having no proper subs) two-sided ideal, $I\mathcal{A} = \mathcal{A}I = I, I + I = I$, then the quotient $\mathcal{A}/I \simeq \mathbb{C}$ where $\mathcal{A}/I = \{a + I; a \in \mathcal{A}\}$.

F.4.2 Gelfand-Naimark theorem

A character of an abelian algebra \mathcal{A}_0 is defined by

$$\chi : \mathcal{A}_0 \rightarrow \mathbb{C} \setminus \{0\}, \quad \chi(ab) = \chi(a)\chi(b), \quad \chi(a + b) = \chi(a) + \chi(b). \quad (\text{F.4.5})$$

If \mathcal{A}_0 is unitary with identity 1 the $\chi(1) = \chi(1)^2 \Rightarrow \chi(1) = 1$.

Thus $\chi(\alpha a) = \alpha\chi(a) \quad \forall \alpha \in \mathbb{C}$. This coincides with the definition of the eigenvalue and generalizes the fact that any two commuting operators can be simultaneously diagonalized (ie. have a common set of eigenvectors).

Recall that the spectrum $\sigma(a)$ is given by $\sigma(a) = \{\lambda(a) \in \mathbb{C}, (\lambda(a) - a)^{-1} \nexists\}$ and satisfies $\lambda(a^*) = \overline{\lambda(a)}, \lambda(f(a)) = f(\lambda(a)), |\lambda(a)| \leq \rho(a) \leq \|a\|$, etc. Thus $\lambda : \mathcal{A}_0 \rightarrow \mathbb{C}$ is a character on $\mathcal{A}_0 \setminus \{0\}$ and by uniqueness of λ , λ & χ coincide on $f(a) \quad \forall f$ and hence $\chi(a) \in \sigma(a)$ which means that

$$|\chi(a)| \leq \|a\| \quad \forall a \in \mathcal{A}_0, \quad \forall \chi \in \sigma(\mathcal{A}_0) = \{\mathcal{X} : \mathcal{A}_0 \rightarrow \mathbb{C} \setminus \{0\}\}.$$

$$\rho(a) = \sup_{\chi \in \sigma(\mathcal{A}_0)} |\chi(a)|. \quad (\text{F.4.5})$$

Define the spectrum $\sigma(\mathcal{A}_0) = \{\chi : \mathcal{A}_0 \rightarrow \mathbb{C} \setminus \{0\}\}$ of \mathcal{A}_0 . Then the map $a \mapsto \tilde{a} : \sigma(\mathcal{A}_0) \rightarrow \mathbb{C}$, $\tilde{a}(\chi) = \chi(a)$

isomorphically and isometrically identifies abelian C^* -algebra \mathcal{A}_0 with the commutative product algebra $\mathcal{F}_\mu(\sigma(\mathcal{A}_0))$ of complex functions $\mathcal{F}(\sigma(\mathcal{A}_0))$ since

$$\begin{aligned}\chi(ab) &= \chi(a)\chi(b) = \tilde{a}(\chi)\tilde{b}(\chi) = (\tilde{a}\tilde{b})(\chi), \\ \chi(a+b) &= \chi(a) + \chi(b) = \tilde{a}(\chi) + \tilde{b}(\chi) = (\tilde{a} + \tilde{b})(\chi).\end{aligned}\tag{F.4.5}$$

That is,

$$\begin{aligned}\mathcal{A}_0 &\simeq \tilde{\mathcal{A}}_0 \simeq \mathcal{F}(\sigma(\mathcal{A}_0)) \simeq \mathcal{F}_\mu(\sigma(\mathcal{A}_0)), \\ \tilde{\mathcal{A}}_0 &= \{\tilde{a}, a \in \mathcal{A}_0\}\end{aligned}\tag{F.4.5}$$

and also the spectrum $\sigma(\mu_{\tilde{a}}) = \sigma(a)$ since if $(\lambda - a)^{-1} \notin \mathcal{A}$ then $\chi((\lambda - a)^{-1}b) \notin \mathbb{C} \forall \chi \in \sigma(\mathcal{A}_0)$, $0 \neq b \in \mathcal{A}$ where

$$\begin{aligned}\chi((\lambda - a)^{-1}b) &= (\chi(\lambda) - \chi(a))^{-1}\chi(b) = (\lambda - \tilde{a}(\chi))^{-1}\tilde{b}(\chi) \\ &= ((\lambda - \mu_{\tilde{a}})^{-1}\tilde{b})(\chi) \notin \mathbb{C} \quad \forall \chi, \tilde{b}\end{aligned}$$

and vice versa.

Thus $\rho(\mu_{\tilde{a}}) = \rho(a)$.

$$\chi(a^*) = \overline{\chi(a)} = \overline{\tilde{a}(\chi)} = \tilde{a}^*(\chi).\tag{F.4.4}$$

For the function multiplication algebra $\mathcal{F}_\mu(X)$ the spectrum of the multiplication operator is

$$\begin{aligned}\sigma(\mu_f) &= \{f(x), x \in X\} = f(X), \\ \|\mu_f\| &= \sup_{x \in X} |f(x)|, \quad \|\mu_f \mu_g\| \leq \|\mu_f\| \|\mu_g\|, \\ \|\mu_f^* \mu_f\| &= \|\mu_f\|^2.\end{aligned}\tag{F.4.3}$$

Similarly,

$$\begin{aligned}
\sigma(\mu_{\tilde{a}}) &= \{\tilde{a}(\chi), \chi \in \sigma(\mathcal{A}_0)\} = \{\chi(a), \chi \in \sigma(\mathcal{A}_0)\} = \tilde{a}(\sigma(\mathcal{A}_0)) \simeq \sigma(\mathcal{A}_0), \\
&\Rightarrow \sigma(a) = \sigma(\mu_{\tilde{a}}) \simeq \sigma(\mathcal{A}_0), \\
\|\mu_{\tilde{a}}\| &= \sup_{\chi \in \sigma(\mathcal{A}_0)} |\tilde{a}(\chi)| = \sup_{\chi \in \sigma(\mathcal{A}_0)} |\chi(a)|, \quad \|\mu_{\tilde{a}}\mu_{\tilde{b}}\| \leq \|\mu_{\tilde{a}}\| \|\mu_{\tilde{b}}\|, \\
\|\mu_{\tilde{a}}^*\mu_{\tilde{a}}\| &= \|\mu_{\tilde{a}}\|^2, \quad \mu_{\tilde{a}}^* = \mu_{\tilde{a}^*}.
\end{aligned} \tag{F.4.1}$$

To check that the map $a \rightarrow \mu_{\tilde{a}}$ is an isometry

$$\begin{aligned}
\|\mu_{\tilde{a}}\|^2 &= \|\mu_{\tilde{a}}^*\mu_{\tilde{a}}\| = \rho(\mu_{\tilde{a}}^*\mu_{\tilde{a}}) \\
&= \rho(\mu_{\tilde{a}^*a}) = \rho(\mu_{\widetilde{a^*a}}) = \rho(a^*a) = \|a\|^2, \\
&\Rightarrow \|\mu_{\tilde{a}}\| = \|a\|.
\end{aligned} \tag{F.4.0}$$

F.5 Ideals and Identities

Given an algebra \mathcal{A} , the concept of its ideals (or its invariant subalgebras in general) is a generalization of the zero element meanwhile the concept of its identities (or its symmetry groups in general) is a generalization of the unit element. Let $\mathcal{P}(\mathcal{A}) = \{S \subseteq \mathcal{A}\}$ ($P(\mathcal{A}) = \{S \subset \mathcal{A}\} = \mathcal{P}(\mathcal{A}) \setminus \{\mathcal{A}\}$) be the set of all subsets (proper subsets) of \mathcal{A} .

Definition: Consider $A, B \in \mathcal{P}(\mathcal{A})$ and define $AB = \{ab, a \in A, b \in B\}$.

It **follows** that $Ab, aB \subseteq AB \quad \forall a \in A, b \in B$. Also $A + B = \{a + b, a \in A, b \in B\}$ from which it **follows** that $A + b, a + B \subseteq A + B \quad \forall a \in A, b \in B$.

Definition: A is a left (right) ideal, denote it I_l (I_r) $\in P(\mathcal{A})$, if

$$I_l \mathcal{A} = I_l, \quad I_l + I_l = I_l \quad (\mathcal{A} I_r = I_r, \quad I_r + I_r = I_r). \tag{F.5.1}$$

It follows that

$$I_l S \subseteq I_l \quad \forall S \in \mathcal{P}(\mathcal{A}), \quad (S I_r \subseteq I_r, \quad \forall S \in \mathcal{P}(\mathcal{A})). \quad (\text{F.5.2})$$

That is $I_l(I_r)$ is a left(right) \mathcal{A} -invariant abelian (ie. additive) proper subgroup (a proper subset that is closed under addition).

Definition: An ideal is two-sided if it is both a left and a right ideal.

Definition: A subset of \mathcal{A} is said to be nonideal if it is not a subset of any ideal.

Definition: Similarly A is a left (right) identity, denote it $E_l (E_r) \in P(\mathcal{A})$, if

$$E_l \mathcal{A} = \mathcal{A}, \quad E_l E_l = E_l \quad (\mathcal{A} E_r = \mathcal{A}, \quad E_r E_r = E_r). \quad (\text{F.5.3})$$

That is $E_l(E_r)$ is a multiplicative proper subgroup (a proper subset that is closed under multiplication) under which \mathcal{A} is left(right)-invariant.

Definition: An identity is two-sided if it is both a left and a right identity.

Definition: A subset of \mathcal{A} is said to be nonidentity if it is not a subset of any identity.

Observe that by definitions (F.5.1) and (F.5.3)

$$I_l(\mathcal{A}) \cap E_l(\mathcal{A}) = \{\} = I_r(\mathcal{A}) \cap E_r(\mathcal{A}) \quad (\text{F.5.4})$$

where $I(\mathcal{A})$ is the set of all ideals in $P(\mathcal{A})$ and $E(\mathcal{A})$ is the set of all identities in $P(\mathcal{A})$.

Definition: A multiplicative left(right) inverse $\tilde{S}^l \in P(\mathcal{A})$ ($\tilde{S}^r \in P(\mathcal{A})$) of a subset $S \in \mathcal{P}(\mathcal{A})$ is any subset such that $\tilde{S}^l S$ ($S \tilde{S}^r$) is a left (right) identity; ie. $\tilde{S}^l S \in E_l(\mathcal{A})$ ($S \tilde{S}^r \in E_r(\mathcal{A})$), where $E_l(\mathcal{A})$ ($E_r(\mathcal{A})$) is the set of left(right) identities of \mathcal{A} .

Observe(1) that \tilde{z}_l^l may $\nexists \quad \forall z_l \in \mathcal{P}(I_l), \quad \forall I_r \in I_r(\mathcal{A})$ (\tilde{z}_r^r may $\nexists \quad \forall z_r \in \mathcal{P}(I_r), \quad \forall I_r \in I_r(\mathcal{A})$) by (F.5.2) and (F.5.4). This is because $z_l S \subseteq I_l \quad \forall S \subseteq \mathcal{A}$ by

definition and if it has a left inverse \tilde{z}_l^l then one can find a left identity E_l^z such that $E_l^z = \tilde{z}_l^l z_l$. That is, one has the two conditions $\tilde{z}_l^l z_l \mathcal{A} = \mathcal{A}$ and $z_l S \subseteq I_l \quad \forall S \subseteq \mathcal{A}$ but $z_l \mathcal{A} \subseteq I_l \Rightarrow \tilde{z}_l^l z_l \mathcal{A} \subseteq \tilde{z}_l^l I_l$ and so for \tilde{z}_l^l (\tilde{z}_r^r) to exist we must have $\tilde{z}_l^l I_l = \mathcal{A}$ ($I_r \tilde{z}_r^r = \mathcal{A}$). In particular \tilde{z}_l^l (\tilde{z}_r^r) cannot exist if I_l (I_r) is also a left ideal [ie. if I_l (I_r) is a two-sided ideal].

Observe(2) that \tilde{z}_l^r may $\nexists \quad \forall z_l \in \mathcal{P}(I_l), \quad \forall I_l \in I_l(\mathcal{A})$ (\tilde{z}_r^l may $\nexists \quad \forall z_r \in \mathcal{P}(I_r), \quad \forall I_r \in I_r(\mathcal{A})$) by (F.5.2) and (F.5.4). This is because $z_l S \subseteq I_l \quad \forall S \subseteq \mathcal{A}$ by definition and if it has a right inverse \tilde{z}_l^r then one can find a right identity E_r^z such that $E_r^z = z_l \tilde{z}_l^r$. That is, one has the two conditions $\mathcal{A} z_l \tilde{z}_l^r = \mathcal{A}$ and $z_l S \subseteq I_l \quad \forall S \subseteq \mathcal{A}$ which together imply that $\mathcal{A} I_l = \mathcal{A}$ ($I_r \mathcal{A} = \mathcal{A}$). Thus for \tilde{z}_l^r (\tilde{z}_r^l) to exist I_l (I_r) must also be a right ideal and thus a two-sided ideal. Thus \tilde{z}_l^r (\tilde{z}_r^l) cannot exist unless I_l (I_r) is a two-sided ideal. It follows that if \tilde{z}_l^r (\tilde{z}_r^l) can exist then \tilde{z}_l^l (\tilde{z}_r^r) cannot exist. Putting results together and removing labels one finds that **a subset z of a two-sided ideal I cannot have an inverse.**

Definition: An ideal I is maximal if it is not a subset of any other ideal; ie. if $I \notin \mathcal{P}(I') \quad \forall I' \in I(\mathcal{A})$.

Definition: Also an ideal is simple if it contains no proper subideal(s).

The spectrum of an element is defined as

$$\sigma(a) = \{\lambda \in \mathbb{C}, \quad (a - \lambda 1)^{-1} \nexists\}. \quad (\text{F.5.5})$$

Thus obviously, if $0 \in \sigma(a)$ then $(a - 0)^{-1} \nexists = a^{-1} \nexists$. That is, inversion of an element a (even if $a \neq 0$, which is all that is required for the elements of \mathbb{C}) is not possible whenever the spectrum $\sigma(a)$ contains 0. For the abelian, ie. \mathcal{A}_0 , case where the spectrum of an element is

$$\sigma(a) \simeq \sigma(\mu_{\bar{a}}) = \tilde{a}(\sigma(\mathcal{A}_0)) \simeq \sigma(\mathcal{A}_0), \quad (\text{F.5.6})$$

one can quotient \mathcal{A}_0 for one chosen character χ , by (ie. remove) those elements $I_\chi(\mathcal{A}_0) = Ker_{\mathcal{A}_0}(\chi) = \{a \in \mathcal{A}_0, \tilde{a}(\chi) = 0\}$ that can take on zero values at χ . One can check that I_χ is a maximal ideal in \mathcal{A}_0 for any $\chi \in \sigma(\mathcal{A}_0)$. The quotienting is consistent only if I_χ is an ideal and this is the case for abelian algebras. The space

$$\mathcal{A}_0/I_\chi = \{c = a + I_\chi, a \in \mathcal{A}_0\}, \quad (\text{F.5.7})$$

in which every nonideal (ie. non- I_χ) element $I_\chi \neq c \in \mathcal{A}_0/I_\chi$ is now invertible, is by the Gelfand-Mazur theorem equivalent to \mathbb{C} . ie. $\mathcal{A}_0/I_\chi \simeq \mathbb{C} \quad \forall \chi \in \sigma(\mathcal{A}_0)$.

One needs to check that every element $I_\chi \neq a + I_\chi \in \mathcal{A}_0/I_\chi$ is invertible. That is

$$0 \notin \sigma(a + I_\chi) = \tilde{a}(\sigma(\mathcal{A}_0)) + I_\chi(\sigma(\mathcal{A}_0)). \quad (\text{F.5.8})$$

Assume on the contrary that $\exists y \in \sigma(\mathcal{A}_0), y \neq \chi$ such that $\tilde{a}(y) + I_\chi(y) = 0, \tilde{a}(\chi) \neq 0$. Since I_χ is a “large” set and this must hold for all its elements the only possibility is $\tilde{a}(y) = 0, I_\chi(y) = 0$ which in turn means that $\{a, I_\chi\} \subseteq I_y$ in contradiction to the fact that I_χ is a maximal ideal. Therefore each character χ is uniquely specified in $\sigma(\mathcal{A}_0)$ by the maximal ideal I_χ .

Therefore given \mathcal{A}_0 , all one needs is knowledge of (a means to construct) the space $\mathcal{I}(\mathcal{A}_0) = \{I\}$ of its (maximal) ideals, from which characters can then be defined as projections

$$\sigma(\mathcal{A}_0) = \{\chi_I : \mathcal{A}_0 \rightarrow \frac{\mathcal{A}_0}{I} \setminus \{0\}, I \in \mathcal{I}(\mathcal{A}_0)\}. \quad (\text{F.5.9})$$

Thus the maximal ideals of an arbitrary C^* -algebra \mathcal{A} may be used to define its (noncommutative) point-like topology/geometry. The space of maximal ideals $\mathcal{I}(\mathcal{A})$ may be written as

$$\mathcal{I}(\mathcal{A}) = \{I_u \subset \mathcal{A}; \mathcal{A}I_u = I_u = I_u I_u = I_u + I_u, \bigcup_u I_u = \mathcal{A}, I_u \not\subset I_v, \forall u, v \in S\},$$

where S is a parameter space ($S \simeq \sigma(\mathcal{A}_0)$ in the commutative algebra \mathcal{A}_0 case).

F.6 GNS construction

A state on \mathcal{A} is a (normalized) positive linear functional

$$\phi(a^*a) \geq 0, \quad \phi(1) = 1. \quad (\text{F.6.1})$$

It follows that

$$|\phi(a^*b)|^2 \leq \phi(a^*a)\phi(b^*b). \quad (\text{F.6.2})$$

Any null element $n \in \mathcal{A}$, $\phi(n^*n) = 0$ is completely orthogonal to \mathcal{A} with respect to \mathcal{A} since (F.6.2) implies that

$$|\phi(n^*b)|^2 \leq \phi(n^*n)\phi(b^*b) = 0 \quad \forall b \in \mathcal{A}. \quad (\text{F.6.3})$$

That is, $\phi(n^*a) = 0 = \phi(a^*n) \quad \forall a \in \mathcal{A}$ or simply $\phi(\mathcal{A}n) = 0$ or

$$\phi(\mathcal{A}N_\phi) = 0, \quad N_\phi = N_\phi(\mathcal{A}) = \{n \in \mathcal{A}, \phi(\mathcal{A}n) = 0\}. \quad (\text{F.6.4})$$

Thus N_ϕ is a left ideal ($\mathcal{A}N_\phi = N_\phi$) in \mathcal{A} and

$\mathcal{H}_1 = \mathcal{A}/N_\phi(\mathcal{A}) = \{\xi = a + N_\phi(\mathcal{A}), a \in \mathcal{A}\}$ is a prehilbert space (to be completed to a hilbert space \mathcal{H}_ϕ) with inner product

$$\phi(\xi^*\eta) \equiv \langle \xi | \eta \rangle = \langle a + N_\phi(\mathcal{A}) | b + N_\phi(\mathcal{A}) \rangle = \phi(a^*b). \quad (\text{F.6.5})$$

This induces the norm

$$\begin{aligned} \|\xi\| &= \sqrt{\langle \xi | \xi \rangle} = \sqrt{\phi(b^*b)}, \quad \xi = b + N_\phi. \\ |\langle \eta | \xi \rangle|^2 &\leq \|\eta\| \|\xi\| \end{aligned} \quad (\text{F.6.5})$$

which gives the operator norm

$$\begin{aligned} \|\pi_\phi(a)\| &= \sup_{\xi \in \mathcal{H}_\phi} \frac{\|\pi_\phi(a)\xi\|}{\|\xi\|}, \\ \Rightarrow \|\pi_\phi(a)\xi\| &\leq \|\pi_\phi(a)\| \|\xi\| \quad \forall \xi \in \mathcal{H} \end{aligned} \quad (\text{F.6.5})$$

can be witten.

Thus one can define a representation

$$\pi_\phi : \mathcal{A} \rightarrow B(\mathcal{H}_\phi), \quad \pi_\phi(\mathcal{A})\xi_\phi \simeq \mathcal{H}_\phi, \quad \xi_\phi = 1_{\mathcal{A}} + N_\phi(\mathcal{A}) \quad (\text{F.6.6})$$

such as that provided by the left action

$$\pi_\phi(a) = L_a : b + N_\phi(\mathcal{A}) \mapsto L_a(b + N_\phi(\mathcal{A})) = a(b + N_\phi(\mathcal{A})) = ab + N_\phi(\mathcal{A}),$$

$$\langle \xi_\phi | \pi_\phi(a) \xi_\phi \rangle = \phi(a), \quad (\text{F.6.6})$$

where the boundedness of L_a needs to be checked. From the definition of the operator norm

$$\|L_a \eta\|^2 \leq \|a\|^2 \|\eta\|^2 \quad \Rightarrow \quad \|L_a\| = \sup_{\eta \in \mathcal{H}_\phi} \frac{\|L_a \eta\|^2}{\|\eta\|^2} \leq \|a\|. \quad (\text{F.6.7})$$

The system $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$, up to unitary isomorphisms, is unique due to cyclicity of the vector ξ_ϕ . The unitary isomorphism with any new system (π, \mathcal{H}, ξ) may be written as

$$\begin{aligned} U : \mathcal{H}_\phi &\rightarrow \mathcal{H}, \quad \pi(a) = U\pi_\phi(a)U^*, \quad \xi = U(\xi_\phi), \\ \pi : \mathcal{A} &\rightarrow B(\mathcal{H}). \end{aligned} \quad (\text{F.6.7})$$

F.7 Algebra Homomorphisms (Representations)

Let $\pi : \mathcal{A} \rightarrow \mathcal{A}'$, $\pi(ab) = \pi(a)\pi(b)$, $\pi^*(a) = \pi(a^*)$. The expansion $\sigma(\pi(a)) \subseteq \sigma(a)$ since $(\lambda 1' - \pi(a))^{-1} \# = (\lambda \pi(1) - \pi(a))^{-1} \# = \pi((\lambda 1 - a)^{-1}) \#$ shows that if $\lambda \in \sigma(\pi(a))$ then $\lambda \in \sigma(a)$ also.

Therefore

$$\begin{aligned}
\|\pi(a)\|^2 &= \|\pi^*(a)\pi(a)\| = \|\pi(a^*)\pi(a)\| = \|\pi(a^*a)\| = \rho(\pi(a^*a)) \\
&\leq \rho(a^*a) \leq \|a^*a\| = \|a\|^2. \\
\text{ie. } \|\pi(a)\| &\leq \|a\|.
\end{aligned} \tag{F.7.-1}$$

F.8 Geometry/algebra dictionary

GEOMETRY	ALGEBRA
points $X = \{x\}$ of a topological space	characters $\mathcal{X} = \{\lambda : \mathcal{F}(X) \rightarrow \mathbb{C} \setminus \{0\}\}$
group $X = (G, \circ)$, $\circ : G \times G \rightarrow G = \{x\}$	characters (\mathcal{X}, \circ) , $\mathcal{X} = \{\lambda : \mathcal{F}(G) \rightarrow \mathbb{C} \setminus \{0\}\}$
complex functions $\mathcal{F}(G) = \{f : G \rightarrow \mathbb{C}\}$, $f(x \circ x') = \langle f x \circ x' \rangle = \langle f \circ(x, x') \rangle = \langle \Delta(f) (x, x') \rangle$ $= \sum_{\alpha} f_{\alpha}(x) f^{\alpha}(x') \equiv \mu_{\mathbb{C}} \sum_{\alpha} (f_{\alpha} \otimes f^{\alpha})(x \otimes x')$	$(\mathcal{F}(G), \Delta, \text{pt-wise-conv})$, $\Delta : \mathcal{F}(G) \rightarrow \mathcal{F}(G) \otimes \mathcal{F}(G)$ $\mu_{\mathcal{B}} : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \quad \forall \mathcal{B}$, $\langle fg x \circ x' \rangle = \langle \Delta(fg) (x, x') \rangle = \langle \Delta(f)\Delta(g) (x, x') \rangle$
complex functions $\mathcal{F}(X) = \{f : X \rightarrow \mathbb{C}\}$	function $*$ -algebra $(\mathcal{F}(X), \text{pt-wise})$
map $m : X \rightarrow Y$	$*$ -homomorphism $h : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$
symmetry of $S : X \rightarrow X$	$*$ -automorphism $U : \mathcal{F}(X) \rightarrow \mathcal{F}(X)$
direct product $X \times Y$	tensor product $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} = (\mathcal{F}(X), \text{pt-wise})$, $\mathcal{B} = (\mathcal{F}(Y), \text{pt-wise})$
probability measures	normalized positive linear functionals
sections $\Gamma(E) = \{s : X \rightarrow E \simeq X \times V\}$ of a vector bundle E over X .	projective module $M(\mathcal{A})$ over $\mathcal{A} = (\mathcal{F}(X), \text{pt-wise})$, $\mathcal{A} \times M(\mathcal{A}) \rightarrow M(\mathcal{A})$
directed (Lie) differential $L_{\xi} : \Gamma(E) \rightarrow \Gamma(E)$ along a smooth vector field $\xi : X \rightarrow V$ $L_{\xi} = _{\delta x \rightarrow \xi} \circ \delta \equiv \delta _{\delta x \rightarrow \xi}$, $\delta s(x) = s(x + \delta x) - s(x)$	Liebnitz differential $D : M(\mathcal{A}) \rightarrow M(\mathcal{A})$, $D(mm') = D(m)m' + mD(m')$
differential forms $\Omega^n(X) = \{\omega_n : L(V/X)^n \rightarrow \mathcal{F}(X)\}$. $L(V/X) = \{L_{\xi}, \xi : X \rightarrow V\}$	$\Omega^n(\mathcal{A}) = \{\omega_n : \text{Der}(\mathcal{A})^n \rightarrow \mathcal{A}, \mathcal{A} = (\mathcal{F}(X), \text{pt-wise})\}$, $\text{Der}(\mathcal{A}) = \{D : M(\mathcal{A}) \rightarrow M(\mathcal{A})\}$
exterior differentials $d, d^* : \Omega^n(X) \rightarrow \Omega^{n \pm 1}(X)$	graded differentials. $d_g, d_g^* : \Omega^n(\mathcal{A}) \rightarrow \Omega^{n \pm 1}(\mathcal{A})$ $d_g(mm') = d_g(m) m' + \pi_g(m) d_g(m')$, $\pi : G \times M \rightarrow M$, $(g, m) \mapsto \pi(g, m) \equiv \pi_g(m)$, $G = S_n$, $\Delta(d) = d \otimes id + \pi \otimes d$

Appendix G

Sets and Physical Logic

G.1 Exclusive sets

A (an exclusive) set S is a selection or conditional collection of objects

$$S = S(X) = \{x \in X; C_S(x)\} \quad (\text{G.1.1})$$

where $C_S : X \rightarrow \{\text{True}, \text{False}, \text{Unsure}\} \subset X$, $x \mapsto C_S(x)$ is a condition that $x \in X$ needs to satisfy in order to be a member of the set S . That is, $x \in S$ iff $C_S(x) = \text{True}$ and $x \notin S$ iff $C_S(x) = \text{False}$. The collection X can be arbitrary or not. We will simply write “ F ” for “False”, “ T ” for “True” and “ U ” for “Unsure”. The result “Unsure” is obtained whenever $C_S(x)$ neither evaluates to T nor to F due to whatever reasons all of which we will refer to as Uncertainty.

The *complement* or *negation* S^\sim of the set S is given by

$$S^\sim = S^\sim(X) = \{x \in X; C^\sim_S(x)\}. \quad (\text{G.1.2})$$

where C^\sim_S is the statement that evaluates to F whenever C_S evaluates to T and vice versa. That is we write $F^\sim = T$, $T^\sim = F$, $U^\sim = U$.

In anticipation of situations where it can be much more difficult to determine when two sets are equal than to determine when one includes the other, one could introduce inclusion \subseteq where $A \subseteq B$ iff $x \in A \Rightarrow x \in B$. In terms of intersection and sum/union

$$\begin{aligned}
AB &= \{x \in A; C_B(x)\} = \{x \in B; C_A(x)\} \\
&= \{x \in X; C_B(x)C_A(x)\}, \\
A + B &= \{x \in X; C_B(x) + C_A(x)\}, \\
A \cap B &= AB, \\
A \cup B &= A + B + AB = \{x \in X; C_B(x) + C_A(x) + C_B(x)C_A(x)\},
\end{aligned}$$

where we have introduced point-wise multiplication/addition of conditions; ie. $(C_1C_2)(x) = C_1(x)C_2(x)$, $(C_1 + C_2)(x) = C_1(x) + C_2(x)$.

We have the following conditions

$$\begin{aligned}
A \subseteq B &\text{ iff } AB = A \text{ iff } A + B = B \\
&\text{ iff } C_A + C_B = C_B \text{ iff } C_AC_B = C_A. \\
A = B &\text{ iff } (A \subseteq A)(B \subseteq A). \tag{G.1.-4}
\end{aligned}$$

When the condition of a set evaluates to either F or U for all $x \in X$ for example in $S \sim S = \{x \in X; C_S \sim(x)C_S(x)\}$ we leave the set blank and call it the empty set denoted $\{\}$.

$$\begin{aligned}
S \sim S &= \{x \in X; C_S \sim(x)C_S(x)\} = \{x \in X; C(x) = U\} \equiv \{\}, \\
S \sim + S &= \{x \in X; C_S \sim(x) + C_S(x)\} = \{x \in X; C(x) \neq U\}, \\
S \sim \cap S &= S \sim S, \quad S \sim \cup S = S \sim + S. \tag{G.1.-5}
\end{aligned}$$

G.1.1 Conditional algebra

All *statements* are (composite) conditions involving *implication* $C_1 \Rightarrow C_2$ and *equality* $C_1 \iff C_2$, *negations* and so on. The operations such as implication, equality, negation and so on, may be written in terms of an algebra system on the set of conditions \mathcal{C} .

In order to compare, compose, decompose, ..., sets one needs to have a means to do similar manipulations on the set of conditions. Let

$$\mathcal{C} = \{C \in X; C : X \rightarrow \{T, F, U\}\}$$

be the set of conditions. Then the value set $\{T, F, U\}$ behaves as follows:

Boolean system:

$$\begin{aligned} TT = T, \quad FT = F, \quad FF = F, \\ T + T = T, \quad F + T = T, \quad F + F = F. \end{aligned}$$

(G.1.-6)

Now if $C_1(x)$ is True but $C_2(x)$ is Unsure then intersection is Unsure meanwhile sum or union is True; ie.

$$TU = U, \quad T + U = T. \tag{G.1.-5}$$

If $C_1(x)$ is False but $C_2(x)$ is Unsure then intersection is False meanwhile sum or union is Unsure; ie.

$$FU = F, \quad F + U = U. \tag{G.1.-4}$$

Finally if both $C_1(x), C_2(x)$ are Unsure then both intersection and sum/union are Unsure; ie.

$$UU = U, \quad U + U = U. \tag{G.1.-3}$$

Thus the summary of the operations is as follows:

$$\begin{aligned}
TT &= T, & FT &= F, & FF &= F, \\
T + T &= T, & F + T &= T, & F + F &= F. \\
TU &= U, & T + U &= T. \\
FU &= F, & F + U &= U. \\
UU &= U, & U + U &= U. \\
F^\sim &= T, & T^\sim &= F, & U^\sim &= U.
\end{aligned} \tag{G.1.-7}$$

One may write the algebra system as $\mathcal{A} = (\{T, F, U\}, +, \cdot, \sim)$ where multiplication \cdot may be thought of as the \sim -conjugate $+^\sim$ of addition $+$ since $(a + b)^\sim = a^\sim b^\sim$, $(ab)^\sim = a^\sim + b^\sim \quad \forall a, b \in \mathcal{A}$. This special property will be lost when an arbitrary \sim -algebra system is considered. One also has the “exclusive or” operation \oplus

$$a \oplus b = ab^\sim + a^\sim b, \quad (a \oplus b)^\sim = ab + a^\sim b^\sim, \quad a, b \in \mathcal{A}. \tag{G.1.-6}$$

Thus **implication** $C_1 \Rightarrow C_2$ is equivalent to $C_1 C_2 = C_2$ or $(C_1 C_2)(x) = C_2(x) \quad \forall x$ and **equality** $C_1 \iff C_2$ is equivalent to $C_1 = C_2$ or $C_1(x) = C_2(x) \quad \forall x$.

The algebra of sets has now been reduced to the algebra of the corresponding set generation conditions.

G.1.2 Maps and bundling

Given two sets A, B one can form another set $C = A \times B$ by pairing elements thus

$$C = \{c; c = (a, b), a \in A, b \in B\} \equiv A \times B = \{(a, b); a \in A, b \in B\}.$$

This operations can be iterated to form $A_1 \times A_2 \times A_3 \times \dots$ given A_1, A_2, A_3, \dots
 In general one can form

$$A \star B = \{x \in X; C_\star(x, C_A, C_B)\}, \quad (\text{G.1.-6})$$

where $C_\star(x, C_A, C_B)$ can for example consist of the sequence of conditions
 $x = (a, b), a \in A, b \in B$ or $x = (a, b), C_A(a)C_B(b)$ corresponding to the direct
 product $A \times B$. That is, we have the conditions
 $C_\star(x, C_A, C_B) \mapsto x = (a, b), C_A(a)C_B(b)$ iff $A \star B \mapsto AB$. We can also have the
 conditions $C_\star(x, C_A, C_B) \mapsto C_A(x)C_B(x)$ iff $A \star B \mapsto AB$ and similarly for the
 sum we have $C_\star(x, C_A, C_B) \mapsto C_A(x) + C_B(x)$ iff $A \star B \mapsto A + B$. This general
 product can be iterated as well.

If $M(\mathcal{A}) = \{m \in X; m : \mathcal{A} \rightarrow X\}$ is the space of maps on \mathcal{A} and
 $P(\mathcal{A}) = \{A \in X; A \subseteq \mathcal{A}\}$ is the set of subsets of \mathcal{A} then one can define a bundle
 twisting map

$$[\] : \mathcal{A} \times M(\mathcal{A}) \rightarrow P(\mathcal{A}), (a, m) \mapsto [a]_m = \{b \in \mathcal{A}; m(a) = m(b)\}$$

which makes a twisted bundle

$$\begin{aligned} [\](\mathcal{A} \times M(\mathcal{A})) &= [\mathcal{A}]_{M(\mathcal{A})} \simeq \mathcal{A} \times [\]_{M(\mathcal{A})} \equiv [\mathcal{A}] \times M(\mathcal{A}), \\ [\]_{M(\mathcal{A})} : \mathcal{A} &\rightarrow (P(\mathcal{A}))^{M(\mathcal{A})}, a \mapsto [a]_{M(\mathcal{A})}, \\ [\]_m : \mathcal{A} &\rightarrow P(\mathcal{A}), a \mapsto [a]_m \quad \forall m \in M(\mathcal{A}), \\ [\mathcal{A}] : M(\mathcal{A}) &\rightarrow (P(\mathcal{A}))^{|\mathcal{A}|}, m \mapsto [\mathcal{A}]_m, \\ [a] : M(\mathcal{A}) &\rightarrow (P(\mathcal{A})), m \mapsto [a]_m \quad \forall a \in \mathcal{A} \end{aligned} \quad (\text{G.1.-10})$$

from the trivial bundle $\mathcal{A} \times M(\mathcal{A})$.

In general,

$$\begin{aligned}
m &: A \times B \rightarrow C, (a, b) \mapsto m(a, b) = c, \\
A \times B &\xrightarrow{m} C, (a, b) \xrightarrow{m} m(a, b) = c, \\
m &: A \times B|_{(a,b)} \mapsto C|_{c=m(a,b)}, \\
A \times B|_{(a,b)} &\xrightarrow{m} C|_{c=m(a,b)}
\end{aligned}$$

may be written as

$$\begin{aligned}
m &: A \rightarrow M(B), a \mapsto m(a, \cdot) : B \rightarrow C, b \mapsto m(a, \cdot)(b) = m(a, b) = c, \\
m &: B \rightarrow M(A), b \mapsto m(\cdot, b) : A \rightarrow C, a \mapsto m(\cdot, b)(a) = m(a, b) = c, \\
A|_a &\xrightarrow{m} M(B)|_{m(a, \cdot)} \xrightarrow{B|_b} C|_{c=m(a,b)}, \\
B|_b &\xrightarrow{m} M(A)|_{m(\cdot, b)} \xrightarrow{A|_a} C|_{c=m(a,b)}, \\
M(A \times B)|_{m=m(\cdot, \cdot)} &\xrightarrow{A|_a} M(B)|_{m(a, \cdot)} \xrightarrow{B|_b} C|_{c=m(a,b)}.
\end{aligned}$$

In bundle form

$$\begin{aligned}
m(A \times B) &\simeq A \times m^B \equiv m_A \times B, \\
m^B &\subseteq M(B \rightarrow C) \equiv C/B \equiv M(B, C) \subset M(B), \\
m_A &\subseteq M(A \rightarrow C) \equiv C/A \equiv M(A, C) \subset M(A), \\
m^B &: A \rightarrow C^{|B|}, \quad m_A : B \rightarrow C^{|A|}, \\
m^b &: A \rightarrow C, a \mapsto m(a, b) \quad \forall b \in B, \\
m_a &: B \rightarrow C, b \mapsto m(a, b) \quad \forall a \in A,
\end{aligned}$$

where $|A|, |B|$ are the number of elements in A, B respectively.

G.1.3 Counting isomorphisms ?

If $|A|$ denotes the number of elements in the set A then the number $|I(A, B)|$ of isomorphic maps $I(A, B) \subseteq \mathcal{F}(A, B) = \{m; m : A \rightarrow \mathcal{D} \subseteq B, |\mathcal{D}| = |A|\}$ is

$$|I(A, B)| = \frac{\Gamma(|B|)}{\Gamma(|A|)\Gamma(|B| - |A|)} |I(A, \mathcal{D})| = \frac{\Gamma(|B|)}{\Gamma(|B| - |A|)},$$

$$\Gamma(n) = n\Gamma(n - 1), \quad \Gamma(1) = 1,$$

$$|I(A, \mathcal{D})| = |I(A, A)| = |(\mathcal{D}, \mathcal{D})| = \Gamma(|A|). \quad (\text{G.1.-26})$$

G.2 Nonexclusive sets: Generalizations

The sets we have defined so far have absolute or rigid rules for choosing their members and thus we can only have members and nonmembers. However in practice there can be intermediate situations with different levels or steps of membership. Therefore we will consider sets for which the set generation conditions (sgc's) can take values in an arbitrary $*$ -algebra system¹ \mathcal{A} .

The operations of multiplication and addition will simply parallel those of the $*$ -algebra system \mathcal{A} . Note that the $*$ -algebra system \mathcal{A} may *neither* be *commutative nor associative in general* and the sets will directly inherit these properties as well. However we will assume associativity, but not commutativity, for simplicity.

¹Other examples of algebra systems include natural numbers \mathbb{N} (which arose due to the need to count things), fractional numbers \mathbb{Q} (which arose due to the need to compare countings) and real or continuous numbers \mathbb{R} (which arose due to the need to compare uncountable characteristics such as lengths). Various products of these number systems (or number sets) also arose due to the need to compare (geometric) shapes and sizes of things.

A set S is a selection or conditional collection of objects

$$S = S(X) = \{x \in X; C_S(x)\} \quad (\text{G.2.1})$$

where $C_S : X \rightarrow \mathcal{A} \subset X$, $x \mapsto C_S(x)$ is a condition that determines the degree (or probability amplitude) of membership in S of each and every $x \in X$.

G.2.1 G1

In one means of generalization we suppose that $x \in S$ with degree or amplitude of membership (aom) $a \in \mathcal{A}$ iff $C_S(x) = a$ and $x \notin S$ with degree or amplitude of nonmembership (aon) $a^\sim \in \mathcal{A}$ iff $C_S(x) = a^\sim$. The collection X can be arbitrary.

Each $a \in \mathcal{A}$ corresponds to a selection of elements $[a]^S = \{x \in X; C_S(x) = a\}$ so that $S = \bigcup_{a \in \mathcal{A}} [a]^S$. Whether \mathcal{A} is represented as an algebra of operators on a Hilbert space, ie. $\mathcal{A} \rightarrow O(\mathcal{H})$, or not one may use the characters $\mathcal{X}(\mathcal{A}) = \{\lambda \in \mathcal{A}^*; \lambda : \mathcal{A} \rightarrow \mathbb{C} \setminus \{0\}\}$ to measure the degrees or amplitude of membership (aom) or nonmembership (aon) carried by each $a \in \mathcal{A}$. The uncertain elements which are those with the property $a^\sim = a$ have a degree of uncertainty or unsureness of membership and their values may be conveniently measured with the help of real linear functionals $\mathcal{A}_R^* = \{\phi \in \mathcal{A}^*; a^\sim = a \Rightarrow \phi(a) \in \mathbb{R}\}$. [Note that in the Boolean algebra system $\mathcal{A} = \{T, F\}$ the elements $a = T, F$ obey $a^2 = a$ and so the characters are given by $\lambda(a^2) = \lambda(a)^2 = \lambda(a) \Rightarrow \lambda(a) = 0, 1$ and one usually chooses $\lambda_1(T) = 1, \lambda_1(F) = 0$ although the only other alternative choice $\lambda_2(T) = 0, \lambda_2(F) = 1$ is equally valid.]

Observe that for a real algebra system where $a^\sim = a \forall a \in \mathcal{A}$ membership of a set is completely determined by the degree or amplitude of unsureness (aou) of membership.

The **complement** or **negation** S^\sim of the set S is given by

$$S^\sim = S^\sim(X) = \{x \in X; C^\sim_S(x)\}. \quad (\text{G.2.2})$$

where C^\sim_S is the statement that evaluates to a^\sim whenever C_S evaluates to a and vice versa. That is we have $a^{\sim\sim} = a$. One should not confuse the logic operation \sim with the $*$ operation with properties

$$a^{**} = a, \quad (ab)^* = b^*a^*, \quad (a + b)^* = a^* + b^*. \quad (\text{G.2.3})$$

[A \sim -algebra system \mathcal{A} with the properties

$$(a + b)^\sim = a^\sim b^\sim, \quad (ab)^\sim = a^\sim + b^\sim \quad \forall a, b \in \mathcal{A}$$

such as the **commutant operation** in **set commutant algebra**, and similar types of analysis, is closer to that of exclusive set theory. A set $\mathcal{S} = \{A\}$ consisting of Von Neumann algebras for example has such properties:

$$A'' = A, \quad (A \cap B)' = A' \cup B', \quad (A \cup B)' = A' \cap B'. \quad] \quad (\text{G.2.4})$$

The set operations are as before given by

$$\begin{aligned} AB &= \{x \in A; C_B(x)\} = \{x \in B; C_A(x)\} \\ &= \{x \in X; C_B(x)C_A(x)\}, \\ A + B &= \{x \in X; C_B(x) + C_A(x)\}, \\ A \cap B &= AB, \\ A \cup B &= A + B + AB = \{x \in X; C_B(x) + C_A(x) + C_B(x)C_A(x)\}, \end{aligned}$$

where the point-wise multiplication/addition of conditions,

$(C_1C_2)(x) = C_1(x)C_2(x)$, $(C_1 + C_2)(x) = C_1(x) + C_2(x)$, is used but this time $(C_1C_2)(x) \neq (C_2C_1)(x)$ in general.

G.2.2 G2

Here we maintain that in $S = \{x \in X; C_S(x)\}$ each and every $x \in X$ is a member of S with degree or amplitude of membership (dom or aom) $C_S(x)$ and degree or amplitude of nonmembership (don or aon) $C_{\tilde{S}}(x)$. That is, ***there is no “sharp” distinction between “members” and “nonmembers”***. There will be uncertainty, with degree or amplitude of uncertainty (dou or aou) $C_S(x)$, in the membership of x if $C_{\tilde{S}}(x) = C_S(x)$ even when $C_{\tilde{S}} \neq C$ on all of X .

To illustrate, let $C_S \in \mathcal{A}$ where \mathcal{A} is an arbitrary $*$ -algebra and $X = \mathcal{A}_{1+}^* \subseteq A_{1+}^* \subseteq A^*$ be the set of normalized positive linear functionals (nplf's) of \mathcal{A} . Then every element $a \in \mathcal{A}$ represents a set generation condition (sgc) for an associated set $S_a = \{\phi \in \mathcal{A}_{1+}^*; \phi(a)\}$ and each $\phi \in \mathcal{A}_{1+}^*$ is a member of S_a with aom $\phi(a)$ and aon $\phi(a^\sim)$. To any such ϕ satisfying $\phi(a^\sim) = \phi(a)$ even when $a^\sim \neq a$ we rather associate an aou $\phi(a)$.

It is important to mention that the set S_a actually corresponds to an equivalence class $[a] = \{b \in \mathcal{A}; \phi(b) = \phi(a) \ \forall \phi \in \mathcal{A}_{1+}^*\}$ of sgc's since every member of $[a]$ generates exactly the same set. That is $S_a \equiv S_{[a]}$.

Set addition and multiplication are straightforward and given by

$$\begin{aligned} S_a S_b &= \{\phi \in \mathcal{A}_{1+}^*; \phi(ab)\} = S_{ab} \equiv S_a \cap S_b, \\ S_a + S_b &= \{\phi \in \mathcal{A}_{1+}^*; \phi(a+b)\} = S_{a+b}, \\ S_a \cup S_b &= \{\phi \in \mathcal{A}_{1+}^*; \phi(a+b+ab)\} = S_{a+b+ab} \end{aligned} \quad (\text{G.2.-2})$$

and the \sim -complement or conjugate S^\sim of S is given by

$$S_a^\sim = S_{a^\sim} = \{\phi \in \mathcal{A}_{1+}^*; \phi(a^\sim)\}. \quad (\text{G.2.-1})$$

Regarding set inclusion, we would like that a set includes itself in which case we must have $S_a S_a = S_{a^2} = S_a$. Thus a further condition for $a \in \mathcal{A}$ to be

a “*pure set*” generation condition (psgc) is for it to be a projector $a^2 = a$. Consequently one has pure and impure sets. If one has a collection of projectors $\mathcal{P} = \{p \in \mathcal{A}; p^2 = p\}$ such that $p_1 p_2 \in \mathcal{P} \quad \forall p_1, p_2 \in \mathcal{P}$, that is $(p_1 p_2)^2 = p_1 p_2$ so that the product of any two sets gives another set, then one has a closed ² system

²One notes that for any given projector $p \in \mathcal{A}$, vpv is another projector $\forall u, v \in \mathcal{A}$ such that $uv = 1$. One also has partial projectors: if p is a projector then $\forall u \in \mathcal{A}$, q in the relation $pu = uq$, $q = q_{(u,v)}$ is a partial projector. Thus corresponding to any projector p is the class of projectors $\mathcal{P}_p(\mathcal{A}) = \{vpv; u, v \in \mathcal{A}, uv = 1\}$ noting that given $\{u_1, u_1, \dots, u_n\} \subset \mathcal{A}$ and $\{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_n\} \subset \mathcal{A}$ such that $u_i \tilde{u}_i = 1 \quad \forall i$ one has $U_n V_n = 1$ where $U_n = \prod_{j=1}^n u_j$, $V_n = \prod_{i=n}^1 \tilde{u}_i$.

Also, given a projector p , $\tilde{p}p$ is a projector for any $\tilde{p} \in p' = \{a \in \mathcal{A}; [a, p] = 0\}$, the commutant of p .

For any given $a \in \mathcal{A}$, $p_a^L = aa_L^{-1}$, $p_a^R = a_R^{-1}a$ are projectors, where $a_L^{-1}a = 1_{\mathcal{A}} = aa_R^{-1}$. Also $a(1_{\mathcal{A}} - p_a^R) = 0 = (1_{\mathcal{A}} - p_a^L)a$.

Given $\phi \in \mathcal{A}_+^*$ a system of projectors $P = \{p_i \in \mathcal{A}; p_i^2 = p_i \quad \forall i\}$ is **right ϕ -measurable** iff $\phi(a) = \sum_i \phi(ap_i) \quad \forall a \in \mathcal{A}$. One may refer to an orthogonal system of projectors $P_1 = \{p_i; p_i p_j = \delta_{ij} p_j \quad \forall i, j\}$ as a **partition**. In a complete system of projector $P = \{p_i \in \mathcal{A}; p_i^2 = p_i\}$ any given $a \in \mathcal{A}$ may be expanded as

$$a = \alpha + \alpha^i p_i + \alpha^{ij} p_i p_j + \dots + \alpha^{i_1 \dots i_k} p_{i_1} \dots p_{i_k} + \dots = \sum_k \alpha^{i_1 \dots i_k} p_{i_1} \dots p_{i_k},$$

$$\alpha^{i_1 \dots i_k} \in \mathbb{C}. \tag{G.2.0}$$

One may orthogonalize a given system of projections $\Pi = \{\pi_i \in \mathcal{A}; \pi_i^2 = \pi_i, \pi_i^* = \pi_i \quad \forall i\}$ when \mathcal{A} is represented on a Hilbert space \mathcal{H} , $\mathcal{A} \rightarrow O(\mathcal{H}) \simeq \mathcal{H} \otimes \mathcal{H}$ where one can write

$\pi_i = \frac{|\xi_i\rangle\langle\xi_i|}{\langle\xi_i|\xi_i\rangle}$ and the set $\{|\xi_i\rangle\}$ can then be orthogonalized. Corresponding to $\{|\xi_i\rangle\}$ is the dual set $\{|\xi_i^*\rangle = |\xi_j\rangle\langle\xi_j|\xi_i\rangle^{-1[j,i]}\}$, with $\langle\xi_i|\xi_j^*\rangle = \delta_{ij}$, from which one obtains the orthogonal set $\{|\hat{\xi}_i\rangle = |\xi_j\rangle\langle\xi_j|\xi_i\rangle^{-\frac{1}{2}[j,i]}\}$ with $\langle\hat{\xi}_i|\hat{\xi}_j\rangle = \delta_{ij}$. Hence $\hat{\pi}_i = |\hat{\xi}_i\rangle\langle\hat{\xi}_i|$ will satisfy $\hat{\pi}_i \hat{\pi}_j = \delta_{ij} \hat{\pi}_j$. Similarly for projectors written as $p_i = \frac{|\xi_i\rangle\langle\eta_i|}{\langle\eta_i|\xi_i\rangle}$ corresponds the orthogonal system $\hat{p}_i = |\xi_k\rangle\langle\eta_k|\xi_i\rangle^{-\frac{1}{2}[k,i]}\langle\eta_i|\xi_l\rangle^{-\frac{1}{2}[i,l]}\langle\eta_l|$, $\hat{p}_i \hat{p}_j = \delta_{ij} \hat{p}_j$. Summation convention is used and the inverse (and square root) is partial in that they are of the matrix in the index types i, j, k, \dots only.

The representation of the projectors p_i in terms of subsets of the Hilbert space \mathcal{H} will take the

of sets.

In a commutative algebra where $\phi(ab) = \phi(a)\phi(b)$, one has that for a projector

general form

$$\begin{aligned}
p_i &= \sum_{(\eta, \xi) \in \mathcal{H}_1 \times \mathcal{H}_2} |\xi_i\rangle \langle \eta_i | \xi_i \rangle^{-1 [\eta, \xi]} \langle \eta_i |, \quad \mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{H}, \\
\hat{p}_i &= \sum_{(\eta, \xi) \in \mathcal{H}_1 \times \mathcal{H}_2} |\xi_k\rangle \langle \eta_k | \xi_i \rangle^{-\frac{1}{2} [k, i] [\eta, \xi]} \langle \eta_i | \xi_l \rangle^{-\frac{1}{2} [i, l] [\eta, \xi]} \langle \eta_l |, \\
&\equiv \sum_{(\eta, \xi) \in \mathcal{H}_1 \times \mathcal{H}_2} |\xi_k\rangle \langle \eta_k | \xi_i \rangle^{-\frac{1}{2}} \langle \eta_i | \xi_l \rangle^{-\frac{1}{2}} \langle \eta_l |, \quad \hat{p}_i \hat{p}_j = \delta_{ij} \hat{p} \quad (\text{G.2.1})
\end{aligned}$$

where $[\eta, \xi]$ simply indicates a partial inverse which is that of a $|\mathcal{H}_1| \times |\mathcal{H}_2|$ matrix, $|\mathcal{H}_1|$ being the size of \mathcal{H}_1 and $[i, j]$ has a similar meaning meanwhile $[i, j][\eta, \xi]$ indicates a full inverse where both index types are involved. The case $\mathcal{H}_1 = \mathcal{H}_2$ corresponds to projections or equivalently $\mathcal{H}_1 = \mathcal{H}_2^*$, $\langle \xi_i | \eta_i \rangle = \delta(\eta - \xi) \langle \xi_i | \xi_i \rangle$. Thus **projectors correspond to subspaces of $\mathcal{H}^2 = \mathcal{H} \times \mathcal{H}$ meanwhile projections (real projectors) correspond to subspaces of the Hilbert space \mathcal{H} . The sum of any number of orthogonal projectors is also a projector.** An orthogonal system of projectors $\{\hat{p}_i\}$ spans a commutative algebra with elements $c = \sum_i \tilde{c}_i \hat{p}_i$, $\text{Tr}(\hat{p}_i) = 1 \Rightarrow \tilde{c}_i = \text{Tr}(c \hat{p}_i)$.

Corresponding to each projector p with additional property $\text{Tr}(pp^*) = 1$ one can define a state ϕ_p given by

$$\begin{aligned}
\text{Tr}(a) &= \sum_{\xi \in \mathcal{H}} \frac{\langle \xi | a | \xi \rangle}{\langle \xi | \xi \rangle}, \quad \phi_p(a) = \text{Tr}(pap^*), \\
\text{Tr}(|u\rangle \langle v|) &= \sum_{\xi \in \mathcal{H}} \frac{\langle \xi | u \rangle \langle v | \xi \rangle}{\langle \xi | \xi \rangle} = \sum_{\xi \in \mathcal{H}} \frac{\langle v | \xi \rangle \langle \xi | u \rangle}{\langle \xi | \xi \rangle} = \langle v | \sum_{\xi \in \mathcal{H}} \frac{|\xi\rangle \langle \xi|}{\langle \xi | \xi \rangle} |u\rangle = \langle v | u \rangle, \\
\text{Tr}_p([a, b]_p) &= 0, \quad \text{where } [a, b]_p := apb - bpa, \quad \text{Tr}_p(a) = \text{Tr}(pa). \quad (\text{G.2.2})
\end{aligned}$$

A projector p can be written as a sum $p = \pi_1 + \pi_2$, $\pi_1^* = \pi_1$, $\pi_2^* = -\pi_2$ of a hermitian and an antihermitian operator with the following properties

$$\begin{aligned}
\pi_1^2 &= \pi_1 - \pi_2^2, \quad \pi_1 \pi_2 + \pi_2 \pi_1 = \pi_2 \\
&\Rightarrow \pi_2 \pi_1 \pi_2 = \pi_1 (1 - \pi_1)^2. \quad (\text{G.2.3})
\end{aligned}$$

A tensor product $p_1 \otimes p_2 \otimes \dots$ of projectors p_1, p_2, \dots is also a projector. In general one may form

$p, \phi(p^2) = \phi(p)\phi(p) = \phi(p) \Rightarrow \phi(p) = 0, 1 \quad \forall \phi$. That is, **membership is of the exclusive type for a projector (projective sgc) in a commutative algebra** \mathcal{A}_0 . Thus projectors indeed generalize sgc's from exclusive (ie. commutative) to nonexclusive (ie. noncommutative) logic. The generalization of the logic operation \sim is $p^\sim = 1_{\mathcal{A}} - p$.

Addition/union of sets is possible but not essential since every set contains the same elements without any exclusions and two sets can only differ in the aom, aon or aou of individual elements. That is, in this sense, all sets are already united.

We may now say that A is a left (right) or two-sided subset of B iff AB (BA)

a Δ -deformed tensor product (compare with G.1.-6) of two sets as

$$\begin{aligned}
S_a \otimes_{\Delta} S_b &= \{\phi \in \mathcal{A}_{1+}^*; \pi_2 \Delta(\phi)(a \otimes b)\} \equiv S_{a \otimes_{\Delta} b}, \quad \Delta : (\mathcal{A}^*)^n \rightarrow \bigoplus_{k=0}^N (\mathcal{A}^*)^{\otimes k}, \quad n, N \in \mathbb{N}, \\
(id \otimes \pi_2 \Delta) \circ \pi_2 \Delta &= (\pi_2 \Delta \otimes id) \circ \pi_2 \Delta \quad (\text{may not necessarily hold in general}), \\
\pi_2 \Delta(\phi)(a \otimes b) &= \langle \phi_{\alpha} \otimes \phi^{\alpha} | a \otimes b \rangle = \phi_{\alpha}(a) \phi^{\alpha}(b). \\
\pi_n \Delta &= ((id \otimes)^k \pi_2 \Delta (\otimes id)^{n-k-1}) \circ \pi_{n-1} \Delta^{n-1} \quad \forall 1 \leq k \leq n-1, \\
\pi_3 \Delta(\phi) &= (id \otimes \pi_2 \Delta) \circ \pi_2 \Delta(\phi) = (id \otimes \pi_2 \Delta)(\phi_{\alpha} \otimes \phi^{\alpha}) = \phi_{\alpha} \otimes \pi_2 \Delta(\phi^{\alpha}) \\
&= \phi_{\alpha} \otimes (\phi^{\alpha})_{\beta} \otimes (\phi^{\alpha})^{\beta}. \\
\pi_1 \Delta(\phi)(a) &= \phi(a), \quad \pi_3 \Delta(\phi)(a \otimes b \otimes c) = \phi_{\alpha}(a) (\phi^{\alpha})_{\beta}(b) (\phi^{\alpha})^{\beta}(c), \quad \dots \tag{G.2.4}
\end{aligned}$$

The tensor product that corresponds (ie. is dual) to the product in \mathcal{A} is a particular case Δ_1 of Δ defined by $\phi(ab) = \langle \phi | ab \rangle = \langle \pi_2 \Delta_1(\phi) | a \otimes b \rangle \equiv \pi_2 \Delta_1(\phi)(a \otimes b)$. On the other end the Δ that corresponds to the usual (undeformed) tensor product \otimes is given by

$\pi_2 \Delta_0(\phi)(a \otimes b) = \langle \pi_2 \Delta_0(\phi) | a \otimes b \rangle = \langle \phi \otimes \phi | a \otimes b \rangle = \phi(a)\phi(b)$. Thus possible Δ 's interpolate between Δ_0 and $\Delta_1 \equiv \Delta_0^*$. The actual Δ 's to be considered may be determined by the way one or more physical systems behave (or evolve) relative to (ie. interact or correlate with) one another. S_a may be interpreted as the amplitude distribution or "painting" in \mathcal{A}^* of the system represented by $a \in \mathcal{A}$.

is more related to A than it is to B or any other set. ie.

$$AB \simeq A \quad (BA \simeq A) \quad \text{or} \quad AB \simeq BA \simeq A. \quad (\text{G.2.5})$$

In particular for each given projector $p \in \mathcal{A}$ which is a sgc for S_p any other projector $p' \in \mathcal{A}$ such that $p'p = p'$ or $pp' = p'$ or $p'p = pp' = p'$ generates a subset $S_{p'}$ of S_p with $S_{p'}S_p = S_{p'}$ or $S_pS_{p'} = S_{p'}$ or $S_{p'}S_p = S_pS_{p'} = S_{p'}$ respectively.

Since every set now has the same members one may introduce a measure on the sets and compare their sizes as for example

$$\mu_1(S_a) = \sqrt{\sum_{\phi \in \mathcal{A}_{1+}^*} |\phi(a)|^2}, \quad \mu_2(S_a) = \max_{\phi \in \mathcal{A}_{1+}^*} |\phi(a)|, \quad \dots \quad (\text{G.2.6})$$

We will define a **family of open sets** to be one in which the intersection (ie. product) of any number of open sets is another open set. Since summation/union is not necessary so is the concept of a cover for a space \mathcal{S} unnecessary. The existence of one or more “closed” or “complete” systems of projectors (ie. the existence of one or more families of open sets) in \mathcal{A} as described above is sufficient to account for results that could require completeness/compactness in terms of covers. Any closed collection of projectors

$\mathcal{P} = \{a \in \mathcal{A}; a^2 = a\}$, $\mathcal{P}\mathcal{P} = \{ab; a, b \in \mathcal{P}\} = \mathcal{P}$ generates a family of open sets which may be considered to define a topology on \mathcal{A}^* (one can choose to work with the whole set of linear functionals). Thus the number of such \mathcal{P} collections will give the number of possible topologies available for one to work with. Due to the duality between \mathcal{A} and \mathcal{A}^* any given topology on \mathcal{A}^* automatically induces an equivalent topology on \mathcal{A} .

G.3 Physics: The logic of quantum theory

At any given time, the *conditional presence or state* S_a of a physical system in \mathcal{A}^* is determined (or generated) by the *creation or existence condition* $a \in \mathcal{A}$ of the physical system. That is ***a physical system, with conditional presence or state S_a in \mathcal{A}^* , is defined (by a community of physical observers) by specifying a creation or existence condition $a \in \mathcal{A}$ for the physical system .***

We will consider the set generating projectors $p \in \mathcal{A}$ to represent creation or existence conditions of actual physical systems living or operating in the space \mathcal{A}^* and each closed collection of projectors \mathcal{P} will represent a collection of basic or elementary physical systems [eps's] (where the systems are basic or elementary in that the product of any two of them gives another). The set S_p , or equivalently $\phi(p)$, $\forall \phi \in \mathcal{A}^*$, determines the amplitude distribution, at a given time, of the elementary physical system (eps) represented by $p \in \mathcal{A}$. That is S_a is interpreted as the (probability) amplitude distribution or “painting” in \mathcal{A}^* of the system represented by $a \in \mathcal{A}$.

As time progresses the eps can change $p = p(t)$ and thus its amplitude distribution $S_{p(t)}$ changes and maps out a “path” (time parametrized set of amplitude distributions) in the space \mathcal{A}^* . For $p(t)$ to remain a projector (ie. for system to remain an eps) during the time evolution the time evolution needs to be in the form $p(t) = U(t, t_0)p(t_0)U^{-1}(t, t_0)$, $U(t, t) = 1_{\mathcal{A}} \quad \forall t$ (More generally $p(t) = U(t, t_0)p(t_0)V(t, t_0)$, $V(t, t_0)U(t, t_0) \in Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$). Moreover, for the product $p_1(t)p_2(t)$ of any $p_1(t), p_2(t) \in \mathcal{P}$ to also remain in \mathcal{P} (ie. $(p_1(t)p_2(t))^2 = p_1(t)p_2(t)$) the time evolution $U(t, t_0)$ must be common to all elements of \mathcal{P} (ie. for all eps's). An infinitesimal time evolution, for such a ***pure or***

elementarity preserving time evolution, may be effected using a directional derivation along a hermitian variable $h(t) \in \mathcal{A}$, $h(t)^* = h(t)$ which generates unitary time evolution; ie. with $U^*(t, t_0) = U^{-1}(t, t_0)$.

$$\begin{aligned} \left[\frac{d}{dt}, p(t)\right]a(t) &= -i[h(t), p(t)]a(t) \quad \forall a : \mathbb{R} \rightarrow \mathcal{A}, \\ \frac{dp(t)}{dt} &= -i[h(t), p(t)] = -iD_h p(t), \\ \frac{d\phi(p(t))}{dt} &= -i\phi([h(t), p(t)]). \end{aligned} \tag{G.3.-1}$$

Even though these equations were derived by considering projective classes, non-projective solutions may be possible and all possible solutions can be physically significant as **any given solution either describes pure time evolution or describes impure time evolution**.

Although the actual eps is described by $p(t)$, different observers experimenting on the eps may use different methods and/or parameters (or coordinates) to construct or represent $p(t)$ and $h(t)$. In addition measurements are carried out during experiments and the measurement parameters are the functionals $\phi \in \mathcal{A}^*$ and consequently **different observers may also use different functionals**.

For a particular observer, if we imagine the projector $p(t)$ and $h(t)$ to be constructed from **auxiliary variables** $q(t)$, $q : \mathbb{R} \rightarrow \mathcal{A}^N = \mathcal{A} \times \mathcal{A}^{N-1}$, which we will refer to as coordinates, $p(t) = P(t, q)$, $h(t) = H(t, q)$ then we have

$$\begin{aligned} \frac{\partial P(t, q)}{\partial t} &= -iD_H P(t, q) = -i[H(t, q), P(t, q)] \quad \forall P \Rightarrow \\ \frac{dq^i(t)}{dt} &= -iD_H q^i(t) = -i[H(t, q), q^i(t)], \quad i = 1, 2, \dots, N. \end{aligned} \tag{G.3.-1}$$

It is important to realize that there can be more than one choice of the variables $q^i(t)$, say $q_1^i(t)$ and $q_2^i(t)$ as well as the choice of functionals, say ϕ_1 and ϕ_2 , that give the same projector $P(t, q)$ and same $H(t, q)$. The transformation

$q_1^i(t) \rightarrow q_2^i(t)$, $\phi_1 \rightarrow \phi_2$ is a symmetry of $P(t, q)$ and $H(t, q)$, or simply a symmetry of the eps that p represents. The center $\mathcal{Z}(G_*) = G_* \cap G'_*$ of the algebra G_* of the symmetry group $G \subset \mathcal{A}$ of the transformations commutes with all of $G_* \subset \mathcal{A}$. However, when two transformations commute **they share the same spectrum** and are therefore equivalent in a sense. For this reason, the spectrum of the center $Z(G_*)$ **represents properties that are shared by all** of G_* and hence by all observers and in particular $Z(G_*)$ may therefore be considered to be intimately related to the most important (ie. basic or elementary) physical (ie. observer independent) characteristics of the eps. The spectrum of $Z(G_*)$ (ie. its spectral orbit in \mathcal{A}^*) can be used to predict, including yet unobserved, basic or elementary characteristics which the eps will eventually display under suitable conditions and which each and every observer will be able to detect even with their different coordinate or parameter systems.

From the point of view of the community of observers, **specifying an eps is equivalent to specifying its elementary physical properties** (epp's). Hence elementary time evolution (ete) of the eps must also preserve any symmetry group (or equivalently any symmetry group of the eps should preserve the ete of the eps) in order that the epp's be maintained.

Possible conditions that can be imposed by physical observations on the variables $q^i(t)$ include

$$\begin{aligned}
 (1) \quad [q^i(t), q^j(t)] &= i\Omega^{ij} \in Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}', \\
 (2) \quad [q^i(t), q^j(t)] &= C^{ij}_k q^k(t), \quad C^{ij}_k \in Z(\mathcal{A}) \\
 &etc. \tag{G.3.-2}
 \end{aligned}$$

One notes that it is also possible to have **more general impure time evolution during which a physical system can tunnel from one pure sgc class to**

a different pure sgc class in a dynamically projective manner. That is, intermediate stages of time evolution involve impure sgc's (ie. nonprojective sgc's). Thus *different pure sgc classes may be associated with inequivalent physical vacua.* The form of the infinitesimal time evolution equation in this case can be more general (*nonlinear*) than the simple (*linear*) form considered so far. To see how, consider the *linear ansatz*

$$\begin{aligned} \dot{p} &= hp + ph_1, \quad p^2 = p \quad \Rightarrow \quad p\dot{p} + \dot{p}p = \dot{p} \quad (\Rightarrow \quad p\dot{p}p = 0) \\ &\Rightarrow \quad php + ph_1p = 0 \quad \Rightarrow \quad h_1 = -h. \end{aligned} \quad (\text{G.3.-2})$$

[Note: These results show that $p + \dot{p}p$ and $p + p\dot{p}$ are also projectors for any given projector p . One notes also that $p^2 = p \Rightarrow p\dot{p} + \dot{p}p = \dot{p}$ but $p\dot{p} + \dot{p}p = \dot{p} \not\Rightarrow p^2 = p$ and so we will simply consider the operators obeying $p\dot{p} + \dot{p}p = \dot{p}$ as a dynamical generalization of those obeying $p^2 = p$ and refer to them as *dynamical projectors*.]

The *nonlinear ansatz* $\dot{p} = h_1p - h_2p + ph_3p$ implies $p(h_1 - h_2 + h_3)p = 0$ and so

$$\frac{dp(t)}{dt} = h_1(t)p(t) - p(t)h_2(t) + p(t)(h_2(t) - h_1(t))p(t) \quad (\text{G.3.-1})$$

$$= [h_1, p] + p(-\delta h_1 + \delta h_1 p), \quad \delta h_1 = h_2 - h_1,$$

$$\frac{dp_1}{dt} = -p_1V_1 + p_1V_1p_1, \quad V_1 = U_1^{-1}\delta h_1U_1, \quad p_1 = U_1^{-1}pU_1, \quad U_1 = T(e^{\int^t h_1}),$$

$$\frac{dp_2}{dt} = V_2p_2 - p_2V_2p_2, \quad \frac{d\mathcal{T}_{12}}{dt} = \mathcal{T}_{12}V_{21}\mathcal{T}_{12}, \quad (\text{G.3.-2})$$

$$\mathcal{T}_{12} = U_1^{-1}pU_2, \quad V_{21} = U_2^{-1}(h_2 - h_1)U_1, \quad U_1 = T(e^{\int^t h_1}), \quad U_2 = T(e^{\int^t h_2})$$

will describe dynamically projective impure time evolution of p . [One notes once more that *dynamically nonprojective solutions* (which are of course impure) *are possible*.]

Thus

$$\begin{aligned}\frac{d[p]_\alpha}{dt} &= V_{0\alpha} [p]_\alpha - [p]_\alpha V_{0\alpha} [p]_\alpha, \\ V_{0\alpha} &= U_\alpha^{-1}(h_0 - h_\alpha)U_\alpha, \quad U_\alpha = T(e^{\int^t h_\alpha})\end{aligned}\quad (\text{G.3.-3})$$

describes tunneling between any given pure sgc class $[p]_\alpha$ and a reference pure sgc class $[p]_0$ with $V_{0\alpha}$ being the “tunneling potential”. $V_{0\alpha} = 0$ corresponds to zero tunneling or pure time evolution in the class $[p]_\alpha$.

Since $dQ^{-1} = -Q^{-1}dQ Q^{-1}$ the solution to $\frac{d\mathcal{T}_{12}}{dt} = \mathcal{T}_{12}V_{21}\mathcal{T}_{12}$ is

$$\begin{aligned}\mathcal{T}_{12} &= -\left(\int^t V_{21}\right)^{-1} \equiv U_1^{-1}pU_2 \quad \Rightarrow \\ p &= U_1\mathcal{T}_{12}U_2^{-1} = -U_1\left(\int^t V_{21}\right)^{-1}U_2^{-1} = -U_1\frac{1}{\int^t U_2^{-1}(h_2 - h_1)U_1}U_2^{-1}, \\ U_1 &= T(e^{\int^t h_1}), \quad U_2 = T(e^{\int^t h_2}).\end{aligned}\quad (\text{G.3.-4})$$

In the limit $h_2 \rightarrow h_1 \equiv h$ one obtains the linear solution

$$p = Up_0U^{-1}, \quad p_0 = \lim_{h_2 \rightarrow h_1 \equiv h} \frac{-1}{\int^t U_2^{-1}(h_2 - h_1)U_1} = p_0(h), \quad (\text{G.3.-3})$$

where one may check that $\frac{dp_0}{dt} = 0$.

Writing $p = U_1p_{12}U_1^{-1} = U_2p_{21}U_2^{-1}$ one identifies the “**directed**” **tunneling operators**

$$\begin{aligned}p_{12} &= -\frac{1}{\int^t U_2^{-1}(h_2 - h_1)U_1}U_2^{-1}U_1, \\ p_{21} &= -U_2^{-1}U_1\frac{1}{\int^t U_2^{-1}(h_2 - h_1)U_1}.\end{aligned}\quad (\text{G.3.-3})$$

Represented on a Hilbert space, a particular projective solution of

$\dot{p} = h_1 p - p h_2 + p(h_2 - h_1)p$ takes the form

$$p = \frac{|\eta\rangle\langle\xi|}{\langle\xi|\eta\rangle}, \quad \frac{d|\eta\rangle}{dt} = h_1|\eta\rangle, \quad \frac{d\langle\xi|}{dt} = -\langle\xi|h_2.$$

$$\text{Tr } p^* p = \frac{\langle\eta|\eta\rangle\langle\xi|\xi\rangle}{|\langle\xi|\eta\rangle|^2} \equiv \frac{1}{\cos\theta_{\eta\xi}}. \quad (\text{G.3.-3})$$

When one wishes that p^* be described by the same equation as p (which is not necessary if p^* describes an independent [anti-]system) we must have

$$h_1^* = -h_2, \quad h_2^* = -h_1.$$

The possible kinds of dynamics may be classified as follows:

dynamics (time evolution)	
linear $\dot{p} = [h, p]$	projective (pure) $p^2 = p$ nonprojective (impure) $p^2 \neq p$
nonlinear $\dot{p} = h_1 p - p h_2 + p(h_2 - h_1)p$	dynamically projective (pure/impure) $\dot{p} p + p \dot{p} = \dot{p}$ dynamically nonprojective (impure) $\dot{p} p + p \dot{p} \neq \dot{p}$

where one notes that the projective linear dynamics is always dynamically projective, and also that the linear nonprojective dynamics can either be dynamically projective or dynamically nonprojective.

One may also regard *interactions* within/without a given physical system as *some kind of tunneling* where $\delta h = h - h_0 = h_I$ is the interaction Hamiltonian. However one should emphasize that this is only a particular case which can exhaust neither the applicability of the nonlinear tunneling equation

$\dot{p} = h_1 p - p h_2 + p(h_2 - h_1)p$ nor that of any possible generalizations of the equation.

If one defines a *finite evolution process* as the (time) ordered (tensor) product

$Q_{if}[p] = \prod_{t=t_i}^{t_f} \otimes p(t)$ where one can also multiply/add processes to obtain new ones, then the amplitude A^ϕ of involvement or participation of a particular functional $\phi \in \mathcal{A}^*$ in the process is

$$A_{if}^\phi = \Delta(\phi)(Q_{if}[p]) = \Delta(\phi)\left(\prod_{t=t_i}^{t_f} \otimes p(t)\right), \quad (\text{G.3.-2})$$

where $p(t)$ may be interpreted as an **instantaneous evolution process** an **infinite evolution process** will involve an infinite time interval. The overall process amplitude is

$$A_{if} = \sum_{\phi} A_{if}^\phi = \sum_{\phi} \Delta(\phi)\left(\prod_{t=t_i}^{t_f} \otimes p(t)\right). \quad (\text{G.3.-1})$$

An **example** of a physical process amplitude (**in coordinate representation**) is the **path integral** in quantum theory.

One notes that an evolution process may involve the switching on and off of interactions in specific time intervals $[t_r, t_s]$: eg. in the case of linear time evolution one may have

$$h(t) = h_0(t) + \sum_{rs} \theta(t - t_r)\theta(t_s - t) V_{rs}(t) \equiv h_0(t) + h_I(t). \quad (\text{G.3.0})$$

Process classes can be named according to the class of dynamics that determines $p(t) \forall t$.

One may also relate h_1 and h_2 by imposing either the conservation of h_1

$$\begin{aligned} \dot{h}_1 &= h_1 h_1 - h_1 h_2 + h_1 (h_2 - h_1) h_1 = 0 \\ \Rightarrow h_1 &= h_2 \text{ or } h_1 = 1 \end{aligned} \quad (\text{G.3.0})$$

or the conservation of h_2

$$\begin{aligned} \dot{h}_2 &= h_1 h_2 - h_2 h_2 + h_2 (h_2 - h_1) h_2 = 0 \\ \Rightarrow h_1 &= h_2 \text{ or } h_2 = 1. \end{aligned} \quad (\text{G.3.0})$$

One can similarly derive an evolution equation for a system of sgc's satisfying $p_i p_j = f_{ij}{}^k p_k$.

$$p_i p_j = f_{ij}{}^k p_k \quad \Rightarrow \quad \dot{p}_i p_j + p_i \dot{p}_j = f_{ij}{}^k \dot{p}_k, \quad (\text{G.3.1})$$

Then linear time evolution

$$\dot{p}_i = h p_i - p_i h \quad (\text{G.3.2})$$

needs no modification. However, nonlinear evolution will be in the form

$$\dot{p}_i = h_1 p_i - p_i h_2 + c_i{}^{jk} p_j (h_2 - h_1) p_k \quad (\text{G.3.3})$$

where the c 's obey some contraction identities with the f 's.

One can have more general nonlinear time evolutions (which would describe tunneling from a given vacuum into more than one different vacua simultaneously) as for example:

$$\dot{p} = h_1 p + p h_2 + p h_3 p + h_4 p h_5 p + p h_6 p h_7, \quad (\text{G.3.4})$$

$$\dot{p} p + p \dot{p} = \dot{p} \quad \Rightarrow$$

$$h_1 + h_2 + h_3 = 0, \quad h_4 = h_6, \quad h_5 + h_7 = 0$$

$$\text{or} \quad h_1 + h_2 + h_3 = 0, \quad h_4 + h_6 = 0, \quad h_5 = h_7$$

and

$$\dot{p} = h_1 p + p h_2 + p h_3 p + h_4 p h_5 p + p h_6 p h_7 + p h_8 p h_9 p, \quad (\text{G.3.2})$$

$$\dot{p} p + p \dot{p} = \dot{p} \quad \Rightarrow$$

$$h_1 + h_2 + h_3 = 0, \quad h_4 = h_6 = h_8, \quad h_5 + h_7 + h_9 = 0$$

$$\text{or} \quad h_1 + h_2 + h_3 = 0, \quad h_4 + h_6 + h_8 = 0, \quad h_5 = h_7 = h_9$$

and so on.

G.3.1 Coordinate types

We will consider only linear time evolution.

The “mechanical” choice of coordinates,

$$q : \mathbb{R} \rightarrow \mathcal{A}^N, \quad t \mapsto q(t) = \{q^i(t); i = 1, 2, \dots, N\} \equiv (q^1(t), q^2(t), \dots, q^N(t))$$

or $q : \mathbb{N}_N \times \mathbb{R} \rightarrow \mathcal{A}$, $(i, t) \mapsto q^i(t)$, $\mathbb{N}_N = \{1, 2, \dots, N\}$, which we made earlier is of course only for illustration. In principle both the number and choices of coordinates (ie. observers), and hence of the corresponding symmetries, is arbitrarily diverse.

The following are a few other examples of coordinate choices:

- Scalar fields

$$\begin{aligned} q : \mathbb{R}^d \times \mathbb{R} &\equiv \mathbb{R}^{d+1} \rightarrow \mathcal{A}, \quad x = (t, \vec{x}) \mapsto q^{\vec{x}}(t) \equiv q(t, \vec{x}). \\ \frac{\partial q(t, \vec{x})}{\partial t} &= -i[H(t, q), q(t, \vec{x})] = -iD_H q(t, \vec{x}). \end{aligned} \quad (\text{G.3.-1})$$

- Vector fields

$$\begin{aligned} q : \mathbb{N}_{d+1} \times \mathbb{R}^{d+1} &\rightarrow \mathcal{A}, \quad (\mu, x) \mapsto q^\mu(t, \vec{x}). \\ \frac{\partial q^\mu(t, \vec{x})}{\partial t} &= -i[H(t, q), q^\mu(t, \vec{x})] = -iD_H q^\mu(t, \vec{x}). \end{aligned} \quad (\text{G.3.-1})$$

- p -Tensor fields

$$\begin{aligned} q : (\mathbb{N}_{d+1})^p \times \mathbb{R}^{d+1} &\rightarrow \mathcal{A}, \quad (\alpha, x) \mapsto q^\alpha(t, \vec{x}). \\ \frac{\partial q^\alpha(t, \vec{x})}{\partial t} &= -i[H(t, q), q^\alpha(t, \vec{x})] = -iD_H q^\alpha(t, \vec{x}). \end{aligned} \quad (\text{G.3.-1})$$

- Spinor fields

$$\begin{aligned} q : \mathbb{N}_{\frac{d}{2}} \times \mathbb{R}^{d+1} &\rightarrow \mathcal{A}, \quad (\sigma, x) \mapsto q^\sigma(t, \vec{x}). \\ \frac{\partial q^\sigma(t, \vec{x})}{\partial t} &= -i[H(t, q), q^\sigma(t, \vec{x})] = -i(D_H)^\sigma{}_{\sigma'} q^{\sigma'}(t, \vec{x}), \\ (D_H)^\sigma{}_{\sigma'} &= (\gamma^\mu)^\sigma{}_{\sigma'} D_{H^\mu} = D_{H^\mu} (\gamma^\mu)^\sigma{}_{\sigma'}, \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \\ \gamma^\mu &= \gamma^\mu(t, \vec{x}), \quad g^{\mu\nu} = g^{\mu\nu}(t, \vec{x}). \end{aligned} \quad (\text{G.3.-3})$$

- r -Gauge fields

$$q : (\mathbb{N}_{d+1})^m \times (\mathbb{N}_{\frac{d}{2}})^n \times (\mathbb{N}_N)^r \times \mathbb{R}^{d+1} \rightarrow \mathcal{A}, \quad (u, x) \mapsto q^u(t, \vec{x}).$$

$$\frac{\partial q^u(t, \vec{x})}{\partial t} = -i[H(t, q), q^u(t, \vec{x})], \quad u = (\mu_1, \dots, \mu_m, \sigma_1, \dots, \sigma_n, a_1, \dots, a_r).$$

- Composite coordinates: In general $q = (q_1, q_2, \dots)$ can be made up of one or more of the coordinate systems above meaning H depends on the whole composite as well;

$$H = H(t, q) = H(t, q_1, q_2, \dots).$$

The (more basic) coordinate types q above are thought to correspond to irreducible representations of a symmetry group whose action may be expressed in a coordinate dependent way as

$$q'^u(x') = U^{-1}(\Lambda, b)q^u(x)U(\Lambda, b) = S^u_v(\Lambda, b) q^v(\Lambda x + b),$$

$$b \in \mathbb{R}^{d+1}, \quad \Lambda \in \mathbb{R}^{d+1} \otimes \mathbb{R}^{d+1},$$

or in a coordinate independent way as

$$U(\Lambda, b)U(\Lambda', b') = U(\Lambda\Lambda', \Lambda b' + b). \quad (\text{G.3.-6})$$

This is also the isometry group of \mathbb{R}^{d+1} as a metric space

$$\begin{aligned}
\mathcal{H}(\mathbb{R}^{d+1}) &= (\text{Der}(\mathbb{R}^{d+1}), \langle, \rangle = \mu_{\mathbb{C}} \circ \eta), \quad \eta \in \text{Der}^*(\mathbb{R}^{d+1}) \otimes \text{Der}^*(\mathbb{R}^{d+1}), \\
\langle U(\Lambda, b)\xi | U(\Lambda, b)\xi_1 \rangle &= \langle \xi | \xi_1 \rangle \quad \forall \xi, \xi_1 \in \mathcal{H}(\mathbb{R}^{d+1}), \\
\text{Der}(\mathbb{R}^{d+1}) &= \{D : \mathcal{F}(\mathbb{C}, \mathbb{R}^{d+1}) \rightarrow \mathcal{F}(\mathbb{C}, \mathbb{R}^{d+1}), D(f+h) = D(f) + D(h), \\
&D(fh) = D(f)h + fD(h) \quad \forall f, h \in \mathcal{F}(\mathbb{C}, \mathbb{R}^{d+1})\} \\
&\equiv \{D_v; v \in \mathcal{F}(\mathbb{C}, \mathbb{R}^{d+1})^{d+1}\}, \quad (D_v(f))(x) = v^i(x)\partial_i f(x), \\
(\langle D_u | D_v \rangle)(x) &= \eta_{ij} u^i(x)v^j(x), \\
(\langle U(\Lambda, b)D_u | U(\Lambda, b)D_v \rangle)(x) &= \eta_{ij} u^i(\Lambda x + b)v^j(\Lambda x + b) = \eta_{ij} u^i(x)v^j(x) \\
\Rightarrow u^i(x) = v^i(x) = dx^i & \quad (\Rightarrow \text{Der}(\mathbb{R}^{d+1}) \simeq \mathbb{R}^{d+1}), \quad \eta_{\alpha\beta}\Lambda^\alpha_i \Lambda^\beta_j = \eta_{ij},
\end{aligned}$$

where $\text{Der}(\mathbb{R}^{d+1})$ is the space of all directional derivatives in \mathbb{R}^{d+1} .

Dynamically (ie. $p(t) = P(t, q)$), p is determined by $H = H(t, q)$ and therefore the possible types of dynamics (including interactions) of various physical systems are described, by observers, by specifying various functional forms of $H(t, q)$.

G.3.2 On Gravity

One may want to “enlarge” the isometry group of \mathbb{R}^{d+1} to that of an \mathbb{R}^{d+1} -manifold $\mathcal{M}(\mathbb{R}^{d+1})$ in order to treat gravity which is believed to be related to the metric/curvature of some \mathbb{R}^{d+1} -manifold. That is, gravity is related to the isometry group

($U(\varphi)U(\varphi') = U(\varphi \circ \varphi')$, $\varphi, \varphi' : \mathcal{D} \subseteq \mathcal{M} \rightarrow \mathcal{C} \subseteq \mathcal{M}$) of $\mathcal{M} = \mathcal{M}(\mathbb{R}^{d+1})$ with its

tangent fiber as a metric space

$$\begin{aligned}
\mathcal{H}(\mathcal{M}) &= (\text{Der}(\mathcal{M}), \langle, \rangle = \mu_{\mathbb{C}} \circ g), \quad g \in \text{Der}^*(\mathcal{M}) \otimes \text{Der}^*(\mathcal{M}), \\
\langle U(\varphi)\xi | U(\varphi)\xi_1 \rangle &= \langle \xi | \xi_1 \rangle \quad \forall \xi, \xi_1 \in \mathcal{H}(\mathcal{M}), \\
\text{Der}(\mathcal{M}) &= \{D : \mathcal{F}(\mathbb{C}, \mathcal{M}) \rightarrow \mathcal{F}(\mathbb{C}, \mathcal{M}), D(f+h) = D(f) + D(h), \\
&\quad D(fh) = D(f)h + fD(h) \quad \forall f, h \in \mathcal{F}(\mathbb{C}, \mathcal{M})\} \\
&\equiv \{D_v; v \in \mathcal{F}(\mathbb{C}, \mathcal{M})^{d+1}\}, \quad (D_v(f))(x) = v^i(x)\partial_i f(x), \\
(\langle D_u | D_v \rangle)(x) &= g_{ij}(x) u^i(x)v^j(x) \quad \forall u, v, \\
(\langle U(\varphi)D_u | U(\varphi)D_v \rangle)(x) &= g_{ij}(\varphi^{-1}(x)) u^i(\varphi(x))v^j(\varphi(x)) = g_{ij}(x) u^i(x)v^j(x) \\
&= \langle D_u | D_v \rangle(x), \tag{G.3.-20}
\end{aligned}$$

where $\text{Der}(\mathcal{M})$ is the space of all directional derivatives on \mathcal{M} . One can apply the functional operator $Q_{\alpha\beta} = \int d\mu(x)d\mu(z) \frac{\delta}{\delta u^\alpha(y)} \frac{\delta}{\delta u^\beta(z)}$, $d\mu(x) = \sqrt{\det g(x)}d^{d+1}x$ on the equation $g_{ij}(\varphi^{-1}(x)) u^i(\varphi(x))v^j(\varphi(x)) = g_{ij}(x) u^i(x)v^j(x) \quad \forall u, v$ to remove the u, v dependence (Check the infinitesimal form $\varphi^i(x) = x^i + \delta x^i(x) \equiv x^i + \xi^i(x)$). There is the constraint

$$g_{ij}(\varphi^{-1}(x)) \det g(\varphi(x)) = g_{ij}(x) \det g(x) \Rightarrow \det g(\varphi^{-1}(x)) = \det g(x).$$

Once the metric g has been determined (eg. by postulating the matter energy momentum tensor as the source of the curvature R generated by g , or by some other means) then the transformations φ , and hence the irreducible representations of

$U(\varphi)U(\varphi') = U(\varphi \circ \varphi') = U(\varphi \circ \varphi' \circ \varphi^{-1} \circ \varphi'^{-1}) U(\varphi')U(\varphi)$ can then be determined as well.

However this “enlargement” effect can also be realized in different ways: by dimensional increase/reduction, coordinate spectrum increase/decrease (eg. by

making x^i noncommutative), etc. Then gravity can arise as a physical effect induced by dimensional reduction, coordinate spectrum increase, etc. And since the usual (general relativistic) gravity theory is acceptable as an effective theory, any other fundamental theory of gravity needs to be compatible with it.

The *time evolution of a gravitational system may involve tunneling between inequivalent physical vacua* and hence the nonlinear impure time evolution equation (G.3.-1) may be more suitable for describing a physical gravitational system.

G.3.3 Projectors on Self Hilbert Spaces

$$\begin{aligned}\mathcal{H}^\phi &= \mathcal{H}^\phi(\mathcal{A}) = (\mathcal{A}, \langle, \rangle_\phi = \phi \circ \mu_{\mathcal{A}} \circ (* \otimes id)) \equiv \{|\xi\rangle_\phi; \xi \in \mathcal{A}\}, \quad \phi \in \mathcal{A}^*, \\ \mu_{\mathcal{A}} \circ (* \otimes id) &: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}, \quad a \otimes b \mapsto a^*b. \\ \mathcal{H}^{\mathcal{A}^*} &= \mathcal{H}^{\mathcal{A}^*}(\mathcal{A}) = (\mathcal{A}, \langle, \rangle_{\mathcal{A}^*} = \mathcal{A}^* \circ \mu_{\mathcal{A}} \circ (* \otimes id)) \simeq \mathcal{A} \times \mathcal{A}^*.\end{aligned}$$

One can have multiplication operator representations:

$$\begin{aligned}m^L : \mathcal{A} &\rightarrow O(\mathcal{H}^\phi(\mathcal{A})), \quad a \rightarrow m_a^L : \mathcal{H}^\phi(\mathcal{A}) \rightarrow \mathcal{H}^\phi(\mathcal{A}), \quad |\xi\rangle \mapsto |a\xi\rangle. \\ m^R : \mathcal{A} &\rightarrow O(\mathcal{H}^\phi(\mathcal{A})), \quad a \rightarrow m_a^R : \mathcal{H}^\phi(\mathcal{A}) \rightarrow \mathcal{H}^\phi(\mathcal{A}), \quad |\xi\rangle \mapsto |\xi a\rangle.\end{aligned}$$

and/or matrix representations:

$$\begin{aligned}\pi^\phi : \mathcal{A} &\rightarrow O(\mathcal{H}^\phi(\mathcal{A})), \quad a \mapsto \pi^\phi(a) = \sum_{(\xi, \eta) \in \mathcal{H}_1^\phi \times \mathcal{H}_2^\phi} |\xi_\alpha\rangle a^{\alpha\beta} \langle \eta_\beta| \\ &\equiv \sum_{(\xi, \eta) \in \mathcal{H}_1^\phi \times \mathcal{H}_2^\phi} a^{\alpha\beta} \xi_\alpha \otimes \eta_\beta^*, \\ p^\phi &= \pi^\phi(p) = \sum_{(\xi, \eta) \in \mathcal{H}_1^\phi \times \mathcal{H}_2^\phi} |\xi_\alpha\rangle \langle \eta_\alpha | \xi_\beta \rangle_\phi^{-1} \langle \eta_\beta|. \\ p^{\phi_i}(t_i) &= U^{-1}(t_i, t) p^\phi(t) U(t_i, t) = U^{-1}(t_i, t_f) p^{\phi_f}(t_f) U(t_i, t_f).\end{aligned}$$

For example:

$$\begin{aligned}
\mathcal{A} &= \mathcal{A}_\theta(\mathbb{R}^D) = \{a_f = W(f) = \sum_{x \in \mathbb{R}^D} f(x) \hat{\delta}_x; f : \mathbb{R}^D \rightarrow \mathbb{C}\}, \\
\mathcal{A}^* &= \mathcal{A}_\theta^*(\mathbb{R}^D) = \{\phi_x = \text{Tr} \circ m_{\hat{\delta}_x}; x \in \mathbb{R}^D\}, \\
\hat{\delta}_x &= \sum_{k \in \mathbb{R}^D} e^{ik(\hat{x}-x)}, \quad [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \\
\mathcal{A}_\delta &= \{W(\delta_y) = \sum_{x \in \mathbb{R}^D} \delta_y(x) \hat{\delta}_x = \sum_{x \in \mathbb{R}^D} \delta(y-x) \hat{\delta}_x = \hat{\delta}_y; y \in \mathbb{R}^D\} \subset \mathcal{A}, \\
\mathcal{A}_e &= \{W(e_k) = \sum_{x \in \mathbb{R}^D} e_k(x) \hat{\delta}_x = \sum_{x \in \mathbb{R}^D} e^{ikx} \hat{\delta}_x = e^{ik\hat{x}}; k \in \mathbb{R}^D\} \subset \mathcal{A}. \\
\mathcal{H}^{\phi_u}(\mathcal{A}) &= (\mathcal{A}, \langle, \rangle_{\phi_u}), \quad \mathcal{H}_{e,\delta}^{\phi_u}(\mathcal{A}) = (\mathcal{A}_{e,\delta}, \langle, \rangle_{\phi_u}) \subset \mathcal{H}^{\phi_u}(\mathcal{A}). \\
\langle, \rangle_{\phi_u} &= \phi_u \circ \mu_{\mathcal{A}} \circ (* \otimes id) = \text{Tr} \circ m_{\hat{\delta}_u} \circ \mu_{\mathcal{A}} \circ (* \otimes id). \tag{G.3.-35}
\end{aligned}$$

G.4 Primitivity: The logic of human society

The logic can be exclusive, nonexclusive or both.

The analysis in the previous section (Physics: The logic of quantum theory) is a reflection of the primitivity or science of human society. The algebra \mathcal{A} is the collection of all possible human emotions (the language of *Eternity* or *Greed*, known otherwise as *God*). The number field, such as the field of complex numbers \mathbb{C} , in which the linear functionals $\phi \in \mathcal{A}^*$ take values is the set of all possible *Gold* (or money) amplitudes or potentials. Here $\phi \in \mathcal{A}^*$ represents an individual being and $\phi(a)$ is the gold amplitude of ϕ to the primitive system represented by the emotion $a \in \mathcal{A}$. A high gold amplitude is supposedly a blessing from Eternity meanwhile a low gold amplitude would mean Eternity's disapproval.

At any given time, the *emotional presence or state* S_a of a primitive system in \mathcal{A}^* is determined (or generated) by the *creation or existence emotion* $a \in \mathcal{A}$ of the

primitive system. That is *a primitive system, with emotional presence or state S_a in \mathcal{A}^* , is defined (by a community of primitive observers [explicitly or implicitly prophets/messengers of Eternity]) by specifying a creation or existence emotion $a \in \mathcal{A}$ for the primitive system.*

We will consider the set generating projectors $p \in \mathcal{A}$ to represent creation or existence emotions of actual primitive systems living or operating in the space \mathcal{A}^* and each closed collection of projectors \mathcal{P} will represent a collection of basic or elementary primitive systems [eps's] (where the systems are basic or elementary in that the product of any two of them gives another). The set S_p , or equivalently $\phi(p)$, $\forall \phi \in \mathcal{A}^*$, determines the amplitude distribution (or configuration), at a given time, of the elementary primitive system (eps) represented by $p \in \mathcal{A}$. That is S_a is interpreted as the (probability) amplitude distribution or configuration in \mathcal{A}^* of the system represented by $a \in \mathcal{A}$.

The dynamics of a primitive system may be described in parallel to the previous section with the following replacements:

physics \rightarrow primitivity, physical \rightarrow primitive, condition \rightarrow emotion,
 observer \rightarrow prophet/messenger of Eternity, and so on.

In the dynamics of primitive systems there can be interactions (a case of tunneling), involving one or more primitive systems. During an interaction process the prophets or messengers of Eternity (the observers) make various readjustments or redefinitions (known as “offerings or sacrifices” willed by Eternity) of the primitive systems. Thus an interaction may result in the conversion (as decided by Eternity through the observers) of some of the initial primitive systems involved into other primitive systems which were not involved initially.

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