QUILLEN BUNDLE AND GEOMETRIC PREQUANTIZATION OF NON-ABELIAN VORTICES ON A RIEMANN SURFACE

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Abstract. In this paper we prequantize the moduli space of non-abelian vortices. We explicitly calculate the symplectic form arising from the L^2 metric and we construct a prequantum line bundle whose curvature is proportional to this symplectic form. The prequantum line bundle turns out to be Quillen's determinant line bundle with a modified Quillen metric. Next, as in the case of abelian vortices, we construct Quillen line bundles over the moduli space whose curvatures form a family of symplectic forms which are parametrised by Ψ_0, a section of a certain bundle.

1. Introduction

Geometric prequantization of a symplectic manifold $(\mathcal{M}, \Omega)$ is a construction of a prequantum line bundle $\mathcal{P}$ on $\mathcal{M}$ so that its curvature is proportional to the symplectic form. The line bundle has to come equipped with a metric so that the Hilbert space of the prequantization is the space of the square integrable sections of $\mathcal{P}$. To every $f \in C^\infty(\mathcal{M})$ we associate an operator acting on the Hilbert space, namely, $\hat{f} = -i\hbar[\nabla^\theta_{X_f}] + f$ where X_f is the vector field defined by $\Omega(X_f, \cdot) = -df$ and θ is a 1-form such that locally, $d\theta = \Omega$ and $\nabla^\theta_{X_f}$ is the covariant derivative with respect to $-\frac{i}{\hbar}\theta$ in the direction X_f. Then if $f_1, f_2 \in C^\infty(\mathcal{M})$ and $f_3 = \{f_1, f_2\}_\Omega$, the Poisson bracket of the two induced by the symplectic form, then $[\hat{f}_1, \hat{f}_2] = -i\hbar\hat{f}_3$, [17].

The step from prequantization to quantization involves choice of a polarization and construction of operators which take polarised sections to polarised sections and which satisfy the axioms of a deformation quantization, i.e. one may have to relax the condition that $[\hat{f}_1, \hat{f}_2] = -i\hbar\hat{f}_3$. Instead one might have $[\hat{f}_1, \hat{f}_2] = -i\hbar\hat{f}_3 + o(\hbar^2)$. Construction of Toepliz operators out of projections to holomorphic sections of the prequatum line bundle (when the latter makes sense) and Berezin-Toeplitz deformation quantization has been carried out by Andersen in [1], [2].

The non-abelian vortices were first introduced in the literature perhaps by Bradlow [8]. The non-abelian vortex equations that we are considering were first studied in [4], [9] and subsequently studied by Baptista, [3]. Let M be a compact Riemann surface and let $\omega = h^2 dz \wedge d\bar{z}$ be the purely imaginary volume form on it, (i.e. h is real). Let V be a complex vector bundle associated to P. Let A be a unitary connection on a V i.e. A is a 1-form such that $A^* = -A$, and $A = A^{(0,1)} + A^{(1,0)}$ i.e. $A^{(0,1)*} = -A^{(1,0)}$. Let $E = \oplus^N V$ i.e. direct sums of N copies of V (with N being the rank of V). Let Ψ be a section of E, i.e. $\Psi \in \Gamma(M, E)$. In this case the section Ψ can be regarded locally as a function on M having values on the $N \times N$ matrices. The hermitian conjugate Ψ^* is defined w.r.t. a Hermitian metric H on E, so that in a unitary trivialization of E it is represented by the Hermitian conjugate matrix of Ψ.

The pair (A, Ψ) will be said to satisfy the non-abelian vortex equations if

$$(1) \qquad F(A) = (\tau I - \Psi\Psi^*)\omega,$$
$$(2) \qquad \bar{\partial}_A \Psi = 0,$$

where $F(A)$ is the curvature of the connection A and $d_A = \partial_A + \bar{\partial}_A$ is the decomposition of the covariant derivative operator into $(1,0)$ and $(0,1)$ pieces. Note we take $e = 1$ in [3]. Let $\mathcal{S}$ be the space of solutions to (1) and (2). There is a gauge group G acting on the space of (A, Ψ) which leaves the equations invariant. We take the group G to be $U(n)$ and locally it looks like Maps$(M, U(n))$. If g is an $U(n)$ gauge transformation then (A_1, Ψ_1) and (A_2, Ψ_2) are gauge equivalent if $A_2 = g^{-1}dg + g^{-1}A_1 g$ and $\Psi_2 = g^{-1}\Psi_1$. Taking the quotient by the gauge group of $\mathcal{S}$ gives the moduli space of solutions to these equations and is denoted by $\mathcal{M}$. In the case where E is of this special form, $\mathcal{M}$ is known to be a smooth manifold for big values of τ , more precisely when $\tau > 2\pi d/nVol(\Sigma)$ (see references in [3]) . It is known that there is a natural metric on the moduli space $\mathcal{M}$ and in fact the metric is Kähler. In this paper, we show the metric explicitly and write down the symplectic (in fact, the Kähler form) Ω arising from this metric and the complex structure.

As done in the case of abelian vortex moduli space by Dey, [11], we show here that there exists a holomorphic prequantum line bundle, namely, a determinant line bundle, which has a modified Quillen metric such that the Quillen curvature is proportional to the sympletic form Ω. This method was first used in constructing determinant line bundles over stable triples by Biswas and Raghavendra, [6]. Also, we should mention that Biswas and Schumacher, [7], has used the Quillen metric on moduli spaces of coupled vortex equations on a complex projective variety and shown that the Kähler form is the Chern form of a Quillen metric on a certain determinant line bundle.

Next, in this paper, as in [10], we construct line bundles over the moduli space whose curvatures are proportional to a family of symplectic forms parametrised by Ψ_0, a section of $E \otimes K$.

In the general case, when E is an arbitrary vector bundle associated to a $U(n)$ principal bundle P the moduli space of non-abelian vortices is not known to be non-empty or smooth.However, the calculations done in this paper will still be valid outside the singular locus.

Note: In this paper, many details of proofs are skipped because they coincide with [10], [11] and an expository paper, [12].

2. Metric and symplectc forms

Let $\mathcal{A}$ be the space of all unitary connections on P and $\Gamma(M, E)$ be sections of E. Let $\mathcal{C} = \mathcal{A} \times \Gamma(M, E)$ be the configuration space on which equations (1) and (2) are imposed. Let $p = (A, \Psi) \in \mathcal{C}$, $X = (\alpha_1, \beta)$, $Y = (\alpha_2, \eta) \in T_p\mathcal{C} \equiv \Omega^1(M, u(n)) \times \Gamma(M, E)$ i.e. $\alpha_i = \alpha_i^{(0,1)} + \alpha_i^{(1,0)}$ such that $\alpha_i^{(0,1)*} = -\alpha_i^{(1,0)}, i = 1, 2$. On $\mathcal{C}$ one can define a metric

$$\mathcal{G}(X, Y) = \int_M \mathrm{Tr}(*_1\alpha_1 \wedge \alpha_2) + 2i \int_M \mathrm{Tr}(\frac{\beta\eta^* + \beta^*\eta}{2})\omega$$

and an almost complex structure $\mathcal{I} = \begin{bmatrix} *_1 & 0 \\ 0 & i \end{bmatrix} : T_p\mathcal{C} \to T_p\mathcal{C}$ where $*_1 : \Omega^1 \to \Omega^1$ is the Hodge star operator on M such that $*_1(\alpha^{(1,0)}) = -i\alpha^{(1,0)}$ and $*_1\alpha^{(0,1)} = i\alpha^{(0,1)}$ (i.e. $(0,1)$ forms are holomorphic w.r.t. this.)

It is easy to check that $\mathcal{G}$ is the L^2 metric on $\mathcal{C}$.

2.1. **The symplectic forms Ω and Ω_{Ψ_0}.** We define

$$\begin{aligned}\Omega(X,Y) &= -\int_M \mathrm{Tr}(\alpha_1 \wedge \alpha_2) + 2i\int_M \mathrm{Tr}(\frac{i\beta\eta^* - i\beta^*\eta}{2})\omega \\ &= -\int_M \mathrm{Tr}(\alpha_1 \wedge \alpha_2) - \int_M \mathrm{Tr}(\beta\eta^* - \beta^*\eta)\omega\end{aligned}$$

such that $\mathcal{G}(\mathcal{I}X, Y) = \Omega(X,Y)$.

Let $\zeta \in \mathrm{Maps}(M, u(n))$ be the Lie algebra of the gauge group (the gauge group element being $g = e^{\zeta}$); note that $\zeta^* = -\zeta$. It generates a vector field X_ζ on $\mathcal{C}$ as follows :

$$X_\zeta(A,\Psi) = (d\zeta, -\zeta\Psi) \in T_p\mathcal{C}$$

where $p = (A,\Psi) \in \mathcal{C}$.

We show next that X_ζ is Hamiltonian. Namely, define $H_\zeta : \mathcal{C} \to \mathbb{C}$ as follows:

$$H_\zeta(p) = \int_M \mathrm{Tr}[\zeta \cdot (F_A - (1 - \Psi\Psi^*)\omega)].$$

Then for $X = (\alpha, \beta) \in T_p\mathcal{C}$,

$$\begin{aligned}dH_\zeta(X) &= \int_M \mathrm{Tr}(\zeta d\alpha) + \int_M \mathrm{Tr}[\zeta(\Psi\beta^* + \beta\Psi^*)]\omega \\ &= -\int_M \mathrm{Tr}[(d\zeta) \wedge \alpha] - \int_M \mathrm{Tr}[\beta\Psi^*(-\zeta) - \beta^*\zeta\Psi]\omega \\ &= -\int_M \mathrm{Tr}[(d\zeta) \wedge \alpha] - \int_M \mathrm{Tr}[\beta(\zeta\Psi)^* - \beta^*(\zeta\Psi)]\omega \\ &= -\int_M \mathrm{Tr}[(d\zeta) \wedge \alpha] - \int_M \mathrm{Tr}[(-\zeta\Psi)\beta^* - (-\zeta\Psi)^*\beta]\omega \\ &= \Omega(X_\zeta, X),\end{aligned}$$

where we use that $\zeta^* = -\zeta$.

Thus we can define the moment map $\mu : \mathcal{C} \to \Omega^2(M, u(n)) = \mathcal{G}^*$ (the dual of the Lie algebra of the gauge group) to be

$$\mu(A,\Psi) \doteq (F(A) - (1 - \Psi\Psi^*)\omega).$$

Thus equation (1) is $\mu = 0$.

It can be shown exactly along lines of Dey, [10] that Ω, $\mathcal{G}$ and $\mathcal{I}$ descend to $\mathcal{M}$ so that the latter is symplectic and almost complex.

Next we will define a family of symplectic forms on $\mathcal{M}$.

Let $\Psi_0 \in \Gamma(M, E \otimes K)$ where K is the canonical bundle such that Ψ_0 has zero only in a set of measure zero.

$$\begin{aligned}\Omega_{\Psi_0}(X,Y) &= -[\int_M \mathrm{Tr}(\alpha_1 \wedge \alpha_2) - \int_M \mathrm{Tr}[(\eta^*\beta - \beta^*\eta)\Psi_0^* \wedge \Psi_0] \\ &= -[\int_M \mathrm{Tr}(\alpha_1 \wedge \alpha_2) + \int_M \mathrm{Tr}\{(\eta^*\beta - \beta^*\eta)f(\Psi_0)\}\omega]\end{aligned}$$

where $f(\Psi_0)\bar{\omega} = \Psi_0^* \wedge \Psi_0$ is zero only in a set of measure zero, $f(\Psi_0)$ being the matrix A^*A where $\Psi_0 = Adz$. We used the fact that $\mathrm{Tr}(\Psi_0 \beta^* \eta \wedge \Psi_0^*) = -\mathrm{Tr}(\beta^* \eta \Psi_0^* \wedge \Psi_0)$ has to cross over a Ψ_0^* which is a matrix-valued $(0,1)$ form.

This is non-degenerate and is a symplectic form on $\mathcal{M}$.

This follows from the fact that

$$\Omega_{\Psi_0}(\mathcal{I}(\alpha_1,\beta),(\alpha_1,\beta)) = -4\int_M |a|^2 dx \wedge dy - 4\int_M \mathrm{Tr}(\beta^*\beta A^* A) h^2 dx \wedge dy$$

where $\omega = -2ih^2 dx \wedge dy$ and $\alpha_1 = adz - a^* d\bar{z} \in \Omega^1(M, u(n))$ and $*_1\alpha_1 = -i(adz + a^* d\bar{z})$. $\mathrm{Tr}(\beta^*\beta A^* A)$ is positive definite because it is $\mathrm{Tr}(A\beta^*\beta A^*) = \mathrm{Tr}((A\beta^*)(A\beta^*)^*)$ which is of the form $\mathrm{Tr}(CC^*)$ which is obviously positive definite outside maybe a set of measure zero.

3. Prequantum line bundle

In this section we briefly review the Quillen construction of the determinant line bundle of the Cauchy Riemann operator $\bar{\partial}_A = \bar{\partial} + A^{(0,1)}$, [16], which enables us to construct prequantum line bundle on the vortex moduli space.

3.1. **Determinant line bundle of Quillen.**

First let us note that a connection A on a $U(n)$-principal bundle induces a connection on any associated line bundle E. We will denote this connection also by A since the same " Lie-algebra valued 1-form" A (modulo representations) gives a covariant derivative operator enabling you to take derivatives of sections of E.

A very clear description of the determinant line bundle can be found in [16] and [5]. We also give more details in [12].

$\mathcal{A} =$ space of unitary connections on a vector bundle E associated to a principal G bundle on a Riemann surface. $A = A^{1,0} + A^{0,1}$ with $A^{1,0*} = -A^{0,1}$. Thus identify $\mathcal{A} = \mathcal{A}^{0,1}$.

Construct a line bundle $\mathcal{L}$ on $\mathcal{A}^{0,1}$ as follows. The fiber on top of $A^{0,1}$ is

$$\det(\bar{\partial}_A) = \wedge^{\mathrm{top}}(\mathrm{Ker}\bar{\partial}_A)^* \otimes \wedge^{\mathrm{top}}(\mathrm{Coker}\bar{\partial}_A)$$

Quillen's ingenious construction: $\mathcal{L}$ carries a metric and a connection s.t. the curvature is exactly $\Omega(\alpha,\beta) = \int_\Sigma \mathrm{Tr}(\alpha \wedge \beta)$ on $\mathcal{A}^{0,1}$.

Let

$$\Delta_A = *\bar{\partial}_A * \bar{\partial}_A$$

be the Laplacian of the $\bar{\partial}_A$ operator.

$K_0^a =$ sum of eigenspaces of Δ_A for eigenvalues less than a and $\bar{\partial}_A K_0^a = K_1^a$.

Consider the exact sequence

$0 \to \mathrm{Ker}\bar{\partial}_A \to K_0^a \stackrel{\bar{\partial}_A}{\to} K_1^a \to \mathrm{Coker}\bar{\partial}_A \to 0$

$\lambda = \wedge^{top}(\mathrm{Ker}\bar{\partial}_A)^* \otimes \wedge^{top}(\mathrm{Coker}\bar{\partial}_A)$ can be identfied with $\lambda^a = \wedge^{top}(K_0^a)^* \otimes \wedge^{top}(K_1^a)$ over the open set $U^a = \{a \notin \mathrm{spec}\Delta_A\} \subset \mathcal{A}$.

λ^a is a smooth line bundle over U^a. For if $a, b \notin \mathrm{spec}\Delta_A$, $a < b$, $K_0^{(a,b)} =$ union of eigenspaces of Δ_A corresponding to eigenvalues μ within $a < \mu < b$.

Let $K_1^{(a,b)} = \bar{\partial}_A(K_0^{(a,b)})$.

$\lambda^{(a,b)} = \wedge^{top}(K_0^{(a,b)})^* \otimes \wedge^{top}(K_1^{(a,b)})$ over $U^a \cap U^b$

Let $\bar{\partial}_A^{(a,b)} = \bar{\partial}_A|_{K_0^{(a,b)}}$ then $\lambda^b = \lambda^a \otimes \lambda^{(a,b)}$.

The identification of λ^a and λ^b via λ is given by the mapping $s \in \lambda^a \to s \otimes \det(\bar{\partial}_A^{(a,b)}) \in \lambda^b$.

Now under the gauge transformation, $\bar{\partial}_A = \bar{\partial} + A^{(0,1)} \to g(\partial + A^{0,1})g^{-1}$, $\Delta_g = g\Delta_A g^{-1}$.

There is an isomorphism of eigenspaces $s \to gs$ (with the same eigenvalues). Thus when one identifies

$\lambda = \wedge^{top}(\mathrm{Ker}\bar{\partial}_A)^* \otimes \wedge^{top}(\mathrm{Coker}\bar{\partial}_A)$ with

$\lambda^a = \wedge^{top}(K_0^a)^* \otimes \wedge^{top}(K_1^a)$ there is an isomorphism of fibers over U^a:

$\wedge^{top}(K_0^a(\Delta_A))^* \otimes \wedge^{top}(K_1^a(\Delta_A)) \equiv \wedge^{top}(K_0^a(\Delta_{A_g}))^* \otimes \wedge^{top}(K_1^a(\Delta_{A_g}))$. The fiber over U^a/G is the equivalence class of this fiber.

Like this we can define the line bundle on $\mathcal{A}/G$.

3.2. **Quillen metric.** Using the Hermitian structure on E (the vector bundle on the Riemann surface Σ) and therefore the one on $\Omega^{p,q}(E)$ one can define Hermitian metrics on K_0^a (K_1^a) over U^a and $K_0^{(a,b)}$ ($K_1^{(a,b)}$)over $U^a \cap U^b$. The bundles λ^a and $\lambda^{(a,b)}$ are then naturally endowed with metric $|\cdot|^a$ and $|\cdot|^{(a,b)}$.

Over $U^a \cap U^b$, $|s \otimes \mathrm{det}\bar{\partial}_A^{(a,b)}|^b = |s|^a|\mathrm{det}\bar{\partial}_A^{(a,b)}|^{(a,b)}$

When identifying λ with λ^a or λ^b, the metrics $|\cdot|^a$ and $|\cdot|^b$ are related to each other by

$|\cdot|^b = |\cdot|^a|\mathrm{det}\bar{\partial}_A^{(a,b)}|^{(a,b)}$.

To correct this discrepancy, Quillen does a zeta function regularisation:

Let $\zeta_A^a(s)$ is exactly the zeta function of the operator Δ_A restricted to eigenspaces whose eigenvalues are larger than a.

Then for $0 < a < b < \infty$ we can also define $\zeta_A^{(a,b)}(s)$. Clearly, $\zeta_A^a(s) = \zeta_A^{(a,b)}(s) + \zeta_A^b(s)$.

Also, $|\mathrm{det}\bar{\partial}_A^{(a,b)}|^{(a,b)} = \exp\{-\frac{1}{2}\frac{\partial\zeta_A^{(a,b)}}{\partial s}(0)\} = \exp\{-\frac{1}{2}\zeta_A'^{(a,b)}(0)\}$.

Thus one can define $||\cdot||^a$ to be the metric on λ^a which is such that if $l \in \lambda^a$

$||l||^a = |l|^a\exp\{-\frac{1}{2}\frac{\partial\zeta_A^a}{\partial s}(0)\} = |l|^a\exp\{-\frac{1}{2}\zeta_A'^a(0)\}$.

Thus under the canonical identification of λ with λ^a over U^a, the metrics $||\cdot||^a$ patch into a smooth metric $||\cdot||$ on λ.

3.3. **The curvature formula.** This metric induced a connection and Quillen computed the curvature of the determinant line bundle λ. For details, see [16], [12]. In fact the Kähler potential for this symplectic form is proportional to $\zeta_A'(0)$

The curvature turned out to be proportional to

$$\frac{i}{2\pi}\Omega_0(\alpha,\beta) = -\frac{i}{2\pi}\int_\Sigma \mathrm{Tr}(\alpha \wedge \beta),$$

the symplectic form, [16].

Thus we are in the situation for geometric prequantization of the affine space $\mathcal{A}$.

4. Prequantum bundle on $\mathcal{M}$

We first define the Quillen's determinant bundle $\mathcal{P} = \mathrm{det}(\bar{\partial}_A)$ on the affine space $\mathcal{C} = \mathcal{A} \times \Gamma(M, E)$, i.e. over each point (A, Ψ) we define the fiber as that of $\mathrm{det}(\bar{\partial}_A)$ independent of Ψ. However we modify the Quillen metric to $e^{-\zeta_A'(0) - \frac{i}{2\pi}\int_M \mathrm{Tr}(\Psi\Psi^*)\omega}$, which now depends on both A and Ψ.

This descends to $\mathcal{C}/G$ because under the gauge transformation, $\bar{\partial}_A = \bar{\partial} + A^{(0,1)} \to g(\partial + A^{0,1})g^{-1}$, $\Delta_g = g\Delta_A g^{-1}$.

There is an isomorphism of eigenspaces $s \to gs$ (with the same eigenvalues). Thus when one identifies

$\lambda = \wedge^{top}(\mathrm{Ker}\bar{\partial}_A)^* \otimes \wedge^{top}(\mathrm{Coker}\bar{\partial}_A)$ with

$\lambda^a = \wedge^{top}(K_0^a)^* \otimes \wedge^{top}(K_1^a)$ there is an isomorphism of fibers over $\tilde{U}^a = U^a \times V$, where U^a is as before an open set containing A and V is an open set in $\Gamma(M, E)$ containing Ψ.

$\wedge^{top}(K_0^a(\Delta_A))^* \otimes \wedge^{top}(K_1^a(\Delta_A)) \equiv \wedge^{top}(K_0^a(\Delta_{(A,g)}))^* \otimes \wedge^{top}(K_1^a(\Delta_{(A,g)}))$. The fiber over $\tilde{U}^a/G$ is the equivalence class of this fiber.

Like this we can define the line bundle on $\mathcal{C}/G$.

Then we restrict it to the moduli space $\mathcal{M} \subset \mathcal{C}/G$.

Curvature and symplectic form:

Following [6], we give $\mathcal{P}$ a modified Quillen metric, namely, we multiply the Quillen metric $e^{-\zeta_A'(0)}$ by the factor $e^{-\frac{i}{2\pi}\int_M \mathrm{Tr}(\Psi\Psi^*)\omega}$, where recall $\zeta_A(s)$ is the *zeta*-function corresponding to the Laplacian of the $\bar{\partial} + A^{0,1}$ operator. We calculate the curvature for this modified metric on the affine space. The *zeta* part of the metric contributes $-\frac{i}{2\pi}\int_M \mathrm{Tr}(\alpha_1 \wedge \alpha_2)$ to the curvature Ω as is well known, [16]. The second part $e^{-\frac{i}{2\pi}\int_M \mathrm{Tr}(\Psi\Psi^*)\omega}$, contributes to the second part of the curvature form Ω, namely, $-\frac{i}{2\pi}\int_M \mathrm{Tr}(\beta\eta^* - \eta^*\beta)\omega$ as follows:

Let $N = \int_M \mathrm{Tr}(\Psi(z,\bar{z})\Psi^*(z,\bar{z}))\omega(z,\bar{z})$, where $\omega(z,\bar{z})$ is the volume form on the Riemann surface.

$$\begin{aligned}
\tau &= \int_M [\int_M \mathrm{Tr}[\frac{\delta^2(\Psi(z,\bar{z})\Psi^*(z,\bar{z}))}{\delta\Psi(z',\bar{z}')\delta\Psi^*(z',\bar{z}')}\delta\Psi(z',\bar{z}') \wedge \delta\Psi^*(z',\bar{z}')]\omega(z',\bar{z}')]\omega(z,\bar{z}) \\
&= \int_M [\int_M \mathrm{Tr}[\delta(z-z')\delta(\bar{z}-\bar{z'})(\delta\Psi(z',\bar{z}'), \wedge\delta\Psi^*(z',\bar{z}'))\omega(z',\bar{z}')]]\omega(z,\bar{z}) \\
&= \int_M \mathrm{Tr}[\delta\Psi(z,\bar{z}) \wedge \delta\Psi^*(z,\bar{z})]\omega(z,\bar{z})
\end{aligned}$$

Then τ is a two form on the affine space $\Gamma(M,E)$ such that $\tau(\beta,\eta) = \int_M \mathrm{Tr}(\beta\eta^* - \eta^*\beta)\omega$.

The addition to the curvature two form due to the modification of the metric by $exp(-\frac{i}{2\pi}N)$ is $\partial_\Psi\partial_{\Psi^*} log(exp(-\frac{i}{2\pi}N)) = \partial_\Psi\partial_{\Psi^*}(-\frac{i}{2\pi}N)$ which is $-\frac{i}{2\pi}\tau$.

Thus the curvature of $\mathcal{P}$ with the modified Quillen metric is indeed

$$\frac{i}{2\pi}\Omega((\alpha_1,\beta),(\alpha_2,\eta)) = -\frac{i}{2\pi}[\int_M \mathrm{Tr}(\alpha_1 \wedge \alpha_2) + \int_M \mathrm{Tr}(\beta\eta^* - \beta^*\eta)\omega]$$

the symplectic form on $\mathcal{M}$

Polarization: In passing from prequantization to quantization, one needs a polarization. It can be shown that the almost complex structure $\mathcal{I}$ is integrable on $\mathcal{M}$, [3]. In fact, Ω is a Kähler form and $\mathcal{G}(X,Y) = \Omega(X, \mathcal{I}Y)$ is a Kähler metric on the moduli space (since it is positive definite). $\mathcal{P}$ is a holomorphic line bundle on $\mathcal{M}$. Thus we can take holomorphic square integrable sections of $\mathcal{P}$ as our Hilbert space. The dimension of the Hilbert space is not easy to compute. (For instance, the holomorphic sections of the determinant line bundle on the moduli space of flat connections for $SU(2)$ gauge group is the Verlinde dimension of the space of conformal blocks in a certain conformal field theory). This would be a topic for future work.

5. Alternate way of prequantizing the moduli space

Here we modify the method used in [10], [11] to accommodate non-abelian vortices.

5.1. Modified determinant line bundles on the moduli space. Let us modify the determinant line bundle construction on a different affine space.

Let $\mathcal{C}_+ = \mathcal{A} + \mathcal{B}$ where $\mathcal{A}$ is the space of connections (its $(0,1)$ part only) on E, as in [16], and $\mathcal{B}$ subspace of $\Omega^1(\Sigma, u(n))$,i.e. a subspace of the Lie-algebra valued $(0,1)$ forms which transforms as $B_g = gBg^{-1}$. $B = B^{(1,0)} + B^{(0,1)}$ where $B^{(1,0)*} = -B^{(0,1)}$

We define coordinates on this affine space as $w = A^{0,1} + B^{0,1}$ which is holomorphic with respect to the usual Hodge star operator which takes $*(\alpha^{(0,1)} + \beta^{(0,1)}) = i(\alpha^{(0,1)} + \beta^{(0,1)})$ and $*(\alpha^{(1,0)} + \beta^{(1,0)}) = -i(\alpha^{(1,0)} + \beta^{(1,0)})$.

Let the Cauchy-Riemann operator $\bar{\partial}_{(A,B,+)}$ be defined locally on the Riemann surface Σ as $\bar{\partial} + A^{0,1} + B^{0,1}$ and it differentiates sections of E. Note if $B = 0$, we get back Quillen's construction, [16].

One can show that one can define the line bundle

$$\mathcal{L}_+ = \det\bar{\partial}_{(A,B,+)} = \wedge^{\text{top}}(\text{Ker}\bar{\partial}_{(A,B,+)})^* \otimes \wedge^{\text{top}}(\text{Coker}\bar{\partial}_{(A,B,+)})$$

on $\mathcal{C}_+$, and subsequently on $(\mathcal{A} \times \mathcal{B})/G$, see [12] for details.

The curvature corresponding to this line bundle is

$$-\frac{i}{2\pi}\int_\Sigma \text{Tr}[(\alpha_1 + \beta_1) \wedge (\alpha_2 + \beta_2)].$$

Exactly analogous to the case of $\mathcal{C}_+$, we define the line bundle on $\mathcal{C}_- = \mathcal{A} - \mathcal{B}$ to be

$$\mathcal{L}_- = \det\bar{\partial}_{(A,B,-)} = \wedge^{\text{top}}(\text{Ker}\bar{\partial}_{(A,B,-)})^* \otimes \wedge^{\text{top}}(\text{Coker}\bar{\partial}_{(A,B,-)})$$

where now $\bar{\partial}_{(A,B,-)} = \bar{\partial} + A^{0,1} - B^{0,1}$. As described in [12] it is easy to define it on $(\mathcal{A} \times \mathcal{B})/G$.

The curvature formula now becomes:

$$-\frac{i}{2\pi}\int_\Sigma \text{Tr}[(\alpha_1 - \beta_1) \wedge (\alpha_2 - \beta_2)]$$

5.2. Special values of B. Let $\Psi_0 \in \Gamma(M, E \otimes K)$ where K is the canonical bundle such that Ψ_0 has zeroes only in a set of measure zero on M.

Now let us take $B = B^{(0,1)} + B^{(1,0)} = \Psi\Psi_0^* - \Psi_0\Psi^*$

Then $\mathcal{L}_+$ defined on $(\mathcal{A} \times \mathcal{B})/G$ will have curvature:
$\frac{-i}{2\pi}\int_M \text{Tr}[(\alpha_1 + \beta\Psi_0^* - \Psi_0\beta^*) \wedge (\alpha_2 + \eta\Psi_0^* - \Psi_0\eta^*)]$
and $\mathcal{L}_-$ also defined on $(\mathcal{A} \times \mathcal{B})/G$ will have curvature:
$\frac{-i}{2\pi}\int_M \text{Tr}[(\alpha_1 - \beta\Psi_0^* + \Psi_0\beta^*) \wedge (\alpha_2 - \eta\Psi_0^* + \Psi_0\eta^*)]$

An easy calculation shows that $\mathcal{P}_{\Psi_0} = \mathcal{L}_+ \otimes \mathcal{L}_-$ has curvature, by adding the above two, $\frac{i}{\pi}\Omega_{\Psi_0}(X,Y)$, see section (2.1).

As described in [12], the line bundle $\mathcal{P}_{\Psi_0}$ is well defined on $(\mathcal{A} \times \mathcal{B})/G$ and hence on $(\mathcal{A} \times \Gamma(M,E))/G$ and hence on $\mathcal{M} \subset (\mathcal{A} \times \Gamma(M,E))/G$ and the curvature formula descends to the moduli space.

As was mentioned before Ω_{Ψ_0} is a family of symplectic forms parametrised by Ψ_0.

Remark: Thus we have two candidates for the Hilbert space of quantization, namely, holomorphic sections of $\mathcal{P}$ and $\mathcal{P}_{\Psi_0}$. It would be interesting to see if they give equivalent quantization.

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