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Many-particle mechanics with $D(2,1;\alpha)$ superconformal symmetry

Sergey Krivonos* and Olaf Lechtenfeld+

*Bogoliubov Laboratory of Theoretical Physics, JINR
141980 Dubna, Russia

+Institut für Theoretische Physik, Leibniz Universität Hannover
Appelstrasse 2, 30167 Hannover, Germany

Abstract

We overcome the barrier of constructing $\mathcal{N}$=4 superconformal models in one space dimension for more than three particles. The $D(2,1;\alpha)$ superalgebra of our systems is realized on the coordinates and momenta of the particles, their superpartners and one complex pair of harmonic variables. The models are determined by two prepotentials, F and U, which must obey the WDVV and a Killing-type equation plus homogeneity conditions. We investigate permutation-symmetric solutions, with and without translation invariance. Models based on deformed A_n and BCD_n root systems are constructed for any value of α, and exceptional F_n-type and super root systems admit solutions as well. Translation-invariant mechanics occurs for any number of particles at $\alpha=-\frac{1}{2}$ ($osp(4|2)$ invariance as a degenerate limit) and for four particles at arbitrary α (three series).

1 Introduction

It has proved surprisingly difficult to construct $\mathcal{N}$=4 superconformal mechanics for more than three particles [1]–[10].[1] The Hamiltonian (or the action) of such models is determined by two scalar prepotentials, F and U, which are functions of the bosonic particle coordinates $\{x^i\}$ and obey two nonlinear differential equations, namely the celebrated WDVV equation for F and a Killing-type equation for U in the F background. On top of this, conformal invariance imposes some homogeneity conditions on U' and F'''. Each solution to all equations produces a consistent many-particle model.

In the one dimension (time) of mechanical systems, four supercharges implies invariance under the exceptional superalgebra $D(2,1;\alpha)$, for some value of the real parameter α.[2] For the special cases of $D(2,1;0) \simeq su(1,1|2) \ni su(2)$ and $D(2,1;1) \simeq D(2,1;-\frac{1}{2}) \simeq osp(4|2)$ some results were obtained [8, 9]. On the one hand, by gauging the U(n) isometry of matrix superfield models, one can construct U(2) spin-extended mechanics for arbitrary values of α [8, 14]. However, the particle coordinates parametrize a non-flat target space, except for $\alpha{=}-\frac{1}{2}$, i.e. the $osp(4|2)$ case. On the other hand, for α=0 a superspace approach produced an alternative formulation, which allowed for the construction of a few nontrivial four-particle solutions [9]. In this case, the 'structure equations' (writing W instead of U here) can be cast in the form [10]

$$\widehat{F}\wedge\widehat{F} \;=\; 0 \qquad \text{and} \qquad (\widehat{\mathrm{d}}-\widehat{F})\,|W\rangle \;=\; 0\;, \tag{1.1}$$

with the shorthand notation ($i,j,k,\ldots$ label the particles)

$$\widehat{F} = \mathrm{d}x^k\widehat{F}_k = \big(\mathrm{d}x^k F_{ijk}\big)\;, \qquad \widehat{\mathrm{d}} = \mathrm{d}x^k\widehat{\partial}_k = \big(\mathrm{d}x^k\partial_k\delta_{ij}\big) \qquad \text{and} \qquad |W\rangle = \big(W_i\big) \tag{1.2}$$

packaging the derivatives of F and W in a matrix-valued one-form and a (ket) vector, respectively. The question remains whether it is possible to construct $D(2,1;\alpha)$ invariant models with more than three particles for other values of α and perhaps without a spin extension.

In this paper we answer this question in the affirmative. By introducing just a single set of bosonic spin variables $\{u^a,\bar{u}_a\,|\,a{=}1,2\}$ with Poisson brackets

$$\big\{u^a,\bar{u}_b\big\} = -\mathrm{i}\,\delta^a_b \qquad \text{and} \qquad su(2)\ \text{currents} \quad J^{ab} = \tfrac{\mathrm{i}}{2}\left(u^a\bar{u}^b + u^b\bar{u}^a\right)\;, \tag{1.3}$$

we slightly generalize the ansatz of [6] for the supersymmetry generators. As a consequence, the structure equations (for any α) get modified to

$$\widehat{F}\wedge\widehat{F} \;=\; 0 \qquad \text{and} \qquad (\widehat{\mathrm{d}}-\widehat{F})\,|U\rangle \;=\; (\widehat{\mathrm{d}}U)\,|U\rangle\;, \tag{1.4}$$

where $|U\rangle = \big(U_i\big)$ is distinguished from $|W\rangle$. The WDVV equation is unchanged, and the integrability of the Killing-type equation still yields $(\widehat{F}\wedge\widehat{F})|U\rangle = 0$. Despite being nonlinear, the new term on the right-hand side is not a nuisance but actually a benefit, because it greatly enhances the solvability of the equation! In addition, one still has (α-dependent) homogeneity conditions for F and U. The new spin variables appear in the Hamiltonian merely via its $su(2)$ currents J^{ab}.

Since we aim at describing a collection of identical particles, we are not interested in arbitrary solutions of (1.4), but only those which are invariant under permutations of the particle labels. We do allow for translation non-invariance, however, because of the canonical relation between a translation-invariant system of n+1 particles to the reduced n-dimensional system of their relative coordinates, after decoupling the center of mass. In the next section, we derive the generic formulae for our new models, present the universal ansatz for F and U in terms of a collection of (co)vectors and their orbits under the permutation group and outline our strategy for solving (1.4). The following four sections present our explicit solutions (F,U) for the A-type, BCD-type, EF-type and non-Coxeter-type series of known WDVV configurations F. There exist families of solutions as well as sporadic ones. Finally, we conclude with a summary and some observations. All irreducible four-particle solutions are collected in an Appendix.

[1]For recent results on three-particle systems, see [11]. In principle, one may add also harmonic potentials [12].
[2]Permuting the three $sl(2)$ subalgebras of $D(2,1;\alpha)$ relates $\alpha \leftrightarrow -1{-}\alpha \leftrightarrow \frac{1}{\alpha} \leftrightarrow \frac{-1-\alpha}{\alpha} \leftrightarrow \frac{-1}{1+\alpha} \leftrightarrow \frac{-\alpha}{1+\alpha}$ [13].

2 $D(2,1;\alpha)$ invariant many-particle system

We consider $n{+}1$ particles on a real line, with (bosonic) coordinates and momenta $\{x_i, p_i \,|\, i{=}1,\ldots,n{+}1\}$ as well as associated complex pairs of fermionic variables $\{\psi_i^a, \bar\psi_{ai} \,|\, i{=}1,\ldots,n{+}1,\ a{=}1,2\}$.[3] In addition, we introduce one set of (bosonic) spin variables $\{u^a, \bar u_a \,|\, a{=}1,2\}$ parametrizing an internal two-sphere. The basic Poisson brackets read

$$\{x_i, p_j\} = \delta_{ij} \ , \qquad \{\psi_i^a, \bar\psi_{bj}\} = \tfrac{\mathrm{i}}{2}\delta^a{}_b\delta_{ij} \ , \qquad \{u^a, \bar u_b\} = -\mathrm{i}\,\delta^a_b \ . \tag{2.1}$$

We would like to realize the $\mathcal{N}{=}4$ superconformal algebra $D(2,1;\alpha)$ on the (classical) phase space of this mechanical system, thereby severely restricting the particle interactions. It is convenient to start with an ansatz for the supercharges Q^a and $\bar Q_a$. Previously [6], they were chosen (in our normalization) as

$$Q^a = p_i\psi_i^a + \mathrm{i}W_i(x)\psi_i^a + \mathrm{i}F_{ijk}(x)\psi_i^b\psi_{bj}\bar\psi_k^a \qquad \text{and} \qquad \bar Q_a = p_i\bar\psi_{ai} - \mathrm{i}W_i(x)\bar\psi_{ai} + \mathrm{i}F_{ijk}(x)\bar\psi_{bi}\bar\psi_j^b\psi_{ak} \tag{2.2}$$

with F_{ijk} being totally symmetric. This ansatz was successful for the algebra $D(2,1;0) \simeq su(1,1|2) \ni su(2)$ with a central charge C, upon solving some integrability conditions for W_i and F_{ijk} including the celebrated WDVV equation [15, 16]. However, it turned out to be very hard to generate explicit solutions for W_i beyond three particles [9].

Here, we utilize the spin variables to slightly generalize this ansatz to

$$Q^a = p_i\psi_i^a + U_i(x)J^{ab}\psi_{bi} + \mathrm{i}F_{ijk}(x)\psi_i^b\psi_{bj}\bar\psi_k^a \qquad \text{and} \qquad \bar Q_a = p_i\bar\psi_{ai} - U_i(x)J_{ab}\bar\psi_i^b + \mathrm{i}F_{ijk}(x)\bar\psi_{bi}\bar\psi_j^b\psi_{ak} \tag{2.3}$$

with the $su(2)$ currents

$$J^{ab} = \tfrac{\mathrm{i}}{2}\left(u^a\bar u^b + u^b\bar u^a\right) \qquad \Rightarrow \qquad \{J^{ab}, J^{cd}\} = -\epsilon^{ac}J^{bd} - \epsilon^{bd}J^{ac} \ . \tag{2.4}$$

The spin variables just serve to produce these currents and do not appear by themselves.

Let us try to build the $D(2,1;\alpha)$ algebra based on (2.3). Firstly, the $\mathcal{N}{=}4$ super-Poincaré subalgebra

$$\{Q^a, Q^b\} = 0 \qquad \text{and} \qquad \{Q^a, \bar Q_b\} = 2\mathrm{i}\,\delta^a{}_b H \tag{2.5}$$

defines a Hamiltonian H and enforces the following conditions on our functions V_i and F_{ijk},

$$\partial_i U_j - \partial_j U_i = 0 \ , \quad \partial_i F_{jkl} - \partial_j F_{ikl} = 0 \ , \tag{2.6}$$

$$F_{kim}F_{mj\ell} - F_{kjm}F_{mi\ell} = 0 \ , \tag{2.7}$$

$$-\partial_i U_j + U_iU_j + F_{ijk}U_k = 0 \ . \tag{2.8}$$

The integrability conditions (2.6) are solved by

$$U_i = \partial_i U \qquad \text{and} \qquad F_{ijk} = \partial_i\partial_j\partial_k F \tag{2.9}$$

with two scalar prepotentials $F(x)$ and $U(x)$, and hence we read subscripts on U and F as derivatives.[4] Thus, the other two conditions become nonlinear differential equations for $F(x)$ and $U(x)$, whose solutions define the various possible models. With the above conditions fulfilled, the Hamiltonian acquires the form

$$H \ = \ \tfrac14 p_ip_i + \tfrac18 J^{ab}J_{ab}\,U_iU_i - \mathrm{i}U_{ij}\,J^{ab}\,\psi_{ai}\bar\psi_{bj} - \tfrac12 F_{ijk\ell}\,\psi_i^a\psi_{aj}\,\bar\psi_{bk}\bar\psi_\ell^b \ . \tag{2.10}$$

One may check that $[H, J^{ab}J_{ab}] = 0$, and thus the Casimir $J^{ab}J_{ab} =: g^2$ appears as a coupling constant in the bosonic potential

$$V \ = \ \tfrac{g^2}{8}\,U_iU_i \ . \tag{2.11}$$

Secondly, for the full $D(1,2;\alpha)$ superconformal invariance one has to realize the additional generators. This can be done via

$$D = -\tfrac12 x_ip_i \ , \qquad K = x_ix_i \ , \qquad S^a = -2x_i\psi_i^a \ , \qquad \bar S_a = -2x_i\bar\psi_{ai} \ , \tag{2.12}$$

[3] Viewed as a one-particle system, the bosonic target is $\mathbb{R}^{n+1}$. Its metric (δ_{ij}) allows us to pull down all particle indices. Spinor indices are raised and lowered with the invariant tensor ε^{ab} and its inverse ε_{ba}, respectively.

[4] Note that $U(x)$ and $F(x)$ are defined only up to polynomials of degree zero and two, respectively.

together with two sets of composite $su(2)$ currents,

$$\mathcal{J}^{ab} = J^{ab} + 2\mathrm{i}\psi_i^{(a}\bar\psi_i^{b)} \qquad \text{and} \qquad I^{11} = \mathrm{i}\psi_i^a\psi_{ai}\ , \quad I^{22} = -\mathrm{i}\bar\psi_{ai}\bar\psi_i^a\ , \quad I^{12} = \mathrm{i}\psi_i^a\bar\psi_{ai}\ , \tag{2.13}$$

in the notation of [14]. Now, dilatation invariance requires homogeneity,

$$(x_i\partial_i + 1)\,U_j = \partial_j(x_iU_i) = 0 \qquad \text{and} \qquad (x_i\partial_i + 1)\,F_{jk\ell} = \partial_j(x_iF_{ik\ell}) = 0\ . \tag{2.14}$$

Thirdly, the remaining superalgebra commutators only fix the integration constants to

$$x_iU_i = 2\alpha \qquad \text{and} \qquad x_iF_{ijk} = -(1{+}2\alpha)\,\delta_{jk} \qquad \Rightarrow \qquad (x_i\partial_i - 2)\,F = -\tfrac12(1{+}2\alpha)\,x_ix_i\ . \tag{2.15}$$

It is instructive to compare our equations with the ones obtained in [6] with the ansatz (2.2) for the case of α=0. The integrability condition (2.6) and the WDVV equation (2.7) emerged these as well, but the Killing-type equation lacked the term quadratic in U. Still, we may map our equation to theirs by defining

$$W = \mathrm{e}^{-U} \qquad \Rightarrow \qquad W_{ij} - F_{ijk}W_k = 0\ . \tag{2.16}$$

The inhomogeneities of U and F_{jk} are also related: At α=0 we may introduce a central charge C by extending the first equation of (2.15) to

$$x_iU_i = C\,\mathrm{e}^U \qquad \Leftrightarrow \qquad x_iW_i = -C\ . \tag{2.17}$$

Here, $U \equiv 0$ is an option via C=0, but not so for $\alpha\neq0$.

For $\alpha\neq0$, no central charge is allowed, and the prepotentials take the form

$$U(x)\ \sim\ \alpha\ln x^2\ +\ U_0(x) \qquad \text{and} \qquad F(x)\ \sim\ -\tfrac14\,(1{+}2\alpha)\,x^2\ln x^2\ +\ F_0(x)\ , \tag{2.18}$$

where U_0 and F_0 are homogeneous of degree 0 and 2, respectively. Clearly, the prepotentials F for any two values of α are related by a mere rescaling as long as $\alpha\neq-\frac12$. The mathematical literature usually does not introduce a euclidean metric δ_{jk} but defines an induced metric $G_{jk} = -x_iF_{ijk}$ which is constant and nondegenerate. Hence, for any $\alpha\neq-\frac12$ we can import all known WDVV solutions [17]–[21] up to constant coordinate transformations. The special case of $D(2,1;-\frac12) \simeq D(2,1;1) \simeq osp(4|2)$ only appears as a singular limit, where F can no longer be 'normalized' via (2.15) and the induced metric degenerates.

A global $SO(n{+}1)$ coordinate transformation does not change the structure of a given model but its physical interpretation, since x_i denote the particle locations. For a system of identical particles in the absence of an external potential we should also demand invariance under permutations of the x_i as well as global translation invariance, $x_i \to x_i + \xi$. The latter is related to the decoupling of the (free) center-of-mass motion. Introducing center-of-mass and relative coordinates

$$X = \textstyle\sum_i x_i =: \rho\cdot x \qquad \text{and} \qquad x_i^\perp = x_i - \frac{1}{n+1}X \qquad \text{so that} \quad \sum_i x_i^\perp = 0\ , \tag{2.19}$$

we can project out the center-of-mass degree of freedom with

$$P^\parallel = \tfrac{1}{n+1}\,\rho\otimes\rho \qquad \text{and} \qquad P^\perp = \tfrac{1}{n+1}\begin{pmatrix} n & -1 & \dots & -1 \\ -1 & n & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n \end{pmatrix}\ . \tag{2.20}$$

One finds that

$$U(x) = U^\perp(x^\perp) \qquad \text{and} \qquad F(x) = F^\parallel(X) + F^\perp(x^\perp) \qquad \text{with} \qquad F^\parallel = -\tfrac14\tfrac{1+2\alpha}{n+1}\,X^2\ln X^2\ , \tag{2.21}$$

and our equations (2.6)–(2.8) are also valid for $U^\perp$ and $F^\perp$, while (2.15) projects to

$$x_iU_i^\perp = 2\alpha \qquad \text{and} \qquad x_iF^\perp_{ijk} = -(1{+}2\alpha)\,P^\perp_{jk} \qquad \Rightarrow \qquad (x_i\partial_i - 2)\,F^\perp = -\tfrac12(1{+}2\alpha)\,x\,P^\perp x\ , \tag{2.22}$$

because $\sum_i U_i^\perp = 0$ and $\sum_i F^\perp_{ijk} = 0$.

However, the set $\{x_i^\perp\}$ is linearly dependent. In order to select n independent relative coordinates, one should apply an $SO(n{+}1)$ transformation which rotates $\rho = (1,1,\ldots,1,1)$ to $(0,0,\ldots,0,\sqrt{n{+}1})$. A possible (but by no means unique) choice for the resulting relative coordinates $y = \{y_i \,|\, i=1,\ldots,n\}$ is

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1\cdot 2}} & \frac{-1}{\sqrt{1\cdot 2}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{2\cdot 3}} & \frac{1}{\sqrt{2\cdot 3}} & \frac{-2}{\sqrt{2\cdot 3}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n\cdot(n+1)}} & \frac{1}{\sqrt{n\cdot(n+1)}} & \frac{1}{\sqrt{n\cdot(n+1)}} & \cdots & \frac{-n}{\sqrt{n\cdot(n+1)}} \\ \frac{1}{\sqrt{n+1}} & \frac{1}{\sqrt{n+1}} & \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{pmatrix} , \tag{2.23}$$

where the center of mass $y_{n+1} \equiv y_0 = \frac{1}{\sqrt{n+1}} X$ has been added again. Any global $\mathrm{SO}(n)$ rotation of the y_i yields an equivalent description. The relative-coordinate parametrization of our $n{+}1$-particle model in terms of y_i offers a second interpretation, by reading the y_i as the *absolute* coordinates of an n-particle system *without* translation invariance. By a slight abuse of notation, we denote

$$U^\perp\big(x^\perp(y)\big) = U(y) \qquad \text{and} \qquad F^\perp\big(x^\perp(y)\big) = F(y) \ . \tag{2.24}$$

Surely, it is also possible to 'oxidize' an n-particle system without translation invariance to a translation-invariant $n{+}1$-particle system, by adding a y_0 coordinate and embedding into $\mathbb{R}^{n+1}$ via (2.23). Since the WDVV equation trivializes for $n \le 2$, it is relatively easy to write down translation-non-invariant two-particle models or translation-invariant three-particle models. In fact, there is a functional freedom in the choice [9]. Therefore, in this paper we concentrate on the nontrivial cases of $n \ge 3$.

All known WDVV solutions are of the form [5]

$$F(x) \;=\; \sum_\beta f_\beta\, K(\beta\cdot x) \qquad \text{with} \qquad f_\beta \in \mathbb{R} \qquad \text{and} \qquad \beta\cdot x = \beta^i x_i \ , \tag{2.25}$$

where the sum runs over a collection $\{\beta\}$ of p non-parallel (co)vectors, and the function K is universal up to a quadratic polynomial,

$$K'''(z) = -\frac{1}{z} \qquad \Rightarrow \qquad K(z) = -\tfrac14 z^2 \ln z^2 \qquad \text{in the rational case ,} \tag{2.26}$$

$$K'''(z) = -\cot z \qquad \Rightarrow \qquad K(z) = -\tfrac14 \mathrm{Li}_3(\mathrm{e}^{2\mathrm{i}z}) + \tfrac{\mathrm{i}}{6} z^3 \qquad \text{in the trigonometric case ,} \tag{2.27}$$

$$K'''(z) = -\frac{\vartheta_1'(\frac{z}{\pi}|\tau)}{\pi\,\vartheta_1(\frac{z}{\pi}|\tau)} \qquad \Rightarrow \qquad K(z) = -\tfrac14 \mathcal{L}i_3(\mathrm{e}^{2\mathrm{i}z}|\tau) \qquad \text{in the elliptic case ,} \tag{2.28}$$

where Li_3 is the trilogarithm and $\mathcal{L}i_3$ an elliptic generalization [23]–[26]. For the prepotential U we make an ansatz which matches the form of F,[6]

$$U(x) \;=\; \sum_\beta u_\beta\, L(\beta\cdot x) \qquad \text{with} \qquad u_\beta \in \mathbb{R} \qquad \text{and} \qquad L'(z) = -K'''(z) \ , \tag{2.29}$$

$$\text{thus} \qquad L_{\mathrm{rat}} = \tfrac12 \ln z^2 \ , \qquad L_{\mathrm{tri}} = \tfrac12 \ln \sin^2 z \ , \qquad L_{\mathrm{ell}} = \ln \vartheta_1(\tfrac{z}{\pi}|\tau) \ . \tag{2.30}$$

Note that not all vectors from $\{\beta\}$ need to appear in F or U, because some f_β or u_β may vanish. For illustration, we have given the 'universal functions' K and L also for the trigonometric and elliptic models. However, the normalization conditions (2.15) imposed by conformal invariance can only be satisfied in the rational case, and this is the only one treated in this paper. Then, the normalizations (2.15) translate into simple conditions for the coefficients u_β and f_β,

$$\sum_\beta u_\beta \;=\; 2\alpha \qquad \text{and} \qquad \sum_\beta f_\beta\, \beta_j \beta_k \;=\; (1{+}2\alpha)\,\delta_{jk} \qquad \Rightarrow \qquad \sum_\beta \beta^2 f_\beta \;=\; (1{+}2\alpha)\, n \ , \tag{2.31}$$

[5] We disregard here the possibility of 'radial' terms, where the argument of K is $\sqrt{\sum_i x_i^2}$ or $\sqrt{\sum_{i<j}(x_i{-}x_j)^2}$ [6, 22].

[6] This may be very restrictive, as the solutions found in [9] demonstrate.

and the bosonic potential becomes

$$V(x) \;=\; \tfrac{g^2}{8}\sum_{\beta,\gamma} u_\beta u_\gamma \, \frac{\beta\cdot\gamma}{\beta{\cdot}x\;\gamma{\cdot}x}\;. \tag{2.32}$$

We can actually employ the Killing-type equation (2.8) to solve for u_β in terms of f_β. When inserting the forms (2.25) and (2.29) into (2.8), the vanishing of each double pole $(\beta{\cdot}x)^{-2}$ yields

$$u_\beta(u_\beta{+}1) \;=\; \beta^2 f_\beta\, u_\beta \qquad\Rightarrow\qquad u_\beta = 0 \qquad\text{or}\qquad u_\beta = \beta^2 f_\beta - 1 \tag{2.33}$$

for each (co)vector β, with $\beta^2 \equiv \beta{\cdot}\beta$. Inserting this into the 'sum rule' (2.31) for u_β, we obtain a second necessary condition for $\{f_\beta\}$, namely

$$\sum_\beta \delta_\beta\,(\beta^2 f_\beta - 1) \;=\; 2\alpha \qquad\text{with}\qquad \delta_\beta \in \{0,1\}\;. \tag{2.34}$$

It restricts the F solutions to those which may admit a U solution as well. However, by no means it guarantees that the single-pole terms in (2.8) work out as well.

Since we are only interested in permutation-invariant models, we demand that the collection $\{\beta\}$ of p (co)vectors is closed under permutations and that the coefficients f_β and u_β depend only on the orbit $[\beta]$ of β. This suggests the notation (with square brackets)

$$K_{n+1}[\beta{\cdot}x] \;:=\; \sum_\pi K\bigl(\pi(\beta){\cdot}x\bigr) \qquad\text{and}\qquad K_n[\beta{\cdot}y] \;:=\; \sum_\pi K\bigl(\pi(\beta){\cdot}y\bigr) \tag{2.35}$$

where the index indicates the particle number, and sum runs over all permutations which alter $\pm\beta$. To indicate a particular orbit in the square-bracket argument, we insert a typical representative and omit the coordinate labels, e.g.

$$K_3[3y{-}y{-}y] \;:=\; K(3y_1{-}y_2{-}y_3) + K(3y_2{-}y_3{-}y_1) + K(3y_3{-}y_1{-}y_2)\;, \tag{2.36}$$

and likewise for the function L.

Finally, we comment on our solution strategy for the 'structure equations' (2.6)–(2.8). Guided by the known WDVV solutions [20, 21], we select a (co)vector collection $\{\beta\}$, which gives us F and U via (2.25) and (2.29), with undetermined coefficients f_β and u_β. Importing from the literature a particular solution for $\{f_\beta\}$, one remains with algebraic relations for $\{u_\beta\}$ which are very intricate, however. Therefore, it is preferable not to start with some solution F, but to pick a structure only for U via (2.29) and then to regard the WDVV and Killing-type equations (2.7) and (2.8) as algebraic equations for the functions F_{ijk}, disregarding their integrability condition for the moment. Beyond three particles, the (algebraic) WDVV equation (2.7) becomes rather involved. Therefore, as a detour, we first solve a simpler (linear) equation which follows from it and (2.8), namely [7]

$$(U_{ij} - U_iU_j)\,F_{jk\ell} \;-\; (U_{kj} - U_kU_j)\,F_{ji\ell} \;=\; 0\;. \tag{2.37}$$

When $\{F_{ijk}\}$ has been constrained to obey this relation and also (2.8), it is much easier to completely solve the WDVV equation. In fact, given $\{\beta\}$ and $\{u_\beta\}$, one can always construct a solution $\{F_{ijk}\}$ in the form

$$F_{ijk}(x) = -\sum_\beta \frac{f_{ijk}}{\beta{\cdot}x}\;. \tag{2.38}$$

The crucial point then is the integrability of those functions, i.e.

$$\partial_{[i}F_{j]k\ell} = 0 \qquad\Leftrightarrow\qquad f_{jk\ell} = \sum_\beta f_\beta\,\beta_j\beta_k\beta_\ell\;, \tag{2.39}$$

see (2.6) or (2.9). It is a very restrictive requirement, which in many cases completely rules out any solution $\{u_\beta, f_\beta\}$. In other cases, it removes any freedom in these coefficients (coming from WDVV-solution moduli) and may even fix the value of the parameter α.

[7] We thank A. Galajinsky for a similar suggestion.

3 A-type models

The simplest and most symmetric WDVV solutions take (the positive part of) the A_n root system for $\{\beta\}$. The canonical representation lives in $\mathbb{R}^{n+1}$ in the hyperplane orthogonal to ρ,

$$F^{\perp}(x) \;=\; \tfrac{1+2\alpha}{n+1}\,K_{n+1}[x{-}x]\;. \tag{3.1}$$

Unfortunately, for $n>3$ we could not find a U solution, except when

$$2\alpha=n: \qquad F^{\perp}(x) \;=\; K_{n+1}[x{-}x] \qquad\text{and}\qquad U^{\perp}(x) \;=\; \sum_{i=1}^{n} L(x_i{-}x_{n+1})\;, \tag{3.2}$$

which is not fully permutation invariant however.

It is known [19] that (3.1) is a special point in an n-parameter family of WDVV solutions based on particular deformations of the A_n root system. These deformations also break the permutation invariance of $F^{\perp}$, but there exists a one-parameter subfamily which is permutation symmetric in the *relative* coordinates y_i. Therefore, let us reduce the description to $\mathbb{R}^n$ and search for translation non-invariant n-particle solutions. With the abbreviations $Y=\sum_i y_i$ and $\delta^2=\frac{n+1}{1+nt}$, the WDVV solution reads

$$F_t(y) \;=\; \tfrac{1+2\alpha}{n+1}\Big\{(1{-}t)\,K_n\big[y-y\big] \;+\; \tfrac{1+nt}{n^2}\,K_n\big[(ny-(1{+}\delta)Y\big]\Big\}\;, \tag{3.3}$$

where the first term contains an A_{n-1} subsystem and the remaining n roots are deformed by changing their component in the direction of the subsystems center-of-mass vector $\rho=(1,1,\dots,1)$. As $t\in[-\frac{1}{n},\infty]$, we cover the four cases

$t=-\frac{1}{n}$	$t=0$	$t=1$	$t=\infty$
$\delta=\infty$	$\delta=\sqrt{n{+}1}$	$\delta=1$	$\delta=0$
$A_{n-1}\oplus A_1$	A_n	A_1^n	$A_{n-1}\oplus\underline{n}$

where $\underline{n}$ denotes the fundamental (quark) weights of A_{n-1}.

Which of these F backgrounds admit a U solution? Permutation invariance applied to (2.34) leaves us with three options, corresponding to the choices of δ_β: U may contain either only the first type of (co)vectors from (3.3), or only the second type of (co)vectors, or both. In each case (2.31) yields an equation of the form $h_n(t,\alpha)=0$. Numerical analysis reveals that only the second option fully works out,

$$U(y) \;=\; u\,L_n\big[(ny-(1{+}\delta)Y\big] \qquad\text{with}\quad u=\tfrac{1-t}{1+nt}=\tfrac{2\alpha}{n}\;. \tag{3.4}$$

The corresponding condition from (2.34) is

$$(1{+}2\alpha)(nt+1)=n+1 \qquad\Leftrightarrow\qquad t=\frac{1}{n}\frac{n{-}2\alpha}{1{+}2\alpha} \qquad\Leftrightarrow\qquad 1{+}2\alpha=\frac{n{+}1}{nt{+}1}\;. \tag{3.5}$$

For $t{=}0$ ($\alpha{=}\frac{n}{2}$) this yields an undeformed A_n solution where in U only n of the roots appear:

$$\begin{aligned} F_0(y) \;&=\; K_n\big[y-y\big] \;+\; \tfrac{1}{n^2}\,K_n\big[(ny-(1{+}\sqrt{n{+}1})Y\big]\;,\\ U_0(y) \;&=\; L_n\big[(ny-(1{+}\sqrt{n{+}1})Y\big]\;, \end{aligned} \tag{3.6}$$

which is the reduced form of (3.2). For $n{=}3$ it simplifies to

$$F_0(y) \;=\; K_3[y-y] \;+\; K_3[y+y] \qquad\text{and}\qquad U_0(y) \;=\; L_3[y+y]\;. \tag{3.7}$$

The limit $t\to\infty$ ($\delta\to0$) deserves special attention, because the induced metric G degenerates. For this reason, this boundary of the WDVV solution space is normally excluded in the mathematical literature (see, however, [26]). Also, (3.3) tells us that we need to tune

$$1{+}2\alpha=0: \qquad F_\infty(y) \;\propto\; -K_n[y-y] \;+\; \tfrac{1}{n}\,K_n[ny-Y] \qquad\text{and}\qquad U_\infty(y) \;=\; -\tfrac{1}{n}\,L_n[ny-Y]\;, \tag{3.8}$$

where the scale of F_∞ is undetermined. In this limit, our deformed root system fits in the hyperplane orthogonal to ρ, thus we recover translation invariance! Therefore, only for $osp(4|2)$ symmetry we have an n-particle solution which meets all physical requirements. Its bosonic potential

$$V(y) \;=\; \frac{g^2}{8}\Big\{\sum_i \frac{1}{(ny_i-Y)^2} \;-\; \frac{1}{n}\Big(\sum_i \frac{1}{(ny_i-Y)}\Big)^2\Big\} \tag{3.9}$$

however, is not of the Calogero type, because only the fundamental weights contribute to it.

Due to the isometry $A_3 \simeq D_3$, the reduction of the $A_4(\infty)$ solution to $\mathbb{R}^3$ remains permutation invariant (see the following section),

$$F_\infty(y) \;\propto\; -K_4[y{-}y] \;+\; \tfrac14 K_4[3y{-}y{-}y{-}y] \qquad\longrightarrow\qquad -K_3[y{\pm}y] \;+\; K_3[y{\pm}y{\pm}y]\ , \tag{3.10}$$

$$U_\infty(y) \;=\; -\tfrac14 L_4[3y{-}y{-}y{-}y] \qquad\longrightarrow\qquad -\tfrac14 L_3[y{\pm}y{\pm}y]\ . \tag{3.11}$$

One may wonder whether further solutions can be produced by admitting other weights to the ansätze (2.25) and (2.29). This is not the case, except for $n \le 4$, where accidents happen due to the existence of F_4 and the isometries $A_3 \simeq D_3$ and $A_3 \oplus \underline{6} \simeq B_3$. These cases are more naturally described in the following sections.

4 BCD-type models

The B_n, C_n and D_n root systems do not yield permutation symmetric models in $\mathbb{R}^{n+1}$, but are naturally formulated in the *relative* coordinates $y_i \in \mathbb{R}^n$. Therefore, we consider the reduced description, which trades translation invariance for permutation invariance. The WDVV equation (2.7) has an n-parameter family of solutions based on deformed BCD_n roots [19],

$$F_{t,\vec{s}}(y) \;=\; \frac{1{+}2\alpha}{2(s^2{-}\delta^2)}\Big\{\sum_{i<j} K_n\big(s_j y_i {\pm} s_i y_j\big) \;+\; 2\sum_i K_n\big(\sqrt{s_i^2{-}\delta^2}\,y_i\big)\Big\} \qquad \text{with} \quad \delta^2 = \frac{1{-}nt}{1{-}t}\ , \tag{4.1}$$

where $t, s_i \in \mathbb{R}$ for $i = 1, \dots, n$, and one parameter is redundant. To retain permutation symmetry, we keep the roots undeformed, $s_i = 1$, but allow t to vary, so that

$$F_t(y) \;=\; \lim_{s_i\to 1} F_{t,\vec{s}}(y) \;=\; (1{+}2\alpha)\Big\{\tfrac{1-t}{2n-2}\,K_n[y{\pm}y] \;+\; t\,K_n[y]\Big\}\ . \tag{4.2}$$

By changing the parameter t, we reach four special cases in $BCD_n(t)$:

$t=0$	$t=\frac{1}{2n-1}$	$t=\frac{2}{n+1}$	$t=1$
D_n	B_n	C_n	A_1^n

Similarly to the $A_n(t)$ deformation, it turns out that most U solutions carry only the short roots. For this case, (2.34) yields the relation [8]

$$(1{+}2\alpha)(nt-1) = n-1 \qquad\Leftrightarrow\qquad t = \frac1n\,\frac{n{+}2\alpha}{1{+}2\alpha} \qquad\Leftrightarrow\qquad 1{+}2\alpha = \frac{n{-}1}{nt{-}1}\ , \tag{4.3}$$

which may be used to fix $t = t(\alpha)$ in the solution

$$F(y) \;=\; \tfrac1n\Big\{\alpha\,K_n[y{\pm}y] \;+\; (n{+}2\alpha)\,K_n[y]\Big\} \qquad\text{and}\qquad U(y) \;=\; \tfrac{2\alpha}{n}\,L_n[y]\ . \tag{4.4}$$

Again, only n of the roots appear in U. Of course, we may instead fix $\alpha = \alpha(t)$ and read off solutions for

$$B_n:\quad 1{+}2\alpha = 1{-}2n \quad\Rightarrow\quad F(y) \;=\; -K_n[y{\pm}y] - K_n[y] \quad\text{and}\quad U(y) \;=\; -2L_n[y]\ , \tag{4.5}$$

$$C_n:\quad 1{+}2\alpha = 1{+}n \quad\Rightarrow\quad F(y) \;=\; \tfrac12 K_n[y{\pm}y] + 2\,K_n[y] \quad\text{and}\quad U(y) \;=\; L_n[y]\ , \tag{4.6}$$

$$D_n:\quad 1{+}2\alpha = 1{-}n \quad\Rightarrow\quad F(y) \;=\; -\tfrac12 K_n[y{\pm}y] \quad\text{and}\quad U(y) \;=\; -L_n[y]\ , \tag{4.7}$$

[8] It is remarkable that this $BCD_n(t)$ relations is obtained from the $A_n(t)$ relation (3.5) by $n \to -n$.

where the D_n case employs the vector weights (v) in U.

For n=4 the triality automorphism T of D_4 generates additional solutions from the ones above, via

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_0 \end{pmatrix} \quad \stackrel{T}{\longmapsto} \quad \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \qquad \text{with} \quad T^3 = -\mathbb{1} \ , \tag{4.8}$$

under which the vector (v), spinor (s) and conjugate spinor (c) representations get cycled around. This allows us to find a couple of further solutions, in which U carries incomplete Weyl orbits of roots. Since the fully deformed WDVV solution (4.1) breaks the Weyl symmetry of the B_n system, we may ignore it even in the limit $s_i \to 1$. Making use of this freedom, we single out y_1 and find the n=4 solutions

$$1{+}2\alpha = 6 : \qquad F(y) \;=\; K_4[y{\pm}y] \qquad \text{and} \qquad U(y) \;=\; L_4[y_1{\pm}y] - L_4(y_1) \ , \tag{4.9}$$

$$1{+}2\alpha = 7 : \qquad F(y) \;=\; K_4[y{\pm}y] + K_4[y] \qquad \text{and} \qquad U(y) \;=\; L_4[y_1{\pm}y] \ , \tag{4.10}$$

where $[y_1{\pm}y]$ stands for $\{y_1{\pm}y_2, y_1{\pm}y_3, y_1{\pm}y_4\}$. Acting with a triality transformation (4.8), we obtain two permutation-invariant solutions with p=13 and p=16, respectively:

$$1{+}2\alpha = 6 : \qquad F(y) = K_4[y{\pm}y] \qquad \text{and} \qquad U(y) = L_4[y{+}y] - L_4(Y) \ , \tag{4.11}$$

$$1{+}2\alpha = 7 : \qquad F(y) = K_4[y{\pm}y] + \tfrac{1}{4}K_4[y{+}y{\pm}y{\pm}y]_+ \qquad \text{and} \qquad U(y) = L_4[y{+}y] \ , \tag{4.12}$$

where $[\ldots]_+$ indicates an even number of minuses in the bracket.

The case of n=3 is also special. For α=1 it admits an isolated additional solution,

$$F(y) \;=\; \tfrac{2}{3}\,K_3[y{\pm}y] \;+\; \tfrac{1}{3}\,K_3[y] \qquad \text{and} \qquad U(y) \;=\; \tfrac{1}{3}\,L_3[y{\pm}y] \ . \tag{4.13}$$

Due to the isomorphy $D_3 \simeq A_3$, the oxidation of the BCD_3 models to $\mathbb{R}^4$ can be made permutation invariant, by reading the D_3 weights as A_3 weights. With an SO(3) rotation built from the triality map (4.8), the explicit coordinate relation reads ($y_4 \equiv y_0$)

$$y_i \;=\; T_{ij}\,x_j \ , \tag{4.14}$$

and we obtain the following translation table,

representation	$\underline{4}$	$\underline{6}$	$\underline{15}$	$\underline{45}$	$\underline{64}$	$\cdots$
length of orbit	4	3	6	12	12	$\cdots$
argument $[\beta{\cdot}y]$	$[y{\pm}y{\pm}y]$	$[y]$	$[y{\pm}y]$	$[2y{\pm}y{\pm}y]$	$[2y{\pm}y]$	$\cdots$
argument $[\beta{\cdot}x]$	$\frac{1}{2}[3x{-}x{-}x{-}x]$	$\frac{1}{2}[x{+}x{-}x{-}x]$	$[x{-}x]$	$[2x{-}x{-}x]$	$\frac{1}{2}[3x{-}3x{+}x{-}x]$	$\cdots$

By oxidizing the $\underline{15}$ and $\underline{6}$ weights and using $K_n[\lambda\beta{\cdot}x] \simeq \lambda^2 K_n[\beta{\cdot}x]$ and $L_n[\lambda\beta{\cdot}x] \simeq L_n[\beta{\cdot}x]$, we can formulate translation-invariant BCD_3 models,

$$\text{any } \alpha: \qquad F^{\perp}(x) = \tfrac{\alpha}{3}K_4[x{-}x] - \tfrac{3+2\alpha}{12}K_4[x{+}x{-}x{-}x] \quad \text{and} \quad U^{\perp}(x) = \tfrac{2\alpha}{3}L_4[x{+}x{-}x{-}x] \ , \tag{4.15}$$

$$\alpha{=}1 : \qquad F^{\perp}(x) = \tfrac{2}{3}K_4[x{-}x] + \tfrac{1}{12}K_4[x{+}x{-}x{-}x] \qquad \text{and} \quad U^{\perp}(x) = \tfrac{1}{3}L_4[x{-}x] \ . \tag{4.16}$$

Note that only the last model, which is invariant under $D(2,1;1) \simeq osp(4|2)$, gives rise to a Calogero potential V. Some D_n spinor weights occur in further solutions, but these are more naturally obtained within the F-type models, which derive from E_8 and are discussed next.

5 EF-type models

Further WDVV solutions are based on the root systems of the exceptional simple Lie algebras. For rank $n > 2$, they can all be obtained by reducing the E_8 system in particular ways. Also exceptional deformed root systems appear in this way, as projections of E_n or $F_4(t)$ along some parabolic subgroup [20].[9] Among this variety, we restrict ourselves to permutation-symmetric models for physical reasons. This leaves us with the following possibilities.

With p=120 (co)vectors the E_8 system is the largest exceptional one,

$$F(y)_\pm \;=\; \tfrac{1+2\alpha}{30}\Big\{K_8[y\pm y] \;+\; \tfrac14 K_8[y+y+y+y\pm y\pm y\pm y\pm y]_\pm\Big\}\;, \tag{5.1}$$

where the '±' subscript indicates an even or odd number of minuses. The two solutions are related by the standard spinor helicity flip. The E_7 system with p=63 is more naturally formulated as a translation-invariant eight-particle model,

$$F^\perp(x) \;=\; \tfrac{1+2\alpha}{18}\Big\{K_8[x-x] \;+\; \tfrac14 K_8[x+x+x+x-x-x-x-x]\Big\}\;. \tag{5.2}$$

E_6 is not permutation symmetric. Neither case allows for a U solution, so no such models exist.

Therefore, we pass to the F_n series as defined in [20] by the projection of the E_8 system along its D_{8-n} subgroup, for $n=3,4,5,6$. First, $F_6\simeq(D_8,A_1^2)$ with p=68 yields

$$F(y) \;=\; \tfrac{1+2\alpha}{30}\Big\{K_6[y\pm y] \;+\; 4K_6[y] \;+\; \tfrac12 K_6[y+y+y\pm y\pm y\pm y]\Big\}\;, \tag{5.3}$$

$$2\alpha{+}1=15:\quad U(y)_\pm \;=\; L_6[y] \;+\; \tfrac12 L_6[y+y+y\pm y\pm y\pm y]_\pm\;, \tag{5.4}$$

with the same notation as above. Second, $F_5\simeq(E_8,A_3)$ has p=41 and produces

$$F(y) \;=\; \tfrac{1+2\alpha}{30}\Big\{K_5[y\pm y] \;+\; 6K_5[y] \;+\; K_5[y+y+y\pm y\pm y]\Big\}\;, \tag{5.5}$$

$$2\alpha{+}1=\tfrac{15}{2}:\quad U(y) \;=\; \tfrac12 L_5[y] \;+\; \tfrac14 L_5[y+y+y\pm y\pm y]\;. \tag{5.6}$$

In the next reduction step, we meet $(E_8,D_4)\simeq F_4$ with p=24 (co)vectors, which as a Lie algebra with two Weyl orbits allows for a one-parameter deformation $F_4(t)$,

$$F_t(y) \;=\; (1{+}2\alpha)\Big\{\tfrac{1-t}{6}K_4[y\pm y] \;+\; \tfrac{t}{3}K_4[y] \;+\; \tfrac{t}{12}K_4[y+y\pm y\pm y]\Big\}\;. \tag{5.7}$$

It is invariant under the exchange of its two D_4 subsystems while $t\to1{-}t$. Special values of t are

$t=0$	$t=\frac13$	$t=1$
D_4	F_4	D_4'
p=12	p=24	p=12

where the D_4 and D_4' systems are formed by the long and short roots of F_4, respectively. The three types of roots allow for more options in U than was the case in the A_n or BCD_n models. On the corresponding curves $h_n(t,\alpha)=0$, howver, only isolated U solutions occur:[10]

$$1{+}2\alpha=-3 \quad\&\quad t=0:\qquad U(y) \;=\; -L_4[y] \quad\text{or}\quad -L_4[y+y\pm y\pm y]_\pm\;, \tag{5.8}$$

$$1{+}2\alpha=+5 \quad\&\quad t=\tfrac15:\qquad U(y) \;=\; \tfrac13 L_4[y\pm y]\;, \tag{5.9}$$

$$1{+}2\alpha=+5 \quad\&\quad t=\tfrac45:\qquad U(y) \;=\; \tfrac13 L_4[y]+\tfrac13 L_4[y+y\pm y\pm y]\;, \tag{5.10}$$

$$1{+}2\alpha=+9 \quad\&\quad t=\tfrac23:\qquad U(y) \;=\; L_4[y]+L_4[y+y\pm y\pm y]_\pm \quad\text{or}\quad L_4[y+y\pm y\pm y]\;. \tag{5.11}$$

The three solutions in the first line and also in the fourth one are related by triality (the very first solution occurred already under D_4). The $D_4{\leftrightarrow}D_4'$ flip applied to lines one or four yields permutation non-invariant configurations (which we ignore here), but relates the solutions in the second and third lines, which are triality invariant by themselves.

[9] Non-crystallographic Coxeter root systems do not produce permutation-invariant systems.

[10] We do expect families of solutions whose generic members, however, will not be permutation invariant. The corresponding curves are $(1{+}2\alpha)(4t{-}3)=9$, $(1{+}2\alpha)(4t{-}3)=-11$, $(1{+}2\alpha)(4t{-}1)=11$ and $(1{+}2\alpha)(8t{-}3)=21$, respectively.

The final reduction yields the $F_3(t)$ family with p=13 and

$$F_t(y) \;=\; (1{+}2\alpha)\Big\{\tfrac{1-t}{6}K_3[y{\pm}y] \;+\; \tfrac13 K_3[y] \;+\; \tfrac{t}{6}K_3[y{\pm}y{\pm}y]\Big\}\ , \tag{5.12}$$

which connects the $BCD_3(t)$ family to the $A_4(t)$ one,

$t=0$	$t=1$	$t=\infty$
$BCD_3(\frac13)$	$D(2,1;\alpha)$	$A_4(\infty)=A_3\oplus\underline{4}$
p=9	p=7	p=10

In this case, U solutions occur in *two* subfamilies and one sporadic case:

$$(1{+}2\alpha)\,t=3: \qquad U(y) \;=\; \tfrac{2\alpha-2}{3}L_3[y] \;+\; \tfrac12 L_3[y{\pm}y{\pm}y]\ , \tag{5.13}$$

$$(1{+}2\alpha)(2t{-}1)=3: \qquad U(y) \;=\; \tfrac{\alpha}{2}L_3[y{\pm}y{\pm}y]\ , \tag{5.14}$$

$$t=\tfrac15 \quad \& \quad 1{+}2\alpha=5: \qquad U(y) \;=\; \tfrac13 L_3[y{\pm}y] \;+\; \tfrac23 L_3[y]\ . \tag{5.15}$$

Indeed, at (t=0 , 1+2α=∞) the first family matches to the $BCD_3(\frac13)$ solution, and at (t=∞ , 1+2α=0) the second family agrees with the $A_4(\infty)$ one. Employing the embedding (4.14) and the subsequent translation table, we can oxidize these systems to translation-invariant four-particle models,[11]

$$\begin{cases} F^\perp(x) \;=\; \frac{\alpha-1}{3}K_4[x{-}x] \;+\; \frac{1+2\alpha}{12}K_4[x{+}x{-}x{-}x] \;+\; \frac18 K_4[3x{-}x{-}x{-}x] \\ U^\perp(x) \;=\; \frac{2\alpha-2}{3}L_4[x{+}x{-}x{-}x] \;+\; \frac12 L_4[3x{-}x{-}x{-}x] \end{cases}, \tag{5.16}$$

$$\begin{cases} F^\perp(x) \;=\; \frac{\alpha-1}{6}K_4[x{-}x] \;+\; \frac{1+2\alpha}{12}K_4[x{+}x{-}x{-}x] \;+\; \frac{2+\alpha}{24}K_4[3x{-}x{-}x{-}x] \\ U^\perp(x) \;=\; \frac{\alpha}{2}L_4[3x{-}x{-}x{-}x] \end{cases}, \tag{5.17}$$

$$\begin{cases} F^\perp(x) \;=\; \frac23 K_4[x{-}x] \;+\; \frac{5}{12}K_4[x{+}x{-}x{-}x] \;+\; \frac{1}{24}K_4[3x{-}x{-}x{-}x] \\ U^\perp(x) \;=\; \frac13 L_4[x{-}x] \;+\; \frac23 L_4[x{+}x{-}x{-}x] \end{cases}. \tag{5.18}$$

6 Non-Coxeter-type models

It is known that the root systems of some Lie superalgebras also give rise to WDVV solutions [20, 27]. Of interest are one-parameter deformations of $AB(1,3)$ and $G(1,2)$ and a two-parameter deformation of $D(2,1;\alpha)$. The $AB(1,3)$ family admits two inequivalent reductions to n=3, one of which yields a permutation-symmetric solution with p=10:

$$F_t(y) = \tfrac{1+2\alpha}{27(t^2+1)}\Big\{9K_3[y{-}y] + K_3[ty{+}ty{-}2ty{+}2wY] + 2K_3[ty{+}ty{-}2ty{-}wY] + \tfrac92(t^2{-}1)K_3[Y]\Big\} \tag{6.1}$$

with $w^2=\frac14(t^2{+}3)$ for $t\in\mathbb{R}_+$. At t=1 there exists a full solution for (F,U):

$$1{+}2\alpha=6: \qquad F(y) \;=\; K_3[y{\pm}y]+2K_3[y] \qquad \text{and} \qquad U(y) \;=\; L_3[y{+}y]+L_3[y]-L_3[Y]\ . \tag{6.2}$$

We know of no other non-Coxeter-type permutation-invariant solutions.

7 Conclusions

By adding to the particle coordinates and their superpartners a single harmonic variable (parametrizing a two-sphere), we have overcome the technical barrier for constructing $\mathcal{N}$=4 superconformal mechanics models with more than three particles. The structure equations determining the two prepotentials F and U admit simple solutions based on deformed root systems, for an arbitrary number of particles and for the superconformal symmetry algebra $D(2,1;\alpha)$ at any value of α. We have restricted ourselves

[11] For α=0, the first one yields a C=0 four-particle solution to (2.16), $W=\mathrm{e}^{-U}=\prod(x{+}x{-}x{-}x)^{2/3}\prod(3x{-}x{-}x{-}x)^{-1/2}$ in obvious notation, which was missed in [9].

to permutation-invariant prepotentials and performed a numerical survey of all permutation-symmetric (deformed) root configurations, with and without translation invariance.

In each moduli space of WDVV solutions F based on a deformed A_n or BCD_n root system, we have identified a permutation-invariant one-parameter (t) subfamily. It turns out that the solutions of the Killing-type equation for the second prepotential U in the background of a given WDVV solution F live on a curve $h_n(t,\alpha)=0$ in the (t,α) plane. The $A_n(t{=}\infty)$ model, built on the roots and fundamental weights of A_{n-1}, is degenerate but distinguished by its translation invariance. Since $h_n(\infty,-\frac{1}{2})=0$, this solution exists only for the $osp(4|2)$ case. Also, its bosonic potential is not of Calogero-type. All other solutions lack translation invariance. Of course, one may reinterpret their variables as *relative* particle coordinates and add the center of mass to reclaim translation invariance, but permutation symmetry will usually be lost in the new variables.

An exception occurs at $n{=}3$ because of the $A_3 \simeq D_3$ isometry. Inside the F_3 family (with parameter t) of WDVV solutions (a reduction of the F_4 family), we have identified two curves $h_3^{(1,2)}(t,\alpha)=0$ and one isolated point $(\hat{t},\hat{\alpha})$ for U solutions. The corresponding models all lift to translation-invariant four-particle systems. For the $n{>}3$ exceptional root systems (F_4 and E_n and reductions thereof) and also for some super root system (a reduction of AB_4), only sporadic solutions for particular values of α and without translation invariance occur. We did not discuss the $n{=}2$ systems, because (for $\alpha{=}0$) they have already been investigated thoroughly and are much less restrictive. We have also constructed some solutions for the trigonometric case, but not displayed them here.

Obviously lacking is a geometrical understanding of the 'zoo' of solutions. It would be nice to find *sufficient* conditions on α or on (t,α) for the existence of U solutions in a given F background. This may become more transparent if the requirement of permutation symmetry is dropped, so that further (F,U) solutions can be revealed. Although this requirement is physically reasonable (and this only for the full translation-invariant system), it is mathematically unnatural. Perhaps a superspace reformulation of our models will shed more light on this question.

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Appendix

For an overview of the 'zoo' of irreducible $n{=}3$ solutions, we collect them all, namely (3.7), (4.4), (4.13), (5.13)–(5.15) and (6.2), in the following table,

		coefficients $f_{[\beta]}$ for $[\beta]=\dots$					coefficients $u_{[\beta]}$ for $[\beta]=\dots$				
system	α	$[y{-}y]$	$[y{+}y]$	$[y]$	$[y{\pm}y{\pm}y]$	$[Y]$	$[y{-}y]$	$[y{+}y]$	$[y]$	$[y{\pm}y{\pm}y]$	$[Y]$
$A_3(0)$	$\frac{3}{2}$	1	1	0	0	0	0	1	0	0	0
$A_4(\infty)$	$-\frac{1}{2}$	$-\lambda$	$-\lambda$	0	λ	0	0	0	0	$-\frac{1}{4}$	0
BCD_3	α	$\frac{\alpha}{3}$	$\frac{\alpha}{3}$	$\frac{2\alpha+3}{3}$	0	0	0	0	$\frac{2\alpha}{3}$	0	0
$BCD_3(\frac{5}{9})$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
$F_3^{(1)}$	α	$\frac{\alpha-1}{3}$	$\frac{\alpha-1}{3}$	$\frac{2\alpha+1}{3}$	$\frac{1}{2}$	0	0	0	$\frac{2\alpha-2}{3}$	$\frac{1}{2}$	0
$F_3^{(2)}$	α	$\frac{\alpha-1}{6}$	$\frac{\alpha-1}{6}$	$\frac{2\alpha+1}{3}$	$\frac{\alpha+2}{6}$	0	0	0	0	$\frac{\alpha}{2}$	0
$F_3(\frac{1}{5})$	2	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{5}{3}$	$\frac{1}{6}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0
$AB_3(1)$	$\frac{5}{2}$	1	1	2	0	0	0	1	1	0	-1

Except for the first and last lines, all systems can be 'oxidized' to translation-invariant four-particle models, see (4.15), (4.16) and (5.16)–(5.18).

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