The most general ELKOs in torsional $f(R)$-theories

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Abstract

We study $f(R)$-gravity with torsion in presence of the most general ELKO matter. We check the consistency of the conservation laws with the matter field equations; we discuss some mathematical features of the field equations.

Keywords: $f(R)$-theory, ELKO field, conservation laws
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1 Introduction

In the last decades, General Relativity has been extended toward several directions in order to solve some the problems left open by Einstein's theory in both the ultra-violet and the infra-red regime; among them one of the simplest is given by the so-called $f(R)$-theories: they consist in considering the gravitational Lagrangian to be a general function of the Ricci scalar R. This approach has acquired great interest in cosmology and astrophysics, where $f(R)$-theories turned out to be useful in addressing cosmological and astrophysical puzzles such as dark energy and dark matter: for example, they lead to possible explanations of the accelerated behaviour of the universe as well as the missing matter at galactic scales. General Relativity is also enlarged by considering torsion: that is the Ricci scalar R is written in terms of the most general metric-compatible connection which carries torsional degrees of freedom. This geometry is enlarged enough to permit a corresponding generalization of physics, since having the background endowed with curvature and torsion allows the dynamics to couple energy and spin: this is

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essential, because in the general theory of fields it is well-known that both energy and spin play an equally fundamental role.

The generalization of Einstein's theory obtained by introducing torsion was achieved by Cartan, and in the same way in which Einstein wrote the field equations coupling curvature to energy Sciama and Kibble wrote the field equations coupling torsion to spin; the resulting theory known as Einstein-Cartan-Sciama-Kibble (ECSK) theory is variationally described by a gravitational Lagrangian linear in the Ricci scalar R. Further generalization giving us an ECSK-like theory is the one for which the gravitational Lagrangian is non-linear in the Ricci scalar R: then, Einstein theory is in relationship with the ECSK theory in the same way in which the metric $f(R)$-theory is in relationship with the metric-torsional $f(R)$-theory [1, 4]. Although in this last case torsion is present even without spin, nonetheless the matter fields that best exploit the coupling between torsion and spin density tensor are those having spin, that is the spinor fields; the simplest case is the spin-$\frac{1}{2}$ field, which in the case of Dirac fields it has been studied in [5].

However, recently a new form of spin-$\frac{1}{2}$ spinor field called ELKO has been defined; this form of matter gets its name from the acronym of the German *Eigenspinoren des LadungsKonjugationsOperators* meaning "eigenspinors of the charge conjugation operator" defined as λ for which $\gamma^2\lambda^* = \pm\lambda$ respectively for self- and antiself-conjugated fields [6, 7]: as a consequence of their definition they turned out to be fermions of mass dimension 1 therefore described by scalar-like field equations [8, 9]. That ELKOs are fermions undergoing second-order derivative field equations is a fact that could lead to potential damages for the foundations of their dynamics; however the fundamental problems about acausality and singularities have been solved by showing that actually neither acausal propagation have place nor singularity formation occurs [10, 11, 12]; as a consequence it makes sense to pursue the study of their dynamical properties by employing them in physical applications. In fact they too have gained a lot of interest in cosmology and astrophysics, where models with ELKOs were useful for the solution of cosmological and astrophysical issues such as dark matter and inflation: for instance, within these models there are promising explanations of the exponential expansion during the inflation of the universe and the constant velocity in the rotation curves of galaxies [13, 29]. Quite recently, this theory of ELKOs has been generalized up to its most general structure [30].

In the present paper, we shall build the theory of $f(R)$ gravity with torsion coupled to fields of matter described by ELKOs in their most general form; our approach will face the problem of the consistency of matter field equations with the conservation laws, in the same way it has been done in [5, 30]. As it may be expected, the field equations of the theory will result to be complicated, especially in the problems of the inversion of the energy and spin density tensor that has to be done in order for the field equations to be decomposed; in our discussion, we shall point out all the problems that could be faced. In this scenario, what we see to be one of the best advantages of having both ELKO and $f(R)$-theories is that it is possible to ascribe to two different sources the two different dark components of the universe, whose apparently opposite behaviour, attractive for dark matter and repulsive for dark energy, suggests that they are likely to have independent explanations. By encompassing these two theories into a single one may be fruitful for cosmology and astrophysics.

2 Geometrical Foundations

In this paper, we shall indicate spacetime indices by Latin letters. A metric tensor on the spacetime is denoted by g_{ij} and a connection by $\Gamma_{ij}{}^{h}$; metric-compatible connections are those whose covariant derivative applied on the metric tensor vanishes, where covariant derivatives are defined as

$$\nabla_i V_j = \partial_i V_j - \Gamma_{ij}{}^{h} V_h \tag{2.1}$$

for any generic vector V_k. Given a connection $\Gamma_{ij}{}^{h}$, the associated torsion and Riemann curvature tensors are

$$T_{ij}{}^{h} = \Gamma_{ij}{}^{h} - \Gamma_{ji}{}^{h} \tag{2.2a}$$

$$R^{h}{}_{kij} = \partial_i \Gamma_{jk}{}^{h} - \partial_j \Gamma_{ik}{}^{h} + \Gamma_{ip}{}^{h}\Gamma_{jk}{}^{p} - \Gamma_{jp}{}^{h}\Gamma_{ik}{}^{p} \tag{2.2b}$$

where contractions $T_i = T_{ij}{}^{j}$, $R_{ij} = R^{h}{}_{ihj}$ and $R = R_{ij}g^{ij}$ are called respectively the torsion vector, the Ricci tensor and the Ricci scalar curvature, and the commutator of covariant derivatives is expressed in terms of torsion and curvature as

$$[\nabla_i, \nabla_j]V_k = -T_{ij}{}^{h}\nabla_h V_k - R^{a}{}_{kij}V_a \tag{2.3}$$

for any generic vector V_k. By considering the commutators of commutators in cyclic permutation and employing the Jacobi identities one obtains the Bianchi identities

$$\begin{aligned} \nabla_c T_{ij}{}^{h} - T_{ij}{}^{a}T_{ca}{}^{h} - R^{h}{}_{cij} + \nabla_i T_{jc}{}^{h} - T_{jc}{}^{a}T_{ia}{}^{h} - R^{h}{}_{ijc} + \\ + \nabla_j T_{ci}{}^{h} - T_{ci}{}^{a}T_{ja}{}^{h} - R^{h}{}_{jci} = 0 \end{aligned} \tag{2.4a}$$

$$\begin{aligned} \nabla_c R^{p}{}_{kij} - T_{ij}{}^{a}R^{p}{}_{kca} + \nabla_i R^{p}{}_{kjc} - T_{jc}{}^{a}R^{p}{}_{kia} + \\ + \nabla_j R^{p}{}_{kci} - T_{ci}{}^{a}R^{p}{}_{kja} = 0. \end{aligned} \tag{2.4b}$$

Given a metric tensor g_{ij} every metric g-compatible connection can be decomposed as

$$\Gamma_{ij}{}^{h} = \tilde{\Gamma}_{ij}{}^{h} - K_{ij}{}^{h} \tag{2.5}$$

so that

$$K_{ij}{}^{h} = \frac{1}{2}\left(-T_{ij}{}^{h} + T_{j}{}^{h}{}_{i} - T^{h}{}_{ij}\right) \tag{2.6}$$

where $\tilde{\Gamma}_{ij}{}^{h}$ is the symmetric Levi–Civita connection written in terms of the metric g_{ij} alone and $K_{ij}{}^{h}$ is called the contorsion tensor, whose contraction $K_{i}{}^{ij} = K^{j}$ is such that $K_i = -T_i$; with the contorsion we can decompose the covariant derivative of the full connection as

$$\nabla_i V_j = \tilde{\nabla}_i V_j + K_{ij}{}^h V_h \tag{2.7}$$

where $\tilde{\nabla}$ is the covariant derivative of the Levi–Civita connection and we can decompose the Riemann curvature of the full connection as

$$R^k{}_{ihj} = \tilde{R}^k{}_{ihj} + \tilde{\nabla}_j K_{hi}{}^k - \tilde{\nabla}_h K_{ji}{}^k + K_{ji}{}^p K_{hp}{}^k - K_{hi}{}^p K_{jp}{}^k \tag{2.8}$$

in terms of the Riemann curvature of the Levi–Civita connection $\tilde{R}_{ij}$ identically.

In the next sections we shall consider spinor fields; as it is known, the most suitable variables to describe fermion fields are tetrad and spin-connections. Tetrad fields possess Lorentz indices denoted by Greek letters as well as spacetime indices denoted as usual with Latin letters. They are defined by $e^\mu = e^\mu_i\, dx^i$ together with their dual $e_\mu = e^i_\mu \frac{\partial}{\partial x^i}$, where $e^j_\mu e^\mu_i = \delta^j_i$ and $e^j_\mu e^\nu_j = \delta^\nu_\mu$, and the spin-connections are defined as 1-forms $\omega^\mu{}_\nu = \omega_i{}^\mu{}_\nu\, dx^i$; metric compatibility conditions are assumed and they are defined by the requirement that the covariant derivatives of tetrads and Minkowskian metric vanish, respectively implying that

$$\Gamma_{ij}{}^h = \omega_i{}^\mu{}_\nu e^h_\mu e^\nu_j + e^h_\mu \partial_i e^\mu_j \tag{2.9}$$

and the antisymmetry $\omega_i{}^{\mu\nu} = -\omega_i{}^{\nu\mu}$ of the spin-connection. In terms of the tetrads and the spin-connection, the associated torsion and curvature tensors are

$$T^\mu_{ij} = \partial_i e^\mu_j - \partial_j e^\mu_i + \omega_i{}^\mu{}_\lambda e^\lambda_j - \omega_j{}^\mu{}_\lambda e^\lambda_i \tag{2.10a}$$

$$R_{ij}{}^{\mu\nu} = \partial_i \omega_j{}^{\mu\nu} - \partial_j \omega_i{}^{\mu\nu} + \omega_i{}^\mu{}_\lambda \omega_j{}^{\lambda\nu} - \omega_j{}^\mu{}_\lambda \omega_i{}^{\lambda\nu} \tag{2.10b}$$

whose relationships with the world tensors defined in equations (2.2) are given by the formulas $T_{ij}{}^h := T_{ij}{}^\alpha e^h_\alpha$ and $R^h{}_{kij} = R_{ij}{}^\mu{}_\nu e^h_\mu e^\nu_k$ respectively.

3 Torsional $f(R)$-theories and conservation laws

The torsional $f(R)$-theories can be formulated in the metric-affine approach [1] or in the tetrad-affine one [2]; in the first case, the gravitational dynamical fields are represented by the metric g and a metric compatible connection Γ while in the second case, the gravitational dynamical fields are given by a tetrad field e^μ_i and a spin-connection $\omega_i{}^{\mu\nu}$ on the spacetime. Field equations are derived variationally through a Lagrangian of the kind

$$\mathcal{L} = f(R)\sqrt{|g|}dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - \mathcal{L}_m \tag{3.1}$$

where $f(R)$ is a real function of the Ricci curvature scalar R written in terms of the metric and connection, or equivalently tetrad and spin-connection, and $\mathcal{L}_m$ indicates a suitable matter Lagrangian.

In general the metric-affine formulation is preferred when the spin vanishes, although we have non-vanishing torsion even if the spin density is zero; on the other hand, in case of coupling with spin, for instance spinor fields, the tetrad-affine formulation is more suitable. In this case the corresponding field equations are given by

$$f'(R)R_{\mu\sigma}{}^{\lambda\sigma}e_{\lambda}^{i}-\frac{1}{2}e_{\mu}^{i}f(R)=\Sigma_{\mu}^{i} \tag{3.2a}$$

$$f'(R)\left(T_{ts}^{\alpha}-T_{t\sigma}^{\sigma}e_{s}^{\alpha}+T_{s\sigma}^{\sigma}e_{t}^{\alpha}\right)=\frac{\partial f'(R)}{\partial x^{t}}e_{s}^{\alpha}-\frac{\partial f'(R)}{\partial x^{s}}e_{t}^{\alpha}+S_{ts}^{\alpha} \tag{3.2b}$$

where $\Sigma^{i}{}_{\mu}:=-\frac{1}{2e}\frac{\partial\mathcal{L}_m}{\partial e_{i}^{\mu}}$ and $S_{ts}{}^{\alpha}:=\frac{1}{2e}\frac{\partial\mathcal{L}_m}{\partial\omega_{i}{}^{\mu\nu}}e_{t}^{\mu}e_{s}^{\nu}e_{i}^{\alpha}$ are the stress-energy and spin density tensors of the matter field. From equation (3.2b) it is seen that, there are two sources of torsion given by the spin density $S_{ts}{}^{\alpha}$ and the non-linearity of the gravitational Lagrangian (for the derivation of the field equations and discussion about special properties of particular cases we refer to the works [1, 4]).

It is then possible to write the field equations (3.2) in their equivalent spacetime form as

$$f'R_{ij}-\frac{1}{2}g_{ij}f=\Sigma_{ij} \tag{3.3a}$$

$$f'\left(T_{ijh}+T_{j}g_{hi}-T_{i}g_{jh}\right)+g_{hi}\frac{\partial f'}{\partial x^{j}}-g_{jh}\frac{\partial f'}{\partial x^{i}}=S_{ijh} \tag{3.3b}$$

where $R^{i}{}_{j}:=R_{\mu\sigma}{}^{\lambda\sigma}e_{\lambda}^{i}e_{j}^{\mu}$, $\Sigma^{i}{}_{j}:=\Sigma_{\mu}^{i}e_{j}^{\mu}$, $T_{ij}{}^{h}:=T_{ij}{}^{\alpha}e_{\alpha}^{h}$, $S_{ij}{}^{h}:=S_{ij}{}^{\alpha}e_{\alpha}^{h}$, which give the Ricci curvature tensor and torsion tensor in terms of the energy and spin densities. Notice that in equations (3.3a) one should distinguish the order of the indices since in general R_{ij} and Σ_{ij} are not symmetric.

Making use of the identities (2.4) it is possible to work out the field equations (3.3) to get the conservation laws of the theory

$$\nabla_{a}\Sigma^{ai}+T_{a}\Sigma^{ai}-\Sigma_{ca}T^{ica}-\frac{1}{2}S_{spq}R^{spqi}=0 \tag{3.4a}$$

$$\nabla_{h}S^{ijh}+T_{h}S^{ijh}+\Sigma^{ij}-\Sigma^{ji}=0 \tag{3.4b}$$

under which the stress-energy and spin density tensors of the matter fields must undergo once the matter field equations are assigned. Notice that the antisymmetric part of the energy tensor is the source of the spin dynamics whereas the spin-curvature coupling is the source of the energy dynamics, where by source we mean the source of matter the make the divergence fail to vanish (for the derivation of these conservation laws see [5]).

In the next section we shall investigate in detail the coupling to the ELKOs.

4 Coupling to most general ELKOs and consistency of field equations

Let us consider $f(R)$-theories with torsion coupled to spinor fields, in the simplest spin-$\frac{1}{2}$ spin content; in [5] we have already studied the Dirac field, here we would like to study the ELKO field. Because ELKO fields have the same spin content of the Dirac field, that is they have the transformation law of any spin-$\frac{1}{2}$ particle, their spinorial covariant derivatives are defined in the same way by

$$D_i\lambda = \partial_i\lambda + \omega_i{}^{\mu\nu} S_{\mu\nu}\lambda \tag{4.1}$$

with commutator of the derivatives given by

$$[D_i, D_j]\lambda = -T_{ij}{}^h D_h\lambda + R_{ij}{}^{\mu\nu} S_{\mu\nu}\lambda \tag{4.2}$$

where $S_{\mu\nu} = \frac{1}{8}[\gamma_\mu, \gamma_\nu]$ and the gamma matrices γ^μ satisfy the anticommutation relationships given by the Clifford algebra, and we define $\gamma^i = \gamma^\mu e^i_\mu$ as usual.

The most general Lagrangian for ELKOs in expressed as

$$\mathcal{L}_m = \left(D_i \overset{\neg}{\lambda} (g^{ij} + aS^{ij}) D_j\lambda - m^2 \overset{\neg}{\lambda} \lambda\right) e \; dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \tag{4.3}$$

in terms of the coefficient a and where m is the mass of the ELKO. By varying (4.3) with respect to the matter field we obtain the matter field equations

$$\left(D^2\lambda + T^i D_i\lambda\right) + a\left(S^{ij} D_i D_j\lambda + T_k S^{kj} D_j\lambda\right) + m^2\lambda = 0 \tag{4.4}$$

in terms of the mass m of the matter field itself; by varying with respect to tetrads and spin-connection we get field equations (3.2) where the stress-energy and spin density tensors are

$$\begin{aligned}\Sigma_{kj} = \tfrac{1}{2}\left(D_j \overset{\neg}{\lambda} D_k\lambda + D_k \overset{\neg}{\lambda} D_j\lambda - g_{jk} D_i \overset{\neg}{\lambda} D^i\lambda\right) + \\ +\tfrac{a}{2}\left(D_j \overset{\neg}{\lambda} S_{ka} D^a\lambda + D^a \overset{\neg}{\lambda} S_{ak} D_j\lambda - g_{jk} D_i \overset{\neg}{\lambda} S^{ia} D_a\lambda\right) + \tfrac{1}{2} g_{jk} m^2 \overset{\neg}{\lambda} \lambda\end{aligned} \tag{4.5a}$$

$$S_{kij} = \left(D_j \overset{\neg}{\lambda} S_{ki}\lambda - \overset{\neg}{\lambda} S_{ki} D_j\lambda\right) + a\left(D^p \overset{\neg}{\lambda} S_{pj} S_{ki}\lambda - \overset{\neg}{\lambda} S_{ki} S_{jp} D^p\lambda\right). \tag{4.5b}$$

We shall now show that the matter field equations (4.4) are consistent with the conservation laws (3.4); to see this, we calculate the divergences of the conserved quantities

$$\begin{aligned}D_k\Sigma^{kj} = \tfrac{1}{2}(D^k D^j \overset{\neg}{\lambda} D_k\lambda + D_k \overset{\neg}{\lambda} D^k D^j\lambda) + \\ +\tfrac{a}{2}(D^k D^j \overset{\neg}{\lambda} S_{ka} D^a\lambda + D^a \overset{\neg}{\lambda} S_{ak} D^k D^j\lambda) + \\ +\tfrac{1}{2}(D^j \overset{\neg}{\lambda} D^2\lambda + D^2 \overset{\neg}{\lambda} D^j\lambda) + \\ +\tfrac{a}{2}(D^j \overset{\neg}{\lambda} S^{ka} D_k D_a\lambda + D_k D_a \overset{\neg}{\lambda} S^{ak} D^j\lambda) + \\ +D^j(\tfrac{1}{2}m^2 \overset{\neg}{\lambda} \lambda - \tfrac{1}{2} D_i \overset{\neg}{\lambda} D^i\lambda - \tfrac{a}{2} D_i \overset{\neg}{\lambda} S^{ia} D_a\lambda)\end{aligned} \tag{4.6a}$$

$$
\begin{aligned}
D_j S^{kij} = {} & (D^2 \bar{\lambda} S^{ki}\lambda - \bar{\lambda} S^{ki} D^2\lambda) + \\
& + a(D_j D_p \bar{\lambda} S^{pj} S^{ki}\lambda - \bar{\lambda} S^{ki} S^{jp} D_j D_p \lambda) + \\
& + (D^j \bar{\lambda} S^{ki} D_j \lambda - D_j \bar{\lambda} S^{ki} D^j \lambda) + \\
& + a(D_p \bar{\lambda} S^{pj} S^{ki} D_j \lambda - D_j \bar{\lambda} S^{ki} S^{jp} D_p \lambda).
\end{aligned} \tag{4.6b}
$$

By employing the matter field equations (4.4), equations (4.6) simplify to

$$
\begin{aligned}
D_k \Sigma^{kj} = {} & \tfrac{1}{2}(D^k D^j \bar{\lambda} D_k \lambda + D_k \bar{\lambda} D^k D^j \lambda) + \\
& + \tfrac{a}{2}(D^k D^j \bar{\lambda} S_{ka} D^a \lambda + D^a \bar{\lambda} S_{ak} D^k D^j \lambda) + \\
& + D^j(-\tfrac{1}{2} D_i \bar{\lambda} D^i \lambda - \tfrac{a}{2} D_i \bar{\lambda} S^{ia} D_a \lambda) - \\
& - \tfrac{1}{2} T_k (D^j \bar{\lambda} D^k \lambda + D^k \bar{\lambda} D^j \lambda) - \tfrac{a}{2} T_k (D^j \bar{\lambda} S^{ka} D_a \lambda + D_a \bar{\lambda} S^{ak} D^j \lambda)
\end{aligned} \tag{4.7a}
$$

$$
\begin{aligned}
D_j S^{kij} = {} & a(D_p \bar{\lambda} S^{pj} S^{ki} D_j \lambda - D_j \bar{\lambda} S^{ki} S^{jp} D_p \lambda) + \\
& + T_j(\bar{\lambda} S^{ki} D^j \lambda - D^j \bar{\lambda} S^{ki} \lambda) + a T_j(\bar{\lambda} S^{ki} S^{jb} D_b \lambda - D_b \bar{\lambda} S^{bj} S^{ki} \lambda).
\end{aligned} \tag{4.7b}
$$

We combine together the first three lines of equation (4.7a) and the first line of equation (4.7b) to get

$$
\begin{aligned}
D_k \Sigma^{kj} = {} & \tfrac{1}{2}([D^k, D^j] \bar{\lambda} D_k \lambda + D_k \bar{\lambda} [D^k, D^j]\lambda) + \\
& + \tfrac{a}{2}([D^k, D^j] \bar{\lambda} S_{ka} D^a \lambda + D^a \bar{\lambda} S_{ak} [D^k, D^j]\lambda) - \\
& - \tfrac{1}{2} T_k (D^j \bar{\lambda} D^k \lambda + D^k \bar{\lambda} D^j \lambda) - \tfrac{a}{2} T_k (D^j \bar{\lambda} S^{ka} D_a \lambda + D_a \bar{\lambda} S^{ak} D^j \lambda)
\end{aligned} \tag{4.8a}
$$

$$
\begin{aligned}
D_j S^{kij} = {} & a(D_p \bar{\lambda} [S^{pj}, S^{ki}] D_j \lambda) + \\
& + T_j(\bar{\lambda} S^{ki} D^j \lambda - D^j \bar{\lambda} S^{ki} \lambda) + a T_j(\bar{\lambda} S^{ki} S^{jb} D_b \lambda - D_b \bar{\lambda} S^{bj} S^{ki} \lambda).
\end{aligned} \tag{4.8b}
$$

Now by employing the commutators of spinorial covariant derivatives D_i and the commutator of the generators S_{ij} we obtain

$$
\begin{aligned}
D_k \Sigma^{kj} = {} & -T^{kjh} \tfrac{1}{2}(D_h \bar{\lambda} D_k \lambda + D_k \bar{\lambda} D_h \lambda) - \\
& - T^{kjh} \tfrac{a}{2}(D_h \bar{\lambda} S_{ka} D^a \lambda + D^a \bar{\lambda} S_{ak} D_h \lambda) - \\
& - R^{abkj} \tfrac{1}{2}(\bar{\lambda} S_{ab} D_k \lambda - D_k \bar{\lambda} S_{ab} \lambda) - \\
& - R^{abkj} \tfrac{a}{2}(\bar{\lambda} S_{ab} S_{kp} D^p \lambda - D^p \bar{\lambda} S_{pk} S_{ab} \lambda) - \\
& - \tfrac{1}{2} T_k (D^j \bar{\lambda} D^k \lambda + D^k \bar{\lambda} D^j \lambda) - \tfrac{a}{2} T_k (D^j \bar{\lambda} S^{ka} D_a \lambda + D_a \bar{\lambda} S^{ak} D^j \lambda)
\end{aligned} \tag{4.9a}
$$

$$
\begin{aligned}
D_j S^{kij} = {} & \tfrac{a}{2}(D^k \bar{\lambda} S^{ij} D_j \lambda + D_j \bar{\lambda} S^{ji} D^k \lambda - D^i \bar{\lambda} S^{kj} D_j \lambda - D_j \bar{\lambda} S^{jk} D^i \lambda) + \\
& + T_j(\bar{\lambda} S^{ki} D^j \lambda - D^j \bar{\lambda} S^{ki} \lambda) + a T_j(\bar{\lambda} S^{ki} S^{jb} D_b \lambda - D_b \bar{\lambda} S^{bj} S^{ki} \lambda).
\end{aligned} \tag{4.9b}
$$

According to the definition of the stress-energy and spin density tensors (4.5) we finally obtain

$$D_k\Sigma^{kj} = -T^{kjh}\Sigma_{kh} + \frac{1}{2}R^{abkj}S_{abk} - T_k\Sigma^{kj} \tag{4.10a}$$

$$D_jS^{kij} = (\Sigma^{ik} - \Sigma^{ki}) - T_jS^{kij} \tag{4.10b}$$

showing that the matter field equations are consistent with the conservation laws. So the system of field equations given by the matter field equations (4.4) and the field equations (3.3) with conserved quantities (4.5) describe the most general ELKOs in torsional $f(R)$-theories.

Now in dealing with equations (3.3) and (4.4), the standard procedure consists in decomposing them in torsionless terms and torsional contributions: this procedure is necessary in studying mathematical aspects such as the Cauchy, causality and singularity problems considered in [31, 32, 33, 10, 11, 12]. More in detail, the steps to follow are: firstly, obtaining from the trace of the Einstein-like equations (3.3a), the expression of the Ricci scalar R as a function of metric and matter fields with their derivatives; secondly, inserting the obtained relationship in the equations (3.3b) getting an explicit representation of the torsion tensor, again in terms of metric and matter fields with their derivatives; finally, replacing the expression for the torsion in equations (3.3a) by making use of equations (2.5), (2.6) and (2.8). By proceeding in this way, the theory can be reduced to an Einstein-like theory where the Einstein-like and matter field equations give the dynamics for the metric tensor and the matter fields, while the role of the torsion-spin equation (3.3b) is to define the torsion tensor as an algebraic function of the metric and matter fields with their derivatives.

In the case of ELKOs this procedure works for torsional $f(R) = R$: in the case $a = 0$ this decomposition is given explicitly in [12], while for the most general model $a \neq 0$ the decomposition is not given explicitly, although the fact that it is always possible to achieve is discussed through a constructive approach in [30]; however for generic torsional $f(R)$-theories this procedure does not work. This situation is due to the fact that both the stress-energy and spin density tensors involve the covariant derivative of the spinors: as a consequence, when replacing the scalar curvature as function of the matter trace in the torsion-spin equation, we no longer have an algebraic but a differential equation for torsion; in other words, we now have a dynamical equation for torsion which is then a genuinely dynamical variable. This is a feature that clearly distinguishes ELKOs from other matter fields, such as the Dirac field, scalar field, electrodynamic field or perfect fluid, within torsional $f(R)$ gravity.

5 Conclusions

In this paper, we have considered $f(R)$-theories of gravitation with Ricci scalar written in terms of connections having both metric and torsional degrees of freedom, in the case in which the matter field was described by ELKOs in their most general dynamics: we have seen that the general conservation laws obtained in [5] are satisfied for the stress-energy and spin density tensors of ELKO once ELKO matter field equations are used; we have discussed general differences between ELKO and other matter fields in torsional $f(R)$-theories.

It is known that both ELKO and $f(R)$-theories are very promising in explaining many of the open problems of cosmology and astrophysics, and ELKO in $f(R)$-theories of gravitation could give two different sources for the two complementary dark components of the universe; it is our conviction that further studies will reveal very intriguing consequences.

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