Group-Theoretical Classification of BPS and Possibly Protected States in D=4 Conformal Supersymmetry

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Abstract

We use earlier results on the classification of the positive energy unitary irreducible representations of the N-extended D=4 conformal superalgebras su(2,2/N). These results are applied to the reduction of supersymmetries and to the classification of BPS and possibly protected states.

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1 Introduction

Recently, superconformal field theories in various dimensions are attracting more interest, especially in view of their applications in string theory. Thus, the classification of the UIRs of the conformal superalgebras is of great importance. For some time such classification was known only for the $D = 4$ superconformal algebras $su(2,2/1)$ [1] and $su(2,2/N)$ for arbitrary N [2], (see also [3,4]). Then, more progress was made with the classification for $D = 3$ (for even N), $D = 5$, and $D = 6$ (for $N = 1, 2$) in [5] (some results being conjectural), then for the $D = 6$ case (for arbitrary N) was finalized in [6]. Finally, the cases $D = 9, 10, 11$ were treated by finding the UIRs of $osp(1/2n)$, [7].

Once we know the UIRs of a (super-)algebra the next question is to find their characters, since these give the spectrum which is important for the applications. This problem was addressed in [8] for the UIRs of $D = 4$ conformal superalgebras $su(2,2/N)$. From the mathematical point of view this question is clear only for representations with conformal dimension above the unitarity threshold viewed as irreps of the corresponding complex superalgebra $sl(4/N)$. But for $su(2,2/N)$ even the UIRs above the unitarity threshold are truncated for small values of spin and isospin. More than that, in the applications the most important role is played by the representations with "quantized" conformal dimensions at the unitarity threshold and at discrete points below. In the quantum field or string theory framework some of these correspond to operators with "protected" scaling dimension and therefore imply "non-renormalization theorems" at the quantum level, cf., e.g., [9,10]. Especially important in this context are the so-called BPS states, cf., [10–17].

Finding the characters involves also deeper knowledge of the structure of the UIRs. Fortunately, most of the needed information is contained in [2–4,18]. We use also more explicit results on the decompositions of long superfields as they descend to the unitarity threshold [8].

In the present paper the above results are applied first to the reduction of supersymmetries in Section 3, and then to the classification of BPS and possibly protected states in Section 4.

2 Preliminaries

2.1 Representations of D=4 conformal supersymmetry

The conformal superalgebras in $D = 4$ are $\mathcal{G} = su(2,2/N)$. The even subalgebra of $\mathcal{G}$ is the algebra $\mathcal{G}_0 = su(2,2) \oplus u(1) \oplus su(N)$. We label their physically relevant representations of $\mathcal{G}$ by the signature:

$$\chi = [\, d\, ;\, j_1\, ,\, j_2\, ;\, z\, ;\, r_1\, , \ldots , r_{N-1}\,] \tag{2.1}$$

where d is the conformal weight, j_1, j_2 are non-negative (half-)integers which are Dynkin labels of the finite-dimensional irreps of the $D = 4$ Lorentz subalgebra $so(3,1)$ of dimension $(2j_1 + 1)(2j_2 + 1)$, z represents the $u(1)$ subalgebra which is central for $\mathcal{G}_0$ (and is central for $\mathcal{G}$ itself when $N = 4$), and $r_1, \ldots, r_{N-1}$ are non-negative integers which are Dynkin labels of the finite-dimensional irreps of the internal (or R) symmetry algebra $su(N)$.

We recall the root system of the complexification $\mathcal{G}^{\mathbb{C}}$ of $\mathcal{G}$ (as used in [4]). The positive root system Δ^+ is comprised of α_{ij}, $1 \leq i < j \leq 4 + N$. The even positive root system $\Delta^+_{\bar{0}}$ is comprised of α_{ij}, with $i, j \leq 4$ and $i, j \geq 5$; the odd positive root system $\Delta^+_{\bar{1}}$ is comprised of α_{ij}, with $i \leq 4, j \geq 5$. The simple roots are chosen as in (2.4) of [4]:

$$\gamma_1 = \alpha_{12}\ ,\ \gamma_2 = \alpha_{34}\ ,\ \gamma_3 = \alpha_{25}\ ,\ \gamma_4 = \alpha_{4,4+N}\ ,\ \ \gamma_k = \alpha_{k,k+1}\ ,\ \ 5 \leq k \leq 3 + N. \tag{2.2}$$

Thus, the Dynkin diagram is:

$$\underset{1}{\bigcirc} --- \underset{3}{\bigotimes} --- \underset{5}{\bigcirc} --- \cdots --- \underset{3+N}{\bigcirc} --- \underset{4}{\bigotimes} --- \underset{2}{\bigcirc} \tag{2.3}$$

This is a non-distinguished simple root system with two odd simple roots [20].

Sometimes we shall use another way of writing the signature related to the above enumeration of simple roots, cf. [4] and (1.16) of [8]:

$$\chi = (2j_1\, ;\, (\Lambda, \gamma_3)\, ;\, r_1, \ldots, r_{N-1}\, ;\, (\Lambda, \gamma_4)\, ;\, 2j_2)\ , \tag{2.4}$$

(where (Λ, γ_3), (Λ, γ_4) are definite linear combinations of all quantum numbers), or even giving only the Lorentz and $SU(N)$ signatures:

$$\chi_N = \{\, 2j_1\, ;\, r_1, \ldots, r_{N-1}\, ;\, 2j_2\, \}\ . \tag{2.5}$$

Remark: We recall that the group-theoretical approach to $D = 4$ conformal supersymmetry developed in [2–4] involves two related constructions - on function spaces and as Verma modules. The first realization employs the explicit construction of induced representations of $\mathcal{G}$ (and of the corresponding supergroup $G = SU(2,2/N)$) in spaces of functions (superfields) over superspace which are called elementary representations (ER). The UIRs of $\mathcal{G}$ are realized as irreducible components of ERs, and then they coincide with the usually used superfields in indexless notation. The Verma module realization is also very useful as it provides simpler and more intuitive picture for the relation between reducible ERs, for the construction of the irreps, in particular, of the UIRs. For the latter the main tool is an adaptation of the Shapovalov form [19] to the Verma modules [2, 18]. Here we shall need only the second - Verma module - construction. ♢

We use lowest weight Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where the lowest weight Λ is characterized by its values on the Cartan subalgebra $\mathcal{H}$ and is in 1-to-1 correspondence with the signature χ. If a Verma module V^Λ is irreducible then it gives the lowest weight irrep L_Λ with the same weight. If a Verma module V^Λ is reducible then it contains a maximal invariant submodule I^Λ and the lowest weight irrep L_Λ with the same weight is given by factorization: $L_\Lambda = V^\Lambda / I^\Lambda$ [21]. The reducibility conditions were given by Kac [21].

There are submodules which are generated by the singular vectors related to the even simple roots $\gamma_1, \gamma_2, \gamma_5, \ldots, \gamma_{N+3}$ [4]. These generate an even invariant submodule I_c^Λ present in all Verma modules that we consider and which must be factored out. Thus, instead of V^Λ we shall consider the factor-modules:

$$\tilde{V}^\Lambda = V^\Lambda / I_c^\Lambda \tag{2.6}$$

The Verma module reducibility conditions for the $4N$ odd positive roots of $\mathcal{G}^{\mathbb{C}}$ were derived in [3, 4] adapting the results of Kac [21]:

$$\begin{aligned} d &= d^1_{Nk} - z\delta_{N4} \\ d^1_{Nk} &\equiv 4 - 2k + 2j_2 + z + 2m_k - 2m/N \end{aligned} \tag{2.7a}$$

$$\begin{aligned} d &= d^2_{Nk} - z\delta_{N4} \\ d^2_{Nk} &\equiv 2 - 2k - 2j_2 + z + 2m_k - 2m/N \end{aligned} \tag{2.7b}$$

$$d = d^3_{Nk} + z\delta_{N4} \qquad (2.7c)$$
$$d^3_{Nk} \equiv 2 + 2k - 2N + 2j_1 - z - 2m_k + 2m/N$$

$$d = d^4_{Nk} + z\delta_{N4} \qquad (2.7d)$$
$$d^4_{Nk} \equiv 2k - 2N - 2j_1 - z - 2m_k + 2m/N$$

where in all four cases of (2.7) $k = 1, \ldots, N$, $m_N \equiv 0$, and

$$m_k \equiv \sum_{i=k}^{N-1} r_i \ , \quad m \equiv \sum_{k=1}^{N-1} m_k = \sum_{k=1}^{N-1} k r_k \qquad (2.8)$$

Note that we shall use also the quantity m^* which is conjugate to m :

$$m^* \equiv \sum_{k=1}^{N-1} k r_{N-k} = \sum_{k=1}^{N-1} (N-k) r_k \ , \qquad (2.9)$$
$$m + m^* = N m_1 \ . \qquad (2.10)$$

We need the result of [2] (cf. part (i) of the Theorem there) that the following is the complete list of lowest weight (positive energy) UIRs of $su(2,2/N)$:

$$d \geq d_{\max} = \max(d^1_{N1}, d^3_{NN}) \ , \qquad (2.11a)$$
$$d = d^4_{NN} \geq d^1_{N1} \ , \quad j_1 = 0 \ , \qquad (2.11b)$$
$$d = d^2_{N1} \geq d^3_{NN} \ , \quad j_2 = 0 \ , \qquad (2.11c)$$
$$d = d^2_{N1} = d^4_{NN} \ , \quad j_1 = j_2 = 0 \ , \qquad (2.11d)$$

where $d_{\max}$ is the threshold of the continuous unitary spectrum. Note that in case (d) we have $d = m_1$, $z = 2m/N - m_1$, and that it is trivial for $N = 1$.

Next we note that if $d > d_{\max}$ the factorized Verma modules are irreducible and coincide with the UIRs L_Λ. These UIRs are called **long** in the modern literature, cf., e.g., [10, 17, 22–26]. Analogously, we shall use for the cases when $d = d_{\max}$, i.e., (2.11a), the terminology of **semi-short** UIRs, introduced in [10,22], while the cases (2.11b,c,d) are also called **short** UIRs, cf., e.g., [10, 17, 23–26].

Next consider in more detail the UIRs at the four distinguished reducibility points determining the UIRs list above: d^1_{N1}, d^2_{N1}, d^3_{NN}, d^4_{NN}. The above reducibilities occur for the following odd roots, resp.:

$$\alpha_{3,4+N} = \gamma_2 + \gamma_4 \ , \quad \alpha_{4,4+N} = \gamma_4 \ , \quad \alpha_{15} = \gamma_1 + \gamma_3 \ , \quad \alpha_{25} = \gamma_3 \ . \qquad (2.12)$$

We note a partial ordering of these four points:

$$d^1_{N1} \ > \ d^2_{N1} \ , \qquad d^3_{NN} \ > \ d^4_{NN} \ . \tag{2.13}$$

Due to this ordering *at most two* of these four points may coincide.

First we consider the situations in which *no two* of the distinguished four points coincide. There are four such situations:

$$\begin{aligned}
&\mathbf{a}: && d = d_{\max} = d^1_{N1} = d^a \equiv 2 + 2j_2 + z + 2m_1 - 2m/N > d^3_{NN} && (2.14a)\\
&\mathbf{b}: && d \ = \ d^2_{N1} = d^b \equiv z - 2j_2 + 2m_1 - 2m/N > d^3_{NN} \ , \quad j_2 = 0 && (2.14b)\\
&\mathbf{c}: && d \ = \ d_{\max} \ = \ d^3_{NN} \ = \ d^c \ \equiv \ 2 + 2j_1 - z + 2m/N > d^1_{N1} && (2.14c)\\
&\mathbf{d}: && d \ = \ d^4_{NN} = d^d \equiv 2m/N - 2j_1 - z > d^1_{N1} \ , \quad j_1 = 0 && (2.14d)
\end{aligned}$$

where for future use we have introduced notations d^a, d^b, d^c, d^d, the definitions including also the corresponding inequality.

We shall call these cases **single-reducibility-condition (SRC)** Verma modules or UIRs, depending on the context. In addition, as already stated, we use for the cases when $d = d_{\max}$, i.e., (2.14a,c), the terminology of semi-short UIRs, while the cases (2.14b,d), are also called short UIRs.

The factorized Verma modules $\tilde{V}^\Lambda$ with the unitary signatures from (2.14) have only one invariant odd submodule which has to be factorized in order to obtain the UIRs. These odd embeddings and factorizations are given as follows:

$$\tilde{V}^\Lambda \ \to \ \tilde{V}^{\Lambda+\beta} \ , \qquad L_\Lambda \ = \ \tilde{V}^\Lambda / I^\beta \ , \tag{2.15}$$

where we use the convention [3] that arrows point to the oddly embedded module, and we give only the cases for β that we shall use later:

$$\begin{aligned}
\beta \quad &= \ \alpha_{3,4+N} \ , \quad \text{for (2.14}a\text{)}, \quad j_2 > 0, && (2.16a)\\
&= \ \alpha_{3,4+N} + \alpha_{4,4+N} \ , \quad \text{for (2.14}a\text{)}, \quad j_2 = 0, && (2.16b)\\
&= \ \alpha_{15} \ , \quad \text{for (2.14}c\text{)}, \quad j_1 > 0, && (2.16c)\\
&= \ \alpha_{15} + \alpha_{25} \ , \quad \text{for (2.14}c\text{)}, \quad j_1 = 0 && (2.16d)
\end{aligned}$$

We consider now the four situations in which *two* distinguished points coincide:

$$\begin{aligned}
&\mathbf{ac}: && d \ = \ d_{\max} \ = \ d^{ac} \ \equiv \ 2 + j_1 + j_2 + m_1 = d^1_{N1} = d^3_{NN} && (2.17a)\\
&\mathbf{ad}: && d \ = \ d^{ac} \ \equiv \ = \ 1 + j_2 + m_1 \ = \ d^1_{N1} = d^4_{NN} \ , \quad j_1 = 0 && (2.17b)\\
&\mathbf{bc}: && d \ = \ d^{bc} \ \equiv \ = \ 1 + j_1 + m_1 \ = \ d^2_{N1} = d^3_{NN} \ , \quad j_2 = 0 && (2.17c)\\
&\mathbf{bd}: && d \ = \ d^{bd} \ \equiv \ = \ m_1 \ = \ d^2_{N1} = d^4_{NN} \ , \quad j_1 = j_2 = 0 && (2.17d)
\end{aligned}$$

We shall call these **double-reducibility-condition (DRC)** Verma modules or UIRs. The cases in (2.17a) are semi-short UIR, while the other cases are short.

The odd embedding diagrams and factorizations for the DRC modules are [3]:

$$\begin{array}{ccc} \tilde{V}^{\Lambda+\beta'} & & \\ \uparrow & & \\ \tilde{V}^{\Lambda} & \rightarrow & \tilde{V}^{\Lambda+\beta} \end{array} \qquad L_\Lambda = \tilde{V}^\Lambda / I^{\beta,\beta'} \ , \quad I^{\beta,\beta'} = I^\beta \cup I^{\beta'} \tag{2.18}$$

and we give only the cases for β, β' to be used later:

$$\begin{aligned} (\beta,\beta') &= (\alpha_{15}, \alpha_{3,4+N}), \quad \text{for (2.17}a), \quad j_1 j_2 > 0 && \text{(2.19a)} \\ &= (\alpha_{15}, \alpha_{3,4+N} + \alpha_{3,4+N}), \quad \text{for (2.17}b), \quad j_1 > 0,\ j_2 = 0 && \text{(2.19b)} \\ &= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N}), \quad \text{for (2.17}c), \quad j_1 = 0,\ j_2 > 0 && \text{(2.19c)} \\ &= (\alpha_{15} + \alpha_{25}, \alpha_{3,4+N} + \alpha_{3,4+N}), \quad \text{for (2.17}d), \quad j_1 = j_2 = 0 && \text{(2.19d)} \end{aligned}$$

2.2 Decompositions of long superfields

First we present the results on decompositions of long irreps as they descend to the unitarity threshold [8].

In the SRC cases we have established that for $d = d_{\max}$ there hold the two-term decompositions:

$$\left(\hat{L}_{\rm long}\right)_{|d=d_{\max}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta} \ , \qquad r_1 + r_{N-1} > 0 \ , \tag{2.20}$$

where Λ is a semi-short SRC designated as type **a** (then $r_1 > 0$) or **c** (then $r_{N-1} > 0$) and there are four possibilities for β depending on the values of j_1, j_2 as given in (2.16). In cases (2.16a,c) also the second UIR on the RHS of (2.20) is semi-short, while in cases (2.16b,d) the second UIR on the RHS of (2.20) is short of type **b**, **d**, resp.

In the DRC cases we have established that for $N > 1$ and $d = d_{\max} = d^{ac}$ hold the four-term decompositions:

$$\left(\hat{L}_{\rm long}\right)_{|d=d^{ac}} = \hat{L}_\Lambda \oplus \hat{L}_{\Lambda+\beta} \oplus \hat{L}_{\Lambda+\beta'} \oplus \hat{L}_{\Lambda+\beta+\beta'} \ , \qquad r_1 r_{N-1} > 0 \ , \tag{2.21}$$

where Λ is the semi-short DRC designated as type **ac** and there are four possibilities for β, β' depending on the values of j_1, j_2 as given in (2.19a,b,c,d). Note that in case (2.19a) all UIRs in the RHS of (2.21) are semi-short. In the case (2.19b) the first two UIRs in the RHS of (2.21) are semi-short, the last two UIRs are short of type **bc**. In the case (2.19c) the first two UIRs in the RHS of (2.21) are semi-short, the last two UIRs are short of type **ad**. In the case (2.19d) the first UIR in the RHS of (2.21) is semi-short, the other three UIRs are short of types **bc**, **ad**, **bd**, resp.

Next we note that for $N = 1$ all SRC cases enter some decomposition, while no DRC cases enter any decomposition. For $N > 1$ the situation is more diverse and so we give the list of UIRs that do **not** enter decompositions together with the restrictions on the R-symmetry quantum numbers:

- **SRC cases:**

•**a** $d = d^a \ , \qquad r_1 = 0 \ .$

•**b** $d = d^b \ , \qquad r_1 \leq 2 \ .$

•**c** $d = d^c \ , \qquad r_{N-1} = 0 \ .$

•**d** $d = d^d \ , \qquad r_{N-1} \leq 2 \ .$

- **DRC cases:**

all non-trivial cases for $N = 1$, while for $N > 1$ the list is:

•**ac** $d = d^{ac} \ , \qquad r_1 r_{N-1} = 0 \ .$

•**ad** $d = d^{ad} \ , \qquad r_{N-1} \leq 2 \ , \quad r_1 = 0$ for $N > 2$.

•**bc** $d = d^{bc} \ , \qquad r_1 \leq 2 \ , \quad r_{N-1} = 0$ for $N > 2$.

•**bd** $d = d^{bd} \ , \qquad r_1, r_{N-1} \leq 2$ for $N > 2$, $\quad 1 \leq r_1 \leq 4$ for $N = 2$.

For further use we recall that $d > d_{\max}$ the factorized Verma modules are irreducible and coincide with the UIRs L_Λ [2–4]. These UIRs are called **long** in the modern literature, cf., e.g., [10, 17, 22–26]. Analogously, we shall use for the cases when $d = d_{\max}$, the terminology of **semi-short** UIRs, cf. [10,22], while the reducible cases when $d < d_{\max}$ are called **short** UIRs, cf., e.g., [10, 17, 23–26].

3 Reduction of supersymmetry in short and semi-short UIRs

Our first task in this paper is to present explicitly the reduction of the supersymmetries in the irreducible UIRs. This means to give explicitly the number κ of odd generators which are eliminated from the corresponding lowest weight module, (or equivalently, the number of super-derivatives that annihilate the corresponding superfield).

3.1 R-symmetry scalars

We start with the simpler cases of R-symmetry scalars when $r_i = 0$ for all i, which means also that $m_1 = m = m^* = 0$. These cases are valid also for $N = 1$. More explicitly:

$$\bullet\ \mathbf{a} \qquad \begin{aligned} d &= d^a_{|m=0} = 2 + 2j_2 + z\ , \quad j_1 \text{ arbitrary,} \\ \kappa &= N + (1-N)\delta_{j_2,0}\ , \quad \text{or casewise :} \\ \kappa &= N, \quad j_2 > 0, \\ \kappa &= 1, \quad j_2 = 0 \end{aligned} \tag{3.1}$$

Here, κ is the number of anti-chiral generators $X^+_{3,4+k}$, $k = 1, \ldots, \kappa$, that are eliminated. Thus, in the cases when $\kappa = N$ the semi-short UIRs may be called semi-chiral since they lack half of the anti-chiral generators.

$$\bullet\ \mathbf{b} \qquad \begin{aligned} d &= d^b_{|m=0} = z\ , \quad j_1 \text{ arbitrary,} \quad j_2 = 0, \\ \kappa &= 2N \end{aligned} \tag{3.2}$$

These short UIRs may be called chiral since they lack all anti-chiral generators $X^+_{3,4+k}\ ,\ X^+_{4,4+k}\ ,\ k = 1, \ldots, N$.

$$\bullet\ \mathbf{c} \qquad \begin{aligned} d &= d^c_{|m=0} = 2 + 2j_1 - z\ , \quad j_2 \text{ arbitrary,} \\ \kappa &= N + (1-N)\delta_{j_1,0}\ , \quad \text{or casewise :} \\ \kappa &= N, \quad j_1 > 0, \\ \kappa &= 1, \quad j_1 = 0 \end{aligned} \tag{3.3}$$

Here, κ is the number of chiral generators $X^+_{1,4+k}$, $k = 1, \ldots, \kappa$, that are eliminated. Thus, in the cases when $\kappa = N$ the semi-short UIRs may be called semi–anti-chiral since they lack half of the chiral generators.

$$\bullet\ \mathbf{d} \qquad d \ = \ d^{d}_{|m=0} \ = \ -z\ , \quad j_2 \text{ arbitrary}, \quad j_1 = 0, \qquad \kappa = 2N \tag{3.4}$$

These short UIRs may be called anti-chiral since they lack all chiral generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k = 1, \ldots, N$.

$$\begin{aligned} \bullet\ \mathbf{ac} \qquad & d \ = \ d^{ac}_{|m=0} \ = \ 2 + j_1 + j_2\ , \quad z \ = \ j_1 - j_2\ , \\ & \kappa \ = \ 2N + (1-N)(\delta_{j_1,0} + \delta_{j_2,0}), \quad \text{or casewise :} \\ & \kappa = 2N,\ if j_1, j_2 > 0, \\ & \kappa = N + 1,\ if j_1 > 0,\ j_2 = 0, \\ & \kappa = N + 1,\ if j_1 = 0,\ j_2 > 0, \\ & \kappa = 2,\ if j_1 = j_2 = 0. \end{aligned} \tag{3.5}$$

Here, κ is the number of mixed elimination: chiral generators $X^+_{1,4+k}$, and anti-chiral generators $X^+_{3,4+k}$. Thus, in the cases when $\kappa = 2N$ the semi-short UIRs may be called semi–chiral–anti-chiral since they lack half of the chiral and half of the anti-chiral generators. (They may be called Grassmann-analytic following [10].)

$$\begin{aligned} \bullet\ \mathbf{ad} \qquad & d \ = \ d^{ad}_{|m=0} \ = \ 1 + j_2 \ = \ -z\ , \quad j_1 = 0, \\ & \kappa \ = \ 3N + (1-N)\delta_{j_2,0}, \quad \text{or casewise :} \\ & \kappa = 3N, \quad j_2 > 0, \\ & \kappa = 2N + 1, \quad j_2 = 0. \end{aligned} \tag{3.6}$$

Here, κ is the number of mixed elimination: chiral generators $X^+_{1,4+k}$, and both types anti-chiral generators $X^+_{3,4+k}$, $X^+_{4,4+k}$. Thus, in the cases when $\kappa = 3N$ the semi-short UIRs may be called chiral and semi–anti-chiral since they lack half of the chiral and all of the anti-chiral generators.

$$\bullet\ \mathbf{bc} \qquad \begin{aligned} d &= d^{bc}_{|m=0} = 1 + j_1 = z\ ,\quad j_2 = 0, \\ \kappa &= 3N + (1-N)\delta_{j_1,0}, \quad \text{or casewise :} \\ \kappa &= 3N, \quad j_1 > 0, \\ \kappa &= 2N+1, \quad j_1 = 0\ . \end{aligned} \tag{3.7}$$

Here, κ is the number of mixed elimination: both types chiral generators $X^+_{1,4+k}$, $X^+_{2,4+k}$, and anti-chiral generators $X^+_{3,4+k}$. Thus, in the cases when $\kappa = 3N$ the semi-short UIRs may be called semi–chiral and anti-chiral since they lack all the chiral and half of the anti-chiral generators.

The last two cases (ad,bc) form two of the three series of massless states, holomorphic and antiholomorphic [2], see also [4, 8].

The case **•bd** for R-symmetry scalars is trivial, since also all other quantum numbers are zero ($d = j_1 = j_2 = z = 0$).

3.2 R-symmetry non-scalars

Here we need some additional notation. Let $N > 1$ and let i_0 be an integer such that $0 \leq i_0 \leq N-1$, $r_i = 0$ for $i \leq i_0$, and if $i_0 < N-1$ then $r_{i_0+1} > 0$. Let now i'_0 be an integer such that $0 \leq i'_0 \leq N-1$, $r_{N-i} = 0$ for $i \leq i'_0$, and if $i'_0 < N-1$ then $r_{N-1-i'_0} > 0$.[2]

With this notation the cases of R-symmetry scalars occur when $i_0 + i'_0 = N-1$, thus, from now on we have the restriction:

$$0 \leq i_0 + i'_0 \leq N-2 \tag{3.8}$$

Now we can make a list for the values of κ, with the same interpretation as in the previous subsection, only the last case is added here.

$$\bullet\ \mathbf{a} \qquad \begin{aligned} &d = d^a\ ,\quad j_1, j_2 \text{ arbitrary}, \\ &\kappa = 1 + i_0(1-\delta_{j_2,0}) \leq N-1\ . \end{aligned} \tag{3.9}$$

[2]Both definitions are formally valid for $N = 1$ with $i_0 = 0$ since $r_0 \equiv 0$ by convention and with $i'_0 = 0$ since $r_N = 0$ by convention.

• **b** $\quad d \ = \ d^a \ , \quad j_2 = 0 \ , \quad j_1$ arbitrary,

$$\kappa = 2 + 2i_0 \leq 2N - 2 \ . \tag{3.10}$$

• **c** $\quad d \ = \ d^c \ , \quad j_1, j_2$ arbitrary,

$$\kappa = 1 + i'_0(1 - \delta_{j_1,0}) \leq N - 1 \ . \tag{3.11}$$

• **d** $\quad d \ = \ d^d \ , \quad j_1 = 0, \ j_2$ arbitrary,

$$\kappa = 2 + 2i'_0 \leq 2N - 2 \ . \tag{3.12}$$

• **ac** $\quad d \ = \ d^{ac} \ , \quad z \ = \ j_1 - j_2 + 2m/N - m_1 \ , \quad j_1, j_2$ arbitrary,

$$\kappa = 2 + i_0(1 - \delta_{j_2,0}) + i'_0(1 - \delta_{j_1,0}) \leq N \ . \tag{3.13}$$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $k \leq 1 + i_0$.

• **ad** $\quad d \ = \ d^{ad} \ , \quad j_1 = 0 \ , \quad z \ = \ 2m/N - m_1 - 1 - j_2 \ , \quad j_2$ arbitrary,

$$\kappa = 3 + i_0(1 - \delta_{j_2,0}) + 2i'_0 \leq 1 + N + i'_0 \leq 2N - 1 \ . \tag{3.14}$$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $X^+_{4,4+k}$, $k \leq 1 + i_0$.

• **bc** $\quad d \ = \ d^{bc} \ , \quad j_2 = 0 \ , \quad z \ = \ 2m/N - m_1 + 1 + j_1 \ , \quad j_1$ arbitrary,

$$\kappa = 3 + 2i_0 + i'_0(1 - \delta_{j_1,0}) \leq 1 + N + i_0 \leq 2N - 1 \ . \tag{3.15}$$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $k \leq 1 + i_0$.

$$\bullet\ \mathbf{bd} \qquad d = d^{bd} , \quad j_1 = j_2 = 0 , \quad z = 2m/N - m_1 ,$$
$$\kappa = 4 + 2i_0 + 2i'_0 \leq 2N . \tag{3.16}$$

Here, the eliminated chiral generators are $X^+_{1,4+k}$, $X^+_{2,4+k}$, $k \leq 1 + i'_0$, and the eliminated anti-chiral generators are $X^+_{3,4+k}$, $X^+_{3,4+k}$, $k \leq 1 + i_0$.
Note that the case $\kappa = 2N$ is possible exactly when $i_0 + i'_0 = N - 2$, i.e., when there is only one nonzero r_i, namely, $r_{i_0+1} \neq 0$, $i_0 = 0, 1, \ldots, N-2$:

$$\bullet\ \mathbf{bd} \quad \kappa = 2N \ : \ d = m_1 = r_{i_0+1} , \quad j_1 = j_2 = 0 , \quad z = r_{i_0+1} \frac{2 + 2i_0 - N}{N} . \tag{3.17}$$

When $d = m_1 = 1$ these $\frac{1}{2}$-eliminated UIRs form the 'mixed' series of massless representations [2], see also [4, 8].[3]

Remark: In this paper we use the Verma (factor-)module realization of the UIRs. We give here a short remark on what happens with the ER realization of the UIRs. As we know, cf. [4], the ERs are superfields depending on Minkowski space-time and on $4N$ Grassmann coordinates θ^i_a, $\bar{\theta}^k_b$, $a, b = 1, 2$, $i, k = 1, \ldots, N$. There is 1-to-1 correspondence in these dependencies and the odd null conditions. Namely, if the condition $X^+_{a,4+k} \, |\Lambda\rangle = 0$, $a = 1, 2$, holds, then the superfields of the corresponding ER do not depend on the variable θ^k_a, while if the condition $X^+_{a,4+k} \, |\Lambda\rangle = 0$, $a = 3, 4$, holds, then the superfields of the corresponding ER do not depend on the variable $\bar{\theta}^k_{a-2}$. These statements were used in the proof of unitarity for the ERs picture, cf. [18], but were not explicated. They were analyzed in detail in the papers [10–12, 23], using the notions of 'harmonic superspace analyticity' and Grassmann analyticity. $\diamondsuit$

In the next Section we shall use the above classification to the so-called BPS states.

[3] This series is absent for $N = 1$.

4 BPS states

4.1 PSU(2,2/4)

The most interesting case is when $N = 4$. This is related to super-Yang-Mills and contains the so-called BPS states, cf., [10–17]. They are characterized by the number κ of odd generators which annihilate them - then the corresponding state is called $\frac{\kappa}{4N}$-BPS state. Group-theoretically the case $N = 4$ is special since the $u(1)$ subalgebra carrying the quantum number z becomes central and one can invariantly set $z = 0$.

We give now the explicit list of these states:

•a $d = d^1_{41} = 2 + 2j_2 + 2m_1 - \frac{1}{2}m > d^3_{44}$. The last inequality leads to the restriction:

$$2j_2 + r_1 > 2j_1 + r_3 \ . \tag{4.1}$$

In the case of R-symmetry scalars, i.e., $m_1 = 0$, follows that $j_2 > j_1$, i.e., $j_2 > 0$, and then we have:

$$\kappa = 4, \quad m_1 = 0,\ j_2 > 0 \ . \tag{4.2}$$

In the case of R-symmetry non-scalars, i.e., $m_1 \neq 0$, we have the range: $i_0 + i'_0 \leq 2$, and thus:

$$\kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq 3 \ . \tag{4.3}$$

•b $d = d^2_{41} = \frac{1}{2}m^* > d^3_{44}$, $j_2 = 0$. The last inequality leads to the restriction:

$$r_1 > 2 + 2j_1 + r_3 \ . \tag{4.4}$$

The latter means that $r_1 > 2$, i.e., $m_1 \neq 0$, $i_0 = 0$, and thus:

$$\kappa = 2 \ . \tag{4.5}$$

The next two cases are conjugate to the previous two so we present them shortly:

•c $d = d^3_{44} = 2 + 2j_1 + \frac{1}{2}m > d^1_{41} \implies$

$$2j_1 + r_3 > 2j_2 + r_1 \ , \tag{4.6}$$

$m_1 = 0 \quad \Longrightarrow \quad j_1 > j_2 \quad \Longrightarrow \quad j_1 > 0 \quad \Longrightarrow$

$$\kappa = 4, \quad m_1 = 0, \ j_1 > 0 \ . \tag{4.7}$$

$m_1 \neq 0 \quad \Longrightarrow \quad i_0 + i'_0 \leq 2 \quad \Longrightarrow$

$$\kappa = 1 + i'_0(1 - \delta_{j_1,0}) \leq 3 \ . \tag{4.8}$$

•d $\quad d \ = \ d^4_{44} \ = \ \frac{1}{2}m \ > \ d^1_{41} \ , \quad j_1 = 0, \qquad \Longrightarrow$

$$r_3 > 2 + 2j_2 + r_1 \ , \tag{4.9}$$

$\Longrightarrow \quad r_3 > 2 \quad \Longrightarrow \quad m_1 \neq 0, \quad i'_0 = 0 \quad \Longrightarrow$

$$\kappa \ = \ 2 \ . \tag{4.10}$$

•ac $\quad d \ = \ d^{ac} \ = \ 2 + j_1 + j_2 + m_1$. From $z = 0$ follows:

$$2j_2 + r_1 = 2j_1 + r_3 \ . \tag{4.11}$$

In the case of R-symmetry scalars, i.e., $m_1 = 0$, follows that $j_2 = j_1 = j$, and then we have:

$$\kappa \ = \ 8 - 6\delta_{j,0} \ , \quad d = 2 + 2j \ . \tag{4.12}$$

In the case of R-symmetry non-scalars, i.e., $m_1 \neq 0$, $i_0 + i'_0 \leq 2$, and thus:

$$\kappa = 2 + i_0(1 - \delta_{j_2,0}) + i'_0(1 - \delta_{j_1,0}) \leq 4 \ . \tag{4.13}$$

•ad From $z = 0$ follows: $r_3 = 2 + 2j_2 + r_1 \quad \Longrightarrow \quad r_3 \geq 2 \quad \Longrightarrow \quad m_1 \neq 0$, and $i'_0 = 0, \quad i_0 \leq 2 \quad \Longrightarrow$

$$\begin{aligned} \kappa &= 3 + i_0(1 - \delta_{j_2,0}) \leq 5 \ , \\ d &= d^{ad} \ = \ 1 + j_2 + m_1 \ = \ 3 + 3j_2 + 2r_1 + r_2 \ , \\ \chi_4 &= \{\, 0\, ; \, r_1, r_2, 2 + 2j_2 + r_1 \, ; \, 2j_2 \,\} \ . \end{aligned} \tag{4.14}$$

•bc From $z = 0$ follows: $r_1 = 2 + 2j_2 + r_3 \quad \Longrightarrow \quad r_1 \geq 2 \quad \Longrightarrow \quad m_1 \neq 0$, and $i_0 = 0, \quad i'_0 \leq 2 \quad \Longrightarrow$

$$\begin{aligned} \kappa &= 3 + i'_0(1 - \delta_{j_1,0}) \leq 5 \ , \\ d &= d^{bc} \ = \ 1 + j_2 + m_1 \ = \ 3 + 3j_2 + 2r_1 + r_2 \ , \\ \chi_4 &= \{\, 2j_1 \, ; \, 2 + 2j_2 + r_3, r_2, r_3 \, ; \, 0 \,\} \ . \end{aligned} \tag{4.15}$$

•**bd** From $z = 0$ follows: $r_1 = r_3 = r$, thus, $i_0 = i'_0 = 0, 1$ and then we have:

$$\kappa = 4(1 + i_0) \; , \tag{4.16}$$
$$d \;=\; d^{bd} \;=\; m_1 \;=\; 2r + r_2 \;\neq\; 0 \; , \quad r, r_2 \in \mathbb{Z}_+ \; ,$$
$$\chi_4 = \{\, 0 \, ; \, r, r_2, r \, ; \, 0 \,\} \; .$$

From the above BPS states we list now the most interesting ones:

4.1.1 $\frac{1}{2}$-BPS states, $\kappa = 8$

These are possible in case **ac**, cf. (4.55), for R-symmetry scalars and nontrivial vector Lorentz spin:

$$d = 2 + 2j \geq 3 \; , \qquad j_1 = j_2 = j \geq \tfrac{1}{2} \; , \quad m_1 = z = 0 \; , \tag{4.17}$$

or in terms of the signature in (2.5):

$$d = 2 + n \; , \qquad \chi_4 \;=\; \{\, n \, ; \, 0, 0, 0 \, ; \, n \,\} \; , \quad n = 2j \in \mathbb{N} \; . \tag{4.18}$$

They are also possible in case **bd**, cf. (4.60), when $i_0 = i'_0 = 1$, i.e., $r_1 = r_3 = 0$, $r_2 \neq 0$:

$$d = r_2 = r \geq 1 \; , \qquad r_1 = r_3 = j_1 = j_2 = z = 0 \; , \tag{4.19}$$

or

$$d = r \in \mathbb{N} \; , \qquad \chi_4 \;=\; \{\, 0 \, ; \, 0, r, 0 \, ; \, 0 \,\} \; . \tag{4.20}$$

4.1.2 $\frac{1}{4}$-BPS states, $\kappa = 4$

These happen in most cases with appropriate conditions:

Case **a**, cf. (4.2),

$$d \;=\; 2 + 2j_2 \geq 3 \; , \qquad m_1 = z = 0, \; j_2 \geq \tfrac{1}{2} \; . \tag{4.21}$$

or

$$d = 2 + n + k \; , \qquad \chi_4 \;=\; \{\, n \, ; \, 0, 0, 0 \, ; \, n + k \,\} \; , \quad n = 2j_1 \in \mathbb{Z}_+ \; , \quad k \in \mathbb{N} \; . \tag{4.22}$$

Case **c**, cf. (4.50),

$$d \;=\; 2+2j_1 \geq 3 \;, \qquad m_1 = z = 0,\; j_1 \geq \tfrac{1}{2} \;. \tag{4.23}$$

or

$$d = 2+n+k \;, \qquad \chi_4 \;=\; \{\, n+k\,;\, 0,0,0\,;\, n\,\} \;, \quad n = 2j_2 \in \mathbb{Z}_+ \;, \quad k \in \mathbb{N} \;. \tag{4.24}$$

In case **ac** we deal with R-symmetry non-scalars with only one non-zero r_i entry, since we have $i_0 + i'_0 = 2$, thus, we take $r_{1+i_0} \neq 0$, $0 \leq i_0 \leq 2$, with the condition:

$$j_1 - j_2 = m_1 - \tfrac{1}{2}m = \tfrac{1}{2}r_{1+i_0}(1-i_0) \;, \tag{4.25}$$

and then we have:

$$d \;=\; 2+j_1+j_2+r_{i_0} \;=\; 2+2j+r_{i_0} \;, \tag{4.26}$$

where case-wise:

$$\begin{aligned} j \;&=\; j_2 \;,\quad j_1 = j + \tfrac{1}{2}r_1 \;,\quad \chi_4 \;=\; \{\, 2j+r_1\,;\, r_1,0,0\,;\, 2j\,\} \;, \\ j \;&=\; j_1 = j_2 \;,\quad \chi_4 \;=\; \{\, 2j\,;\, 0,r_2,0\,;\, 2j\,\} \;, \\ j \;&=\; j_1 \;,\quad j_2 = j + \tfrac{1}{2}r_3 \;,\quad \chi_4 \;=\; \{\, 2j\,;\, 0,0,r_3\,;\, 2j+r_3\,\} \;. \end{aligned} \tag{4.27}$$

Case **ad**, cf. (4.14),

$$d \;=\; 3+3j_2+r_2 \geq \tfrac{9}{2} \;, \qquad r_3 = 2+2j_2 \geq 3 \;,\;\; j_2 \geq \tfrac{1}{2} \;, \quad r_1 = j_1 = z = 0 \;. \tag{4.28}$$

or

$$d = 3+\tfrac{3}{2}n+r \;, \qquad \chi_4 \;=\; \{\, 0\,;\, 0,r,2+n\,;\, n\,\} \;, \quad r,n \in \mathbb{N} \;. \tag{4.29}$$

Case **bc**, cf. (4.15),

$$d \;=\; 3+3j_1+r_2 \geq \tfrac{9}{2} \;, \qquad r_1 = 2+2j_1 \geq 3 \;,\;\; j_1 \geq \tfrac{1}{2} \;, \quad r_3 = j_2 = z = 0 \;. \tag{4.30}$$

or

$$d = 3+\tfrac{3}{2}n+r \;, \qquad \chi_4 \;=\; \{\, n\,;\, 2+n,r,0\,;\, 0\,\} \;, \quad r,n \in \mathbb{N} \;. \tag{4.31}$$

Case **bd**, cf. (4.60), when $i_0 = i'_0 = 0$, i.e., $r_1 = r_3 = n \neq 0$:

$$d = r+2n \geq 2 \;, \qquad n \geq 1, \;\; r = r_2 \geq 0, \;\; j_1 = j_2 = z = 0 \;. \tag{4.32}$$

or

$$d = r+2n \;, \qquad \chi_4 \;=\; \{\, 0\,;\, n,r,n\,;\, 0\,\} \;, \quad r,n \in \mathbb{N} \;. \tag{4.33}$$

4.1.3 $\frac{1}{8}$-BPS states, $\kappa = 2$

Case **a**, cf. (4.2) with $j_2 > 0$, $i_0 = 1$,

$$d = 2 + 2j_2 + r_2 + \tfrac{1}{2}r_3 , \qquad 2j_2 > 2j_1 + r_3 , \quad r_2 > 0, \quad r_1 = z = 0 . \tag{4.34}$$

or

$$d = 2 + k + 2n + r_2 + \tfrac{3}{2}r_3 , \qquad \chi_4 = \{ n ; 0, r_2, r_3 ; n + r_3 + k \} , \quad k, r_2 \in I\!N, \quad n, r_3 \in Z\!\!Z_+ . \tag{4.35}$$

Case **b**, cf. (4.48),

$$d = \tfrac{1}{2}(3r_1 + 2r_2 + r_3) \equiv \tfrac{1}{2}m^* , \qquad r_1 > 2 + 2j_1 + r_3 , \quad j_2 = z = 0 . \tag{4.36}$$

or

$$d = 3 + 2r_3 + r_2 + \tfrac{3}{2}(n + k), \qquad \chi_4 = \{ n ; 2 + n + r_3 + k, r_2, r_3 ; 0 \} , \quad k \in I\!N, \quad n, r_2, r_3 \in Z\!\!Z_+ . \tag{4.37}$$

Case **c**, cf. (4.50) with $j_1 > 0$, $i'_0 = 1$,

$$d = 2 + 2j_1 + r_2 + \tfrac{1}{2}r_1 , \qquad 2j_1 > 2j_2 + r_1 , \quad r_2 > 0, \quad r_3 = z = 0 , \tag{4.38}$$

or

$$d = 2 + k + 2n + r_2 + \tfrac{3}{2}r_1 , \qquad \chi_4 = \{ n + r_1 + k ; r_1, r_2, 0 ; n \} , \quad k, r_2 \in I\!N, \quad n, r_1 \in Z\!\!Z_+ . \tag{4.39}$$

Case **d**, cf. (4.53),

$$d = \tfrac{1}{2}m = \tfrac{1}{2}(r_1 + 2r_2 + 3r_3) , \qquad r_3 > 2 + 2j_2 + r_1 , \quad j_1 = z = 0 , \tag{4.40}$$

or

$$d = 3 + 2r_1 + r_2 + \tfrac{3}{2}(n + k), \qquad \chi_4 = \{ 0 ; r_1, r_2, 2 + n + r_1 + k ; n \} , \quad k \in I\!N, \quad n, r_2, r_1 \in Z\!\!Z_+ . \tag{4.41}$$

Case **ac**, cf. (4.55),(4.57),

$$d = 2 + m_1 \geq 2 , \qquad j_1 = j_2 = z = 0 . \tag{4.42}$$

or

$$d \ = \ 2 + r_1 + r_2 + r_3 \ , \qquad \chi_4 \ = \ \{\, 0\, ;\, r_1, r_2, r_3\, ;\, 0\, \} \ . \tag{4.43}$$

Some of these BPS-cases are extensively studied in the literature, mostly those listed here as cases **ac,bd**, cf. [10–17].

Finally, we remark that some of the above states would violate the protectedness conditions that we gave in Subsection 2.2. These would be the $\frac{1}{4}$-BPS cases listed as cases **ad,bc**, and in case **bd** for $n > 2$, while for the $\frac{1}{8}$-BPS cases that would be the cases **b,d**, and in case **ac** for $r_1 r_3 \neq 0$.

4.2 SU(2,2/N), $N \leq 3$

We can set $z = 0$ also for $N \neq 4$ though this does not have the same group-theoretical meaning as for $N = 4$. In this Subsection we treat separately the cases $N = 1, 2, 3$, which are more peculiar.

4.2.1 SU(2,2/1)

For $N = 1$ setting $z = 0$ is possible only for three cases **a,c,ac** :

•**a** $d = 2 + 2j_2$, $j_2 > j_1 \geq 0$,
$\kappa = 1$, $\frac{1}{4}$-BPS;

•**c** $d = 2 + 2j_1$, $j_1 > j_2 \geq 0$,
$\kappa = 1$, $\frac{1}{4}$-BPS;

•**ac** $d = 2 + 2j$, $j_1 = j_2 = j$,
$\kappa = 2$, $\frac{1}{2}$-BPS.

Note that according to the result of Subsection 2.2 the first two cases would not be protected.

4.2.2 SU(2,2/2)

For $N = 2$ holds $i_0 = i_0' = 0, 1$. Setting $z = 0$ is possible for four cases **a,c,ac,bd** when we have:

•**a** $d = 2 + 2j_2 + r_1$, $j_2 > j_1 \geq 0$,
$\kappa = 1 + i_0 \leq 2$;

•**c** $d = 2 + 2j_1 + r_1$, $j_1 > j_2 \geq 0$,
$\kappa = 1 + i'_0 \leq 2$;

•**ac** $d = 2 + 2j + r_1$, $j_1 = j_2 = j$,
$\kappa = 2 + 2\delta_{i_0 j,0} \leq 4$;

•**bd** $d = r_1 \neq 0$, $j_1 = j_2 = 0$, (here $z = 0$ holds in all cases),
$\kappa = 4$, $\frac{1}{2}$-BPS.

Note that according to the result of Subsection 2.2 the first three cases would not be protected when $r_1 \neq 0$, i.e., when $i_0 = i'_0 = 0$. In contradistinction, when $r_1 = 0$, i.e., $i_0 = i'_0 = 1$, the first two are $\frac{1}{4}$-BPS, and the third, when $j > 0$, a $\frac{1}{2}$-BPS. The fourth case would not be protected if $r_1 > 4$.

4.2.3 SU(2,2/3)

In fact, the case $N = 3$ is similar in these considerations to $N = 4$, (though some results differ), so we present it telegraphically.

•**a** $d = d^1_{31} = 2 + 2j_2 + 2m_1 - 2m/3 > d^3_{33} \Longrightarrow$

$$j_2 + \tfrac{1}{3}r_1 > j_1 + \tfrac{1}{3}r_2 \ . \tag{4.44}$$

For $m_1 = 0 \Rightarrow j_2 > j_1 \Rightarrow j_2 > 0 \Rightarrow$

$$\kappa = 3, \quad m_1 = 0,\ j_2 > 0 \ . \tag{4.45}$$

For $m_1 \neq 0, \Rightarrow i_0 + i'_0 \leq 1 \Rightarrow$

$$\kappa = 1 + i_0(1 - \delta_{j_2,0}) \leq 2 \ . \tag{4.46}$$

•**b** $d = d^2_{31} = 2m_1 - 2m/3 > d^3_{33}$, $j_2 = 0 \Longrightarrow$

$$r_1 > 3 + 3j_1 + r_2 \quad \Longrightarrow \tag{4.47}$$

$r_1 > 3 \Rightarrow m_1 \neq 0$ and $i_0 = 0 \Longrightarrow$

$$\kappa = 2 \ . \tag{4.48}$$

•**c** $d = d^3_{33} = 2 + 2j_1 + 2m/3 > d^1_{31} \Longrightarrow$

$$j_1 + \tfrac{1}{3}r_2 > j_2 + \tfrac{1}{3}r_1 \ , \tag{4.49}$$

$m_1 = 0 \implies j_1 > j_2 \implies j_1 > 0 \implies$

$$\kappa = 3, \quad m_1 = 0,\ j_1 > 0\ . \tag{4.50}$$

$m_1 \neq 0 \implies i_0 + i'_0 \leq 1 \implies$

$$\kappa = 1 + i'_0(1 - \delta_{j_1,0}) \leq 2\ . \tag{4.51}$$

•d $\quad d = d^4_{33} = 2m/3 > d^1_{31}\,, \quad j_1 = 0, \quad \implies$

$$r_2 > 3 + 3j_2 + r_1\ , \tag{4.52}$$

$\implies r_2 > 3 \implies m_1 \neq 0$, and $i'_0 = 0 \implies$

$$\kappa = 2\ . \tag{4.53}$$

•ac $\quad d = d^1_{31} = d^3_{33} = 2 + j_1 + j_2 + m_1$. From $z = 0$ follows:

$$j_2 + \tfrac{1}{3}r_1 = j_1 + \tfrac{1}{3}r_2\ . \tag{4.54}$$

In the case of R-symmetry scalars, i.e., $m_1 = 0$, follows that $j_2 = j_1 = j$, and then we have:

$$\kappa = 6 - 4\delta_{j,0}\ . \tag{4.55}$$

Thus, for $j \neq 0$ we have $\frac{1}{2}$-BPS state:

$$\kappa = 6, \quad d = 2 + 2j \geq 3\ , \quad \chi_3 = \{\, 2j\,;\, 0, 0\,;\, 2j\,\}\ . \tag{4.56}$$

In the case of R-symmetry non-scalars, i.e., $m_1 \neq 0$, $i_0 + i'_0 \leq 1$, and thus:

$$\kappa = 2 + i_0(1 - \delta_{j_2,0}) + i'_0(1 - \delta_{j_1,0}) \leq 3\ . \tag{4.57}$$

Thus, when $i_0 j_2 \neq 0$ or $i'_0 j_1 \neq 0$ we have $\frac{1}{4}$-BPS state since $\kappa = 3$.

•ad From $z = 0$ follows: $r_2 = 3 + 3j_2 + r_1 \implies r_2 \geq 3 \implies m_1 \neq 0$, and $i'_0 = 0$, $i_0 \leq 1 \implies$

$$\begin{aligned} \kappa &= 3 + i_0(1 - \delta_{j_2,0}) \leq 4\ , \\ d &= d^{ad} = 1 + j_2 + m_1 = 4 + 4j_2 + 2r_1\ , \\ \chi_3 &= \{\, 0\,;\, r_1, 3 + 3j_2 + r_1\,;\, 2j_2\,\}\ . \end{aligned} \tag{4.58}$$

Thus, when $i_0 j_2 = 0$ we have $\frac{1}{4}$-BPS state since $\kappa = 3$.

•bc From $z=0$ follows: $r_1 = 3+3j_1+r_2 \implies r_1 \geq 3 \implies m_1 \neq 0$, and $i_0 = 0$, $i_0' \leq 1 \implies$

$$\begin{aligned} \kappa &= 3 + i_0'(1-\delta_{j_1,0}) \leq 4 \ , \\ d &= d^{bc} \ = \ 1+j_1+m_1 \ = \ 4+4j_1+2r_2 \ , \\ \chi_3 &= \{\, 2j_1\, ;\, 3+3j_1+r_2, r_2\, ;\, 0\,\} \ . \end{aligned} \tag{4.59}$$

Thus, when $i_0' j_1 = 0$ we have $\frac{1}{4}$-BPS state since $\kappa = 3$.

•bd From $z=0$ follows: $r_1 = r_2 = \frac{1}{2}m_1 = r \in I\!N$, thus, $i_0 = i_0' = 0$ and then we have:

$$\kappa = 4\ , \quad d \ = \ d^{bd} \ = \ 2r \ \neq \ 0\ , \quad \chi_3 = \{\, 0\, ;\, r, r\, ;\, 0\,\} \ . \tag{4.60}$$

Note that according to the result of Subsection 2.2 the following cases would not be protected: cases **a,ad** when $r_1 \neq 0$; cases **c,bc** when $r_2 \neq 0$; cases **b,d**; case **ac** when $r_1 r_2 \neq 0$, (i.e., $i_0 = i_0' = 0$); case **bd** when $r > 2$.

4.3 SU(2,2/N), $N \geq 5$

The cases $N \geq 5$ are described adequately by the general exposition in Section 3, though some cases are excluded for $z = 0$. Thus, we shall give only the special cases.

4.3.1 $\frac{1}{2}$-BPS states, $\kappa = 2N$

These are possible only in cases **ac,bd**, and appear as for $N = 4$.

In case **ac** we deal with R-symmetry scalars and $j_1 = j_2 = j \geq \frac{1}{2}$:

$$d = 2+n, \quad n = 2j \in I\!N, \qquad \chi_N \ = \ \{\, n\, ;\, 0, \ldots, 0\, ;\, n\,\} \ . \tag{4.61}$$

In case **bd** this is possible when N is *even*, and there is only one non-zero r_i, namely, the middle one, i.e.,

$$d \ = \ m_1 \ = \ r_{\frac{1}{2}N-1} \ \neq \ 0\ , \qquad \chi_N \ = \ \{\, 0\, ;\, 0, \ldots, 0,\, r_{\frac{1}{2}N-1}, 0, \ldots, 0;\, 0\,\} \ . \tag{4.62}$$

Note that according to the result of Subsection 2.2 these $\frac{1}{2}$-BPS cases would be protected.

4.3.2 $\frac{1}{4}$-BPS states, $\kappa = N$

In case **a** we deal with R-symmetry scalars and $j_2 > j_1$,

$$d = 2 + n + k, \quad n = 2j_1 \in \mathbb{Z}_+, \ k \in \mathbb{N}, \qquad \chi_N \ = \ \{ n \, ; 0, \ldots, 0 \, ; n + k \} \ . \tag{4.63}$$

In the conjugate case **c**, $j_1 > j_2$,

$$d = 2 + n + k, \quad n = 2j_2 \in \mathbb{Z}_+, \ k \in \mathbb{N}, \qquad \chi_N \ = \ \{ n + k \, ; 0, \ldots, 0 \, ; n \} \ . \tag{4.64}$$

In case **b** we would deal with R-symmetry non-scalars for N-*even*, and we must have $i_0 = \frac{N}{2} - 1$; this means that $r_i = 0$ for $i = 1, \ldots, \frac{N}{2} - 1$. On the other hand we must satisfy the condition:

$$1+j_1 < m_1 - 2m/N \ = \ \sum_{k=1}^{\frac{N}{2}-1} (r_k - r_{N-k})(1-2k/N) = - \sum_{k=1}^{\frac{N}{2}-1} r_{N-k}(1-2k/N) \ \leq 0 \, ,$$

which is not possible.

For the same reason the conjugate case **d** is not possible.

In case **ac** we deal with R-symmetry non-scalars with only one non-zero r_i entry, since we have $i_0 + i'_0 = N - 2$, thus, we take $r_{1+i_0} \neq 0, \ 0 \leq i_0 \leq N-2$, with the condition:

$$j_1 - j_2 = m_1 - 2m/N = r_{1+i_0}(1 - \tfrac{2}{N}(1 + i_0)) \ , \tag{4.65}$$

and then we have:

$$d \ = \ 2 + j_1 + j_2 + r_{i_0} \ , \qquad \chi_N \ = \ \{ 2j_1 \, ; 0, \ldots, 0, \, r_{i_0} \, , 0, \ldots, 0; \, 2j_2 \} \ . \tag{4.66}$$

Note that: $i_0 < \frac{N}{2} - 1 \implies j_1 - j_2 > 0$,
$i_0 > \frac{N}{2} - 1 \implies j_1 - j_2 < 0$,
$i_0 = \frac{N}{2} - 1 \implies j_1 - j_2 = 0$, only for N–even.

In case **ad** we deal with R-symmetry non-scalars with two subcases depending whether j_2 is zero or not.

• When $j_2 = 0$ we have $\kappa = N = 3 + 2i'_0$, i.e., N is odd, and $i'_0 = \frac{1}{2}(N-3)$, $(i_0 \leq \frac{1}{2}(N-1))$. On the other hand from $z = 0$ we must satisfy the condition (rescaling by N):

$$N \;=\; 2m{-}Nm_1 \;=\; \sum_{k=1}^{\frac{1}{2}(N-1)} (r_{N-k}{-}r_k)(N{-}2k) \;=\; r_{\frac{1}{2}(N+1)} - \sum_{k=1}^{\frac{1}{2}(N-1)} r_k(N{-}2k)\,.$$

Thus, we have:

$$d \;=\; 1+m_1 \;=\; 1+\sum_{k=1}^{\frac{1}{2}(N+1)} r_k \;=\; 1+N+\sum_{k=\frac{1}{2}(N+1)}^{N-1} r_k(1+N-2k)\;,$$
$$\chi_N \;=\; \{\,0\,;\, r_1,\ldots,0,\, r_{\frac{1}{2}(N+1)}, 0,\ldots,0;\, 0\,\}\,. \tag{4.67}$$

• When $j_2 \neq 0$ we have $\kappa = N = 3 + 2i'_0 + i_0$, from where we follows that $i_0 + i'_0 = N-2$ is not possible, thus $i_0 + i'_0 \leq N-3$, also $i'_0 \leq \frac{1}{2}(N-3)$. Also the following condition must hold:

$$1 + j_2 = \tfrac{2}{N}m - m_1\ .$$

Thus, we have:

$$d \;=\; 1+j_2+m_1 = \tfrac{2}{N}m\ , \quad j_1 = 0\ . \tag{4.68}$$

In the conjugate case **bc** we expose shortly:

• $j_1 = 0 \;\Rightarrow\; \kappa = N = 3 + 2i_0 \;\Rightarrow\; N$ is odd, $\Rightarrow\; i_0 = \frac{1}{2}(N-3)$, $(i'_0 \leq \frac{1}{2}(N-1))$. On the other hand must hold:

$$N \;=\; Nm_1{-}2m \;=\; \sum_{k=1}^{\frac{1}{2}(N-1)} (r_k{-}r_{N-k})(N{-}2k) \;=\; r_{\frac{1}{2}(N-1)} - \sum_{k=1}^{\frac{1}{2}(N-1)} r_{N-k}(N{-}2k)\,.$$

Thus, we have:

$$d \;=\; 1+m_1 \;=\; 1+\sum_{k=\frac{1}{2}(N-1)}^{N-1} r_k \;=\; 1+N+\sum_{k=\frac{1}{2}(N+1)}^{N-1} r_k(1+2k-N)\;,$$
$$\chi_N \;=\; \{\,0\,;\, 0,\ldots,0,\, r_{\frac{1}{2}(N-1)}, r_{\frac{1}{2}(N+1)},\ldots,r_{N-1};\, 0\,\} \tag{4.69}$$

• $j_1 \neq 0 \quad \Rightarrow \quad \kappa = N = 3 + 2i_0 + i_0' \quad \Rightarrow \quad i_0 + i_0' = N - 2$ is not possible, thus $i_0 + i_0' \leq N - 3$, also $i_0 \leq \frac{1}{2}(N-3)$. Also the following condition must hold:

$$1 + j_1 = m_1 - \tfrac{2}{N} m \ .$$

Thus, we have:

$$d \ = \ 1 + j_1 + m_1 = 2m_1 - \tfrac{2}{N} m = \tfrac{2}{N} m^* \ , \quad j_2 = 0 \ . \tag{4.70}$$

Case **bd** is possible only for N–even with R-symmetry non-scalars, and from $\kappa = N$ and $z = 0$ follows:

$$i_0 + i_0' = \tfrac{N}{2} - 2 \ , \qquad m_1 \ = \ \tfrac{2}{N} m \ \neq \ 0 \ .$$

Thus, we have:

$$d \ = \ m_1 \ = \ \sum_{k=1+i_0}^{1+i_0+\frac{N}{2}} r_k \ , \quad j_1 = j_2 = 0 \ . \tag{4.71}$$

Note that according to the result of Subsection 2.2 the following $\frac{1}{4}$-BPS cases would not be protected: case **ad** when $r_1 \neq 0$; case **bc** when $r_{N-1} \neq 0$; case **bd** when $r_1, r_{N-1} > 2$.

4.3.3 $\frac{1}{8}$-BPS states, $\kappa = N/2$, N-even

In all possible cases we deal with R-symmetry non-scalars.

In case **a** to achieve $\kappa = \frac{N}{2}$, we need $j_2 \neq 0$, and $i_0 = \frac{N}{2} - 1$. We also have the defining restriction (with $z = 0$): $j_2 > j_1 + (m - m^*)/N$. Combining all, we have:

$$\begin{aligned} d \ &= \ 2 + 2j_2 + \tfrac{2}{N} m^* \ , \\ \chi_N \ &= \ \{\, 2j_1\, ;\, 0, \ldots, 0,\, r_{\frac{N}{2}}\, , \ldots, r_{N-1}\, ;\, 2j_2 \,\} \ , \\ j_2 \ &> \ j_1 + \sum_{k=\frac{N}{2}+1}^{N-1} (\tfrac{2}{N} k - 1) r_k \ . \end{aligned} \tag{4.72}$$

In case **b** to achieve $\kappa = \frac{N}{2}$, we need $i_0 = \frac{N}{4} - 1$. Thus, this case is possible only if N is divisible by 4. We also have the defining restriction (with $z = 0$): $(m^* - m)/N > j_1 + 1$. Combining all, we have:

$$d = \frac{2}{N}m^* , \qquad \chi_N = \{ 2j_1 ; 0, \ldots, 0, r_{\frac{N}{4}}, \ldots, r_{N-1} ; 0 \} , \tag{4.73}$$
$$\sum_{k=\frac{N}{4}}^{\frac{N}{2}-1} (1 - \tfrac{2}{N}k) r_k > 1 + j_1 + \sum_{k=\frac{N}{2}+1}^{N-1} (\tfrac{2}{N}k - 1) r_k .$$

The conjugated cases **c,d** are presented shortly:

In case **c** : $j_1 \neq 0$, $i_0' = \frac{N}{2} - 1$, $j_1 > j_2 + (m^* - m)/N \implies$

$$d = 2 + 2j_1 + \tfrac{2}{N}m , \tag{4.74}$$
$$\chi_N = \{ 2j_1 ; r_1, \ldots, r_{\frac{N}{2}}, 0, \ldots, 0 ; 2j_2 \} ,$$
$$j_1 > j_2 + \sum_{k=1}^{\frac{N}{2}-1} (1 - \tfrac{2}{N}k) r_k .$$

In case **d** : $i_0' = \frac{N}{4} - 1$, N is divisible by 4, $(m - m^*)/N > j_2 + 1$. Combining all, we have:

$$d = \frac{2}{N}m , \qquad \chi_N = \{ 0 ; r_1, \ldots, 0, r_{\frac{3N}{4}}, 0, \ldots, 0 ; 2j_2 \} , \tag{4.75}$$
$$\sum_{k=\frac{N}{2}+1}^{\frac{3N}{4}} (\tfrac{2}{N}k - 1) r_k > 1 + j_2 + \sum_{k=1}^{\frac{N}{2}-1} (1 - \tfrac{2}{N}k) r_k .$$

The case **ac** has several subcases depending on j_1, j_2 being zero or not:

- The subcase $j_1 = j_2 = 0$ is possible only for $N = 4$ considered above.
- In the subcase $j_1 = 0$, $j_2 \neq 0$ should hold $i_0 = \frac{N}{2} - 2$, $i_0' \leq \frac{N}{2} - 2$.

Altogether we have:

$$d = 2 + j_2 + m_1 , \tag{4.76}$$
$$\chi_N = \{ 0 ; 0, \ldots, 0, r_{\frac{N}{2}-1}, \ldots, r_{N-1} ; 2j_2 \} ,$$
$$j_2 + \tfrac{2}{N} r_{\frac{N}{2}-1} = \sum_{k=\frac{N}{2}+1}^{N-1} r_k (\tfrac{2}{N}k - 1) .$$

• In the conjugate subcase $j_1 \neq 0,\ j_2 = 0$ should hold $i_0 \leq \frac{N}{2} - 2$, $i'_0 = \frac{N}{2} - 2 \implies$

$$\begin{aligned} d &= 2 + j_1 + m_1 \ , \\ \chi_N &= \{\, 2j_1 \,;\, r_1, \dots, r_{\frac{N}{2}+1}\,, 0, \dots, 0 \,;\, 0 \,\} \ , \\ j_1 + \tfrac{2}{N} r_{\frac{N}{2}+1} &= \sum_{k=1}^{\frac{N}{2}-1} r_k (1 - \tfrac{2}{N}k) \ . \end{aligned} \tag{4.77}$$

• In the subcase $j_1 j_2 \neq 0$ should hold $i_0 + i'_0 = \frac{N}{2} - 2$, χ_N is in general position, and we have:

$$\begin{aligned} d &= 2 + j_1 + j_2 + m_1 \ , \\ j_2 - j_1 &= (m - m^*)/N \ . \end{aligned} \tag{4.78}$$

In case **ad** we need $\kappa = \frac{N}{2} = 3 + i_0(1 - \delta_{j_2,0}) + 2i'_0$, while from the condition $z = 0$ follows:

$$1 + j_2 \ = \ (m - m^*)/N \ = \sum_{k=1+i_0}^{N-1-i'_0} r_k (\tfrac{2}{N}k - 1) \ , \tag{4.79}$$

and then we have:

$$d \ = \ \tfrac{2}{N} m \ = \ \tfrac{2}{N} \sum_{k=1+i_0}^{N-1-i'_0} k r_k \ . \tag{4.80}$$

The subcase $j_2 = 0$ leads to the restriction that $N = 6, 10, \ldots$, and $i'_0 = \frac{1}{2}(\frac{N}{2} - 3)$, and then:

$$j_2 = 0 \implies 1 = \sum_{k=1+i_0}^{\frac{1}{4}(3N+2)} r_k (\tfrac{2}{N}k - 1) \ , \quad d \ = \ \tfrac{2}{N} \sum_{k=1+i_0}^{\frac{1}{4}(3N+2)} k r_k \ . \tag{4.81}$$

In case **bc** we need $\kappa = \frac{N}{2} = 3 + i'_0(1 - \delta_{j_1,0}) + 2i_0$, while from the condition $z = 0$ follows:

$$1 + j_1 \ = \ (m^* - m)/N \ = \sum_{k=1+i_0}^{N-1-i'_0} r_k (1 - \tfrac{2}{N}k) \ , \tag{4.82}$$

and then we have:

$$d \;=\; \tfrac{2}{N} m^* \;=\; \tfrac{2}{N} \sum_{k=1+i_0}^{N-1-i_0'} (N-k) r_k \ . \tag{4.83}$$

The subcase $j_1 = 0$ leads to the restriction that $N = 6, 10, \ldots$, and $i_0 = \frac{1}{2}(\frac{N}{2} - 3)$, and then:

$$j_1 = 0 \implies 1 = \sum_{k=\frac{1}{4}(3N+2)}^{N-1-i_0'} r_k (1 - \tfrac{2}{N} k) \ , \quad d \;=\; \tfrac{2}{N} \sum_{k=\frac{1}{4}(3N+2)}^{N-1-i_0'} (N-k) r_k \ . \tag{4.84}$$

In case **bd** we need $\kappa \;=\; \frac{N}{2} \;=\; 4 + 2i_0 + 2i_0'$, thus $i_0 + i_0' = \frac{N}{4} - 2$, thus $N = 8, 12, \ldots$. From $z = 0$ follows that $m = m^* = \frac{N}{2} m_1$, and then

$$d \;=\; m_1 \;=\; \sum_{k=1+i_0}^{1+i_0+\frac{3N}{4}} r_k \ , \quad j_1 = j_2 = 0 \ . \tag{4.85}$$

Note that according to the result of Subsection 2.2 the following $\frac{1}{8}$-BPS cases would not be protected: case **ad** when $r_1 \neq 0$; case **bc** when $r_{N-1} \neq 0$; case **bd** when $r_1, r_{N-1} > 2$.

5 Outlook

In the present paper, we presented the classification of BPS states in D=4 conformal supersymmetry. We gave also the necessary conditions for the protected states. Our considerations are group-theoretic and model-independent.

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