The ground state energy of a charged particle on a Riemann surface

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Abstract

It is shown that the quantum ground state energy of particle of mass m and electric charge e moving on a compact Riemann surface under the influence of a constant magnetic field of strength B is $E_0 = \frac{eB}{2m}$. Remarkably, this formula is completely independent of both the geometry and topology of the Riemann surface. The formula is obtained by reinterpreting the quantum Hamiltonian as the second variation operator of an associated classical variational problem.

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Consider a point particle of mass m and electric charge e moving on a compact Riemann surface Σ under the influence of a uniform magnetic field of strength B. The purpose of this letter is to show that the ground state energy of such a particle, in nonrelativistic quantum mechanics, is

$$E_0 = \frac{eB}{2m} \tag{1}$$

where we have chosen to use natural units ($\hbar = c = 1$). The remarkable thing about this formula is that it is completely independent of the choice of surface Σ; not only is it independent of the *metric* on Σ, and hence of local details of the shape of Σ, it is also independent of the *genus* of Σ.

The equivalent problem on euclidean $\mathbb{R}^2$ is, of course, well understood [5], the whole energy spectrum being easily computed,

$$E_n^{(\mathbb{R}^2)} = \left(n + \frac{1}{2}\right)\frac{eB}{m}, \quad n = 0, 1, 2, \dots. \tag{2}$$

Note that $E_0^{(\mathbb{R}^2)}$, known in condensed matter contexts as the energy of the *first Landau level*, coincides precisely with the ground state energy on a compact domain, (1). So compactifying space leaves the ground state energy completely unchanged. This seems to be a special property of just E_0 which does not hold for E_n with $n \geq 1$. Indeed, in the case where $\Sigma = S^2$ with

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the round metric of radius R, Haldane [3] has exploited the $SO(3)$ symmetry to obtain the full spectrum

$$E_n^{(S^2)} = \left(n + \frac{1}{2}\right)\frac{eB}{m} + \frac{n(n+1)}{2mR^2}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

which, for $n \geq 1$, agrees with $E_n^{(\mathbb{R}^2)}$ only in the limit $R \to \infty$. So there is something special about E_0 which protects it against change even under topology-changing deformations of the domain. It would be interesting to see whether the same formula for E_0 holds on arbitrary complete oriented two-manifolds, without the assumption of compactness. The example of $\Sigma = \mathbb{R}^2$ suggests it may, although our argument relies strongly on compactness of Σ.

Our method is to reinterpret the quantum Hamiltonian as (part of) the second variation of the energy of a related gauge theory, then use a known phase transition in this field theory to deduce the lowest eigenvalue. The phase transition is analogous to that which occurs in a type II superconductor at the upper critical magnetic field H_{c2}, where the normal state becomes stable and energetically preferred over the Abrikosov vortex lattice [7]. To the best of our knowledge this is the first time that the logic of the stability analysis has been inverted in this fashion: usually one uses spectral properties of a differential operator, possibly reinterpreted as a quantum Hamiltonian, to deduce stability properties of the classical system, whereas we argue in exactly the opposite direction.

We begin by defining the quantum Hamiltonian in local coordinates. Let x_1, x_2 be isothermal local coordinates on Σ, so that the metric is locally

$$g = \Omega(x_1, x_2)^2(dx_1^2 + dx_2^2) \tag{4}$$

for some smooth function Ω. The magnetic field is $B = \Omega^{-2}(\partial_1 A_2 - \partial_2 A_1)$ where $A = A_1 \mathrm{d}x_1 + A_2 \mathrm{d}x_2$ is a local gauge potential. The quantum Hamiltonian of a particle of mass m and electric charge e moving on Σ in this background field is

$$H\psi = -\frac{1}{2m\Omega^2}(\partial_i - ieA_i)(\partial_i - ieA_i)\psi. \tag{5}$$

It is this operator, in the case where B is constant, whose lowest eigenvalue we claim is E_0, as in (1).

To proceed further, it is convenient to formulate things in a global, coordinate free language. If $B \neq 0$ then, since Σ is compact, one should not think of A as a one-form on Σ, but rather as the local coordinate expression of a metric connexion ∇ on a hermitian line bundle (L, h) over Σ. The wave function ψ is not a mapping $\Sigma \to \mathbb{C}$, but rather a section of L. Explicitly, let h be the fibre metric on L and ε be a local unit section of L (that is, $|\varepsilon|^2 = h(\varepsilon, \varepsilon) = 1$). Then the connexion ∇ acts on an arbitrary local section $\varphi = f\varepsilon$ as

$$\nabla_X(f\varepsilon) = (X[f] - ieA(X)f)\varepsilon, \tag{6}$$

where $X \in T_p\Sigma$. Reality of A ensures that ∇ is metric compatible, that is, $X[h(\varphi, \psi)] = h(\nabla_X\psi, \psi) + h(\psi, \nabla_X\varphi)$. Associated with ∇ are an exterior differential operator $\mathrm{d}^\nabla : \Omega^p(L) \to \Omega^{p+1}(L)$ and its L^2 adjoint (the coderivative) $\delta^\nabla : \Omega^p(L) \to \Omega^{p-1}(L)$, where $\Omega^p(L)$ denotes the space of p-forms on Σ taking values in L. Explicitly, given any $\varphi \in \Gamma(L)$ and $\lambda \in \Omega^p(\Sigma)$,

$$(\mathrm{d}^\nabla\varphi)(X) = \nabla_X\varphi, \qquad \mathrm{d}^\nabla(\varphi\lambda) = (\mathrm{d}^\nabla\varphi) \wedge \lambda + \varphi\mathrm{d}\lambda, \tag{7}$$

and $\delta^\nabla = -*\mathrm{d}^\nabla *$ where $*$ is the Hodge isomorphism $\Omega^p(\Sigma) \to \Omega^{2-p}(\Sigma)$ induced by the metric g. One sees immediately that in this language

$$H\psi = \frac{1}{2m}\delta^\nabla \mathrm{d}^\nabla \psi = \frac{1}{2m}\Delta^\nabla \psi \tag{8}$$

where Δ^∇ denotes the natural laplacian operator on (L, h, ∇). This laplacian is manifestly non-negative, and is known to be elliptic [1], so its spectrum is discrete, non-negative, and each eigenvalue has finite multiplicity. Hence, it has a lowest eigenvalue $\lambda_0 \geq 0$, and $E_0 = \frac{\lambda_0}{2m}$.

It is not hard to show that $\lambda_0 \geq eB$. The curvature F^∇ of ∇ is $\mathrm{d}^\nabla \mathrm{d}^\nabla \in \Omega^2(End(L))$ which can be identified globally with an imaginary 2-form, coinciding locally with

$$F^\nabla = -ie\mathrm{d}A = -ieB\mathrm{vol}_\Sigma \tag{9}$$

where vol_Σ is the volume form on (Σ, g). It is well known that

$$n = \int_\Sigma \frac{iF^\nabla}{2\pi} = \frac{e}{2\pi}\int_\Sigma B\mathrm{vol}_\Sigma \tag{10}$$

is an integer topological invariant, the degree of the line bundle L. Since B is uniform, this implies

$$B = \frac{2\pi n}{e\mathsf{Area}(\Sigma)} \tag{11}$$

which, for the case $\Sigma = S^2$ with the round metric, coincides with the celebrated Dirac quantization condition (one can interpret B as being the uniform field produced by a magnetic monopole placed at the centre of the sphere). Now, given any section $\psi \in \Gamma(L)$, we can define $\widehat{\psi} = -i\psi\mathrm{vol}_\Sigma \in \Omega^2(L)$. Then

$$\langle \delta^\nabla \widehat{\psi}, \mathrm{d}^\nabla \psi\rangle = \langle \widehat{\psi}, \mathrm{d}^\nabla \mathrm{d}^\nabla \psi\rangle = \langle -i\psi\mathrm{vol}_\Sigma, -ieB\psi\mathrm{vol}_\Sigma\rangle = eB\|\psi\|^2. \tag{12}$$

But, by Cauchy-Schwarz,

$$\langle \delta^\nabla \widehat{\psi}, \mathrm{d}^\nabla \psi\rangle \leq \|\delta^\nabla \widehat{\psi}\|\|\mathrm{d}^\nabla \psi\| = \|\mathrm{d}^\nabla \psi\|^2 = \langle \psi, \Delta^\nabla \psi\rangle. \tag{13}$$

Hence,

$$\langle \psi, \Delta^\nabla \psi\rangle \geq eB\|\psi\|^2, \tag{14}$$

whence $\lambda_0 \geq eB$ as claimed. Formula (1) is equivalent to the statement that the topological lower energy bound (14) is attained, which, in turn, is equivalent to the statement that there exists a nonzero section $\psi \in \Gamma(L)$ with

$$*\mathrm{d}^\nabla \psi = i\mathrm{d}^\nabla \psi, \tag{15}$$

because equality holds in the Cauchy-Schwarz inequality if and only if $\delta^\nabla \widehat{\psi} = c\mathrm{d}^\nabla \psi$ for some $c > 0$, and $\|\delta^\nabla \widehat{\psi}\| = \|\mathrm{d}^\nabla \psi\|$, so $c = 1$. Perhaps a direct proof that (15) has a nontrivial solution is possible, but we shall instead determine λ_0 by an indirect argument.

It is convenient henceforth to allow ∇ to denote a general metric connexion on (L, h), and denote by ∇_0 any metric connexion with uniform B (for $\Sigma \neq S^2$, such connexions are not unique). Consider the variational problem which assigns to a section φ of L and a connexion ∇ the energy

$$E(\varphi, \nabla) = \frac{1}{2}\|\mathrm{d}^\nabla \varphi\|^2 + \frac{1}{2}\|iF^\nabla\|^2 + \frac{1}{8}\|\tau - h(\varphi, \varphi)\|^2 \tag{16}$$

where $\|\cdot\|$ denotes L^2 norm and $\tau > 0$ is a positive constant. This is the abelian Higgs model on Σ and was studied (in a rather more general setting) by Bradlow [2]. The field equations are obtained by demanding that

$$\frac{d}{dt}E(\varphi_t, \nabla_t)\bigg|_{t=0} = 0 \tag{17}$$

for all smooth variations of φ, ∇. Defining $\eta = \partial_t \varphi_t|_{t=0} \in \Gamma(L)$ and $\alpha = i\partial_t \nabla_t|_{t=0} \in \Omega^1(\Sigma)$, we see that

$$\frac{d}{dt}E(\varphi_t, \nabla_t)\bigg|_{t=0} = \langle \delta^\nabla \mathrm{d}^\nabla \varphi, \eta\rangle - \langle j_\varphi, \alpha\rangle + \langle i\delta F^\nabla, \alpha\rangle - \frac{1}{2}\langle(\tau - h(\varphi,\varphi))\varphi, \eta\rangle \tag{18}$$

where $\langle\cdot,\cdot\rangle$ denotes L^2 inner product, j_φ is the "supercurrent" one form

$$j_\varphi(X) = h(\nabla_X \varphi, i\varphi), \tag{19}$$

and $\delta = -*\mathrm{d}*$ is the L^2 adjoint of d. So (φ, ∇) is a critical point of E if and only if

$$\delta^\nabla \mathrm{d}^\nabla \varphi = \frac{1}{2}(\tau - h(\varphi,\varphi))\varphi, \qquad \delta(iF^\nabla) = j_\varphi. \tag{20}$$

Hence $(0, \nabla)$ is a critical point of E for all $\tau > 0$ provided $\delta F^\nabla = 0$, that is, provided B is constant. So $(0, \nabla_0)$ is a critical point of E for all $\tau > 0$.

Let us consider how the stability properties of the critical point $(0, \nabla_0)$ depend on τ. To determine whether a critical point of E is stable, we compute the second variation of E about that critical point [1, 4]. So, let $(\varphi_{s,t}, \nabla_{s,t})$ be a smooth two-parameter variation of $(0, \nabla_0)$, with infinitesimal variations $\eta = \partial_t \varphi_{s,t}|_{(0,0)}, \nu = \partial_s \varphi_{s,t}|_{(0,0)} \in \Gamma(L)$ and $\alpha = i\partial_t \nabla_{s,t}|_{(0,0)}, \beta = i\partial_s \nabla_{s,t}|_{(0,0)} \in \Omega^1(\Sigma)$. Then, from (18) we have

$$\frac{\partial^2 E(\varphi_{s,t}, \nabla_{s,t})}{\partial s \partial t}\bigg|_{s=t=0} = \langle \delta^{\nabla_0}\mathrm{d}^{\nabla_0}\nu, \eta\rangle + \langle \delta \mathrm{d}\beta, \alpha\rangle - \frac{\tau}{2}\langle \nu, \eta\rangle. \tag{21}$$

The critical point $(0, \nabla_0)$ is stable if the associated quadratic form on $\Gamma(L \oplus T^*\Sigma)$,

$$Q(\eta, \alpha) = \langle(\delta^{\nabla_0}\mathrm{d}^{\nabla_0} - \frac{\tau}{2})\eta, \eta\rangle + \langle \delta \mathrm{d}\alpha, \alpha\rangle \tag{22}$$

is non-negative. Clearly $(0, \nabla_0)$ is stable against all variations of ∇, but is stability against variations of φ only while $0 < \tau \leq 2\lambda_0$, becoming unstable when $\tau > 2\lambda_0$.

Now, it is known from work of Bradlow [2] that for all

$$\tau > \tau_0 = \frac{4\pi n}{\mathsf{Area}(\Sigma)} \tag{23}$$

the global minimum of E is attained by a n-vortex solution (a certain section φ and connexion ∇ satisfying a first order system of PDEs, called Bogomol'nyi equations, which imply the field equations). Furthermore, in the limit that $\tau \to \tau_0$ from above, these vortex solutions converge to a uniform solution $(0, \nabla_0)$. Hence, $(0, \nabla_0)$ is stable precisely at $\tau = \tau_0$ (since it globally minimizes E), but becomes unstable for $\tau > \tau_0$ (since the lower energy n-vortex branch bifurcates off at $\tau = \tau_0$). Comparing with our linear stability analysis, we deduce that $\tau_0 = 2\lambda_0$. But recall that the quantum Hamiltonian of interest is $H = \frac{1}{2m}\Delta_0$, whose lowest eigenvalue is thus

$$E_0 = \frac{1}{2m}\lambda_0 = \frac{\tau_0}{4m}. \tag{24}$$

Combining this with (23) and (11) gives the formula claimed (1). As an aside, we note that, since the bound (14) is attained by the ground state wavefunction, it must satisfy (15). This reduces the problem of constructing the ground state wavefunction to solving a first order linear PDE. It would be interesting to see whether recent work by Manton and Romao on the geometry of vortices in the limit $\tau \to \tau_0$ yields any useful information about this ground state [6].

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