

Optimal dividend control for a generalized risk model with
investment incomes and debit interest

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Abstract

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This paper investigates dividend optimization of an insurance corporation under a more realistic model which takes into consideration refinancing or capital injections. The model follows the compound Poisson framework with credit interest for positive reserve, and debit interest for negative reserve. Ruin occurs when the reserve drops below the critical value. The company controls the dividend pay-out dynamically with the objective to maximize the expected total discounted dividends until ruin. We show that the optimal strategy is a band strategy and it is optimal to pay no dividends when the reserve is negative.

Key words Absolute ruin, dividend optimization, stochastic control, value function, viscosity solution.

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1 Introduction

Dividend optimization problems for financial and insurance corporations have attracted extensive attention over the last few decades. One of this type of problems is to find the optimal dividend pay-out scheme, i.e. choosing the times and amounts of dividend payments to maximize the objective function - the expected total discounted dividend pay-outs until the time of ruin.

In the area of non-life insurance, a well established model for the cash reserve is the Cramér-Lundberg model (also called the compound Poisson model or the classical risk model), which is based on Poisson claim-arrivals and linear premium income. Embrechts and Schmidli (1994) claimed that “many of the ‘rules of thumb’ used in practice can be traced back to the classical Cramér-Lundberg model”. However, starting from the middle of 1990’s, a large number of papers dealing with optimization problems for insurance companies, use the diffusion process - a limiting process of the Cramér-Lundberg model, to model the reserve in the absence of the dividends, e.g. Jeanblanc-Picque and Shiryaev (1995), Cadenillas et al. (2006) and Paulsen (2007). Diffusion process modeling of the reserve process allows the use of optimal diffusion control techniques and is therefore more mathematically tractable. A survey of optimal dividend control for diffusion processes can be found in Taksar (2000).

There have been a few attempts to study the dividend optimization problem under the Cramér-Lundberg model. Gerber (1969) considered the dividends optimization problem for a classical Cramér-Lundberg model and proved that the corresponding optimal dividend strategy is a band strategy. Azcue and Muler (2005) considered the Cramér-Lundberg model with reinsurance and dividend payments and proved the optimal dividend payment policy maximizing the expected total discounted dividend pay-outs is also a band strategy. Albrecher and Thonhauser (2008) studied the dividend optimization problem in the Cramér-Lundberg setting including constant force of interest and pointed out that the optimal strategy is also of band type. Kulenko and Schmidli (2008) found that the optimal dividend strategy for the Cramér-Lundberg model with capital injections is a barrier strategy. For applications of stochastic control in insurance, please refer to Schmidli (2008) and references therein. A list of literature on dividend optimization problems under the Cramér-Lundberg model can be found in Albrecher and Thonhauser (2009). For a review of dividend strategies in the actuarial literature, see Avanzi (2009).

There has been extensive work dedicated to the generalization of the classical risk model to suit more realistic situations. One way of generalization is to allow the company to refinance

when the company is in deficit and the deficit is not too large. The idea was developed by Borch (1969), where he proposed that ruin (negative reserve) does not mean the end of game but only the necessity of raising additional money. He argued that “insurance companies get into difficulties fairly regularly and rescue operations are considered in the insurance world, if not daily, at least annually” and that it will be a good investment to rescue a company when the situation is not too serious, and concluded that a company should be rescued if the benefits exceed the cost of the new financing required, e.g. when the deficit is not too large. Since then the “absolute ruin model” has been developed, where the company is allowed to borrow money to settle the claims if the reserve is negative but still above the critical level so that it can continue its business. The company will need to pay interest (debit interest) on the loan and pay back debt interest continuously from the received premiums. The critical level is the value of reserve below which the premiums received are insufficient to cover interest payments on the debt. Absolute ruin occurs when the reserve reaches or drops below the critical level for the first time.

The absolute ruin problem has received considerable attention. Gerber (1971) studied the absolute ruin probability in the compound Poisson model. Embrechts and Schmidli (1994) considered the absolute ruin probability when the reserve process is a piecewise-deterministic Markov process. Dickson and Egídio dos Reis (1997) used simulation to study the Cramér-Lundberg model with absolute ruin. Cai et al. (2006) studied an Ornstein-Uhlenbeck type model with credit and debit interest. Cai (2007) discussed the Gerber-Shiu function in the classical risk model with absolute ruin. Gerber and Yang (2007) investigated the absolute ruin probability based on the classical risk model perturbed by diffusion with investment. Zhu and Yang (2008) studied the asymptotic behavior of the absolute ruin probability in the Cramér-Lundberg model with credit and debit interest. Some other related references are Yuen et al. (2008) and Wang and Yin (2009).

In this paper, we consider the dividend optimization under the the compound Poisson model with credit interest for positive reserve, and debit interest for negative reserve. The paper is organized as follows. Section 2 presents the model and formulates the dividend optimization problem. In section 3, we derive some basic and important properties of the value function, and characterize the value function as the unique nonnegative and nondecreasing viscosity super-solution of the associated Hamilton-Jacobi-Bellman equation that satisfies a linear growth condition and a boundary condition. In section 4, we prove the existence of the optimal dividend strategy and identify the optimal dividend payout scheme as a band strategy. It is shown that the optimal strategy is to pay no dividends when the reserve is negative. A conclusion is provided in section 5.

2 The model and the optimization problem

Consider a continuous time model for the surplus of an insurance company where claims arrive according to a Poisson process with intensity rate λ and premiums are collected continuously at the rate p . The amount of each claim is independent of its arrival time, and is also independent of any other claims. Let S_i denote the arrival epoch of the i th claim and U_i its size. Let $N(t) = \#\{i : S_i \leq t\}$. Then $N(t)$ is the number of claims up to time t and follows a Poisson process with rate λ . The sequence $\{U_i\}$ is assumed to be identically and independently distributed with distribution function $F(\cdot)$ and independent of $\{N(t)\}$. Moreover, the insurance company earns credit interest under a constant force r ($r > 0$) when the surplus is positive, and when the surplus drops below 0, the insurer could borrow money with the amount equal to the deficit under force of debit interest $\alpha > r$. In the mean time, the insurer will repay the debts and the debt interest continuously from the premium incomes. This leads to the following dynamics for the risk reserve process $\{X_t\}_{t \geq 0}$ in the absence of

dividend payments:

$$dX_t = (p + rX_{t-}I\{X_{t-} \geq 0\} + \alpha X_{t-}I\{X_{t-} < 0\})dt - dY_t,$$

where X_t represents the surplus at time t and $Y_t = \sum_{i=1}^{N(t)} U_i$ is the aggregate claim up to time t .

Now suppose the company pays dividends to its shareholders with the accumulative amount of dividends paid up to time t being denoted by L_t . Let \bar{R}_t^L denote the controlled reserve at time t . Then

$$d\bar{R}_t^L = (p + r\bar{R}_{t-}^L I\{\bar{R}_{t-}^L \geq 0\} + \alpha\bar{R}_{t-}^L I\{\bar{R}_{t-}^L < 0\})dt - d\left(\sum_{i=1}^{N(t)} U_i\right) - dL_t. \quad (2.1)$$

The company controls dynamically the dividend pay-outs: the times and the amounts of dividends to be paid out. A control strategy is described by a dividend distribution process $L = \{L_t\}_{t \geq 0}$.

Notice from the above dynamics that the premium incomes will no longer be able to cover the debts when the surplus is less than or equal to $-\frac{p}{\alpha}$. That is, the surplus process will not be able to return to a positive amount whenever the process hits $-\frac{p}{\alpha}$ or any level below that. We call $-\frac{p}{\alpha}$ the critical value and define *the time of ruin* as $T^L = \inf\{t \geq 0 : \bar{R}_t^L \leq -\frac{p}{\alpha}\}$. The time of ruin defined above is also called the time of absolute ruin in the sense that the surplus will no longer be able to return to a positive level.

All our random quantities are defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let \mathcal{N} denote the class of null sets in Ω and define $\mathcal{F}_t = \sigma(X_0, Y_s, 0 \leq s \leq t) \vee \mathcal{N}$. Throughout the paper, we base our study on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

A control strategy is *admissible* if the process $\{L_t\}_{t \geq 0}$ with $L_0 = 0$, is predictable, nondecreasing, left continuous with right limits (cáglád) and satisfies the requirement that paying dividends would not cause ruin immediately. We use Π to denote the set of all admissible strategies.

Define $E_x[\cdot] = \mathbb{E}[\cdot | \bar{R}_0 = x]$. Let δ be the force of discount with $\delta > r$. Given the initial reserve x , the performance of a dividend strategy L is measured by the expectation of the cumulative discounted dividends until ruin, i.e.

$$V_L(x) = \mathbb{E}_x \left[\int_0^{T^L} e^{-\delta s} dL_s \right]. \quad (2.2)$$

The integral here is interpreted path-wise in a Lebesgue-Stieltjes sense. The function $V_L(x)$ is called the *return function*. Obviously, $V_L(x) = 0$ for $x \leq -\frac{p}{\alpha}$.

The objective of the company is to find an optimal dividend payout scheme L in the set of admissible strategies Π such that the expectation of total discounted dividend pay-outs until the time of ruin is maximized.

Define the *value function* (also called the *optimal return function*) by $V(x) = \sup_{L \in \Pi} V_L(x)$. If there exists a control strategy L^* such that $V(x) = V_{L^*}(x)$, then L^* is called the optimal dividend distribution process (the optimal dividend strategy).

It can be seen that T^L is a stopping time. In the paper, we will consider the stopped process $R_t^L = \bar{R}_t^L I\{t < T^L\} - \frac{p}{\alpha} I\{t \geq T^L\}$.

To simplify the notation we will omit the superscripts L in T^L and R^L .

Since the reserve process in the absence of the control variable is a Markov process, the problem here is the optimization problem for a controlled Markov process. As the cumulative dividend process L may not be continuous with respect to time, the optimization problem is a singular control problem. In the context of stochastic control theory, the optimization problem can be associated with a Hamilton-Jacobi-Bellman (HJB) equation derived by using the Dynamic Programming Principle. In this case, the HJB equation is a first-order

integro-differential equation. However, the differentiability of the value function is a question. Actually, even under a specifically predetermined dividend strategy, the differentiability of the corresponding return function can not be guaranteed. It was shown in Zhu and Yang (2009) that the differentiability of the return function under a barrier or threshold dividend strategy depends on the level of smoothness of the claim size distributions. In this paper, we show that the value function is absolutely continuous but may not be differentiable. So we resort to the concept of viscosity solutions.

Based on techniques of probability and Stochastic Control theory, we show that the value function is a viscosity solution of the associated HJB equation and it is the unique solution satisfying certain regularity and boundary conditions. We also prove that the optimal dividend payment strategy exists and is of a band type, and that it is optimal to pay no dividends at all when the surplus is negative. Proofs of some lemmas and theorems are relegated to the appendix.

3 The value function

In this section, we derive some analytical properties of the value function $V(x)$. We show that $V(x)$ is not necessarily differentiable everywhere, but almost everywhere, and that the value function is the viscosity solution to the associated HJB equation but not necessarily the classical solution. It will also be proven that the value function is the unique solution satisfying certain conditions.

Theorem 3.1 *If $r < \delta$, $V(x) \geq x + \frac{p}{\alpha}$ for $x \in \mathbb{R}$, and $V(x) \leq \frac{\delta x + p}{\delta - r} + \frac{p}{\alpha}$ for $x \geq 0$.*

Proof. To prove the lower bound, consider a dividend payout scheme such that the part of initial reserve in excess of the critical value $-\frac{p}{\alpha}$ is paid out immediately as dividends. Then ruin occurs immediately. In this case, the return function given the initial reserve x , is $x + \frac{p}{\alpha}$. So the optimal return function $V(x)$ is always greater than or equal to $x + \frac{p}{\alpha}$.

From (2.1) we can see that given that the initial reserve is nonnegative, the inequality $dR_t \leq (p + rR_{t-})dt$ holds. As a result, given $R_0 = x$ we have $R_t \leq \frac{1}{r}(e^{rt}(p + rx) - p)$ for $x \geq 0$. Hence, by integration by parts, $L_t \leq R_t + \frac{p}{\alpha}$ for any $L \in \Pi$ and the definition (2.2) we can obtain $V_L(x) = \mathbb{E}_x \left[\int_0^T e^{-\delta s} dL_s \right] \leq \mathbb{E}_x \left[\int_0^\infty \delta L_s e^{-\delta s} ds \right] = \frac{\delta x + p}{\delta - r} + \frac{p}{\alpha}$ for $x \geq 0$. \square

Define

$$t_0(x, y) = \begin{cases} \frac{1}{r} \log\left(\frac{ry+p}{rx+p}\right) & y > x \geq 0 \\ \frac{1}{r} \log\left(\frac{ry+p}{p}\right) + \frac{1}{\alpha} \log\left(\frac{p}{\alpha x+p}\right) & y > 0 > x > -\frac{p}{\alpha} \\ \frac{1}{\alpha} \log\left(\frac{\alpha y+p}{\alpha x+p}\right) & 0 \geq y > x > -\frac{p}{\alpha} \end{cases} \quad (3.3)$$

The quantity $t_0(x, y)$ is equivalent to the time it takes for the surplus process with initial value x to reach y ($y > x$) for the first time given that there are no claims and no dividends paid out.

Theorem 3.2 *The value function V satisfies the following inequalities*

$$y - x \leq V(y) - V(x) \leq V(x) \begin{cases} \left(\left(\frac{ry+p}{rx+p} \right)^{\frac{\lambda+\delta}{r}} - 1 \right) & y > x \geq 0 \\ \left(\left(\frac{ry+p}{p} \right)^{\frac{\lambda+\delta}{r}} \left(\frac{p}{\alpha x+p} \right)^{\frac{\lambda+\delta}{\alpha}} - 1 \right) & y \geq 0 > x > -\frac{p}{\alpha} \\ \left(\left(\frac{\alpha y+p}{\alpha x+p} \right)^{\frac{\lambda+\delta}{\alpha}} - 1 \right) & 0 > y > x > -\frac{p}{\alpha} \end{cases}$$

Proof. (i) We first prove the lower bound. For any $\epsilon > 0$, let $L_\epsilon(x)$ denote an admissible ϵ -optimal strategy given the initial reserve x , i.e. $V_{L_\epsilon(x)}(x) \geq V(x) - \epsilon$.

For $y > x > -\frac{p}{\alpha}$, given the initial reserve $R_0 = y$ we use $L(y, x)$ to denote a strategy that pays an amount $y - x$ as dividends immediately and then pays dividends according to the strategy $L_\epsilon(x)$. Then given the initial reserve $R_0 = y > x$, under the strategy $L(y, x)$ we have $V_{L(y,x)}(y) = y - x + V_{L_\epsilon(x)}(x)$. So for any $\epsilon > 0$, $V(y) \geq y - x + V_{L_\epsilon(x)}(x) \geq y - x + V(x) - \epsilon$. Consequently, $V(y) - V(x) \geq y - x$.

(ii) Now, we proceed to prove the upper bounds. For $y > x > -\frac{p}{\alpha}$, for the surplus process with initial reserve x , let $\tau(x, y)$ denote the time it will take for the surplus process to reach up to y for the first time, and define the strategy $\hat{L}(x, y)$ as follows:

- pay out no dividends until the reserve reaches y ,
- then at the moment that the reserve reaches y for the first time ($\tau(x, y)$), treat the reserve process as a new process that starts at this moment with initial capital y , and apply the strategy $L_\epsilon(y)$, i.e. $\theta_{\tau(x,y)}\hat{L}(x, y) = L_\epsilon(y)$.

Note that starting from the initial value $x > -\frac{p}{\alpha}$, ruin will not occur before the arrival of the first claim (S_1), and the reserve will reach y ($y > x$) at time $t_0(x, y)$ if no claims arrive before time $t_0(x, y)$, that is $\tau(x, y) = t_0(x, y)$ on $\{S_1 > t_0(x, y)\}$. Then for $y > x > -\frac{p}{\alpha}$ and for $\epsilon > 0$, by noticing that $V_{L_\epsilon(y)}(y) \geq V(y) - \epsilon$ we have

$$\begin{aligned} V(x) &\geq V_{\hat{L}(x,y)}(x) = \mathbb{E}_x[e^{-\delta\tau(x,y)}V_{L_\epsilon(y)}(y); \tau(x, y) \leq T] \\ &\geq \mathbb{E}_x[e^{-\delta\tau(x,y)}V_{L_\epsilon(y)}(y); S_1 > t_0(x, y)] \geq e^{-(\lambda+\delta)t_0(x,y)}(V(y) - \epsilon). \end{aligned}$$

Hence, $V(y) - V(x) \leq V(x)(e^{(\lambda+\delta)t_0(x,y)} - 1)$. This combined with (3.3) gives the upper bounds. \square

Theorem 3.3 *The value function $V(x)$ is nonnegative, nondecreasing, continuous on $[-\frac{p}{\alpha}, \infty)$ and locally Lipschitz continuous on $(-\frac{p}{\alpha}, \infty)$. Therefore, $V'(x)$ exists almost everywhere on $(-\frac{p}{\alpha}, \infty)$. Furthermore, $V'(x) \geq 1$, if $V'(x)$ exists.*

Proof. All the stated properties of $V(x)$ are direct results of Theorem 3.1 and Theorem 3.2 except for the right continuity of $V(x)$ at $x = -\frac{p}{\alpha}$.

To prove the right continuity, it is sufficient to show that $\limsup_{x \downarrow -\frac{p}{\alpha}} V(x) = 0$. If this is not true, then we can find a sequence $\{x_n\}$ with $x_n \downarrow -\frac{p}{\alpha}$ such that $\lim_{n \rightarrow \infty} V(x_n) > 0$, that is, there exists an $\epsilon_0 > 0$ and N such that $V(x_n) > \epsilon_0$ for all $n \geq N$. Let $L(x, \frac{\epsilon_0}{2})$ denote a $\frac{\epsilon_0}{2}$ -optimal strategy for the reserve process with initial reserve x , that is, $V_{L(x, \frac{\epsilon_0}{2})}(x) \geq V(x) - \frac{\epsilon_0}{2}$. Then, we have

$$V_{L(x_n, \frac{\epsilon_0}{2})}(x_n) \geq V(x_n) - \frac{\epsilon_0}{2} > \frac{\epsilon_0}{2} \quad \text{for } n \geq N. \quad (3.4)$$

Consider a stochastic process $\{R'_t\}$ with dynamics $dR'_t = (p + rR'_t I\{R'_t \geq 0\} + \alpha R'_t I\{R'_t < 0\})dt$. Given $R'_0 = x$, integration yields

$$R'_t \leq \begin{cases} \frac{1}{r}(e^{rt}(p + rx) - p) & x \geq 0 \\ \frac{1}{\alpha}(e^{\alpha t}(p + \alpha x) - p) & x < 0, t \leq t_0(x, 0) \\ \frac{1}{r}(e^{r(t-t_0(x,0))}p - p) & x < 0, t > t_0(x, 0) \end{cases} \quad (3.5)$$

Note that $R_t \leq R'_t$ given that $R_0 = R'_0 > -\frac{p}{\alpha}$. Using the fact that $L_t \leq R_t + \frac{p}{\alpha} \leq R'_t + \frac{p}{\alpha}$ and (3.5), by integration by parts it follows from (2.2) that for $x \in (-\frac{p}{\alpha}, 0)$

$$\begin{aligned} V_L(x) &\leq \mathbb{E}_x\left[\int_0^\infty \delta L_s e^{-\delta s} ds\right] \\ &\leq \delta \int_0^{t_0(x,0)} \frac{1}{\alpha}(e^{\alpha s}(p + \alpha x) - p) e^{-\delta s} ds \\ &\quad + \delta \int_{t_0(x,0)}^\infty \frac{1}{r}(e^{r(s-t_0(x,0))}p - p) e^{-\delta s} ds + \frac{p}{\alpha}. \end{aligned} \quad (3.6)$$

Notice that the expression on the right-hand side of (3.6) has limit 0 as $x \downarrow -\frac{p}{\alpha}$ and does not depend on L . So we can find an N' such that for all $n \geq N'$, $V_L(x_n) < \frac{\epsilon_0}{4}$ holds for all admissible strategy L . Therefore, setting L to be $L^{(x_n, \frac{\epsilon_0}{2})}$ gives $V_{L^{(x_n, \frac{\epsilon_0}{2})}}(x_n) < \frac{\epsilon_0}{4}$ for all $n \geq N'$, which is a contradiction to (3.4). Hence, the value function $V(x)$ is right continuous at $-\frac{p}{\alpha}$. \square

Applying standard arguments from stochastic control theory (e.g. Fleming and Soner (1993)) or an approach similar to that in Azcue and Muler (2005), we can show that the optimal value function fulfils the Dynamic Programming Principle:

$$V(x) = \sup_{L \in \Pi} \mathbb{E}_x \left[\int_0^{\tau \wedge T} e^{-\delta s} dL_s + e^{-\delta(\tau \wedge T)} V(R_{\tau \wedge T}) \right] \quad \text{for any stopping time } \tau,$$

and the associated Hamilton-Jacobi-Bellman (HJB) equation is

$$\max\{1 - V'(x), \mathcal{L}_V(x)\} = 0, \quad (3.7)$$

where \mathcal{L} is a generator defined by

$$\begin{aligned} \mathcal{L}_V(x) &= (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\}) V'(x) \\ &\quad - (\lambda + \delta)V(x) + \lambda \int_0^{x + \frac{p}{\alpha}} V(x - u) dF(u). \end{aligned} \quad (3.8)$$

Although from the last section we know that $V'(x)$ exists almost everywhere, we have no guarantee that $V(x)$ is differentiable for all $x > -\frac{p}{\alpha}$. Therefore, we can not expect $V(x)$ to be a classical solution to the HJB equation. In the following we will show that the value function $V(x)$ is a viscosity solution to the HJB equation (3.7), and that $V(x)$ is the unique nonnegative, nondecreasing and locally Lipschitz continuous viscosity solution of (3.7) satisfying a linear growth condition and the boundary condition $V(-\frac{p}{\alpha}) = 0$.

Definition 3.1 (i) A continuous function $\underline{u} : [-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ is said to be a viscosity sub-solution of (3.7) on $(-\frac{p}{\alpha}, \infty)$ if for any $x \in (-\frac{p}{\alpha}, \infty)$ each continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ with $\phi(x) = \underline{u}(x)$ such that $\underline{u} - \phi$ reaches the maximum at x satisfies

$$\max\{1 - \phi'(x), \mathcal{L}_\phi(x)\} \geq 0.$$

(ii) A continuous function $\bar{u} : [-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ is said to be a viscosity super-solution of (3.7) on $(-\frac{p}{\alpha}, \infty)$ if for any $x \in (-\frac{p}{\alpha}, \infty)$ each continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ with $\phi(x) = \bar{u}(x)$ such that $\bar{u} - \phi$ reaches the minimum at x satisfies

$$\max\{1 - \phi'(x), \mathcal{L}_\phi(x)\} \leq 0.$$

(iii) A continuous function $u : [-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ is a viscosity solution of (3.7) on $(-\frac{p}{\alpha}, \infty)$ if it is both a viscosity sub-solution and a viscosity super-solution on $(-\frac{p}{\alpha}, \infty)$.

For any continuously differentiable function ϕ and any continuous function v , define an operator $\mathcal{L}_{v, \phi}(x) = (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\}) \phi'(x) - (\lambda + \delta)v(x) + \lambda \int_0^{x + \frac{p}{\alpha}} v(x - u) dF(u)$. As has been shown in (Sayah (1991) and Benth et al. (2001)), the definition of viscosity sub and super solutions has the following alternative version.

Definition 3.2 (i) A continuous function $\underline{u} : [-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ is said to be a viscosity sub-solution of (3.7) on $(-\frac{p}{\alpha}, \infty)$ if for any $x \in (-\frac{p}{\alpha}, \infty)$ each continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ with $\phi(x) = \underline{u}(x)$ such that $\underline{u} - \phi$ reaches the maximum at x satisfies

$$\max\{1 - \phi'(x), \mathcal{L}_{\underline{u}, \phi}(x)\} \geq 0.$$

(ii) A continuous function $\bar{u} : [-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ is said to be a viscosity super-solution of (3.7) on $(-\frac{p}{\alpha}, \infty)$ if for any $(-\frac{p}{\alpha}, \infty)$ each continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ with $\phi(x) = \bar{u}(x)$ such that $\bar{u} - \phi$ reaches the minimum at x satisfies

$$\max\{1 - \phi'(x), \mathcal{L}_{\bar{u}, \phi}(x)\} \leq 0.$$

The following remarks are standard in the context of viscosity theory (eg Capuzzo-Dolcetta and Lions (1990) and Crandall et al. (1984)), which will be useful in the proof of our main results.

Remark 3.1 (i) For any viscosity sub-solution \underline{u} on $(-\frac{p}{\alpha}, \infty)$, there exists a continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ such that $\underline{u} - \phi$ reaches a maximum at $x > -\frac{p}{\alpha}$ with $\phi'(x) = q$ if and only if

$$\liminf_{y \uparrow x} \frac{\underline{u}(y) - \underline{u}(x)}{y - x} \geq q \geq \limsup_{y \downarrow x} \frac{\underline{u}(y) - \underline{u}(x)}{y - x}.$$

(ii) For any viscosity super-solution \bar{u} on $(-\frac{p}{\alpha}, \infty)$, there exists a continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ such that $\bar{u} - \phi$ reaches a minimum at $x > -\frac{p}{\alpha}$ with $\phi'(x) = q$ if and only if

$$\liminf_{y \downarrow x} \frac{\bar{u}(y) - \bar{u}(x)}{y - x} \geq q \geq \limsup_{y \uparrow x} \frac{\bar{u}(y) - \bar{u}(x)}{y - x}.$$

For any $t \geq 0$, define a functional \mathcal{M}_t by

$$\begin{aligned} \mathcal{M}_t(\phi) &= \sum_{s \leq t, R_s \neq R_{s-}} (\phi(R_s) - \phi(R_{s-})) e^{-\delta s} \\ &\quad - \lambda \int_0^t e^{-\delta s} ds \int_0^\infty (\phi(R_{s-} - y) - \phi(R_{s-})) dF(y). \end{aligned} \quad (3.9)$$

Then $\{\mathcal{M}_t(\phi)\}$ is a local martingale. If $\phi(\cdot)$ is bounded by a linear function, then $\mathcal{M}_t(\phi)$ is bounded below and therefore a super-martingale by applying Fatou's Lemma.

Consider any nonnegative and nondecreasing function ϕ and any stopping time τ such that $\phi'(R_t)$ exists for all $t \leq \tau$ and

$$\phi'(R_t) \geq 1 \text{ for all } t \leq \tau. \quad (3.10)$$

Let $\{L_t^c\}$ denote the continuous part of $\{L_t\}$. It can be seen that

$$\begin{aligned} \phi(R_\tau) e^{-\delta \tau} - \phi(R_0) &= \int_0^\tau d(\phi(R_t) e^{-\delta t}) \\ &= \int_0^\tau \phi'(R_t) e^{-\delta t} dR_t - \delta \int_0^\tau \phi(R_t) e^{-\delta t} dt \\ &= \int_0^\tau \phi'(R_t) e^{-\delta t} (p + r R_t I\{R_t \geq 0\} + \alpha R_t I\{R_t < 0\}) dt \\ &\quad - \int_0^\tau \phi'(R_t) e^{-\delta t} dL_t^c + \sum_{t \leq \tau, R_t \neq R_{t-}} (\phi(R_t) - \phi(R_{t-})) e^{-\delta t} \\ &\quad + \sum_{t < \tau, R_t \neq R_{t+}} (\phi(R_{t+}) - \phi(R_t)) e^{-\delta t} - \delta \int_0^\tau \phi(R_t) e^{-\delta t} dt, \end{aligned} \quad (3.11)$$

where the last equality follows from the fact that L_t is left-continuous and nondecreasing. Since $R_t \neq R_{t+}$ only occurs at the jumps of L_t and L_t is left-continuous in t , then $R_{t+} - R_t = -(L_{t+} - L_t)$ and

$$\sum_{t < \tau, R_t \neq R_{t+}} (\phi(R_{t+}) - \phi(R_t)) e^{-\delta t} = - \sum_{t < \tau, R_t \neq R_{t+}} e^{-\delta t} \int_0^{L_{t+} - L_t} \phi'(R_t - u) du. \quad (3.12)$$

Then by (3.12) and (3.10) we have

$$\begin{aligned}
& - \int_0^\tau \phi'(R_t) e^{-\delta t} dL_t^c + \sum_{t < \tau, R_t \neq R_{t+}} (\phi(R_{t+}) - \phi(R_t)) e^{-\delta t} \\
& \leq - \int_0^\tau e^{-\delta t} dL_t^c - \sum_{t < \tau, R_t \neq R_{t+}} e^{-\delta t} \left(\int_0^{L_{t+} - L_t} du \right) \\
& = - \int_0^\tau e^{-\delta t} dL_t.
\end{aligned} \tag{3.13}$$

Using (3.8), (3.9), (3.11) and (3.13) and noting that $\phi(x) \geq 0$ for $x \leq -\frac{p}{\alpha}$, we have

$$\begin{aligned}
& \phi(R_\tau) e^{-\delta \tau} - \phi(R_0) \\
& \leq \int_0^\tau \phi'(R_{t-}) e^{-\delta t} (p + r R_{t-} I\{R_{t-} \geq 0\} + \alpha R_{t-} I\{R_{t-} < 0\}) dt \\
& \quad - \int_0^\tau e^{-\delta t} dL_t + \mathcal{M}_\tau(\phi) + \lambda \int_0^\tau e^{-\delta t} dt \int_0^\infty (\phi(R_{t-} - u) - \phi(R_{t-})) dF(u) \\
& \quad - \delta \int_0^\tau \phi(R_t) e^{-\delta t} dt \\
& = \int_0^\tau \mathcal{L}_\phi(R_{t-}) e^{-\delta t} dt - \int_0^\tau e^{-\delta t} dL_t + \mathcal{M}_\tau(\phi).
\end{aligned} \tag{3.14}$$

In the next theorem, we show that the value function V is a viscosity solution of the HJB equation (3.7).

Theorem 3.4 (i) $V(x)$ is a viscosity solution of (3.7) on $(-\frac{p}{\alpha}, +\infty)$.

(ii) Define $D_V(x, h_n) = \frac{V(x+h_n) - V(x)}{h_n}$. If for some $\{h_n\}$ with $h_n > 0$ for all n or $h_n < 0$ for all n and $\lim_{n \rightarrow \infty} h_n = 0$, $\lim_{n \rightarrow \infty} D_V(x, h_n)$ exists, then

$$\begin{aligned}
& \max \left\{ 1 - \lim_{n \rightarrow \infty} D_V(x, h_n), \quad \lim_{n \rightarrow \infty} D_V(x, h_n) (p + r x I\{x \geq 0\} + \alpha x I\{x < 0\}) \right. \\
& \quad \left. - (\lambda + \delta) V(x) + \int_0^{x + \frac{p}{\alpha}} V(x - y) dF(y) \right\} \leq 0.
\end{aligned} \tag{3.15}$$

Proof. We employ a standard technique in the controlled Markov process theory, which has also been used in Benth et al. (2001), Albrecher and Thonhauser (2008) and Azcue and Muler (2005).

First, we show that $V(x)$ is a viscosity super-solution of the HJB equation (3.7) on $(-\frac{p}{\alpha}, +\infty)$. For any fixed $x \in (-\frac{p}{\alpha}, \infty)$. Let ϕ be a continuously differentiable function with $\phi(x) = V(x)$ and $V - \phi$ attaining a minimum at x . For any $h > 0$, define

$$a(x, l, h) = \begin{cases} x e^{rh} + (p - l) \int_0^h e^{r(h-s)} ds, & x > 0 \text{ and } l \geq 0 \text{ or } x = 0 \text{ and } 0 \leq l \leq p, \\ x e^{\alpha h} + (p - l) \int_0^h e^{\alpha(h-s)} ds, & -\frac{p}{\alpha} < x < 0 \text{ and } l \geq 0 \text{ or } x = 0 \text{ and } l > p. \end{cases}$$

For any $l \geq 0$, choose an h small enough such that $a(x, l, h) \in (-\frac{p}{\alpha}, 0) \cup (0, \infty)$. Consider a dividend strategy L' that the insurer pays out dividends continuously at rate l until time $S_1 \wedge h$. Then under the strategy L' , ruin will not occur before the earlier of the arrival time of the first claim S_1 and time h . Notice that given the initial reserve $R_0 = x$, under the strategy L' we have $R_t = a(x, l, t)$ for $t < S_1 \wedge h$ and $R_{S_1} = (a(x, l, S_1) - U_1) \vee (-\frac{p}{\alpha})$ on $\{S_1 \leq h\}$. By the Dynamic Programme Principle, distinguishing two cases $S_1 \geq h$ and $S_1 < h$ and then

conditioning on S_1 we have

$$\begin{aligned}
V(x) &= \sup_{L \in \Pi} \mathbf{E}_x \left[\int_0^{S_1 \wedge h} e^{-\delta s} dL_s + e^{-\delta(S_1 \wedge h)} V(R_{S_1 \wedge h}) \right] \\
&\geq e^{-\lambda h} \left(\int_0^h e^{-\delta s} l ds + e^{-\delta h} V(a(x, l, h)) \right) \\
&\quad + \int_0^h \lambda e^{-\lambda t} dt \left\{ \int_0^t l e^{-\delta s} ds + e^{-\delta t} \int_0^{a(x, l, t) + \frac{p}{\alpha}} V(a(x, l, t) - u) dF(u) \right\}. \quad (3.16)
\end{aligned}$$

By subtracting $V(x)$ from the last inequality and noting that $V(x) = \phi(x)$ and $V(a(x, l, h)) \geq \phi(a(x, l, h))$, we obtain

$$\begin{aligned}
0 &\geq \frac{l}{\delta} e^{-\lambda h} (1 - e^{-\delta h}) + \int_0^h \lambda e^{-\lambda t} dt \int_0^t l e^{-\delta s} ds \\
&\quad + e^{-(\lambda + \delta)h} [\phi(a(x, l, h)) - \phi(x)] + (e^{-(\lambda + \delta)h} - 1)V(x) \\
&\quad + \int_0^h \lambda e^{-(\lambda + \delta)t} dt \int_0^{a(x, l, t) + \frac{p}{\alpha}} V(a(x, l, t) - u) dF(u). \quad (3.17)
\end{aligned}$$

Dividing by $h > 0$ and then letting $h \downarrow 0$ yields

$$\begin{aligned}
0 &\geq l(1 - \phi'(x)) - (\lambda + \delta)V(x) + \lambda \int_0^{x + \frac{p}{\alpha}} V(x - u) dF(u) \\
&\quad + (rxI\{x > 0\} \cup \{x = 0, l \leq p\} + \alpha xI\{x < 0\} \cup \{x = 0, l > p\} + p) \phi'(x).
\end{aligned}$$

Letting $l = 0$ shows $(p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})\phi'(x) - (\lambda + \delta)V(x) + \lambda \int_0^{x + \frac{p}{\alpha}} V(x - u) dF(u) \leq 0$, and letting l be large enough indicates $\phi'(x) \geq 1$.

Next, we will show that $V(x)$ is a viscosity sub-solution of (3.7) on $(-\frac{p}{\alpha}, +\infty)$. To this end, we employ the proof by contradiction. Assume that V is not a viscosity sub-solution of (3.7) at some point x . Then we can find a constant $\eta > 0$ and a continuously differentiable function ψ_0 with $\psi_0(x) = V(x)$, $\psi_0(y) \geq V(y)$ for all y and

$$\max\{1 - \psi_0'(x), \frac{1}{\lambda} \mathcal{L}_{\psi_0}(x)\} < -2\eta. \quad (3.18)$$

Define

$$\psi_1(y) = \psi_0(y) + \eta \left(\frac{x - y}{x + 2\frac{p}{\alpha}} \right)^2. \quad (3.19)$$

Then $\psi_1(y)$ is continuously differentiable with $\psi_1(x) = \psi_0(x) = V(x)$ and $\psi_1'(x) = \psi_0'(x)$, and by (3.8) and (4.62) we have $\frac{1}{\lambda} \mathcal{L}_{\psi_1}(x) = \frac{1}{\lambda} \mathcal{L}_{\psi_0}(x) + \int_0^{x + \frac{p}{\alpha}} \eta \left(\frac{u}{x + 2\frac{p}{\alpha}} \right)^2 dF(u) < -2\eta + \eta = -\eta$, which along with the fact that ψ_1 is nonnegative and continuously differentiable, and $\mathcal{L}_{\psi_1}(x)$ is continuous, indicates that there exists an $h > 0$ such that

$$\max\{1 - \psi_1'(y), \frac{1}{\lambda} \mathcal{L}_{\psi_1}(y)\} < -\frac{\eta}{2} \text{ for } y \in [x - 2h, x + 2h]. \quad (3.20)$$

Let k be a continuously differentiable and nonnegative function with support included in $(-1, 1)$ such that $\int_{-1}^1 k(s) ds = 1$. Define a function $v_n(y) : R \rightarrow [0, \infty)$ as the convolution

$$v_n(y) = \int_{-1}^1 \left(V(y - \frac{s}{n}) + \frac{\eta h^2}{2(x + 2\frac{p}{\alpha})^2} \right) k(s) ds,$$

and another function $v : R \rightarrow [0, \infty)$ by $v(y) = V(y) + \frac{\eta h^2}{2(x + 2\frac{p}{\alpha})^2}$. Since $V(y - \frac{s}{n}) + \frac{\eta h^2}{2(x + 2\frac{p}{\alpha})^2}$ is continuous with respect to (y, s) on $[-\frac{p}{\alpha}, x + h; -1, 1]$, then we conclude that $v_n(y)$ is continuous on $[-\frac{p}{\alpha}, x + h]$. Moreover, $v_n(y)$ is a monotone sequence and by dominated convergence

theorem, it converges to $v(y)$. Therefore, it follows by Dini's theorem that $v_n(y) \rightarrow v(y)$ uniformly on $[-\frac{p}{\alpha}, x+h]$. Hence, there exists an n_0 such that for all $y \in [-\frac{p}{\alpha}, x+h]$,

$$V(y) + \frac{\eta h^2}{(x + 2\frac{p}{\alpha})^2} \geq v_{n_0}(y) \geq V(y) + \frac{\eta h^2}{4(x + 2\frac{p}{\alpha})^2}. \quad (3.21)$$

Define $f_{n_0}(y, s) = \left(V(y - \frac{s}{n_0}) + \frac{\eta h^2}{2(x + 2\frac{p}{\alpha})^2} \right) k(s)$. It can be seen that $f_{n_0}(y, s)$ is continuous on $[-\frac{p}{\alpha}, x+h; -1, 1]$. Let $D = \{y : V(y) \text{ is differentiable}\}$ and $n_0(y-D) = \{n_0(y-s) : s \in D\}$. As V is differentiable almost every where, the complement of D is a null set. Noting that $\frac{\partial}{\partial y} f_{n_0}(y, s) = \frac{\partial}{\partial y} V(y - \frac{s}{n_0}) k(s)$ on $[-\frac{p}{\alpha}, x+h] \times n_0(y-D)$ and that $V'(y)$, if exists, is greater than or equal to 1, it follows that for $y \in [-\frac{p}{\alpha}, x+h]$,

$$v'_{n_0}(y) = \int_{-1 \leq s \leq 1, s \in n_0(y-D)} \frac{\partial}{\partial y} f_{n_0}(y, s) ds \geq \int_{-1}^1 k(s) ds = 1. \quad (3.22)$$

Construct a continuously differentiable function $\omega : R \rightarrow [0, 1]$ such that $\omega(y) = 1$ for $y \in [x-h, x+h]$, $\omega(y) = 0$ for $y \in (-\infty, x-2h) \cup (x+2h, \infty)$, and $\omega'(y) \geq 0$ for $y \in [x-2h, x-h]$. Consider a function ψ defined by

$$\psi(y) = \omega(y)\psi_1(y) + (1 - \omega(y))v_{n_0}(y). \quad (3.23)$$

Obviously, $\psi(x) = \psi_1(x) = V(x)$. Noting that $\psi_0 \geq V$, it follows by (3.19), (3.20), (3.21), (3.22) and (3.23) that

$$\begin{aligned} \psi'(y) &\geq \omega(y)\left(1 + \frac{\eta}{2}\right) + \omega'(y)(\psi_1(y) - v_{n_0}(y)) + (1 - \omega(y)) \\ &\geq 1 + \omega'(y)\frac{\eta}{(x + 2\frac{p}{\alpha})^2}((x-y)^2 - h^2) \geq 1 \text{ for } y \in [-\frac{p}{\alpha}, x-h]. \end{aligned} \quad (3.24)$$

The last inequality follows by noticing $\omega'(y) \geq 0$ and $(x-y)^2 - h^2 \geq 0$ when $-\frac{p}{\alpha} \leq y \leq x-h$.

By (3.19) and (3.21), using the fact that that $V \leq \psi_0$ we obtain that for $-\frac{p}{\alpha} \leq y-u \leq x+h$ and $h \in (0, \frac{p}{2\alpha})$, $v_{n_0}(y-u) - \psi_1(y-u) \leq \eta \frac{h^2}{(x+2\frac{p}{\alpha})^2} - \eta \left(\frac{x-y+u}{x+2\frac{p}{\alpha}}\right)^2 < \frac{\eta}{4}$, which along with the definitions for \mathcal{L} in (3.8) and ψ in (3.23), and the fact that $\psi = \psi_1$ on $[x-h, x+h]$ indicates that for $y \in [x-h, x+h]$,

$$\begin{aligned} \mathcal{L}_\psi(y) &= \mathcal{L}_{\psi_1}(y) + \lambda \int_{y-x+h}^{y+\frac{p}{\alpha}} (1 - \omega(y-u))(v_{n_0}(y-u) - \psi_1(y-u)) dF(u) \\ &< \mathcal{L}_{\psi_1}(y) + \frac{\lambda\eta}{4}. \end{aligned} \quad (3.25)$$

Write $A = 1 + \frac{\delta}{\lambda}$. For any positive

$$\epsilon \leq \min\left\{\frac{\eta h^2}{12(x + 2\frac{p}{\alpha})^2}, \frac{\eta}{8(A-1)}\right\}, \quad (3.26)$$

it follows by (3.20) and (3.25) that

$$\frac{1}{\lambda}\mathcal{L}_\psi(y) \leq -\frac{\eta}{4} \leq -2(A-1)\epsilon \text{ for } y \in [x-h, x+h]. \quad (3.27)$$

From the definitions (3.23) and (3.19) for ψ and ψ_1 , respectively, by noting $0 \leq \omega(y) \leq 1$ and using (3.21) it follows that for any y satisfying $|y-x| \geq h$, we have

$$\begin{aligned} \psi(y) &\geq \omega(y) \left(\psi_0(y) + \eta \left(\frac{x-y}{x + 2\frac{p}{\alpha}} \right)^2 \right) + (1 - \omega(y)) \left(V(y) + \frac{\eta h^2}{4(x + 2\frac{p}{\alpha})^2} \right) \\ &\geq V(y) + 3\epsilon, \end{aligned} \quad (3.28)$$

where the last inequality follows by the fact $\psi_0 \geq V$ and (3.26).

From the definition (3.23) for the function ψ , and the fact that all the functions ψ , ω and v_{n_0} are continuously differentiable, we can see that \mathcal{L}_ψ is continuous. Therefore, there exist a constant $K > 0$ such that

$$\frac{1}{\lambda} \mathcal{L}_\psi(y) \leq K \quad \text{for } y \in [-\frac{p}{\alpha}, x+h]. \quad (3.29)$$

For any fixed σ with

$$0 < \sigma < \min \left\{ \frac{\epsilon}{2\lambda K}, \frac{1}{4\lambda(A-1)}, \frac{1}{\alpha} \log \left(\frac{x + \frac{p}{\alpha} - \frac{h}{2}}{x + \frac{p}{\alpha} - h} \right) \right\}, \quad (3.30)$$

define $\bar{\tau} = \inf\{t > 0 : R_t \geq x+h\}$, $\underline{\tau} = \inf\{t > 0 : R_t \leq x-h\}$, and $\tau^* = \bar{\tau} \wedge (\underline{\tau} + \sigma)$. By (3.28) we have

$$V(R_{\tau^*}) \leq \psi(R_{\tau^*}) - 3\epsilon \quad \text{on } \{\omega : |R_{\tau^*} - x| \geq h\}. \quad (3.31)$$

Note that $R_{\bar{\tau}} = x+h$, as the surplus process has only downward jumps. Then given the initial reserve $R_0 = x$, we have on the set $\{\omega : |R_{\tau^*} - x| < h\}$,

$$x+h > R_{\tau^*} = R_{\underline{\tau}+\sigma} > x-h \geq R_{\underline{\tau}} = R_{\underline{\tau} \wedge \bar{\tau}}. \quad (3.32)$$

Then it follows by (3.30), (3.32) and noticing that $R_{t+\sigma} \leq R_t e^{\alpha\sigma} + \frac{p}{\alpha}(e^{\alpha\sigma} - 1)$ from the dynamics (2.1), that given $R_0 = x$,

$$R_{\tau^*} = R_{\underline{\tau}+\sigma} \leq R_{\underline{\tau}} e^{\alpha\sigma} + \frac{p}{\alpha}(e^{\alpha\sigma} - 1) \leq (x-h + \frac{p}{\alpha})e^{\alpha\sigma} - \frac{p}{\alpha} < x - \frac{h}{2} \quad \text{on } \{\omega : |R_{\tau^*} - x| < h\}.$$

which implies that given $R_0 = x$, $R_{\tau^*} - x < -\frac{h}{2}$ on $\{\omega : |R_{\tau^*} - x| < h\}$. Hence, $(x - R_{\tau^*})^2 > \frac{h^2}{4}$ on the set $\{\omega : |R_{\tau^*} - x| < h\}$. Using this and noticing that $\psi_0 \geq v$ and that from (3.26) we have $\frac{\eta}{(x + 2\frac{p}{\alpha})^2} \geq \frac{12\epsilon}{h^2}$, by the definitions (3.19) and (3.23), we can show that given the initial value $R_0 = x$, $\psi(R_{\tau^*}) = \psi_0(R_{\tau^*}) + \eta \left(\frac{x - R_{\tau^*}}{x + 2\frac{p}{\alpha}} \right)^2 \geq V(R_{\tau^*}) + 3\epsilon$ on $\{\omega : |R_{\tau^*} - x| < h\}$. This along with (3.31) shows

$$V(R_{\tau^*}) \leq \psi(R_{\tau^*}) - 3\epsilon \quad \text{given } R_0 = x. \quad (3.33)$$

Note that from (3.24) we have $\psi'(R_t) \geq 1$ for $R_t \in [-\frac{p}{\alpha}, x-h]$ and that from (3.20) and (3.23) we have $\psi'(R_t) = \psi'_1(R_t) \geq 1$ for $[x-h, x+h]$. Then by noticing that $R_t \in [-\frac{p}{\alpha}, x+h]$ for $t \in [0, \tau^*]$, we conclude that $\psi'(R_t) \geq 1$ for $t \in [0, \tau^*]$. Then by setting ϕ and τ in (3.14) to be ψ and τ^* , respectively, we have

$$\psi(R_{\tau^*})e^{-\delta\tau^*} - \psi(R_0) \leq \int_0^{\tau^*} \mathcal{L}_\psi(R_{t-})e^{-\delta t} dt - \int_0^{\tau^*} e^{-\delta t} dL_t + \mathcal{M}_{\tau^*}(\psi). \quad (3.34)$$

Note that given $R_0 = x$, we have $R_{t-} \in [x-h, x+h]$ for $t \in [0, \bar{\tau} \wedge \underline{\tau}]$ and $R_{t-} \in [-\frac{p}{\alpha}, x+h]$ for $t \in [\bar{\tau} \wedge \underline{\tau}, \tau^*]$. From (3.27) and (3.29), it follows that

$$\begin{aligned} \int_0^{\tau^*} \mathcal{L}_\psi(R_{t-})e^{-\delta t} dt &= \int_0^{\bar{\tau} \wedge \underline{\tau}} \mathcal{L}_\psi(R_{t-})e^{-\delta t} dt + \int_{\bar{\tau} \wedge \underline{\tau}}^{\tau^*} \mathcal{L}_\psi(R_{t-})e^{-\delta t} dt \\ &\leq -(A-1)2\epsilon\lambda \int_0^{\bar{\tau} \wedge \underline{\tau}} e^{-\delta t} dt + \lambda K \int_{\bar{\tau} \wedge \underline{\tau}}^{\tau^*} e^{-\delta t} dt \\ &= -(A-1)2\epsilon\lambda \int_0^{\tau^*} e^{-\delta t} dt + \lambda((A-1)2\epsilon + K) \int_{\bar{\tau} \wedge \underline{\tau}}^{\tau^*} e^{-\delta t} dt \\ &\leq -(A-1)2\epsilon\lambda \int_0^{\tau^*} e^{-\delta t} dt + \lambda((A-1)2\epsilon + K)\sigma \\ &< -(A-1)2\epsilon\lambda \int_0^{\tau^*} e^{-\delta t} dt + \epsilon, \end{aligned} \quad (3.35)$$

where the second last inequality follows by noticing $\tau^* - \bar{\tau} \wedge \underline{\tau} \leq \sigma$ and the last inequality follows by (3.30).

Given the initial reserve $R_0 = x$, it follows from (3.33), (3.34) and (3.35) that

$$\begin{aligned} V(R_{\tau^*})e^{-\delta\tau^*} &< \psi(R_{\tau^*})e^{-\delta\tau^*} - 2\epsilon e^{-\delta\tau^*} \\ &= (\psi(R_{\tau^*})e^{-\delta\tau^*} - \psi(x)) + (\psi(x) - 2\epsilon e^{-\delta\tau^*}) \\ &\leq -(A-1)2\epsilon\lambda \int_0^{\tau^*} e^{-\delta t} dt - \int_0^{\tau^*} e^{-\delta s} dL_t + \mathcal{M}_{\tau^*}(\psi) \\ &\quad + (\psi(x) - 2\epsilon e^{-\delta\tau^*}) + \epsilon. \end{aligned} \quad (3.36)$$

As $\int_0^{\tau^*} e^{-\delta s} ds = \frac{1-e^{-\delta\tau^*}}{\delta}$ and $A = 1 + \frac{\delta}{\lambda}$, from (3.36) we obtain

$$V(R_{\tau^*})e^{-\delta\tau^*} + \int_0^{\tau^*} e^{-\delta s} dL_t \leq \mathcal{M}_{\tau^*}(\psi) + \psi(x) - \epsilon. \quad (3.37)$$

Noting that $\mathcal{M}_t(\psi)$ is a super-martingale with zero-expectation, we have $\mathbb{E}[\mathcal{M}_{\tau^*}(\psi)] \leq 0$. As a result, taking conditional expectation on (3.37) yields $V(x) = \sup_{L \in \Pi} \mathbb{E}_x[\int_0^{\tau^*} e^{-\delta s} dL_t + V(R_{\tau^*})e^{-\delta\tau^*}] \leq \psi(x) - \epsilon$, which contradicts the fact $V(x) = \psi(x)$.

(ii) By Theorem 3.2, it follows that for any h_n with $\lim_{n \rightarrow \infty} h_n = 0$,

$$\lim_{n \rightarrow \infty} D_V(x, h_n) \geq 1. \quad (3.38)$$

Consider a sequence h'_n with $h'_n \downarrow 0$ as $n \rightarrow \infty$ such that $\lim_{n \rightarrow \infty} \frac{V(a(x, l, h'_n)) - V(x)}{h'_n}$ exists. Following the same lines as in the proof for (i), it can be shown that (3.17) also holds when h and $\phi(\cdot)$ there being replaced by h'_n and $V(\cdot)$, respectively. Dividing both sides of the newly obtained inequality by h'_n and then letting $n \rightarrow \infty$ yields for $l \geq 0$,

$$\begin{aligned} 0 \geq & l(1 - \lim_{n \rightarrow \infty} D_V(x, a(x, l, h'_n) - x)) - (\lambda + \delta)V(x) + \lambda \int_0^{x + \frac{p}{\alpha}} V(x - u) dF(u) + [p + \\ & (rI\{x > 0\} \cup \{x = 0, l \leq p\} + \alpha I\{x < 0\} \cup \{x = 0, l > p\})x] \lim_{n \rightarrow \infty} D_V(x, a(x, l, h'_n) - x). \end{aligned}$$

By letting $l = 0$ we have

$$\begin{aligned} & (p + rxI\{x \geq 0\} + \alpha x(\{x < 0\})) \lim_{n \rightarrow \infty} D_V(x, a(x, 0, h'_n) - x) - (\lambda + \delta)V(x) \\ & + \lambda \int_0^{x + \frac{p}{\alpha}} V(x - u) dF(u) \leq 0. \end{aligned} \quad (3.39)$$

For any $\{h_n\}$ with $h_n > 0$ such that $\lim_{n \rightarrow \infty} h_n = 0$, and $\lim_{n \rightarrow \infty} D_V(x, h_n)$ exists, we can find a subsequence $\{h_{n_k}\} \subset \{a(x, l, h'_n) - x\}$. Therefore, $\lim_{n \rightarrow \infty} D_V(x, h_n) = \lim_{k \rightarrow \infty} D_V(x, h_{n_k}) = \lim_{n \rightarrow \infty} D_V(x, a(x, 0, h'_n) - x)$. It follows by (3.38) and (3.39) that

$$\begin{aligned} \max \left\{ 1 - \lim_{n \rightarrow \infty} D_V(x, h_n), \quad \lim_{n \rightarrow \infty} D_V(x, h_n)(p + rxI\{x \geq 0\} + \alpha xI\{x < 0\}) \right. \\ \left. - (\lambda + \delta)V(x) + \int_0^{x + \frac{p}{\alpha}} V(x - y) dF(y) \right\} \leq 0. \end{aligned} \quad (3.40)$$

For any sequence $\{h_n\}$ with $h_n < 0$ such that $\lim_{n \rightarrow \infty} h_n = 0$, and $\lim_{n \rightarrow \infty} D_V(x, h_n)$ exists, by repeating the above argument by replacing all x there by $x - c(x, l, h)$ (i.e., conditioning on the initial reserve $R_0 = x - c(x, l, h)$), where

$$c(x, l, h) = \begin{cases} x(1 - e^{-rh}) + (p - l) \int_0^h e^{-rs} ds & x > 0 \text{ and } l \geq 0 \text{ or } x = 0 \text{ and } 0 \leq l \leq p \\ x(1 - e^{-\alpha h}) + (p - l) \int_0^h e^{-\alpha s} ds & -\frac{p}{\alpha} < x < 0 \text{ and } l \geq 0 \text{ or } x = 0 \text{ and } l > p, \end{cases}$$

and noticing that $a(x - c(x, l, h), l, h) = x$, we can show that (3.40) is also true. \square

Next we will show that the value function $V(x)$ is the unique nonnegative, nondecreasing and locally Lipschitz continuous viscosity solution of (3.7) satisfying a linear growth condition and the boundary condition $V(-\frac{p}{\alpha}) = 0$. We start with the following comparison principle.

Lemma 3.5 *Let $\bar{u}(x)$ and $\underline{u}(x)$ be a nonnegative viscosity super-solution and sub-solution, respectively. Assume that for both $u = \bar{u}(x)$ and $\underline{u}(x)$, the function u is continuous on $[-\frac{p}{\alpha}, \infty)$ and locally Lipschitz continuous on $(-\frac{p}{\alpha}, \infty)$, and satisfies $u(-\frac{p}{\alpha}) = 0$ and $u(x) \leq c_1x + c_2$ for some constants c_1 and c_2 . Then $\underline{u}(x) \leq \bar{u}(x)$ for all $x \geq -\frac{p}{\alpha}$.*

From Theorem 3.1, Theorem 3.3 and Theorem 3.4, we know that the value function $V(x)$ is a nondecreasing and nonnegative viscosity solution of the HJB equation (3.7) that is locally Lipschitz continuous on $(-\frac{p}{\alpha}, \infty)$, satisfies a linear growth condition, and fulfills the boundary condition $V(-\frac{p}{\alpha}) = 0$. Consider any other viscosity solution $W(x)$ of (3.7) that fulfills the same conditions. Since $V(x)$ is also a super-solution and $W(x)$ is also a sub-solution, by Lemma 3.5 we conclude that $V(x) \geq W(x)$ for all $x \geq -\frac{p}{\alpha}$. This leads to the following theorem stating the uniqueness of the value function as a viscosity solution of (3.7).

Theorem 3.6 *The value function $V(x)$ is the unique nondecreasing and nonnegative viscosity solution of the HJB equation (3.7) that*

- i) is locally Lipschitz on $(-\frac{p}{\alpha}, \infty)$,*
- ii) satisfies a linear growth condition, and*
- iii) fulfills the boundary condition $V(-\frac{p}{\alpha}) = 0$.*

As an immediate result of Lemma 3.5, we arrive at the Verification Theorem as follows.

Theorem 3.7 *For any strategy $L \in \Pi$, if V_L is an locally Lipschitz continuous viscosity super-solution of HJB equation (3.7), then $V_L = V$, i.e. L is an optimal dividend strategy.*

Proof. Obviously, V_L is nonnegative and nondecreasing and $V_L(-\frac{p}{\alpha}) = 0$. Since $V_L \leq V$, it is true that V_L also satisfies the linear growth condition. Therefore, by Lemma 3.5 we know that $V_L \geq V$. Consequently, $V_L = V$. \square

Lemma 3.8 *Let $\Pi_{\bar{x}}$ be the set of admissible strategies such that the controlled reserve R_t is less than or equal to \bar{x} for all $t > 0$. If for some $\bar{x} > 0$, $\bar{u}(x)$ is a nonnegative, nondecreasing and locally Lipschitz continuous super-solution of the HJB equation (3.7) on $(-\frac{p}{\alpha}, \bar{x})$, then $\bar{u}(x) \geq \sup_{L \in \Pi_{\bar{x}}} V_L(x)$ for all $x \in [-\frac{p}{\alpha}, \bar{x}]$.*

Proof. i) We can prove this by showing that for any dividend strategy $L \in \Pi_{\bar{x}}$, $V_L(x) \leq \bar{u}(x)$ for $x \in [-\frac{p}{\alpha}, \bar{x}]$. For any continuous super-solution \bar{u} of the HJB equation (3.7) on $(-\frac{p}{\alpha}, \bar{x})$, consider a function $\bar{v}(x)$ with $\bar{v}(x) = 0$ for $x < -\frac{p}{\alpha}$, $\bar{v}(x) = \bar{u}(x)$ for $x \in [-\frac{p}{\alpha}, \bar{x}]$ and $\bar{v}(x) = \bar{u}(\bar{x})$ for $x \geq \bar{x}$. Consider a sequence of nonnegative functions $v_n(x) = \int_{-\infty}^{\infty} \bar{v}(x - y)n\phi(ny)dy$ for $x \in [-\frac{p}{\alpha}, \bar{x}]$, where $\phi(x)$ is a nonnegative, even and continuously differentiable function with its support included in $(-1, 1)$ such that $\int_{-1}^1 \phi(x)ds = 1$. It can be seen that $v_n(x)$ is nonnegative and nondecreasing, and satisfies

$$v_n(x) \leq \bar{u}(\bar{x}), \text{ for } x \in [-\frac{p}{\alpha}, \bar{x}]. \quad (3.41)$$

Using the standard techniques in real analysis (eg Wheeden and Zygmund (1977)), we can show that v_n is continuously differentiable on $[-\frac{p}{\alpha}, \bar{x}]$,

$$v_n(x) \text{ converges to } \bar{u}(x) \text{ uniformly on } [-\frac{p}{\alpha}, \bar{x}]; \text{ and} \quad (3.42)$$

$$v'_n(x) \text{ converges to } \bar{u}'(x) \text{ almost everywhere.} \quad (3.43)$$

Noting from Definition 3.1 (ii), $1 \leq \bar{u}'(x) \leq \frac{\lambda + \delta}{p + rxI\{x \geq 0\} + \alpha xI\{x < 0\}} \bar{u}(x)$ for x such that $\bar{u}'(x)$ exists, we can obtain

$$1 \leq v'_n(x) \leq \frac{\lambda + \delta}{p + rxI\{x \geq 0\} + \alpha xI\{x < 0\}} v_n(x) \quad \text{for } x \in [-\frac{p}{\alpha}, \bar{x}]. \quad (3.44)$$

From now on in the proof of this lemma, we assume $x \in [-\frac{p}{\alpha}, \bar{x}]$. By setting u and τ in (3.14) to be \bar{v}_n and $t \wedge \tau$, respectively, and then taking expectation, we obtain

$$\mathbb{E}_x[v_n(R_{t \wedge T})e^{-\delta(t \wedge T)}] \leq v_n(x) + \mathbb{E}_x \left[\int_0^{t \wedge T} e^{-\delta s} \mathcal{L}_{v_n}(R_{s-}) ds \right] - \mathbb{E}_x \left[\int_0^{t \wedge T} e^{-\delta s} dL_s \right]. \quad (3.45)$$

Notice that under any strategy $L \in \Pi_{\bar{x}}$ the controlled reserve is below or at \bar{x} . Then by (3.41) we have $\int_0^{R_{s-} + \frac{p}{\alpha}} v_n(R_{s-} - y) dF(y) \leq \bar{u}(\bar{x})$. Hence, by (3.8), (3.41) and (3.44), we can obtain

$$\begin{aligned} |\mathcal{L}_{v_n}(R_{s-})| &\leq (p + rR_{s-}I\{R_{s-} \geq 0\} + \alpha R_{s-}I\{R_{s-} < 0\}) v'_n(R_{s-}) \\ &\quad + (\lambda + \delta)v_n(R_{s-}) + \lambda \int_0^{R_{s-} + \frac{p}{\alpha}} v_n(R_{s-} - y) dF(y) \\ &\leq (2\delta + 3\lambda)\bar{u}(\bar{x}). \end{aligned} \quad (3.46)$$

Then it follows by dominated convergence that

$$\lim_{t \rightarrow \infty} \mathbb{E}_x \left[\int_0^{t \wedge T} e^{-\delta s} \mathcal{L}_{v_n}(R_{s-}) ds \right] = \mathbb{E}_x \left[\int_0^T e^{-\delta s} \mathcal{L}_{v_n}(R_{s-}) ds \right].$$

As L_s is nondecreasing, by monotone convergence we obtain

$$\lim_{t \rightarrow \infty} \mathbb{E}_x \left[\int_0^{t \wedge T} e^{-\delta s} dL_s \right] = \mathbb{E}_x \left[\int_0^T e^{-\delta s} dL_s \right] = V_L(x).$$

Letting $t \rightarrow \infty$ on both sides of (3.45) yields

$$0 \leq v_n(x) + \mathbb{E}_x \left[\int_0^T e^{-\delta s} \mathcal{L}_{v_n}(R_s) ds \right] - V_L(x). \quad (3.47)$$

Since under any strategy $L \in \Pi_{\bar{x}}$, the controlled reserve $R_{s-} \leq \bar{x}$, by (3.42) and (3.43) it can be shown that $\mathcal{L}_{v_n}(R_{s-}) \rightarrow \mathcal{L}_{\bar{u}}(R_{s-})$ a.e. - P. Using this and (3.46), by dominated convergence taking limit $n \rightarrow \infty$ on (3.47) yields $0 \leq \bar{u}(x) + \mathbb{E}_x \left[\int_0^T e^{-\delta s} \mathcal{L}_{\bar{u}}(R_{s-}) ds \right] - V_L(x)$. As \bar{u} is a supersolution, so $\mathcal{L}_{\bar{u}}(R_{s-}) \leq 0$ a.e. - P. Then it follows that $V_L(x) \leq \bar{u}(x)$. Consequently, $V(x) = \sup_{L \in \Pi_{\bar{x}}} V_L(x) \leq \bar{u}(x)$. \square

Define an operator \mathcal{G} by

$$\begin{aligned} \mathcal{G}_u(x) &= p + rxI\{x \geq 0\} + \alpha xI\{x < 0\} - (\lambda + \delta)u(x) \\ &\quad + \lambda \int_0^{x + \frac{p}{\alpha}} u(x - y) dF(y). \end{aligned} \quad (3.48)$$

Theorem 3.9 *If for some $\bar{x} \in (-\frac{p}{\alpha}, \infty)$, $\mathcal{G}_V(\bar{x}) = 0$, then $V(x) = \sup_{L \in \Pi_{\bar{x}}} V_L(x)$ on $[-\frac{p}{\alpha}, \bar{x}]$, where $\Pi_{\bar{x}}$ is defined same as in Lemma 3.8.*

Theorem 3.10 *Let $\Pi_{\bar{x}}$ be defined same as in Lemma 3.8. If there exists an $\bar{x} \in (-\frac{p}{\alpha}, \infty)$ such that $V'(\bar{x}) = 1$, then $V(x) = \sup_{L \in \Pi_{\bar{x}}} V_L(x)$ for $x \in [-\frac{p}{\alpha}, \bar{x}]$.*

As an immediate consequence of Theorem 3.8, Theorem 3.9 and Theorem 3.10, we obtain the following theorem.

Theorem 3.11 *If for some $\bar{x} > -\frac{p}{\alpha}$ with either $\mathcal{G}_V(\bar{x}) = 0$ or $V'(\bar{x}) = 1$, then for any nonnegative, nondecreasing and locally Lipschitz continuous super-solution $\bar{u}(x)$ of the HJB equation (3.7) on $(-\frac{p}{\alpha}, \bar{x}]$ which satisfies $\bar{u}(x) \leq c_1 + c_2x$ for some constants c_1 and c_2 , and the boundary condition $\bar{u}(-\frac{p}{\alpha}) = 0$, we have $\bar{u}(x) \geq V(x)$ on $(-\frac{p}{\alpha}, \bar{x}]$. Furthermore, if for some strategy $L \in \Pi_{\bar{x}}$ the function V_L is an absolutely continuous super-solution to the HJB equation (3.7) on $(-\frac{p}{\alpha}, \bar{x}]$, then $V(x) = V_L(x)$ for all $x \in (-\frac{p}{\alpha}, \bar{x}]$.*

For any $y \geq -\frac{p}{\alpha}$, define $G_y(x) = V(x)$ if $x \leq y$ and $G_y(x) = V(y) + x - y$ if $x > y$.

Theorem 3.12 *i) If G_y is a super-solution to the HJB equation (3.7) on (y, ∞) , then $G_y = V$ on $[-\frac{p}{\alpha}, \infty)$.*

ii) If for some $\bar{x} > -\frac{p}{\alpha}$ with either $\mathcal{G}_V(\bar{x}) = 0$ or $V'(\bar{x}) = 1$, and for some $y < \bar{x}$, G_y is a super-solution of the HJB equation (3.7) in $(y, \bar{x}]$, then $G_y(x) = V(x)$ on $[-\frac{p}{\alpha}, \bar{x}]$.

Proof. First we show that G_y is a viscosity super-solution to the HJB equation (3.7) on $(-\frac{p}{\alpha}, y]$. For any fixed $x \in [-\frac{p}{\alpha}, y]$, let ϕ be any continuously differentiable function with $\phi(x) = G_y(x)$ and $G_y - \phi$ reaches minimum at x . Then by Remark 3.1 ii) we obtain

$$\limsup_{h \uparrow 0} \frac{G_y(x) - G_y(x-h)}{h} \leq \lim_{n \rightarrow \infty} D_V(x, a(x, h'_n) - x) \leq \liminf_{h \downarrow 0} \frac{G_y(x+h) - G_y(x)}{h}. \quad (3.49)$$

Notice that $\limsup_{h \uparrow 0} \frac{G_y(x) - G_y(x-h)}{h} = \limsup_{h \uparrow 0} \frac{V(x) - V(x-h)}{h}$ and that $\liminf_{h \downarrow 0} \frac{V_y(x+h) - V_y(x)}{h}$ equals $\liminf_{h \downarrow 0} \frac{G_y(x+h) - G_y(x)}{h}$ if $x \in [-\frac{p}{\alpha}, y)$ and equals 1 if $x = y$. As a result, using (3.49) and $\liminf_{h \downarrow 0} \frac{V_y(x+h) - V_y(x)}{h} \geq 1$ yields $\limsup_{h \downarrow 0} \frac{V(x) - V(x-h)}{h} \leq \phi'(x) \leq \liminf_{h \downarrow 0} \frac{V(x+h) - V(x)}{h}$, which by Remark 3.1 ii) again implies that $V - \phi$ reaches minimum at x . Since V is a viscosity super-solution of (3.7), we have $\max\{1 - \phi'(x), \mathcal{L}_{V, \phi}(x)\} \leq 0$. Hence, by noticing $\mathcal{L}_{V, \phi}(x) = \mathcal{L}_{G_y, \phi}(x)$ for $x \in (-\frac{p}{\alpha}, y]$, we have $\max\{1 - \phi'(x), \mathcal{L}_{G_y, \phi}(x)\} \leq 0$ for $x \in (-\frac{p}{\alpha}, y]$. Consequently, G_y is a viscosity super-solution on $(-\frac{p}{\alpha}, y]$.

i) If G_y is a viscosity super-solution on (y, ∞) , then it is a super-solution on $(-\frac{p}{\alpha}, \infty)$. Also note that G_y satisfies the linear growth condition. Then by Theorem 3.8 i), we have $G_y \geq V$ on $(-\frac{p}{\alpha}, \infty)$. Noticing that $G_y \leq V$, therefore, $G_y = V$ on $[-\frac{p}{\alpha}, \infty)$.

ii) If G_y is a viscosity super-solution on $(y, \bar{x}]$, then it is a super-solution on $(-\frac{p}{\alpha}, \bar{x}]$. By Theorem 3.11, we have $G_y \geq V$ on $(-\frac{p}{\alpha}, \bar{x}]$. Noticing by definition that $G_y \leq V$, therefore, $G_y = V$ on $(-\frac{p}{\alpha}, \bar{x}]$. \square

4 The optimal dividend strategy

In this section, we show that there exists an optimal dividend strategy and the optimal strategy is a band strategy, that is, the optimal strategy at any time is to pay no dividends, pay out at a rate same as the premium incoming rate or a positive lump sum, depending on the current reserve at that time. We also show that under certain condition, when the reserve is negative, the optimal strategy is to pay no dividends.

We start with the following definition for three sets.

Definition 4.1 *Define $\mathcal{A} = \{x \in [-\frac{p}{\alpha}, \infty) : \mathcal{G}_V(x) = 0\}$, $\mathcal{B} = \{x \in [-\frac{p}{\alpha}, \infty) : V'(x) = 1 \text{ and } \mathcal{G}_V(x) < 0\}$, and $\mathcal{C} = (\mathcal{A} \cup \mathcal{B})^c$.*

The sets defined above will play a crucial role in proving the existence of and characterizing the optimal dividend strategy. we can prove the following useful properties of these sets.

Lemma 4.1 *The following properties hold.*

- (a) \mathcal{A} is nonempty and closed.
- (b) \mathcal{B} is nonempty and left-open. And there exists a y such that $(y, \infty) \subset \mathcal{B}$.
- (c) If $(x_0, x_1] \subset \mathcal{B}$ and $x_0 \notin \mathcal{B}$, then $x_0 \in \mathcal{A}$.
- (d) \mathcal{C} is right-open.

Based on the above three sets and their characteristics, we define the following dividend strategy, which will be shown to be the optimal one.

Definition 4.2 *Let L^* be a dividend strategy defined as follows:*

(a) If $R_{t-}^{L^*} \in \mathcal{A}$, the insurer pays out dividends at the same rate as the premium incoming rate, i.e.

$$dL_t^* = p + rR_{t-}I\{R_{t-} \geq 0\} + \alpha R_{t-}I\{R_{t-} < 0\} \text{ if } R_{t-}^{L^*} \in \mathcal{A}.$$

(b) If $R_{t-}^{L^*} \in \mathcal{B}$, then by Lemma 4.1 (c) there exists an $x_0 \in \mathcal{A}$ with $x_0 < R_{t-}^{L^*}$ such that $(x_0, R_{t-}^{L^*}] \subset \mathcal{B}$. At time t , the insurer pays out a lump sum $R_{t-}^{L^*} - x_0$ as dividends, ie

$$L_t^* - L_{t-}^* = R_{t-}^{L^*} - x_0 \text{ if } R_{t-}^{L^*} \in \mathcal{B}, \text{ where } x_0 = \inf\{x : (x, R_{t-}^{L^*}] \subset \mathcal{B}\}.$$

(c) If $R_{t-}^{L^*} \in \mathcal{C}$, then the insurer pays out no dividends at the moment.

In the following, we prove that the strategy L^* constructed above is the optimal dividend strategy.

Theorem 4.2 *The strategy L^* defined in Definition (4.2) is optimal, i.e. $V(x) = V_{L^*}(x)$ for all $x \geq -\frac{p}{\alpha}$.*

Proof. By Lemma 4.1 it follows that there exists some $\underline{x} = \inf\{x : (x, \infty) \subset \mathcal{B}\}$.

Let \mathcal{H} be a set of continuous functions $f : [-\frac{p}{\alpha}, \infty) \rightarrow [0, \infty)$ with $f(x) = x - \underline{x} + f(\underline{x})$ for $x > \underline{x}$.

Define the distance $\rho(f_1, f_2) = \max_{x \geq -\frac{p}{\alpha}} |f_1(x) - f_2(x)|$ for $f_1, f_2 \in \mathcal{H}$.

Define an operator \mathcal{T} as follows:

$$\mathcal{T}_f(x) = \mathbb{E}_x \left[\int_0^{S_1} e^{-\delta s} dL_s^* + e^{-\delta S_1} f(R_{S_1}^{L^*}) \right]. \quad (4.50)$$

Since S_1 is an exponential random variable, by Markov property it follows that for any $t \geq 0$,

$$\mathcal{T}_f(x) = \mathbb{E}_x \left[\int_0^{S_1} e^{-\delta s} dL_s^* + e^{-\delta S_1} f(R_{S_1}^{L^*}); S_1 \leq t \right] + \mathbb{E}_x \left[\int_0^t e^{-\delta s} dL_s^* + e^{-\delta t} \mathcal{T}_f(R_t^{L^*}); S_1 > t \right] \quad (4.51)$$

Note that

$$\begin{aligned} |T_{f_1}(x) - T_{f_2}(x)| &= |\mathbb{E}_x[e^{-\delta S_1}(f_1(R_{S_1}^{L^*}) - f_2(R_{S_1}^{L^*}))]| \\ &\leq \frac{\lambda}{\lambda + \delta} \rho(f_1, f_2). \end{aligned}$$

Noting that for any $x \geq \underline{x}$, we have $(\underline{x}, \infty) \subset \mathcal{B}$ and $\underline{x} \in \mathcal{A}$, by setting $x_0 = \underline{x}$ in (4.52) we get

$$\mathcal{T}_f(x) = x - \underline{x} + \mathcal{T}_f(\underline{x}) \text{ for } x \in \mathcal{B}. \quad (4.52)$$

As a result, $\mathcal{T}_f \in \mathcal{H}$ for all $f \in \mathcal{H}$. Obviously, $T_f \in [0, \infty)$ for all $f \in \mathcal{H}$. Therefore, by (4.52) it follows that \mathcal{T} is a contraction and has a unique fixed point. By the dynamic programming principle and (4.50), it is obvious that V_{L^*} is a fixed point. So to prove $V = V_{L^*}$ it is sufficient to show that $V \in \mathcal{H}$ and V is also a fixed point.

Obviously, $V \in [0, \infty)$. Moreover, since $(\underline{x}, \infty) \subset \mathcal{B}$, then $V'(x) = 1$ for all $x > \underline{x}$. As a result, $V(x) = V(\underline{x}) + x - \underline{x}$ for all $x \geq \underline{x}$. Consequently, we can conclude that $V \in \mathcal{H}$.

Assume $x \in \mathcal{A}$. By the definition of L^* , we can see that $dL_t^* = (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})dt$ for all time t before the arrival S_1 of the first claim. Therefore, by (4.50) we obtain that

$$\begin{aligned} \mathcal{T}_f(x) &= \mathbb{E}_x \left[\int_0^{S_1} (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})e^{-\delta s} ds + e^{-\delta S_1} V(x - U_1) \right] \\ &= \frac{(p + rI\{x \geq 0\} + \alpha I\{x < 0\})}{\lambda + \delta} + \int_0^\infty \lambda e^{-\lambda t} e^{-\delta t} dt \int_0^{x + \frac{x}{\alpha}} V(x - y) dF(y) \\ &= \frac{(p + rI\{x \geq 0\} + \alpha I\{x < 0\}) + \lambda \int_0^{x + \frac{x}{\alpha}} V(x - y) dF(y)}{\lambda + \delta} \text{ for } x \in \mathcal{A}. \end{aligned} \quad (4.53)$$

It follows by (4.53) and the equality $\mathcal{G}_V(x) = 0$ that

$$\mathcal{T}_V(x) = V(x) \text{ for } x \in \mathcal{A}. \quad (4.54)$$

For any $x \in \mathcal{B}$, we can find an $x_0 < x$ such that $(x_0, x] \subset \mathcal{B}$ and $x_0 \in \mathcal{A}$, which implies $V'(y) = 1$ for $y \in (x_0, x]$. Therefore, $V(x) = x - x_0 + V(x_0)$. By the definition of L^* , we know that a lump sum of $x - x_0$ will be paid out as dividends immediately. Then it follows from (4.50) and (4.54) that

$$\mathcal{T}_V(x) = x - x_0 + \mathcal{T}_V(x_0) = x - x_0 + V(x_0) = V(x) \text{ for } x \in \mathcal{B}. \quad (4.55)$$

Now we look at the case $x \in \mathcal{C}$. Since \mathcal{C} is right open, there exists an x_1 such that $(x, x_1) \subset \mathcal{C}$ and $x_1 \notin \mathcal{C}$. As \mathcal{B} is left open, so $x_1 \in \mathcal{A}$. By the definition for L^* we know that given the initial reserve, the insurance company pays out no dividends until the reserves x_1 or the arrival (S_1) of the first claim. Consider a function $a(\cdot)$ which satisfies $da(t) = (p + ra(t)I\{a(t) \geq 0\} + \alpha a(t)I\{a(t) < 0\})dt$ and $a(0) = x$. Recall that $t_0(x, x_1)$ is the time it will take for this dynamics to reach x_1 . It can be seen that given $R_0 = x$, $R_t = a(t)$ for all $x < S_1 \wedge t_0(x, x_1)$, and $R_{S_1} = a(S_1) - U_1$ if $S_1 < t_0(x, x_1)$. Then by setting t and f (4.51) by $t_0(x, x_1)$ and V , respectively, and by noting $\mathcal{T}_V(x_1) = V(x_1)$ because $x_1 \in \mathcal{A}$, it follows that

$$\begin{aligned} \mathcal{T}_V(x) &= \mathbb{E}_x \left[e^{-\delta S_1} V(a(S_1) - U_1) I(S_1 \leq t_0(x, x_1)) + e^{-\delta t_0(x, x_1)} \mathcal{T}_V(x_1) I(S_1 > t_0(x, x_1)) \right] \\ &= \int_0^{t_0(x, x_1)} \lambda e^{-\lambda t} e^{-\delta t} dt \int_0^{a(t) + \frac{x}{\alpha}} V(a(t) - y) dF(y) + e^{-(\lambda + \delta)t_0(x, x_1)} V(x_1). \end{aligned} \quad (4.56)$$

Let $\mathcal{D} = \{x > 0 : V'(x) \text{ exists}\}$ and $t \in \overline{\mathcal{D}} := \{y : a(y) \in \mathcal{D}\}$. As $V(x_n)$ is differentiable almost everywhere, the Lebesgue measure of \mathcal{D}^c is 0. Noting that $V(a(t))$ is differentiable for $a(t) \in \mathcal{D}$, the complement of $\overline{\mathcal{D}}$ has a zero Lebesgue measure, too.

Notice that for any $x \in \mathcal{C}$, if $V'(y)$ exists, then

$$(p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})V'(y) - (\lambda + \delta)V(y) + \lambda \int_0^{y + \frac{y}{\alpha}} V(y - z) dF(z) = 0. \quad (4.57)$$

It follows from (4.56) and (4.57) that

$$\begin{aligned} \mathcal{T}_V(x) &= \int_{\overline{\mathcal{D}} \cap (0, t_0(x, x_1))} e^{-(\lambda + \delta)t} ((\lambda + \delta)V(a(t)) \\ &\quad - (p + ra(t)I\{a(t) \geq 0\} + \alpha a(t)I\{a(t) < 0\})V'(a(t))) dt \\ &\quad + e^{-(\lambda + \delta)t_0(x, x_1)} V(x_1) \\ &= \int_{\overline{\mathcal{D}} \cap (0, t_0(x, x_1))} d(e^{-(\lambda + \delta)t} V(a(t)) + e^{-(\lambda + \delta)t_0(x, x_1)} V(x_1)) \\ &= V(x) - e^{-(\lambda + \delta)t_0(x, x_1)} V(x_1) + e^{-(\lambda + \delta)t_0(x, x_1)} V(x_1) \\ &= V(x), \text{ for } x \in \mathcal{C}. \end{aligned} \quad (4.58)$$

Combining (4.54), (4.55) and (4.58) shows that $V(\cdot)$ is a fixed point. This completes the proof. \square

Now we have shown that like the Cramér-Lundberg cases respectively with and without interest, the optimal strategy is also a band strategy in the absolute ruin case. Intuitively, we would think that under the optimal strategy, there should be no dividends if the company is in deficit. In the following we will prove this rigorously.

Lemma 4.3 *For any fixed $x_0 \in (-\frac{p}{\alpha}, \infty)$, there exists a unique in (x_0, ∞) differentiable, strictly increasing and positive solution u on $[x_0, \infty)$ to the equation*

$$\begin{aligned} 0 &= (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})u'(x) - (\lambda + \alpha)u(x) \\ &\quad + \lambda \int_0^{x-x_0} u(x-y)dF(y) + \lambda \int_{x-x_0}^{x+\frac{p}{\alpha}} V(x-y)dF(y) \end{aligned} \quad (4.59)$$

with boundary condition $u(x_0) = V(x_0)$.

Proof. First we show that there is a such solution on $[x_0, x_0 + h)$ for $h = \frac{p + \alpha x_0 I\{x_0 < 0\}}{z(2\lambda + \delta)}$. Let $\mathcal{H}[x_0, x_0 + h)$ denote the set of continuous, increasing and positive functions on $[x_0, x_0 + h)$. Define an operator \mathcal{T} that for any $u \in \mathcal{H}[x_0, x_0 + h)$,

$$\mathcal{T}_u(x) = \int_{x_0}^x \frac{(\lambda + \delta)u(s) - \lambda \int_0^{s-x_0} u(s-y)dF(y) - \lambda \int_{s-x_0}^{s+\frac{p}{\alpha}} V(s-y)dF(y)}{p + rsI\{s \geq 0\} + \alpha sI\{s < 0\}} ds + V(x_0). \quad (4.60)$$

We will show that \mathcal{T} is a contraction on $\mathcal{H}[x_0, x_0 + h)$.

For any $u \in \mathcal{H}[x_0, x_0 + h)$, as both u and V are increasing and $u(x_0) = V(x_0)$, we get

$$\begin{aligned} &\lambda \int_0^{x-x_0} u(x-y)dF(y) + \lambda \int_{x-x_0}^{x+\frac{p}{\alpha}} V(x-y)dF(y) \\ &\leq \lambda \int_0^{x-x_0} u(x)dF(y) + \lambda \int_{x-x_0}^{x+\frac{p}{\alpha}} u(x)dF(y) \leq \lambda u(x). \end{aligned}$$

Define $\|u\| = \sup_{x \in [x_0, x_0 + h)} |u(x)|$.

It follows by (4.60) that for any $x \in [x_0, x_0 + h)$ and $u_1, u_2 \in \mathcal{H}[x_0, x_0 + h)$,

$$\begin{aligned} |\mathcal{T}_{u_1}(x) - \mathcal{T}_{u_2}(x)| &\leq \int_{x_0}^x \frac{(\lambda + \delta)\|u_1 - u_2\|_{x_0}^{x_0+h} + \lambda\|u_1 - u_2\|_{x_0}^{x_0+h}}{p + rsI\{s \geq 0\} + \alpha sI\{s < 0\}} ds \\ &\leq \frac{(2\lambda + \delta)h\|u_1 - u_2\|_{x_0}^{x_0+h}}{p + \alpha x_0 I\{x_0 < 0\}} \leq \frac{1}{2}\|u_1 - u_2\|_{x_0}^{x_0+h}. \end{aligned}$$

Therefore, \mathcal{T} is a contraction on $\mathcal{H}[x_0, x_0 + \varepsilon)$. As a result, there exists a unique $u \in \mathcal{H}[x_0, x_0 + \varepsilon)$ such that $u(x) = \mathcal{T}_u(x)$, i.e.,

$$u(x) = \int_{x_0}^x \frac{(\lambda + \delta)u(s) - \lambda \int_0^{s-x_0} u(s-y)dF(y) - \lambda \int_{s-x_0}^{s+\frac{p}{\alpha}} V(s-y)dF(y)}{p + rsI\{s \geq 0\} + \alpha sI\{s < 0\}} ds + V(x_0),$$

which implies

$$u'(x) = \frac{(\lambda + \delta)u(x) - \lambda \int_0^{x-x_0} u(x-y)dF(y) - \lambda \int_{x-x_0}^{x+\frac{p}{\alpha}} V(x-y)dF(y)}{p + rxI\{x \geq 0\} + \alpha xI\{x < 0\}} \quad \text{for } x \in [x_0, x_0 + h).$$

This completes the proof of the existence and uniqueness of an positive, increasing and differentiable solution to (6) on $[x_0, x_0 + h)$.

Similarly, we can prove the existence and uniqueness of a solution to (6) on $[x_0+h, x_0+2h]$ fulfilling the required properties. Repeating the above process, we can prove the existence of a unique solution to (6) on $[x_0, \infty)$, which is differentiable, increasing and positive. \square

Theorem 4.4 (i) For any $x \in \mathcal{A}$, $V(x)$ is differentiable and $V'(x) = 1$.

(ii) For any $(x_0, x_1) \subset \mathcal{C}$, $V(x)$ is differentiable on (x_0, x_1) , and $V'(x) > 1$ for $x \in (x_0, x_1)$.

Proof. (i) Consider any $x \in \mathcal{A}$. Choose sequences $h_n^+ > 0$ and $h_n^- < 0$ with $\lim_{n \rightarrow \infty} h_n^\pm = 0$, such that $\lim_{n \rightarrow \infty} \frac{V(x+h_n^+) - V(x)}{h_n^+} = \limsup_{h \downarrow 0} \frac{V(x+h) - V(x)}{h}$ and $\lim_{n \rightarrow \infty} \frac{V(x+h_n^-) - V(x)}{h_n^-} = \limsup_{h \downarrow 0} \frac{V(x+h) - V(x)}{h}$. As $x \in \mathcal{A}$, $\mathcal{L}_v(x) = 0$. Then it follows by (3.8) and Theorem 3.4 (ii) that $\lim_{n \rightarrow \infty} D_V(x, h_n^\pm) \leq 1$, which implies $\limsup_{h \uparrow 0} \frac{V(x+h) - V(x)}{h} \leq 1$ and $\limsup_{h \downarrow 0} \frac{V(x+h) - V(x)}{h} \leq 1$. As $\liminf_{h \rightarrow 0} \frac{V(x+h) - V(x)}{h} \geq 1$, we conclude that $\lim_{h \rightarrow 0} \frac{V(x+h) - V(x)}{h} = 1$. (ii) Use proof by contradiction. Note that for any $x \in \mathcal{C} \cap (-\frac{p}{\alpha}, 0)$, if $V(x)$ is differentiable, then $0 = (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})V'(x) - (\lambda + \delta)V(x) + \lambda \int_0^x V(x-y)dF(y)$, which can be rewritten as

$$0 = (p + rxI\{x \geq 0\} + \alpha xI\{x < 0\})V'(x) - (\lambda + \delta)V(x) + \lambda \int_0^{x-x_0} V(x-y)dF(y) + \lambda \int_{x-x_0}^{x+\frac{p}{\alpha}} V(x-y)dF(y). \quad (4.61)$$

Then by (4.61) and Lemma 4.3, we conclude that $V(x)$ is equal to the unique solution of (4.59) on $(x_0, x_0 + h)$ and therefore is differentiable on (x_0, x_1) .

By Theorem 3.2, we know that for any $x \in \mathcal{C}$, if $V'(x)$ exists, then $V'(x) \geq 1$. By the definition of the set \mathcal{C} , we know that $V'(x)$, if exists, can not be 1. If $V'(x) = 1$, then x belongs to either \mathcal{A} or \mathcal{B} . Therefore, $V'(x) \neq 1$ for all $x \in (x_0, x_1)$. \square

Theorem 4.5 Assume $\alpha > \lambda + \delta$. The following statements hold.

(i) $\mathcal{A} \cap (-\frac{p}{\alpha}, 0)$ consists of isolated points only.

(ii) For any $x_0 \in \mathcal{A} \cap (-\frac{p}{\alpha}, 0)$, we can find an $h > 0$ such that $(x_0, x_0 + h) \subset \mathcal{B}$.

(iii) $\mathcal{B} \cap (-\frac{p}{\alpha}, 0) = \emptyset$.

Proof. Consider any x_1 and x_2 with $-\frac{p}{\alpha} < x_1 < x_2 < 0$ and $[x_1, x_2) \subset \mathcal{A}$ such that $V(x)$ is differentiable on $[x_1, x_2)$, $V'(x_1) = 1$ and

$$(p + \alpha x)V'(x) = (\lambda + \delta)V(x) - \lambda \int_0^{x+\frac{p}{\alpha}} V(x-y)dF(y) \text{ for } x \in [x_0, x_2). \quad (4.62)$$

By setting x in the above equality to be x_1 and $x_1 + \varepsilon$, respectively, and using the newly obtained equations, we can obtain that for any $\varepsilon \in (0, x_2 - x_1)$,

$$\frac{(p + \alpha x_1)(V'(x_1 + \varepsilon) - V'(x_1))}{\varepsilon} = -\alpha V'(x_1 + \varepsilon) + (\lambda + \delta) \frac{V(x_1 + \varepsilon) - V(x_1)}{\varepsilon} - \lambda I(x_1, \varepsilon), \quad (4.63)$$

where $I(x_1, \varepsilon) = \frac{\int_0^{x_1+\varepsilon+\frac{p}{\alpha}} V(x_1+\varepsilon-y)dF(y) - \int_0^{x_1+\frac{p}{\alpha}} V(x_1-y)dF(y)}{\varepsilon}$. By noticing that $V'(x_1) = 1$, $V'(x_1 + \varepsilon) \geq 1$, $I(x_1, \varepsilon) \geq 0$ and $\lambda + \delta < \alpha$, from (4.63) we obtain $(p + \alpha x_1) \frac{V'(x_1 + \varepsilon) - V'(x_1)}{\varepsilon} \leq -\alpha + (\lambda + \delta)(1 + \frac{\alpha(\varepsilon)}{\varepsilon}) < 0$ for small ε . As a result,

$$V'(x_1 + \varepsilon) < V'(x_1) = 1 \text{ for } \varepsilon (\varepsilon > 0) \text{ small enough.} \quad (4.64)$$

(i) Use proof by contradiction. Assume that $x_0 \in \mathcal{A} \cap (-\frac{p}{\alpha}, 0)$ and it is not isolated. Then, as \mathcal{A} is closed, we can find an $h > 0$ such that either $[x_0, x_0 + h] \subset \mathcal{A}$ or $[x_0 - h, x_0] \subset \mathcal{A}$. Use $[x_1, x_2]$ to denote $[x_0 - h, x_0]$ if $[x_0 - h, x_0] \subset \mathcal{A}$, and $[x_0, x_0 + h]$, otherwise. Then $[x_1, x_2] \subset \mathcal{A}$. It follows by Theorem 4.4 (i) that

$$V'(x) = 1 \quad \text{for } x \in [x_1, x_2]. \quad (4.65)$$

Therefore, according to the definition for \mathcal{A} , we have $\mathcal{L}_V(x) = \mathcal{G}_V(x) = 0$ for all $x \in [x_1, x_2]$, which is equivalent to $(p + \alpha x)V'(x) = (\lambda + \delta)V(x) - \lambda \int_0^{x+\frac{p}{\alpha}} V(x-y)dF(y)$ for $x \in [x_1, x_2]$. Then by (4.64) it follows that $V'(x_1 + \varepsilon) < V'(x_1) = 1$ for small positive ε , which is a contradiction to (4.65).

(ii) Assume that there exists an $x_0 \in \mathcal{A} \cap (-\frac{p}{\alpha}, 0)$, such that we can find an $h > 0$ satisfying $(x_0, x_0 + h) \not\subset \mathcal{B}$. Then $(x_0, x_0 + h) \subset \mathcal{C}$, because \mathcal{A} consists of isolated points only and both \mathcal{B} and \mathcal{C} are half open. Hence, it follows by Theorem 4.4 (ii) that $V(x)$ is differentiable on $(x_0, x_0 + h)$ and $V'(x) > 1$ for $x \in (x_0, x_0 + h)$. Hence, V is a solution to the HJB equation (3.7) and therefore, (4.62) holds for $x \in (x_0, x_0 + h)$. As $x_0 \in \mathcal{A}$, we have $V'(x_0) = 1$, which along with the definition for \mathcal{A} implies that (4.61) also holds for $x = x_0$. Then by setting x_1 and x_2 in (4.64) as x_0 and $x_0 + h$, respectively, it follows that $V'(x_0 + \varepsilon) < V'(x_0) = 1$ for small positive ε , which is a contradiction to the fact that $V'(x_0 + \varepsilon) \geq 1$.

(iii) Assume $\mathcal{B} \cap (-\frac{p}{\alpha}, 0) \neq \emptyset$. Then there exist x_0 and x_1 , such that $-\frac{p}{\alpha} < x_0 < x_1 < 0$, $[x_0, x_1] \subset \mathcal{B}$ and $x_0 \in \mathcal{A}$. Therefore, by Theorem 4.4 (i) and the definition for \mathcal{B} , we get $V'(x) = 1$ for $x \in [x_0, x_1)$, which implies $V(x) = x - x_0 + V(x_0)$ for $x \in [x_0, x_1)$. Note that $\mathcal{G}_V(x_0) = 0$. Then for $x \in (x_0, x_1)$,

$$\begin{aligned} \mathcal{G}_V(x) &= \mathcal{G}_V(x) - \mathcal{G}_V(x_0) \\ &= \alpha(x - x_0) - (\lambda + \delta)(x - x_0) + \lambda \int_0^{x+\frac{p}{\alpha}} V(x-y)dF(y) - \lambda \int_0^{x_0+\frac{p}{\alpha}} V(x_0-y)dF(y) \\ &> 0, \end{aligned} \quad (4.66)$$

where the last inequality follows by $\alpha > \lambda + \delta$ and the fact that V is nonnegative and increasing. Since $x \in (x_0, x_1) \subset \mathcal{B}$, we have $\mathcal{G}_V(x) < 0$, which contradicts the inequality (4.66). \square

Theorem 4.6 *If $\alpha > \lambda + \delta$, $(-\frac{p}{\alpha}, 0) \subset \mathcal{C}$.*

Proof. By Theorem 4.5 (iii), it follows that

$$(-\frac{p}{\alpha}, 0) \cap \mathcal{B} = \emptyset. \quad (4.67)$$

So it is sufficient to show that $(-\frac{p}{\alpha}, 0) \cap \mathcal{A} = \emptyset$. If this is not true, then we can find an $x_0 \in \mathcal{A} \cap (-\frac{p}{\alpha}, 0)$. By Theorem 4.5, it follows that there exist an $h > 0$ such that $(x_0, x_0 + h) \subset \mathcal{B}$, which contradicts (4.67) by noting $x_0 + h \in (-\frac{p}{\alpha}, 0)$ for small $h > 0$. \square

Remark 4.1 *Theorem 4.2 and Theorem 4.6 together imply that if $\alpha > \lambda + \delta$, under the optimal strategy L^* the company will pay no dividends when the reserve is negative. In other words, if $\alpha > \lambda + \delta$ it is optimal to pay no dividends when the reserve is negative.*

5 Conclusion

We studied the dividend optimization problem of an insurance corporation, of which the surplus is modeled by a compound Poisson model with credit and debit interest. The company earns interest when the reserve is positive, and can refinance to settle its claims when the reserve is negative but above the critical level. The company controls the dividend pay-out

dynamically and seeks to maximize the expected total discounted dividends until ruin. We proved that the value function is the unique viscosity solution satisfying certain conditions of the associated Hamilton-Jacob-Bellman equation, that the optimal strategy is a band strategy, and that it is optimal to pay no dividends when the reserve is negative. This result provides theoretical justification to the regulation of no dividend payments when the surplus is in deficit.

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APPENDIX

In this appendix, we present the proofs to Lemma 3.5, Theorem 3.9, Theorem 3.10 and Lemma 4.1

Proof of Lemma 3.5 We employ a proof by contradiction.

Assume that there exists a $x_0 \in (-\frac{p}{\alpha}, \infty)$ such that $\underline{u}(x_0) > \bar{u}(x_0)$.

For any constant $\gamma > 0$, define functions for $x \geq -\frac{p}{\alpha}$,

$$\bar{u}_\gamma(x) = e^{-\gamma x} \bar{u}(x) \quad \text{and} \quad \underline{u}_\gamma(x) = e^{-\gamma x} \underline{u}(x).$$

By the fact that both the functions \bar{u} and \underline{u} are locally Lipschitz continuous and bounded by a linear function, it can be easily shown that $\bar{u}_\gamma(x)$ and $\underline{u}_\gamma(x)$ are both bounded and Lipschitz continuous on $(-\frac{p}{\alpha}, \infty)$, too, which implies that there exists some constant $m > 0$ such that

$$\frac{\bar{u}_\gamma(y) - \bar{u}_\gamma(x)}{y - x} \leq m \quad \text{and} \quad \frac{\underline{u}_\gamma(y) - \underline{u}_\gamma(x)}{y - x} \leq m \quad \text{for } x, y \in (-\frac{p}{\alpha}, \infty). \quad (\text{A-1})$$

For $\rho > 0$, consider a function $\phi_\rho : [-\frac{p}{\alpha}, \infty) \times [-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ given by

$$\phi_\rho(x, y) = \underline{u}_\gamma(x) - \bar{u}_\gamma(y) - \frac{\rho}{2}(x - y)^2 - \frac{2m}{\rho^2(y - x)^2 + \rho}. \quad (\text{A-2})$$

Note that we can find a $\gamma_1 > 0$ such that $\underline{u}_\gamma(x_0) - \bar{u}_\gamma(x_0) > 0$ for all $\gamma \in (0, \gamma_1]$, and that $\underline{u}(-\frac{p}{\alpha}) = \bar{u}(-\frac{p}{\alpha}) = 0$ and $\lim_{x \rightarrow \infty} \underline{u}(x) = \lim_{x \rightarrow \infty} \bar{u}(x) = 0$. Then we can define

$$M = \max_{x \geq -\frac{p}{\alpha}} (\underline{u}_\gamma(x) - \bar{u}_\gamma(x)) \quad \text{and} \quad M_\rho = \max_{x, y \geq -\frac{p}{\alpha}} \phi_\rho(x, y). \quad (\text{A-3})$$

Then $0 < M < \infty$ and M has a maximizer denoted by x^* , and M_ρ also has a maximizer, denoted by (x_ρ, y_ρ) here.

Noting that

$$M_\rho \geq \phi_\rho(x^*, x^*) = M - \frac{2m}{\rho}, \quad (\text{A-4})$$

then it follows that

$$\liminf_{\rho \rightarrow \infty} M_\rho \geq M > 0. \quad (\text{A-5})$$

Let $(\rho_n)_{n \in \mathbb{N}}$ be a sequence tending to ∞ as $n \rightarrow \infty$ such that (x_{ρ_n}, y_{ρ_n}) converges as $n \rightarrow \infty$. Use (\bar{x}, \bar{y}) to denote the limit of (x_{ρ_n}, y_{ρ_n}) . We will show in the following that

$$\bar{x} = \bar{y}. \quad (\text{A-6})$$

If this is not true, then $|\bar{x} - \bar{y}| > 0$. By noticing

$$M_{\rho_n} = \underline{u}_\gamma(x_{\rho_n}) - \bar{u}_\gamma(y_{\rho_n}) - \frac{\rho_n}{2}(x_{\rho_n} - y_{\rho_n})^2 - \frac{2m}{\rho_n^2(y_{\rho_n} - x_{\rho_n})^2 + \rho_n}, \quad (\text{A-7})$$

$\lim_{n \rightarrow \infty} \underline{u}_\gamma(x_{\rho_n}) = \underline{u}_\gamma(\bar{x})$ and $\lim_{n \rightarrow \infty} \bar{u}_\gamma(y_{\rho_n}) = \bar{u}_\gamma(\bar{y})$, we obtain

$$\lim_{n \rightarrow \infty} M_{\rho_n} = \underline{u}_\gamma(\bar{x}) - \bar{u}_\gamma(\bar{y}) - \frac{\lim_{n \rightarrow \infty} \rho_n}{2}(\bar{x} - \bar{y})^2 = -\infty,$$

which is a contradiction to (A-5).

Next, we will show that for large n , $x_{\rho_n}, y_{\rho_n} \in (-\frac{p}{\alpha}, \infty)$. For any constant $\hat{x} \geq -\frac{p}{\alpha}$ with $\underline{u}_\gamma(\hat{x}) \leq \bar{u}_\gamma(\hat{x})$, we have $\phi_\rho(\hat{x}, \hat{x}) = \underline{u}_\gamma(\hat{x}) - \bar{u}_\gamma(\hat{x}) - \frac{2m}{\rho} < 0$, which indicates $\bar{x} = \bar{y} \neq \hat{x}$. Note that $\underline{u}(-\frac{p}{\alpha}) = \bar{u}(-\frac{p}{\alpha}) = 0$. We can conclude immediately that

$$\bar{x} = \bar{y} \neq -\frac{p}{\alpha}. \quad (\text{A-8})$$

By observing that

$$\lim_{y \rightarrow \infty} \phi_\rho(x, y) = \lim_{y \rightarrow \infty} \left(\underline{u}_\gamma(x) - \bar{u}_\gamma(y) - \frac{\rho}{2}(x - y)^2 - \frac{2m}{\rho^2(y - x)^2 + \rho} \right) = -\infty,$$

we conclude that

$$\bar{y} < \infty. \quad (\text{A-9})$$

Combining (A-6), (A-8) and (A-9) yields $\bar{x} = \bar{y} \in (-\frac{p}{\alpha}, \infty)$.

As x_{ρ_n} and y_{ρ_n} converge to \bar{x} and \bar{y} , respectively, we can find an N_1 such that for all $n \geq N_1$,

$$x_{\rho_n}, y_{\rho_n} \in (-\frac{p}{\alpha}, \infty). \quad (\text{A-10})$$

Now we introduce two more functions

$$\zeta_\rho(x) = \bar{u}_\gamma(x_\rho) + \frac{\rho}{2}(x - y_\rho)^2 + \frac{2m}{\rho^2(y_\rho - x)^2 + \rho} + \phi_\rho(x_\rho, y_\rho),$$

and

$$\varphi_\rho(y) = \underline{u}_\gamma(y_\rho) - \frac{\rho}{2}(x_\rho - y)^2 - \frac{2m}{\rho^2(x_\rho - y)^2 + \rho} - \phi_\rho(x_\rho, y_\rho).$$

It can be easily shown that for all $n \geq N_1$, ζ_{ρ_n} and φ_{ρ_n} are both continuously differentiable. Furthermore, $\underline{u}_\gamma(x) - \zeta_{\rho_n}(x) = \phi_{\rho_n}(x, y_{\rho_n}) - \phi_{\rho_n}(x_{\rho_n}, y_{\rho_n})$ attains its maximum 0 at x_{ρ_n} , and $\bar{u}_\gamma(y) - \varphi_{\rho_n}(y) = -\phi_{\rho_n}(x_{\rho_n}, y) + \phi_{\rho_n}(x_{\rho_n}, y_{\rho_n})$ reaches its minimum 0 at y_{ρ_n} . Since \underline{u} and \bar{u} are respectively viscosity sub and super-solutions of (3.7), by the definition for viscosity solutions we can see that \underline{u}_γ and \bar{u}_γ are respectively viscosity sub and super-solutions of the following equation

$$\begin{aligned} & \max \{ 1 - e^{\gamma x}(\gamma u(x) + u'(x)), (p + rx)I\{x \geq 0\} + \alpha x I\{x < 0\} \} \times \\ & (\gamma u(x) + u'(x)) - (\lambda + \delta)u(x) + \lambda \int_0^{x + \frac{p}{\alpha}} u(x - y)e^{-\gamma y} dF(y) \} = 0. \end{aligned}$$

Therefore, by Definition 3.2 we can obtain that for $n \geq N_1$,

$$\begin{aligned} & \max \{ 1 - e^{\gamma x_{\rho_n}}(\gamma \underline{u}_\gamma(x_{\rho_n}) + \zeta'_{\rho_n}(x_{\rho_n})), (p + rx_{\rho_n})I\{x_{\rho_n} \geq 0\} + \alpha x_{\rho_n} I\{x_{\rho_n} < 0\} \} \times \\ & (\gamma \underline{u}_\gamma(x_{\rho_n}) + \zeta'_{\rho_n}(x_{\rho_n})) - (\lambda + \delta)\underline{u}_\gamma(x_{\rho_n}) + \lambda \int_0^{x_{\rho_n} + \frac{p}{\alpha}} \underline{u}_\gamma(x_{\rho_n} - y)e^{-\gamma y} dF(y) \} \geq 0, \end{aligned} \quad (\text{A-11})$$

and

$$\begin{aligned} & \max \{1 - e^{\gamma y_{\rho_n}} (\gamma \bar{u}_{\gamma}(y_{\rho_n}) + \varphi'_{\rho_n}(y_{\rho_n})), (p + r y_{\rho_n} I\{y_{\rho_n} \geq 0\} + \alpha y_{\rho_n} I\{y_{\rho_n} < 0\}) \times \\ & (\gamma \bar{u}_{\gamma}(y_{\rho_n}) + \varphi'_{\rho_n}(y_{\rho_n})) - (\lambda + \delta) \bar{u}_{\gamma}(y_{\rho_n}) + \lambda \int_0^{y_{\rho_n} + \frac{p}{\alpha}} \bar{u}_{\gamma}(y_{\rho_n} - y) e^{-\gamma y} dF(y)\} \leq 0. \end{aligned} \quad (\text{A-12})$$

Use B_1, B_2 to represent the first and second terms in the curly brackets on the left-hand side of (A-11), respectively, and D_1, D_2 to represent the first and second terms on the left-hand side of (A-12), respectively. Then $\max\{B_1, B_2\} \geq 0 \geq \max\{D_1, D_2\}$. So at least one of the inequalities $B_1 \geq D_1$ and $B_2 \geq D_2$ holds.

(i) First, assume that $B_2 \geq D_2$ is true. Noticing that

$$\zeta'_{\rho_n}(x_{\rho_n}) = \varphi'_{\rho_n}(y_{\rho_n}) = \rho_n(x_{\rho_n} - y_{\rho_n}) + \frac{4m(y_{\rho_n} - x_{\rho_n})}{(\rho_n(y_{\rho_n} - x_{\rho_n})^2 + 1)^2}, \quad (\text{A-13})$$

by substitutions for $\zeta'_{\rho_n}(x_{\rho_n})$ and $\varphi'_{\rho_n}(y_{\rho_n})$ by (A-13), it follows immediately that

$$\begin{aligned} & (p + r y_{\rho_n} I\{y_{\rho_n} \geq 0\} + \alpha y_{\rho_n} I\{y_{\rho_n} < 0\}) \times \\ & \left(\gamma \bar{u}_{\gamma}(y_{\rho_n}) + \rho_n(x_{\rho_n} - y_{\rho_n}) + \frac{4m(y_{\rho_n} - x_{\rho_n})}{(\rho_n(y_{\rho_n} - x_{\rho_n})^2 + 1)^2} \right) \\ & - (p + r x_{\rho_n} I\{x_{\rho_n} \geq 0\} + \alpha x_{\rho_n} I\{x_{\rho_n} < 0\}) \times \\ & \left(\gamma \underline{u}_{\gamma}(x_{\rho_n}) + \rho_n(x_{\rho_n} - y_{\rho_n}) + \frac{4m(y_{\rho_n} - x_{\rho_n})}{(\rho_n(y_{\rho_n} - x_{\rho_n})^2 + 1)^2} \right) \\ & + (\lambda + \delta) (\underline{u}_{\gamma}(x_{\rho_n}) - \bar{u}_{\gamma}(y_{\rho_n})) \\ & \leq \lambda \left(\int_0^{x_{\rho_n} + \frac{p}{\alpha}} \underline{u}_{\gamma}(x_{\rho_n} - y) e^{-\gamma y} dF(y) - \int_0^{y_{\rho_n} + \frac{p}{\alpha}} \bar{u}_{\gamma}(y_{\rho_n} - y) e^{-\gamma y} dF(y) \right). \end{aligned} \quad (\text{A-14})$$

Notice that $\phi_{\rho_n}(x_{\rho_n}, x_{\rho_n}) + \phi_{\rho_n}(y_{\rho_n}, y_{\rho_n}) \leq 2\phi_{\rho_n}(x_{\rho_n}, y_{\rho_n})$, i.e.

$$\begin{aligned} & \underline{u}_{\gamma}(x_{\rho_n}) - \bar{u}_{\gamma}(x_{\rho_n}) + \underline{u}_{\gamma}(y_{\rho_n}) - \bar{u}_{\gamma}(y_{\rho_n}) - \frac{4m}{\rho_n} \\ & \leq 2 \left(\underline{u}_{\gamma}(x_{\rho_n}) - \bar{u}_{\gamma}(y_{\rho_n}) - \frac{\rho_n}{2}(x_{\rho_n} - y_{\rho_n})^2 - \frac{2m}{\rho_n^2(y_{\rho_n} - x_{\rho_n})^2 + \rho_n} \right). \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} \rho_n(x_{\rho_n} - y_{\rho_n})^2 & \leq \underline{u}_{\gamma}(x_{\rho_n}) - \underline{u}_{\gamma}(y_{\rho_n}) + \bar{u}_{\gamma}(x_{\rho_n}) - \bar{u}_{\gamma}(y_{\rho_n}) + \frac{4m(y_{\rho_n} - x_{\rho_n})^2}{\rho_n(y_{\rho_n} - x_{\rho_n})^2 + 1} \\ & \leq 2m|y_{\rho_n} - x_{\rho_n}| + 4m(y_{\rho_n} - x_{\rho_n})^2, \end{aligned}$$

where the last inequality follows by (A-1). As a result,

$$|y_{\rho_n} - x_{\rho_n}| \leq \frac{2m}{\rho_n - 4m} \text{ for } \rho_n > 4m. \quad (\text{A-15})$$

As \underline{u}_{γ} and \bar{u}_{γ} are both bounded, taking limits on (A-14) yields

$$\begin{aligned} & \gamma (p + r \bar{x} I\{\bar{x} \geq 0\} + \alpha \bar{x} I\{\bar{x} < 0\}) (\bar{u}_{\gamma}(\bar{x}) - \underline{u}_{\gamma}(\bar{x})) \\ & + (\lambda + \delta) (\underline{u}_{\gamma}(\bar{x}) - \bar{u}_{\gamma}(\bar{x})) \\ & \leq \lambda \left(\int_0^{\bar{x} + \frac{p}{\alpha}} (\underline{u}_{\gamma}(\bar{x} - y) - \bar{u}_{\gamma}(\bar{x} - y)) e^{-\gamma y} dF(y) \right) \text{ for } \gamma > 0 \end{aligned} \quad (\text{A-16})$$

$$\leq \lambda M \text{ for } \gamma \in (0, \gamma_1), \quad (\text{A-17})$$

where the last inequality follows from (A-3).

By choosing $\gamma < \min \left\{ \frac{\delta}{2(p+r\bar{x}I\{\bar{x} \geq 0\})}, \gamma_1 \right\}$, it follows immediately from (A-17) that

$$\underline{u}_\gamma(\bar{x}) - \bar{u}_\gamma(\bar{x}) < \frac{\lambda}{\lambda + \frac{\delta}{2}} M < M. \quad (\text{A-18})$$

On the other hand, from (A-5) we get $M \leq \liminf_{\rho \rightarrow \infty} M_\rho \leq \lim_{n \rightarrow \infty} M_{\rho_n} = \underline{u}_\gamma(\bar{x}) - \bar{u}_\gamma(\bar{x})$, which contradicts (A-18). Consequently, $B_2 \geq D_2$ does not hold.

(ii) Now, we look at the case $B_1 \geq D_1$. Then we have

$$e^{\gamma x_{\rho_n}}(\gamma \underline{u}_\gamma(x_{\rho_n}) + \zeta'_{\rho_n}(x_{\rho_n})) \leq e^{\gamma y_{\rho_n}}(\gamma \bar{u}_\gamma(y_{\rho_n}) + \varphi'_{\rho_n}(y_{\rho_n})). \quad (\text{A-19})$$

It follows immediately from (A-13) and (A-19) that

$$e^{\gamma x_{\rho_n}} \underline{u}_\gamma(x_{\rho_n}) - e^{\gamma y_{\rho_n}} \bar{u}_\gamma(y_{\rho_n}) \leq \frac{4m - \rho_n}{\gamma} (y_{\rho_n} - x_{\rho_n})(e^{\gamma y_{\rho_n}} - e^{\gamma x_{\rho_n}}). \quad (\text{A-20})$$

Let $N_2(\epsilon)$ be a positive integer such that for all $n \geq N_2(\epsilon)$, $\rho_n \geq 4m$.

Since $(y_{\rho_n} - x_{\rho_n})(e^{\gamma y_{\rho_n}} - e^{\gamma x_{\rho_n}})$ is always nonnegative, then from (A-20) we can see that for all $n \geq N_2(\epsilon)$,

$$e^{\gamma x_{\rho_n}} \underline{u}_\gamma(x_{\rho_n}) - e^{\gamma y_{\rho_n}} \bar{u}_\gamma(y_{\rho_n}) \leq 0. \quad (\text{A-21})$$

Recall that $x_{\rho_n} \rightarrow \bar{x}$, $y_{\rho_n} \rightarrow \bar{y}$ and $\bar{x} = \bar{y}$. There exists an integer $N_3(\epsilon)$ such that for all $n \geq N_3(\epsilon)$,

$$|e^{\gamma x_{\rho_n}} - e^{\gamma \bar{x}}| < \epsilon, \quad |e^{\gamma y_{\rho_n}} - e^{\gamma \bar{x}}| < \epsilon \quad \text{and} \quad |\bar{u}_\gamma(x_{\rho_n}) - \bar{u}_\gamma(y_{\rho_n})| < \epsilon.$$

Then for $n \geq N_3(\epsilon)$, we have

$$\begin{aligned} & \underline{u}_\gamma(x_{\rho_n})(1 - e^{\gamma x_{\rho_n}}) - \bar{u}_\gamma(y_{\rho_n})(1 - e^{\gamma y_{\rho_n}}) \\ &= \underline{u}_\gamma(x_{\rho_n})(1 - e^{\gamma x_{\rho_n}}) - \bar{u}_\gamma(x_{\rho_n})(1 - e^{\gamma y_{\rho_n}}) + (\bar{u}_\gamma(x_{\rho_n}) - \bar{u}_\gamma(y_{\rho_n}))(1 - e^{\gamma y_{\rho_n}}) \\ &< \underline{u}_\gamma(x_{\rho_n})(1 - e^{\gamma \bar{x}} + \epsilon) - \bar{u}_\gamma(x_{\rho_n})(1 - e^{\gamma \bar{x}} - \epsilon) + \epsilon \\ &\leq M(1 - e^{\gamma \bar{x}}) + (\underline{u}_\gamma(x_{\rho_n}) + \bar{u}_\gamma(x_{\rho_n}) + 1)\epsilon, \end{aligned} \quad (\text{A-22})$$

where the last inequality follows by (A-3).

Since the functions \underline{u}_γ and \bar{u}_γ are bounded, it can be easily shown that

$$\frac{M e^{\gamma \bar{x}}}{\sup_x (\underline{u}_\gamma(x) + \bar{u}_\gamma(x) + 1)} > 0. \quad (\text{A-23})$$

From (A-4), (A-7), (A-21), (A-22) and (A-23), it follows that for any $\epsilon < \frac{M e^{\gamma \bar{x}}}{\sup_x (\underline{u}_\gamma(x) + \bar{u}_\gamma(x) + 1)}$ and $n \geq \max\{N_2(\epsilon), N_3(\epsilon)\}$,

$$\begin{aligned} M \leq M_{\rho_n} + \frac{2m}{\rho_n} &= \underline{u}_\gamma(x_{\rho_n}) - \bar{u}_\gamma(y_{\rho_n}) \\ &= e^{\gamma x_{\rho_n}} \underline{u}_\gamma(x_{\rho_n}) - e^{\gamma y_{\rho_n}} \bar{u}_\gamma(y_{\rho_n}) \\ &\quad + \underline{u}_\gamma(x_{\rho_n})(1 - e^{\gamma x_{\rho_n}}) - \bar{u}_\gamma(y_{\rho_n})(1 - e^{\gamma y_{\rho_n}}) \\ &< M(1 - e^{\gamma \bar{x}}) + (\underline{u}_\gamma(x_{\rho_n}) + \bar{u}_\gamma(x_{\rho_n}) + 1)\epsilon < M, \end{aligned}$$

which is a contradiction. So $B_1 \geq D_1$ does not hold.

Combining (i) and (ii) shows that $B_1 < D_1$ and $B_2 < D_2$. This is a contraction to the fact that at least of the inequalities $B_1 \geq D_1$ and $B_2 \geq D_2$ holds. As a result, $\underline{u}(x) \leq \bar{u}(x)$ for all $x \geq -\frac{p}{\alpha}$. This completes the proof. \square

Proof of Theorem 3.9 Assume that $x \in (-\frac{p}{\alpha}, \bar{x}]$.

i) Let $\Pi(n)$ denote the set of admissible strategies such that if the initial reserve $x < \bar{x}$, the controlled reserve will always stay below or at \bar{x} until the arrival of the n th claim.

We will show that for any $n \in \mathbb{N}$ and $x \in (-\frac{p}{\alpha}, \bar{x}]$, $V(x) = \sup_{L \in \Pi(n)} V_L(x)$ by induction.

Noting that $\Pi(0) = \Pi$, we get $V(x) = \sup_{L \in \Pi(0)} V_L(x)$.

Assume that $V(x) = \sup_{L \in \Pi(n-1)} V_L(x)$ for some $n \geq 1$.

Let $L^{(n-1,x)} \in \Pi(n-1)$ be an $\frac{\epsilon}{2}$ -optimal strategy for the reserve process with the initial value x , that is

$$0 \leq V(x) - V_{L^{(n-1,x)}}(x) \leq \frac{\epsilon}{2}. \quad (\text{A-24})$$

Let τ^L denote the first time that the reserve process under strategy L reaches \bar{x} , and $\hat{\tau}^L$ the arrival time of the next claim occurring after time τ^L .

Then given the initial reserve x ($x \leq \bar{x}$), we can construct an $\frac{\epsilon}{2}$ -optimal strategy $L^{(n,x)} \in \Pi(n)$ as follows. Apply the strategy $L^{(n-1,x)}$ until the first time the controlled reserve reaches \bar{x} , then pays out dividends at a rate equal to the premium incoming rate to keep the reserve at the level \bar{x} until the arrival of the next claim. After that, we apply the strategy $L^{(n-1, R_{\hat{\tau}^L(n,x)}})$ to the shifted process $\theta_{\hat{\tau}^L(n,x)} R$.

Recall that S_1 and \hat{U}_1 are respectively the arrival time and the amount of the first claim. Note that for the case with initial reserve $R_0 = \bar{x}$, under strategy $L^{(n,x)}$ we have

$$\hat{\tau}^{L^{(n,x)}} = S_1, \quad R_{S_1} = (\bar{x} - U_1) \vee (-\frac{p}{\alpha}) \quad \text{and}$$

$$R_t = \bar{x}, \quad dL_t^{(n,\bar{x})} = (p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\})dt \quad \text{for } t < S_1.$$

Hence, by noticing the fact that ruin will not occur before the arrival of the first claim, i.e. $T \geq S_1$, and that $V(-\frac{p}{\alpha}) = 0$, we obtain that given the initial reserve \bar{x} ,

$$\begin{aligned} V_{L^{(n,\bar{x})}}(\bar{x}) &= \mathbb{E}_{\bar{x}} \left[\int_0^{S_1} e^{-\delta s} (p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\}) ds \right. \\ &\quad \left. + e^{-\delta S_1} V_{L^{(n-1, (\bar{x}-U_1) \vee (-\frac{p}{\alpha})}} \left((\bar{x} - U_1) \vee (-\frac{p}{\alpha}) \right) \right] \\ &= \frac{1}{\lambda + \delta} (p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\}) \\ &\quad + \frac{\lambda}{\lambda + \delta} \int_0^{\bar{x} + \frac{p}{\alpha}} V_{L^{(n-1, \bar{x}-y)}}(\bar{x} - y) dF(y) \end{aligned} \quad (\text{A-25})$$

It follows by (3.48), (A-24), (A-25) and assumption that $\mathcal{G}_V(\bar{x}) = 0$ that

$$\begin{aligned} V_{L^{(n,\bar{x})}}(\bar{x}) &\geq \frac{1}{\lambda + \delta} (p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\}) \\ &\quad + \frac{\lambda}{\lambda + \delta} \int_0^{\bar{x} + \frac{p}{\alpha}} (V(\bar{x} - y) - \frac{\epsilon}{2}) dF(y) \\ &\geq V(\bar{x}) - \frac{\epsilon}{2}. \end{aligned} \quad (\text{A-26})$$

Note that for any fixed $x \in [-\frac{p}{\alpha}, \bar{x}]$ and for $k = n - 1$ and n , we have

$$\begin{aligned} V_{L^{(k,x)}}(x) &= \mathbb{E}_x \left[\int_0^{\tau^{L^{(k,x)}}} e^{-\delta s} dL_s^{(k,x)}; \tau^{L^{(k,x)}} < T \right] \\ &\quad + \mathbb{E}_x [e^{-\delta \tau^{L^{(k,x)}}}; \tau^{L^{(k,x)}} < T] V_{L^{(k,\bar{x})}}(\bar{x}) + \mathbb{E}_x [e^{-\delta \tau^{L^{(k,x)}}}; \tau^{L^{(k,x)}} \geq T]. \end{aligned} \quad (\text{A-27})$$

From the construction of the strategies, we can see that given the initial reserve x ,

$$\tau^{L(n,x)} = \tau^{L(n-1,x)} \quad \text{and} \quad L_s^{(n,x)} = L_s^{(n-1,x)} \quad \text{for } s \leq \tau^{L(n,x)}. \quad (\text{A-28})$$

By using (A-27) for $k = n - 1$ and n , and (A-28), we obtain

$$V_{L(n-1,x)}(x) - V_{L(n-1,x)}(x) = \mathbb{E}_x \left[e^{-\delta \tau^{L(n,x)}} ; \tau^{L(n,x)} < T \right] (V_{L(n,\bar{x})}(\bar{x}) - V_{L(n-1,\bar{x})}(\bar{x})). \quad (\text{A-29})$$

Note that by the definition of $L^{(n-1,x)}$ and (A-26) we have

$$V(\bar{x}) \geq V_{L(n-1,\bar{x})}(\bar{x}) \geq V(\bar{x}) - \frac{\epsilon}{2} \quad \text{and} \quad V(\bar{x}) \geq V_{L(n,\bar{x})}(\bar{x}) \geq V(\bar{x}) - \frac{\epsilon}{2}, \quad (\text{A-30})$$

which implies

$$V_{L(n,\bar{x})}(\bar{x}) - V_{L(n-1,\bar{x})}(\bar{x}) \geq -\frac{\epsilon}{2}. \quad (\text{A-31})$$

Combining (A-24), (A-29) and (A-31) gives $V_{L(n,x)}(x) \geq V(x) - \epsilon$. Therefore,

$$V(x) \geq \sup_{L \in \Pi(n)} V_L(x) \geq V_{L(n,x)}(x) \geq V(x) - \epsilon.$$

Consequently, letting $\epsilon \rightarrow 0$ gives us $V(x) = \sup_{L \in \Pi(n)} V_L(x)$ for $x \in [-\frac{p}{\alpha}, \bar{x}]$.

ii) Now we try to find a strategy $\hat{L} \in \Pi_{\bar{x}}$ such that it is ϵ -optimal.

Noting that $V(\bar{x}) > 0$, we can find a t_1 large enough such that

$$e^{-\delta t_1} < \frac{\epsilon}{4V(\bar{x})}. \quad (\text{A-32})$$

Then for this fixed t_1 , choose an n large enough such that

$$\mathbb{P}(N(t_1) \geq n) = \sum_{k \geq n} \frac{e^{-\lambda t_1} (\lambda t_1)^k}{k!} \leq \frac{\epsilon}{4V(\bar{x})}. \quad (\text{A-33})$$

Define σ^L to be the first time that the controlled reserve process under strategy L reaches \bar{x} after the arrival of the n th claim (S_n).

Let $L^{(n,x)}$ be any $\frac{\epsilon}{2}$ -optimal strategy in $\Pi(n)$ given the initial reserve x . Given the initial reserve x , construct an dividend payout strategy $\hat{L}(x)$ such that the strategy $L^{(n,x)}$ is applied before time $\sigma^{L^{(n,x)}}$, then at time $t = \sigma^{L^{(n,x)}}$, a lump sum of $\bar{x} + \frac{p}{\alpha}$ is paid out immediately, and thereafter no dividends will be paid out.

Then, we have

$$\begin{aligned} V_{\hat{L}(x)}(x) &= \mathbb{E}_x \left[\int_0^{\sigma^{L^{(n,x)}}} e^{-\delta s} dL_s^{(n,x)} ; \sigma^{L^{(n,x)}} < T \right] + \mathbb{E}_x [e^{-\delta \sigma^{L^{(n,x)}}} ; \sigma^{L^{(n,x)}} < T] (\bar{x} + \frac{p}{\alpha}) \\ &\quad + \mathbb{E}_x \left[\int_0^T e^{-\delta s} dL_s^{(n,x)} ; \sigma^{L^{(n,x)}} \geq T \right]. \end{aligned} \quad (\text{A-34})$$

Note that for any initial reserve $x \leq \bar{x}$, the strategy $\hat{L}(x)$ is same as $L^{(n,x)}$ until both the controlled reserve under the former strategy reaches \bar{x} for the first time, which implies

$$\sigma^{\hat{L}(x)} = \sigma^{L^{(n,x)}} \quad \text{and} \quad \hat{L}_t(x) = L_t^{(n,x)} \quad \text{for } t \leq \sigma^{L^{(n,x)}}.$$

Noting that (A-27) also holds for the strategy $L^{(n,x)}$ here, by (A-34) we get, for $x \leq \bar{x}$,

$$\begin{aligned} V_{\hat{L}(x)}(x) - V_{L^{(n,x)}}(x) &= \mathbb{E}_x[e^{-\delta\sigma L^{(n,x)}}; \sigma L^{(n,x)} < T](\bar{x} + \frac{p}{\alpha} - V_{L^{(n,\bar{x})}}(\bar{x})) \\ &\geq -\mathbb{E}_x[e^{-\delta\sigma L^{(n,x)}}]V(\bar{x}). \end{aligned} \quad (\text{A-35})$$

As $S_n \leq \sigma L^{(n,x)}$, we have $\{\sigma L^{(n,x)} < t_1\} \subseteq \{S_n < t_1\} \subset \{N(t_1) \geq n\}$ for $x \leq \bar{x}$. Therefore, for any $x \leq \bar{x}$,

$$\{\sigma L^{(n,x)} < \infty\} \subset \{\sigma L^{(n,x)} \geq t_1\} \cup \{N(t_1) \geq n\}. \quad (\text{A-36})$$

Then by (A-32) and (A-33) we have

$$\begin{aligned} \mathbb{E}_x[e^{-\delta\sigma L^{(n,x)}}] &\leq \mathbb{E}_x[e^{-\delta\sigma L^{(n,x)}}; \sigma L^{(n,x)} \geq t_1] + \mathbb{E}_x[e^{-\delta\sigma L^{(n,x)}}; N(t_1) \geq n] \\ &\leq e^{-\delta t_1} + \mathbb{P}(N(t_1) \geq n) \leq \frac{\epsilon}{2V(\bar{x})}. \end{aligned} \quad (\text{A-37})$$

It follows from (A-35) and (A-37) that for $x \in [-\frac{p}{\alpha}, \bar{x}]$

$$V_{\hat{L}(x)}(x) \geq V_{L^{(n,x)}}(x) - \frac{\epsilon}{2} \geq V(x) - \epsilon, \quad (\text{A-38})$$

where the last inequality is due to the fact that $L^{(n,x)}$ is an $\frac{\epsilon}{2}$ -optimal strategy.

Noting that $\hat{L}(x) \in \Pi_{\bar{x}}$, the above inequality implies

$$\sup_{L \in \Pi_{\bar{x}}} V_L(x) \geq V(x), \quad x \in [-\frac{p}{\alpha}, \bar{x}].$$

This concludes the proof. \square

Proof of Theorem 3.10 It is sufficient to show that for any $\epsilon > 0$, there exists a strategy $\bar{L}^{(x)} \in \Pi_{\bar{x}}$ such that

$$V_{\bar{L}^{(x)}}(x) \geq V(x) - \epsilon \text{ for all } x \in (-\frac{p}{\alpha}, \bar{x}]. \quad (\text{A-39})$$

For a positive $\epsilon < 4V(\bar{x})$, define

$$\Delta(\epsilon) = \frac{p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\}}{\delta} \ln \frac{4V(\bar{x})}{\epsilon}, \quad (\text{A-40})$$

$$x_n = \bar{x} - \frac{\Delta(\epsilon)}{n}, \text{ and } h_n = \frac{V(x_n) - V(\bar{x})}{x_n - \bar{x}} - 1. \quad (\text{A-41})$$

It can be shown that $x_n \leq \bar{x}$ and $x_n \rightarrow \bar{x}$. Since $V'(\bar{x}) = 1$, we have $\lim_{n \rightarrow \infty} h_n = 0$. Moreover, notice

$$\begin{aligned} &\lim_{n \rightarrow \infty} \left(\frac{rx_n + p}{r\bar{x} + p} \right)^{\frac{\delta n}{r}} I\{\bar{x} \geq 0\} + \left(\frac{\alpha x_n + p}{\alpha\bar{x} + p} \right)^{\frac{\delta n}{\alpha}} I\{\bar{x} < 0\} \\ &= \exp \left\{ \frac{-\delta\Delta(\epsilon)}{p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\}} \right\}. \end{aligned}$$

Hence we can choose a n_0 such that

$$\begin{aligned} &\left(\frac{rx_{n_0} + p}{r\bar{x} + p} \right)^{\frac{\delta n_0}{r}} I\{\bar{x} \geq 0\} + \left(\frac{\alpha x_{n_0} + p}{\alpha\bar{x} + p} \right)^{\frac{\delta n_0}{\alpha}} I\{\bar{x} < 0\} \\ &\leq \exp \left\{ \frac{-\delta\Delta(\epsilon)}{p + r\bar{x}I\{\bar{x} \geq 0\} + \alpha\bar{x}I\{\bar{x} < 0\}} \right\} + \frac{\epsilon}{4V(\bar{x})}, \end{aligned} \quad (\text{A-42})$$

and

$$h_{n_0} < \frac{\epsilon}{8\Delta(\epsilon)}. \quad (\text{A-43})$$

For any $x \geq -\frac{\rho}{\alpha}$, let $L^{(0,x)}$ be a $\frac{\epsilon}{8n_0}$ -optimal strategy given the initial reserve x , that is

$$V_{L^{(0,x)}}(x) \geq V(x) - \frac{\epsilon}{8n_0}. \quad (\text{A-44})$$

Let τ^L denote the first time that the controlled reserve process under strategy L reaches \bar{x} starting from an initial reserve below \bar{x} .

For $x \leq \bar{x}$, define a sequence of strategies $\{L^{(n,x)}\}_{n \geq 1}$ recursively as follows: $L^{(n,x)}$ is a strategy given the initial reserve x that the insurer pays dividends according to strategy $L^{(n-1,x)}$ until the reserve reaches \bar{x} for the first time ($\tau^{L^{(n-1,x)}}$), pays out a lump sum of $\bar{x} - x_{n_0}$ at time $\tau^{L^{(n-1,x)}}$, and thereafter employs the strategy $L^{(n-1,x_{n_0})}$ to the shifted process $\theta_{\tau^{L^{(n-1,x)}}} R$.

It can be shown that for all n , $\tau^{L^{(n,x)}} = \tau^{L^{(0,x)}}$, and $L_s^{(n,x)} = L_s^{(0,x)}$ for $s \leq \tau^{L^{(0,x)}}$. Then we have for $x \in (-\frac{\rho}{\alpha}, \bar{x}]$ and $n = 1, 2, \dots$,

$$\begin{aligned} V_{L^{(n,x)}}(x) &= \mathbb{E}_x \left[\int_0^{\tau^{L^{(0,x)}}} e^{-\delta s} dL_s^{(0,x)}; \tau^{L^{(0,x)}} < T \right] \\ &\quad + \mathbb{E}_x [e^{-\delta \tau^{L^{(0,x)}}}; \tau^{L^{(0,x)}} < T] (V_{L^{(n-1,x_{n_0})}}(x_{n_0}) + \bar{x} - x_{n_0}) \\ &\quad + \mathbb{E}_x \left[\int_0^T e^{-\delta s} dL_s^{(n-1,x)}; \tau^{L^{(0,x)}} > T \right]. \end{aligned} \quad (\text{A-45})$$

Using (A-44) for $x = x_{n_0}$ and \bar{x} , (A-45) for $n = 1$ and the second equality in (A-41), we get

$$\begin{aligned} &|V_{L^{(1,x)}}(x) - V_{L^{(0,x)}}(x)| \\ &= |\mathbb{E}_x [e^{-\delta \tau^{L^{(0,x)}}}; \tau^{L^{(0,x)}} < T] (\bar{x} - x_{n_0} + V_{L^{(0,x_{n_0})}}(x_{n_0}) - V_{L^{(0,\bar{x})}}(\bar{x}))| \\ &\leq |\bar{x} - x_{n_0} - V(\bar{x}) + V(x_{n_0})| + V(x_{n_0}) - V_{L^{(0,x_{n_0})}}(x_{n_0}) + V(\bar{x}) - V_{L^{(0,\bar{x})}}(\bar{x}) \\ &\leq h_{n_0}(\bar{x} - x_{n_0}) + \frac{\epsilon}{4n_0} \leq \frac{3\epsilon}{8n_0}, \end{aligned} \quad (\text{A-46})$$

where the last inequality follows by the first equality in (A-41) and (A-43).

Therefore, from (A-45) we have for $x \in (-\frac{\rho}{\alpha}, \bar{x}]$ and $n \geq 2$,

$$\begin{aligned} |V_{L^{(n,x)}}(x) - V_{L^{(n-1,x)}}(x)| &\leq \mathbb{E}_x [e^{-\delta \tau^{L^{(0,x)}}}] |V_{L^{(n-1,x_{n_0})}}(x_{n_0}) - V_{L^{(n-2,x_{n_0})}}(x_{n_0})| \\ &\leq |V_{L^{(1,x_{n_0})}}(x_{n_0}) - V_{L^{(0,x_{n_0})}}(x_{n_0})| \leq \frac{3\epsilon}{8n_0}. \end{aligned} \quad (\text{A-47})$$

Consequently, by (A-44) and (A-47)

$$\begin{aligned} |V(x) - V_{L^{(n_0,x)}}(x)| &= |V(x) - V_{L^{(0,x)}}(x) + \sum_{n=1}^{n_0} (V_{L^{(n-1,x)}}(x) - V_{L^{(n,x)}}(x))| \\ &< \frac{\epsilon}{8n_0} + \frac{3\epsilon}{8n_0} \leq \frac{\epsilon}{2}. \end{aligned}$$

Define $\bar{\tau} = \inf\{t > 0 : R_t^{L^{(n_0,x)}} > \bar{x}\}$, where $R_t^{L^{(n_0,x)}}$ represent the controlled reserve process under strategy $L^{(n_0,x)}$.

Under strategy $L^{(n_0,x)}$, in order to exceed \bar{x} , the controlled reserve process with initial reserve x_{n_0} should go from x_{n_0} up to \bar{x} for at least n_0 times. Note from the dynamics (2.1)

that it will take at least $t_0(x_{n_0}, \bar{x})$ (defined in (3.3)) for this reserve process to reach \bar{x} starting from x_{n_0} . Therefore, $\bar{\tau} \geq n_0 t_0(x_{n_0}, \bar{x})$. Consequently, it follows by (3.3) (A-40), (A-41) and (A-42) that

$$\mathbb{E}_{x_{n_0}}[e^{-\delta\bar{\tau}}] \leq \mathbb{E}[e^{-\delta n_0 t_0(x_{n_0}, \bar{x})}] \leq \frac{\epsilon}{2V(\bar{x})}. \quad (\text{A-48})$$

Next, we construct a strategy $\bar{L}(x)$ through $L^{(n_0, x)}$: pays dividends according to the strategy $L^{(n_0, x)}$ before time $\bar{\tau}$ (the time that the reserve process reaches \bar{x} for the first time), pays out a lump sum of $\bar{x} + \frac{p}{\alpha}$ at time $\bar{\tau}$, and thereafter pays no dividends.

Then we have

$$\begin{aligned} V_{\bar{L}(x)}(x) &= \mathbb{E}_x \left[\int_0^{\bar{\tau}} e^{-\delta s} d\bar{L}_s(x); \bar{\tau} < T \right] + \mathbb{E}_x [e^{-\delta\bar{\tau}}; \bar{\tau} < T] \left(\bar{x} + \frac{p}{\alpha} \right) \\ &\quad + \mathbb{E}_x \left[\int_0^T e^{-\delta s} d\bar{L}_s(x); \bar{\tau} > T \right]. \end{aligned} \quad (\text{A-49})$$

Notice that

$$\begin{aligned} V_{L^{(n_0, x)}}(x) &= \mathbb{E}_x \left[\int_0^{\bar{\tau}} e^{-\delta s} dL_s^{(n_0, x)}; \bar{\tau} < T \right] + \mathbb{E}_x \left[\int_{\bar{\tau}}^T e^{-\delta s} dL_s^{(n_0, x)}; \bar{\tau} < T \right] \\ &\quad + \mathbb{E}_x \left[\int_0^T e^{-\delta s} dL^{(n_0, x)}; \bar{\tau} > T \right], \end{aligned} \quad (\text{A-50})$$

and

$$\mathbb{E}_x \left[\int_{\bar{\tau}}^T e^{-\delta s} dL_s^{(n_0, x)}; \bar{\tau} < T \right] \leq \mathbb{E}[e^{-\delta\bar{\tau}}]V(\bar{x}). \quad (\text{A-51})$$

Since $\bar{L}_s(x) = L_s^{(n_0, x)}$ for $s \leq \bar{\tau}$, it follows from (A-49), (A-50) and (A-51) that for $x \in (-\frac{p}{\alpha}, \bar{x}]$,

$$V_{\bar{L}(x)}(x) - V_{L^{(n_0, x)}}(x) \geq \mathbb{E}_x [e^{-\delta\bar{\tau}}] \left(\bar{x} + \frac{p}{\alpha} - V(\bar{x}) \right) \geq -\mathbb{E}_x [e^{-\delta\bar{\tau}}]V(\bar{x}) \geq -\frac{\epsilon}{2},$$

where the last inequality is due to (A-48).

So $\bar{L}(x)$ is the desired strategy. \square

Proof of Lemma 4.1 Apply similar arguments as in Azcue and Muler (2005) and Albrecher and Thonhauser (2008).

(a) Since $\Lambda_V(x)$ is continuous in x , \mathcal{A} is closed.

(b) (i) To prove that \mathcal{B} is left-open, it is sufficient to show that for any $x \in \mathcal{B}$ we can find an $h > 0$ such that for any $y \in (x - h, x)$, $V'(y) = 1$ and $\mathcal{G}_V(y) < 0$.

Note that $V'(x) = 1$ for $x \in \mathcal{B}$ and $G'_{(x-h)}(y) = 1$, and that that $p + ryI\{y \geq 0\} + \alpha yI\{y < 0\}$ is increasing in y . Therefore, it follows from (3.8) that for any $y \in (x - h, x)$,

$$\begin{aligned} \mathcal{L}_{G_{x-h}}(y) &\leq \mathcal{L}_V(x) - (\lambda + \delta)(V(x) - G_{x-h}(y)) + \\ &\quad \lambda \left(\int_0^{y + \frac{p}{\alpha}} G_{x-h}(y - u) dF(u) - \int_0^{x + \frac{p}{\alpha}} V(x - u) dF(u) \right). \end{aligned} \quad (\text{A-52})$$

Noticing $\mathcal{L}_V(x) < 0$, $G_{x-h}(y) = V(y)$ for $y \leq x - h$, and $\lim_{h \rightarrow 0} G_{x-h}(y) \rightarrow V(x)$ for $y \in (x - h, x)$, it follows from (A-52) that

$$\mathcal{L}_{G_{x-h}}(y) < 0 \text{ for small } h > 0. \quad (\text{A-53})$$

This along with the fact that $G'_{x-h}(y) \equiv 1$ for $y > x - h$ implies that G_{x-h} is a viscosity super-solution to (3.7) on $(x - h, x]$. Then by Theorem 3.12 (ii), we have $V(y) = G_{x-h}(y)$ for all $y \in [-\frac{p}{\alpha}, x]$. As a result,

$$V'(y) = G'_{x-h}(y) = 1, \text{ for } y \in (x - h, x]. \quad (\text{A-54})$$

Combining (A-53) and (A-54) implies $(x - h, x] \subset \mathcal{B}$. Therefore, \mathcal{B} is left-open.

(ii) To prove that there exist a y such that $(y, \infty) \subset \mathcal{B}$, it is sufficient to show that we can find a large enough $y > 0$ such that $\mathcal{L}_{G_y}(x) < 0$ for all $x > y$, because if $G_y(x)$ of this kind is a super-solution on (y, ∞) and therefore $V'(x) = G'_y(x) \equiv 1$ for $x > y$.

Noticing that $G_y(x)$ is nondecreasing in x , and that $G_y(x) = x - y + V(y)$ and $V(y) - y > \frac{p}{\alpha}$ for $x > y$, we obtain that for $x > y > 0$,

$$\begin{aligned} \mathcal{L}_{G_y}(x) &= p + rx - (\lambda + \delta)(G_y(x)) + \lambda \int_0^{x+\frac{p}{\alpha}} G_y(x-y) dF(y) \\ &< p - (\delta - r)x - \delta y - \delta \frac{p}{\alpha} < 0 \text{ for large } y, \end{aligned}$$

where the last inequality follow by noticing $\delta > r$.

The existence of y also indicates that \mathcal{B} is not empty.

(c) Noticing that $V(-\frac{p}{\alpha}) = 0$, by the definition of \mathcal{G} (3.48), we obtain $\mathcal{G}_V(-\frac{p}{\alpha}) = 0$, which implies $-\frac{p}{\alpha} \in \mathcal{A}$.

Assume that $x_1 > x_0 > -\frac{p}{\alpha}$, $(x_0, x_1] \subset \mathcal{B}$ and $x_0 \notin \mathcal{B}$. We will show in the following that $x_0 \in \mathcal{A}$.

If $V'(x_0) = 1$, then from the fact that $x_0 \notin \mathcal{B}$ and $\mathcal{L}_V \leq 0$, we know $\mathcal{G}_V(x_0) = 0$. Therefore, $x_0 \in \mathcal{A}$.

Now assume, on the other hand, $V'(x_0) \neq 1$. It follows from the fact $(x_0, x_1] \subset \mathcal{B}$ that $V'(x) = 0$ for all $x \in (x_0, x_1]$, which implies

$$\lim_{x \downarrow x_0} \frac{V(x) - V(x_0)}{x - x_0} = 1. \quad (\text{A-55})$$

Define

$$a = \liminf_{x \uparrow x_0} \frac{V(x) - V(x_0)}{x - x_0}.$$

By Lemma 3.2 we know $a \geq 1$. We distinguish two cases: 1. $a > 1$ and 2. $a = 1$.

Case 1: Assume $a > 1$. Then for any b with $1 < b \leq a$, we have

$$\limsup_{x \downarrow x_0} \frac{V(x) - V(x_0)}{x - x_0} = 1 < b < \liminf_{x \uparrow x_0} \frac{V(x) - V(x_0)}{x - x_0}.$$

Since V is a viscosity sub-solution, by Remark 3.1 (i), it follows that there exists a continuously differentiable function $\phi : (-\frac{p}{\alpha}, \infty) \rightarrow \mathbb{R}$ such that $V - \phi$ reaches a maximum at x_0 with $\phi'(x_0) = b$. Therefore, by Definition 3.2 (i) it follows that

$$\begin{aligned} &\max \left\{ 1 - b, (p + rx_0 I\{x_0 \geq 0\} + \alpha x_0 I\{x_0 < 0\})b - (\lambda + \delta)V(x_0) \right. \\ &\left. + \lambda \int_0^{x_0 + \frac{p}{\alpha}} V(x_0 - y) dF(y) \right\} \geq 0, \end{aligned}$$

which implies

$$(p + rx_0 I\{x_0 \geq 0\} + \alpha x_0 I\{x_0 < 0\})b - (\lambda + \delta)V(x_0) + \lambda \int_0^{x_0 + \frac{p}{\alpha}} V(x_0 - y) dF(y) \geq 0.$$

Taking limits $b \rightarrow 1$ gives $\mathcal{G}_V(x_0) \geq 0$. Since $\mathcal{G}_V(x)$ is continuous in x and $\mathcal{G}_V(x) < 0$ for $x \in (x_0, x_1]$, it can be seen that $\mathcal{G}_V(x_0) = 0$, which implies $x_0 \in \mathcal{A}$.

Case 2: Assume $a = 1$. then we can find a sequence $\{h_n\}$ with $h_n \downarrow 0$ such that

$$\liminf_{n \rightarrow \infty} \frac{V(x_0) - V(x_0 - h_n)}{h_n} = 1. \quad (\text{A-56})$$

Define

$$a_n = \frac{V(x_0) - V(x_0 - h_n)}{h_n} - 1, \text{ and } A_n = \{x \in [0, h_n] : V'(x) \text{ exists and } V'(x) \geq 1 + 2a_n\}.$$

By Theorem 3.2 we know that $a_n \geq 0$.

i) If there exists some n such that $a_n = 0$, then we have

$$V(x_0) - V(x) = x_0 - x \text{ for } x \in [x_0 - h_n, x_0]. \quad (\text{A-57})$$

Otherwise, if for some $x \in [x_0 - h_n, x_0]$, $V(x_0) - V(x) > x_0 - x$, then

$$\begin{aligned} V(x_0) - V(x_0 - h_n) &= V(x_0) - V(x) + V(x) - V(x_0 - h_n) \\ &> x_0 - x + x - (x_0 - h_n) = h_n, \end{aligned}$$

which contradicts the assumption $a_n = 0$.

As a result of (A-55) and (A-57), we have $V'(x_0) = 1$. Therefore, $\mathcal{G}_V(x_0) \geq 0$ follows by noticing $x_0 \in \mathcal{B}$. Notice that $\mathcal{G}_V(x_0) \leq 0$ due to the continuity of \mathcal{G}_V . Therefore, $\mathcal{G}_V(x_0) = 0$, implying $x_0 \in \mathcal{A}$.

ii) Suppose $a_n > 0$ for all n . Since $V(x)$ is differentiable almost everywhere, and $V'(x)$, if exists, is greater than 1, we have

$$\begin{aligned} a_n + 1 &= \frac{\int_0^{h_n} V'(x) dx}{h_n} = \frac{\int_{A_n} V'(x) dx + \int_{[0, h_n] \setminus A_n} V'(x) dx}{h_n} \\ &\geq \frac{|A_n|(1 + 2a_n) + (h_n - |A_n|)}{h_n}, \end{aligned}$$

where $|A_n|$ denotes the Lebesgue measure of the set A_n . It follows from (A-58) that $|A_n| \leq \frac{h_n}{2} \rightarrow 0$. Therefore we can find a sequence $x_n \uparrow x_0$ such that $V'(x_n)$ exist and $1 \leq V'(x_n) < 1 + 2a_n$. Consequently, $\lim_{n \rightarrow \infty} V'(x_n) = 1$.

If there exists a subsequence $\{x_{n_j}\}$ with $x_{n_j} \uparrow x_0$ such that $V'(x_{n_j}) > 1$, then by (3.7) we have $\mathcal{G}_V(x_{n_j}) = 0$. This implies $x_{n_j} \in \mathcal{A}$. Since \mathcal{A} is a closed set, we conclude that $x_0 \in \mathcal{A}$.

Suppose that there is an integer $n_0 > 0$, such that for all $n \geq n_0$, $V'(x_n) = 1$. We will show by Proof by Contradiction that $\mathcal{G}_V(x_0) = 0$. Assume $\mathcal{G}_V(x_0) < 0$. Let n be large enough such that

$$V(x_0) - V(x_n) < -\mathcal{G}_V(x_0)/(\lambda + \delta). \quad (\text{A-58})$$

Note that $V(y) \geq V(x_n) + y - x_n = G_{x_n}(x)$ for all $y \geq x_n$. Then for all $x \in [x_n, x_0]$,

$$\begin{aligned} \mathcal{G}_{G_{x_n}}(x) &= p + r x I\{x \geq 0\} + \alpha x I\{x < 0\} - (\lambda + \delta) G_{x_n}(x) + \lambda \int_0^{x + \frac{x}{\alpha}} G_{x_n}(x - y) dF(y) \\ &\leq \mathcal{G}_V(x_0) + (\lambda + \delta)(V(x_0) - (V(x_n) + x_0 - x_n)) \\ &\leq \mathcal{G}_V(x_0) + (\lambda + \delta)(V(x_0) - V(x_n)) < 0, \end{aligned} \quad (\text{A-59})$$

where the last inequality follows from (A-58).

Noting that $G'_{x_n}(x) = 1$ for $x > x_n$ and $G_{x_n}(x) = V_{x_n}(x)$ for $x \in [0, x_n]$, so by (A-59) it follows $\mathcal{L}_{G_{x_n}}(x) = \mathcal{G}_{G_{x_n}}(x) < 0$ for all $x \in [x_n, x_0]$. Therefore G_{x_n} is a viscosity super-solution on $[x_n, x_0]$.

Recalling that $\mathcal{G}_V(x_0) = 0$, then by Theorem 3.12 (ii), we have $V(x) = G_{x_n}(x)$ for $x \in [0, x_0]$. As a result, $V(x)$ is differentiable at x_0 and

$$V'(x_0) = G'_{x_n}(x_0) = 1. \quad (\text{A-60})$$

Combining (A-59) and (A-60) implies $x_0 \in \mathcal{B}$, which contradicts the fact that $x_0 \notin \mathcal{B}$. Therefore $\mathcal{G}_V(x_0) \geq 0$.

Since V is a viscosity super-solution and $V'(x_0) = 1$, from the definition of viscosity super-solution we can see that $\mathcal{G}_V(x_0) = \mathcal{L}_V(x_0) \leq 0$.

Consequently, $\mathcal{G}_V(x_0) = 0$, which implies $x_0 \in \mathcal{A}$.

(d) For any $x \in \mathcal{C}$, we have $\mathcal{G}_V(x) < 0$. Since $\mathcal{G}_V(x)$ is continuous, we can find a ϵ small enough such that

$$\mathcal{G}_V(y) < 0 \text{ for all } y \in [x, x + \epsilon). \quad (\text{A-61})$$

If for all $y \in (x, x + \epsilon)$, $y \notin \mathcal{B}$, then $[x, x + \epsilon) \subset \mathcal{C}$.

If, on the other hand, there exist an $x_1 \in (x, x + \epsilon)$ such that $x_1 \in \mathcal{B}$, then we can find an x_0 and x_1 with $x_0 < x_1$ such that $x_0 \in \mathcal{A}$ and $(x_0, x_1] \subset \mathcal{B}$. As $x < x_1$ and $x \notin \mathcal{B}$, we conclude that $x_0 \in (x, x_1) \subset (x, x + \epsilon)$, which along with $\mathcal{G}_V(x_0) = 0$ is a contradiction to (A-61). This completes the proof. \square

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