

Minimizing shortfall risk for multiple assets derivatives

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Abstract

The risk minimizing problem $\mathbf{E}[l((H - X_T^{x,\pi})^+)] \xrightarrow{\pi} \min$ in the Black-Scholes framework with correlation is studied. General formulas for the minimal risk function and the cost reduction function for the option H depending on multiple underlying are derived. The case of a linear and a strictly convex loss function l are examined. Explicit computation for $l(x) = x$ and $l(x) = x^p$, with $p > 1$ for digital, quantos, outperformance and spread options are presented. The method is based on the quantile hedging approach presented in [4], [5] and developed for the multidimensional options in [1].

Key words: shortfall risk, multiple assets options, correlated assets, quantile hedging.

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1 Introduction

The paper is devoted to the stochastic control problem arising in the risk analysis of financial markets. Let H be a random variable representing future random payoff which is traded on the market. Denote its price determined by the no arbitrage method by $p(H)$. If the initial capital x of the writer exceeds $p(H)$ then he is able to hedge H perfectly, i.e. he can follow some trading strategy π such that the wealth process at the final time is greater than H , i.e.

$$P(X_T^{x,\pi} \geq H) = 1.$$

If $x < p(H)$ then the above equality fails for each π and as a consequence a shortfall risk appears. The aim of the trader is to find a strategy which is optimal in a sense. Let $l : [0, +\infty) \rightarrow [0, +\infty)$ be a loss function which describes the attitude of the trader to the hedging losses. The goal is to minimize the shortfall risk defined as

$$\mathbf{E}[l((H - X_T^{x,\pi})^+)].$$

This problem was studied with various model settings in many papers. The ones mentioned below do not form a complete list. Existence of the optimal strategy for the case when $l(x) = x$ in the context of complete market with the stock prices modeled by the diffusion processes was shown in [3]. These results were generalized to incomplete markets

in [2] where the existence of solution with the use of dual methods was shown. Existence of the optimal strategy in a general semimartingale model was shown in [10]. More explicit results were presented in [5], where the quantile hedging methods, which were introduced in [4], enabled to obtain a more precise description of the solution. This paper is conceptually close to the latter approach and is based on the results obtained in [4], [5] and their adaptation for the multidimensional market presented in [1]. We work with the multidimensional Black-Scholes model with a correlated Wiener process. A great advantage of such a model is its tractability - all parameters can be easily estimated from data, see [6], p. 104. It is also complete and one can find explicit formula for the density of the martingale measure, see for instance [1]. This enables us to apply the quantile hedging methods from [4], [5], [1] which are based on the Neyman-Pearson technique.

In this paper we study two aspects of the risk minimizing problem for derivatives based on multiple underlying. The first is to minimize the risk for a given initial capital $x \geq 0$. If $x \geq p(H)$ then the risk can be eliminated by a replicating strategy and thus the minimal risk function Φ_1^l , which will be specified later, equals zero, i.e. $\Phi_1^l(x) = 0$. In the opposite case $\Phi_1^l(x)$ is strictly positive and the problem is to find a precise value of $\Phi_1^l(x)$ and the corresponding risk minimizing strategy. The second aspect of the problem is to minimize initial costs for the investor who accepts some level of risk $v \geq 0$. If $v = 0$ then the cost minimizing function Φ_2^l equals to the price of H , i.e. $\Phi_2^l(v) = p(H)$ but can be strictly smaller if $v > 0$. The problem is thus to determine $\Phi_2^l(v)$ and find the cost minimizing strategy. Let us stress the fact that both functions Φ_1^l, Φ_2^l reflect the interplay between hedging risk and trading costs and thus serve as important tools for applications. The aim of this paper is to present explicit computing methods for the functions Φ_1^l, Φ_2^l so that to be close to the practitioners' needs.

The idea of the paper is motivated by the formulas presented in Section 6 of [5] which concern the call option in a one dimensional Black-Scholes model. The key observation is that the solution to the risk minimizing problems can be formulated with the use of two real valued deterministic functions. We generalize this concept to the multidimensional setting and show that the risk minimizing problems can be solved in the same way as well provided regularity of the auxiliary functions. For the linear case we introduce real valued functions Ψ_1, Ψ_2 , see formulas (3.4), (3.5), and show that they are continuous if the Wiener process is not degenerate, see Lemma 3.1 and Corollary 3.3. Moreover, for many derivatives Ψ_1, Ψ_2 are strictly monotone, see Lemma 3.1, Example (c) and comments proceeding formulation of Theorem 3.4. Thus Ψ_1^{-1}, Ψ_2^{-1} exist and roughly speaking

$$\Phi_1^l = \Psi_1 \circ \Psi_2^{-1}, \quad \Phi_2^l = \Psi_2 \circ \Psi_1^{-1},$$

up to the discounting factor and the shift parameter, for precise formulation see Theorem 3.4. For the general case of a strictly convex loss function l we introduce the functions Ψ_1^l, Ψ_2^l , see (3.22), (3.23). They are regular and Theorem 3.5, which is actually a reformulation of Theorem 3.2 in [5], yields

$$\Phi_1^l = \Psi_1^l \circ (\Psi_2^l)^{-1}.$$

The characterization of the function Φ_2^l requires proving of the auxiliary result which is formulated as Theorem 3.6. That result can not be proved with the same method as in [5] since the constraints in the associated problem (3.26) are not linear and this excludes possibility of applying the Neyman-Pearson lemma. We present the proof which is based

on the Lagrange multipliers. Finally, in Theorem 3.7 we show that

$$\Phi_2^l = \Psi_2^l \circ (\Psi_1^l)^{-1}.$$

To summarize, if one is able to find the auxiliary functions $\Psi_1, \Psi_2, \Psi_1^l, \Psi_2^l$, then he is able to find the corresponding functions Φ_1^l, Φ_2^l . We present explicit computation of the auxiliary functions for several derivatives which are widely traded like digital option, quantos, outperformance and spread options. For these examples the auxiliary functions are expressed as integrals of the normal densities with appropriate parameters and can effectively be applied in practice.

The paper is organized as follows. In Section 2 we describe the model settings and strictly formulate the problem. Section 3 contains the main results which consist of two parts concerning a linear and a convex loss function respectively. Section 4 is devoted to presenting explicit calculations for two dimensional model when $l(x) = x$ and $l(x) = \frac{x^p}{p}$ with $p > 1$.

2 Problem formulation

We work with a multidimensional stock price model with dynamics of d stocks given by a standard Black-Scholes model

$$dS_t^i = S_t^i(\alpha_i dt + \sigma_i dW_t^i), \quad i = 1, 2, \dots, d, \quad t \in [0, T],$$

where $\alpha_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1, 2, \dots, d$. Above $W_t = (W_t^1, W_t^2, \dots, W_t^d), t \in [0, T]$, is a sequence of correlated standard Wiener processes. The correlation matrix Q of W , which is assumed to be positive definite, is of the form

$$Q = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \dots & \rho_{1,d} \\ \rho_{2,1} & 1 & \rho_{2,3} & \dots & \rho_{2,d} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{d,1} & \rho_{d,2} & \rho_{d,3} & \dots & 1 \end{bmatrix},$$

where

$$\rho_{i,j} = \text{cor} \left\{ W_1^i, W_1^j \right\}, \quad i, j = 1, 2, \dots, d.$$

The process W as above is called a Q -Wiener process. The dynamics the a money market account is given by

$$dB_t = rB_t dt, \quad t \in [0, T],$$

where r stands for a constant interest rate. It is known that such a market is complete and that the unique martingale measure \tilde{P} is given by the density

$$\frac{d\tilde{P}}{dP} = \tilde{Z}_T := e^{-(Q^{-1}[\frac{\alpha-r\mathbf{1}_d}{\sigma}], W_T) - \frac{1}{2}|Q^{-\frac{1}{2}}[\frac{\alpha-r\mathbf{1}_d}{\sigma}]|^2 T}, \quad t \in [0, T], \quad (2.1)$$

with the notation

$$Q^{-1} \left[\frac{\alpha - r\mathbf{1}_d}{\sigma} \right] := Q^{-1} \begin{bmatrix} \frac{\alpha_1 - r}{\sigma_1} \\ \frac{\alpha_2 - r}{\sigma_2} \\ \vdots \\ \frac{\alpha_d - r}{\sigma_d} \end{bmatrix}, \quad t \in [0, T],$$

for more details see, for instance, [1]. Moreover,

$$\widetilde{W}_t := W_t + \frac{\alpha - r\mathbf{1}_d}{\sigma} t, \quad t \in [0, T],$$

is a Q -Wiener process under \tilde{P} . The dynamics of the prices under the measure \tilde{P} can be written as

$$dS_t^i = S_t^i(rdt + \sigma_i d\widetilde{W}_t^i), \quad i = 1, 2, \dots, d, \quad t \in [0, T].$$

The wealth process corresponding to the initial endowment x and the trading strategy π is given by

$$X_0^{x,\pi} = x, \quad X_t^{x,\pi} := \pi_t^0 B_t + \sum_{i=1}^d \pi_t^i S_t^i, \quad t \in [0, T].$$

Each strategy is assumed to be admissible, i.e. $X_t^{x,\pi} \geq 0$ for each $t \in [0, T]$ almost surely and self-financing, i.e.

$$dX_t^{x,\pi} = \pi_t^0 dB_t + \sum_{i=1}^d \pi_t^i dS_t^i, \quad t \in [0, T].$$

A contingent claim is represented by an \mathcal{F}_T -measurable random variable H which is assumed to be nonnegative, i.e. $H \geq 0$. As the market is complete, the price of H defined by

$$p(H) := \inf \{x : \exists \pi \text{ s.t. } P(X_T^{x,\pi} \geq H) = 1\}$$

is given by $p(H) = \tilde{\mathbf{E}}[e^{-rT} H]$, where the expectation is calculated under the measure \tilde{P} .

The aim of the trader is to minimize the shortfall risk defined by

$$\mathbf{E}[l((H - X_T^{x,\pi})^+)],$$

where $l : [0, +\infty) \rightarrow [0, +\infty)$ is a loss function which is assumed to be increasing with $l(0) = 0$. It is clear that if $x \geq p(H)$ then the risk equals zero for the replicating strategy. In the opposite case the risk is strictly positive and the question under consideration is to find a strategy such that

$$\mathbf{E}[l((H - X_T^{x,\pi})^+)] \xrightarrow{\pi} \min.$$

We will refer the corresponding function $\Phi_1 : [0, +\infty) \rightarrow [0, \mathbf{E}[l(H)]]$ given by

$$\Phi_1^l(x) := \min_{\pi} \mathbf{E}[l((H - X_T^{x,\pi})^+)], \quad (2.2)$$

as the *minimal risk function*. The strategy $\hat{\pi}$ such that $\mathbf{E}[l((H - X_T^{x,\hat{\pi}})^+)] = \Phi_1^l(x)$ will be called the *risk minimizing strategy for x* . If $x \geq p(H)$ then $\Phi_1^l(x) = 0$ and $\Phi_1^l(x) > 0$ otherwise.

We also consider the cost reduction problem. Let $v \geq 0$ be a fixed number describing the level of shortfall risk accepted by the trader. We are searching for a minimal initial cost such that there exists a strategy with the risk not exceeding v , i.e.

$$x \rightarrow \min; \quad \exists \pi \text{ s.t. } \mathbf{E}[l((H - X_T^{x,\pi})^+)] \leq v.$$

The *cost reduction function* $\Phi_2^l : [0, +\infty) \rightarrow [0, p(H)]$ is thus defined by

$$\Phi_2^l(v) := \min \{x : \exists \pi \text{ s.t. } \mathbf{E}[l((H - X_T^{x,\pi})^+)] \leq v\}. \quad (2.3)$$

The strategy $\hat{\pi}$ such that $\mathbf{E}[l((X_T^{\Phi_2^l(r), \hat{\pi}} - H)^+)] \leq v$ will be called *the cost minimizing strategy for v*. Notice that $\Phi_2^l(0) = p(H)$.

In the sequel we examine two cases of the loss functions, i.e. $l(x) = x$ and l a general strictly convex function. The aim is to provide explicit computing methods for the functions Φ_1^l, Φ_2^l .

3 Main results

3.1 Linear loss function

In this section we examine the case when $l(x) = x$ and denote the corresponding functions Φ_1^l, Φ_2^l by Φ_1, Φ_2 respectively. It turns out that the functions Φ_1, Φ_2 can be characterized in terms of two auxiliary functions

$$\Psi_1(c) := \mathbf{E}(H\mathbf{1}_{A_c}), \quad (3.4)$$

$$\Psi_2(c) := \tilde{\mathbf{E}}(H\mathbf{1}_{A_c}), \quad (3.5)$$

where

$$A_c := \{\tilde{Z}_T^{-1} \geq c\}, \quad c \geq 0,$$

and \tilde{Z}_T is given by (2.1).

Let us start with an auxiliary result which establishes regularity properties for the functions Ψ_1, Ψ_2 .

Lemma 3.1 *Let $X \geq 0, Y \geq 0$ be random variables such that $\mathbf{E}X < +\infty$. Then the function $g : [0, +\infty) \rightarrow [0, +\infty)$ given by*

$$g(c) := \mathbf{E}[X\mathbf{1}_{\{Y \geq c\}}]$$

- a) *is left continuous on $(0, +\infty)$ with right limits on $[0, +\infty)$,*
- b) *is right continuous on $[0, +\infty)$ if the distribution function of Y is continuous,*
- c) *is strictly decreasing if for any $0 \leq a < b < +\infty$ holds*

$$P(X > 0, Y \in [a, b]) > 0. \quad (3.6)$$

Proof: The function g is decreasing and thus it has right and left limits. Let us consider the auxiliary probability measure \hat{P} defined by

$$\frac{d\hat{P}}{dP} = \frac{X}{\mathbf{E}[X]}dP,$$

which is absolutely continuous wrt. P , i.e. $\hat{P} \ll P$.

- a) For any $c > 0$ we have

$$\bigcap_n \left\{ c - \frac{1}{n} \leq Y < c \right\} = \emptyset,$$

and thus

$$|g(c - \frac{1}{n}) - g(c)| = \mathbf{E} \left(X \mathbf{1}_{\{c - \frac{1}{n} \leq Y < c\}} \right) = \mathbf{E}[X] \hat{P} \left(c - \frac{1}{n} \leq Y < c \right) \xrightarrow{n} 0.$$

b) For any $c \geq 0$ we have

$$\bigcap_n \left\{ c \leq Y < c + \frac{1}{n} \right\} = \{Y = c\},$$

and thus

$$|g(c) - g(c + \frac{1}{n})| = \mathbf{E} \left(X \mathbf{1}_{\{c \leq Y < c + \frac{1}{n}\}} \right) = \mathbf{E}[X] \hat{P} \left(c \leq Y < c + \frac{1}{n} \right) \xrightarrow{n} \mathbf{E}[X] \hat{P}(Y = c) = 0,$$

as $\hat{P} \ll P$ and $P(Y = c) = 0$.

c) Let us notice that (3.6) is equivalent to the condition

$$\exists \varepsilon > 0 \text{ s.t. } P(X > \varepsilon, Y \in [a, b]) > 0,$$

and thus for $0 \leq a < b < +\infty$ we have

$$\begin{aligned} |g(a) - g(b)| &= \mathbf{E} \left(X \mathbf{1}_{\{a \leq Y < b\}} \right) = \mathbf{E} \left(X \mathbf{1}_{\{a \leq Y < b\}} \mathbf{1}_{\{X=0\}} \right) + \mathbf{E} \left(X \mathbf{1}_{\{a \leq Y < b\}} \mathbf{1}_{\{X>0\}} \right) \\ &\geq \mathbf{E} \left(X \mathbf{1}_{\{a \leq Y < b\}} \mathbf{1}_{\{X>\varepsilon\}} \right) \geq \varepsilon P(X > \varepsilon, a \leq Y < b) > 0. \end{aligned}$$

□

Remark 3.2 *Let us notice that the condition (3.6) implies that Y has a strictly increasing distribution function. Indeed, in the opposite case (3.6) is not satisfied for some $0 \leq a < b < +\infty$.*

Corollary 3.3 *If the distribution function of Y is continuous then the function $g(c)$ in Lemma 3.1 is continuous on $(0, +\infty)$ and right continuous at 0.*

Examples The condition (3.6) is satisfied in the following cases.

- a) $X > 0$ and Y has strictly increasing distribution function.
- b) X, Y are independent, Y has strictly increasing distribution function and $P(X > 0) > 0$.
- c) Let (Z_1, Z_2) be a random vector with nondegenerate normal distribution on a plane. Let f, g be functions such that

$$f : \mathbb{R}^2 \longrightarrow (0, +\infty),$$

$$g : \mathbb{R} \longrightarrow (0, +\infty) \text{ is strictly monotone.}$$

Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ be such that the vectors $(\alpha, \beta), (\gamma, \delta)$ are not parallel, i.e. $(\alpha, \beta) \nparallel (\gamma, \delta)$. Then

$$X := f(Z_1, Z_2) \mathbf{1}_{\{\alpha Z_1 + \beta Z_2 > k\}}, \quad Y := g(\gamma Z_1 + \delta Z_2),$$

where k is some constant, satisfy (3.6). Indeed, we have

$$P(X > 0, Y \in [a, b]) = P\left(\alpha Z_1 + \beta Z_2 > k, g^{-1}(a) \leq \gamma Z_1 + \delta Z_2 < g^{-1}(b)\right)$$

for the case when g is strictly increasing. The probability above is positive because the set

$$\left\{ (x, y) : (\alpha x + \beta y) > k, g^{-1}(a) \leq \gamma x + \delta y < g^{-1}(b) \right\}$$

is of positive Lebesgue measure and (Z_1, Z_2) has nondegenerate distribution. \square

Let us notice that due to the fact that Q is nonsingular the random variable

$$\tilde{Z}_T^{-1} := e^{(Q^{-1}[\frac{\alpha-r\mathbf{1}_d}{\sigma}], W_T) + \frac{1}{2}|Q^{-\frac{1}{2}}[\frac{\alpha-r\mathbf{1}_d}{\sigma}]|^2 T} \quad (3.7)$$

has a continuous distribution function wrt. P and \tilde{P} . Thus it follows from Corollary 3.3 that the functions Ψ_1, Ψ_2 are continuous. As they are decreasing with images $[0, \mathbf{E}[H]]$, $[0, e^{rT}p(H)]$, the equations

$$\begin{aligned} \Psi_1(c) &= x, & x &\in [0, \mathbf{E}[H]], \\ \Psi_2(c) &= x, & x &\in [0, e^{rT}p(H)], \end{aligned}$$

have solutions. Moreover, the solution of the first (resp. second) equation is unique if

$$P\left(H > 0, \tilde{Z}_T^{-1} \in [a, b)\right) > 0, \quad (3.8)$$

resp.

$$\tilde{P}\left(H > 0, \tilde{Z}_T^{-1} \in [a, b)\right) > 0. \quad (3.9)$$

It follows from Example (c) above that (3.8) and (3.9) are satisfied, for instance, when $d = 2$ and

- a) H is a digital option, i.e. $H = K\mathbf{1}_{\{S_T^1 \geq S_T^2\}}$ and $(\sigma_1, -\sigma_2) \not\parallel Q^{-1}[\frac{\alpha-r\mathbf{1}_d}{\sigma}]$,
- b) H is a quanto domestic option, i.e. $H = S_T^2(S_T^1 - K)^+$ and $(\sigma_1, 0) \not\parallel Q^{-1}[\frac{\alpha-r\mathbf{1}_d}{\sigma}]$,
- c) H is a quanto foreign option, i.e. $H = (S_T^1 - \frac{K}{S_T^2})^+$ and $(\sigma_1, \sigma_2) \not\parallel Q^{-1}[\frac{\alpha-r\mathbf{1}_d}{\sigma}]$.

Below we present the description of the functions Φ_1, Φ_2 .

Theorem 3.4 a) Let $c = c(x)$ be a solution of the equation

$$\Psi_2(c) = e^{rT}x, \quad x \in [0, p(H)]. \quad (3.10)$$

Then

$$\Phi_1(x) = \begin{cases} \Psi_1(0) - \Psi_1(c) & \text{for } x \in [0, p(H)), \\ 0 & \text{for } x \geq p(H). \end{cases}$$

Moreover, the replicating strategy for the payoff $H\mathbf{1}_{A_{c(x)}}$ is a risk minimizing strategy for x .

b) Let $c = c(v)$ be a solution of the equation

$$\Psi_1(c) = \Psi_1(0) - v, \quad v \in [0, \mathbf{E}[H]]. \quad (3.11)$$

Then

$$\Phi_2(v) = \begin{cases} e^{-rT}\Psi_2(c) & \text{for } v \in [0, \mathbf{E}[H]), \\ 0 & \text{for } v \geq \mathbf{E}[H]. \end{cases}$$

Moreover, the replicating strategy for the payoff $H\mathbf{1}_{A_{c(v)}}$ is a cost minimizing strategy for v .

Proof: For any admissible strategy (x, π) let us define the success function

$$\varphi_{x,\pi} := \mathbf{1}_{\{X_T^{x,\pi} \geq H\}} + \frac{X_T^{x,\pi}}{H} \mathbf{1}_{\{X_T^{x,\pi} < H\}}.$$

One can check the following identity

$$(H - X_T^{x,\pi})^+ = H - X_T^{x,\pi} \wedge H = H - H\varphi_{x,\pi},$$

which implies that

$$\mathbf{E}[(H - X_T^{x,\pi})^+] = \mathbf{E}[H] - \mathbf{E}[H\varphi_{x,\pi}]. \quad (3.12)$$

a) In view of (3.12) the problem (2.2) of finding $\Phi_1(x)$ is equivalent to that of finding the strategy π satisfying

$$\mathbf{E}[H\varphi_{x,\pi}] \rightarrow \max.$$

If $x \geq p(H)$ then $\varphi_{x,\pi} = 1$ for the replicating strategy and $\Phi_1(x) = 0$, so consider the case $0 \leq x < p(H)$. Let us formulate an auxiliary problem of determining $\varphi \in \mathcal{R}$ solving

$$\begin{cases} \mathbf{E}[H\varphi] \rightarrow \max, \\ \tilde{\mathbf{E}}[e^{-rT} H\varphi] \leq x, \end{cases} \quad (3.13)$$

where

$$\mathcal{R} := \{\varphi : 0 \leq \varphi \leq 1 \text{ and } \varphi \text{ is } \mathcal{F}_T \text{-measurable}\}. \quad (3.14)$$

It is clear that if $\hat{\varphi}$ such that $\tilde{\mathbf{E}}[e^{-rT} H\hat{\varphi}] = x$ is a solution of (3.13) then the replicating strategy $\tilde{\pi}$ for the payoff $H\hat{\varphi}$ is a risk minimizing strategy for x and

$$\Phi_1(x) = \mathbf{E}[(H - X_T^{\tilde{x}, \tilde{\pi}})^+] = \mathbf{E}[H] - \mathbf{E}[H\hat{\varphi}]. \quad (3.15)$$

Thus now let us focus on determining solution $\hat{\varphi}$ of (3.13). To this end introduce two probability measures P_1, P_2 with densities

$$\frac{dP_1}{dP} = \frac{H}{\mathbf{E}[H]}, \quad \frac{dP_2}{dP} = \frac{e^{-rT} \tilde{Z}_T H}{\mathbf{E}[e^{-rT} \tilde{Z}_T H]}.$$

Then (3.13) reads as

$$\begin{cases} \mathbf{E}^{P_1}[\varphi] \rightarrow \max, \\ \mathbf{E}^{P_2}[\varphi] \leq \frac{x}{p(H)}, \end{cases} \quad (3.16)$$

which is a standard problem in the theory of statistical tests. One should try to search for the solution in the class of 0 – 1 valued functions of the form $\mathbf{1}_{A_c}; c \geq 0$, where

$$A_c := \left\{ \frac{dP_1}{dP_2} \geq c \right\} = \left\{ \frac{dP_1}{dP} \frac{dP}{dP_2} \geq c \right\} = \left\{ \frac{H}{\mathbf{E}[H]} \frac{\mathbf{E}[\tilde{Z}_T H]}{\tilde{Z}_T H} \geq c \right\} = \left\{ \tilde{Z}_T^{-1} \geq c \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]} \right\}.$$

For the sake of simplicity we can reparametrize A_c by denoting the constant $c \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]}$ above just by c . Then A_c is of the form

$$A_c := \left\{ \tilde{Z}_T^{-1} \geq c \right\}.$$

It is known by the Neyman-Pearson lemma that if there exists $c = c(x)$ such that

$$\mathbf{E}^{P_2}[\mathbf{1}_{A_c}] = P_2(A_c) = \frac{x}{p(H)}, \quad (3.17)$$

then the solution of (3.16), or equivalently (3.13), is given by $\hat{\varphi} = \mathbf{1}_{A_{c(x)}}$. But let us notice that (3.17) is equivalent to the following

$$\Psi_2(c) = e^{rT} x,$$

and the existence of the required constant c follows from (3.10). Finally, coming back to (3.15) and using definition of Ψ_1 , we obtain

$$\Phi_1(x) = \mathbf{E}[H] - \mathbf{E}[H\hat{\varphi}] = \mathbf{E}[H] - \mathbf{E}[H\mathbf{1}_{A_c}] = \Psi_1(0) - \Psi_1(c).$$

b) If $v \geq \mathbf{E}[H]$ then the cost minimizing strategy is trivial, i.e. $(x = 0, \pi = 0)$ and thus $\Phi_2(v) = 0$. Let us focus on the case when $v \in [0, \mathbf{E}[H]]$. In view of (3.12) the risk minimizing strategy is the one which solves the problem

$$\begin{cases} \mathbf{E}[H\varphi_{x,\pi}] \geq \mathbf{E}[H] - v \\ \tilde{\mathbf{E}}[e^{-rT} H\varphi_{x,\pi}] \longrightarrow \min. \end{cases}$$

We are thus looking for a solution $\hat{\varphi} \in \mathcal{R}$ of the problem

$$\begin{cases} \mathbf{E}[H\hat{\varphi}] \geq \mathbf{E}[H] - v \\ \tilde{\mathbf{E}}[e^{-rT} H\hat{\varphi}] \longrightarrow \min. \end{cases} \quad (3.18)$$

If (3.18) has a solution satisfying $\mathbf{E}[H\hat{\varphi}] = \mathbf{E}[H] - v$ then the cost minimizing strategy is the one which replicates $H\hat{\varphi}$ and the cost minimizing function equals

$$\Phi_2(r) = e^{-rT} \tilde{\mathbf{E}}[H\hat{\varphi}]. \quad (3.19)$$

Let us focus on determining the solution $\hat{\varphi}$ of (3.18). Using notation from the part (a) we can reformulate (3.18) to the form

$$\begin{cases} \mathbf{E}^{P_1}[\varphi] \geq \frac{\mathbf{E}[H]-v}{\mathbf{E}[H]} \\ \mathbf{E}^{P_2}[\varphi] \longrightarrow \min. \end{cases} \quad (3.20)$$

It can be shown in the same way as in the proof of Neyman-Pearson lemma that the solution should be searched in the 0 – 1 valued functions of the form $\mathbf{1}_{B_c}; c \geq 0$, where

$$B_c := \left\{ \frac{dP_2}{dP_1} \leq c \right\} = \left\{ \frac{dP_2}{dP} \frac{dP}{dP_1} \leq c \right\} = \left\{ \tilde{Z}_T^{-1} \geq \frac{1}{c} \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]} \right\}.$$

Denoting, for simplicity, the constant $\frac{1}{c} \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]}$ above by c , we have

$$B_c = \{ \tilde{Z}_T^{-1} \geq c \}.$$

If there exists constant $c = c(v)$ satisfying

$$\mathbf{E}^{P_1}[\mathbf{1}_{B_c}] = P_1(B_c) = \frac{\mathbf{E}[H] - v}{\mathbf{E}[H]} \quad (3.21)$$

then $\hat{\varphi} = \mathbf{1}_{B_c}$ is a solution of (3.20) or, equivalently, (3.18). Let us notice that (3.21) can be written as

$$\Psi_1(c) = \Psi_1(0) - v$$

and existence of the required constant $c(v)$ follows from (3.11). Coming back to (3.19) we obtain

$$\Phi_2(v) = e^{-rT} \tilde{\mathbf{E}}[H \mathbf{1}_{B_c}] = e^{-rT} \Psi_2(c).$$

□

3.2 Convex loss function

In this section we study the case when $l : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing, strictly convex function such that $l(0) = 0$. We assume that $l \in C^2(0, +\infty)$ and that l' is strictly increasing with $l'(0+) = 0$, $l'(+\infty) = +\infty$. The inverse of the first derivative will be denoted by I , i.e.

$$I = (l')^{-1}.$$

The functions Φ_1^l, Φ_2^l can be characterized in terms of the functions

$$\Psi_1^l(c) := \mathbf{E}[l((1 - \varphi_c)H)] \quad (3.22)$$

$$\Psi_2^l(c) := \tilde{\mathbf{E}}[H\varphi_c]. \quad (3.23)$$

where φ_c is defined by

$$\varphi_c := \left\{ 1 - \left(\frac{I(c\tilde{Z}_T)}{H} \wedge 1 \right) \right\} \mathbf{1}_{\{H > 0\}}, \quad c \geq 0. \quad (3.24)$$

It was shown in [5], Theorem 5.1, that the problem of determining Φ_1^l is equivalent to finding the solution $\tilde{\varphi}$ of the problem

$$\begin{cases} \mathbf{E}[l((1 - \varphi)H)] \xrightarrow{\varphi \in \mathcal{R}} \min \\ \tilde{\mathbf{E}}[e^{-rT} H \varphi] \leq x, \end{cases} \quad (3.25)$$

where \mathcal{R} is defined in (3.14). Then $\Phi_1^l(x) = \mathbf{E}[l((1 - \tilde{\varphi})H)]$ and the risk minimizing strategy is the one which replicates $H\tilde{\varphi}$. Moreover, since the function Ψ_2^l is continuous with the image $[0, e^{rT}p(H)]$, see the proof of Theorem 5.1 in [5], it follows that for any $x \in [0, e^{rT}p(H)]$ there exists constant c such that $\Psi_2^l(c) = \tilde{\mathbf{E}}[H\varphi_c] = e^{rT}x$. Such φ_c solves the auxiliary problem (3.25) and thus

$$\Phi_1^l(x) = \mathbf{E}[l((1 - \varphi_c)H)],$$

and the minimal risk strategy is that replicating the payoff $H\varphi_c$, see Theorem 3.2 in [5]. Thus the results from [5] can be expressed in our notation as follows.

Theorem 3.5 *Let $c = c(x)$ be a solution of the equation*

$$\Psi_2^l(c) = e^{rT}x, \quad x \in [0, p(H)].$$

Then

$$\Phi_1^l(x) = \begin{cases} \Psi_1^l(c) & \text{for } x \in [0, p(H)), \\ 0 & \text{for } x \geq p(H). \end{cases}$$

Although Theorem 3.5 is only a reformulation of Theorem 3.2 in [5], it provides an effective method for practical applications if one is able to derive the functions Ψ_1^l, Ψ_2^l for concrete derivatives.

In the sequel we will show that the function Φ_2^l can be characterized in terms of the functions Ψ_1^l, Ψ_2^l as well. It is easy to show that the cost reduction problem is equivalent to that of finding $\varphi \in \mathcal{R}$ such that

$$\begin{cases} \mathbf{E}[l((1 - \varphi)H)] \leq v \\ \tilde{\mathbf{E}}[e^{-rT}H\varphi] \longrightarrow \min. \end{cases} \quad (3.26)$$

Let us notice that (3.26) can not be solved with the same method as (3.25). In (3.25) the constraints are linear and thus the solution could be found via Neyman-Pearson approach to the variational problem, see the proof of Theorem 5.1 in [5] and p.210 in [9]. The constraints in (3.26) are no longer linear and the method above fails. Below we present the proof based on the Lagrange multipliers.

Theorem 3.6 *The solution of the problem (3.26) is of the form*

$$\tilde{\varphi} := \left\{ 1 - \left(\frac{I(c\tilde{Z}_T)}{H} \wedge 1 \right) \right\} \mathbf{1}_{\{H>0\}}$$

where c is such that $\mathbf{E}[l((1 - \tilde{\varphi})H)] = v$.

Proof: First let us notice that if $\varphi \in \mathcal{R}$ is a solution to (3.26) then necessarily $\mathbf{E}[l((1 - \varphi)H)] = v$. Indeed, assume to the contrary that φ is a solution to (3.26) with $\mathbf{E}[l((1 - \varphi)H)] < v$ and consider a family of random variables $\varphi_\alpha := \varphi \wedge \alpha; \alpha \in [0, 1]$. Then the function $\alpha \rightarrow \mathbf{E}[l((1 - \varphi_\alpha)H)]$ is continuously decreasing from $\mathbf{E}[l(H)]$ to 0. Thus there exists $\tilde{\alpha} \in [0, 1]$ such that $\mathbf{E}[l((1 - \varphi_{\tilde{\alpha}})H)] = v$. Then $\varphi_{\tilde{\alpha}} \leq \varphi$ and thus $\tilde{\mathbf{E}}[H\varphi_{\tilde{\alpha}}] < \tilde{\mathbf{E}}[H\varphi]$, which is a contradiction.

Let $\varphi \neq \tilde{\varphi}$ be any element of \mathcal{R} such that $\mathbf{E}[l((1 - \varphi)H)] = v$. We need to show that $\tilde{\mathbf{E}}[H\tilde{\varphi}] \leq \tilde{\mathbf{E}}[H\varphi]$. Let us define φ_ε by

$$\varphi_\varepsilon := (1 - \varepsilon)\tilde{\varphi} + \varepsilon\varphi, \quad \varepsilon \in [0, 1],$$

and the function

$$F_\varphi(\varepsilon) := \tilde{\mathbf{E}}(H\varphi_\varepsilon) = \mathbf{E}(\tilde{Z}_T H\varphi_\varepsilon).$$

We need to show that $F_\varphi(0) \leq F_\varphi(1)$. We will show that F_φ has minimum at 0. Let us define the auxiliary function

$$G_\varphi(\varepsilon) := \mathbf{E}[l((1 - \varphi_\varepsilon)H)],$$

and notice that due to the convexity of l we have $G_\varphi(\varepsilon) \leq v$ for each $\varepsilon \in [0, 1]$. Thus the problem of minimizing F_φ on $[0, 1]$ is equivalent to the following

$$\begin{cases} F_\varphi(\varepsilon) \longrightarrow \min \\ G_\varphi(\varepsilon) \leq v, \\ \varepsilon \geq 0, \\ 1 - \varepsilon \geq 0. \end{cases} \quad (3.27)$$

Both functions F_φ, G_φ are smooth with

$$\begin{aligned} F'_\varphi(\varepsilon) &\equiv \mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H], \\ G'_\varphi(\varepsilon) &= \mathbf{E}[l'((1 - \varphi_\varepsilon)H) \cdot (\tilde{\varphi} - \varphi)H], \\ G''_\varphi(\varepsilon) &= \mathbf{E}[l''((1 - \varphi_\varepsilon)H) \cdot (\tilde{\varphi} - \varphi)^2 H^2], \end{aligned}$$

and thus the Lagrange function for (3.27) is of the form

$$L(\varepsilon, \lambda_1, \lambda_2, \lambda_3) = F_\varphi(\varepsilon) - \lambda_1(v - G_\varphi(\varepsilon)) - \lambda_2\varepsilon - \lambda_3(1 - \varepsilon).$$

As the function F_φ is linear, it attains its minimal value at 0 or 1. We will show that the first and the second order differential conditions are satisfied for $\varepsilon = 0$.

The first order conditions are

$$L'_\varepsilon(\varepsilon, \lambda_1, \lambda_2, \lambda_3) = \mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H] + \lambda_1\mathbf{E}[l'((1 - \varphi_\varepsilon)H) \cdot (\tilde{\varphi} - \varphi)H] - \lambda_2 + \lambda_3 = 0 \quad (3.28)$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0, \quad \lambda_1(v - G_\varphi(\varepsilon)) = 0, \quad \lambda_2\varepsilon = 0, \quad \lambda_3(1 - \varepsilon) = 0. \quad (3.29)$$

By the definition of $\tilde{\varphi}$ we have

$$\begin{aligned} \tilde{\varphi} &= 1 - \frac{I(c\tilde{Z}_T)}{H} \quad \text{and} \quad c\tilde{Z}_T = l'((1 - \tilde{\varphi})H) \quad \text{on } A \\ & \\ & \tilde{\varphi} = 0 \quad \text{on } A^c, \end{aligned}$$

where $A := \{c\tilde{Z}_T < l'(H)\}$ and A^c stands for the compliment of A . For $\varepsilon = 0$ it follows from (3.29) that $\lambda_3 = 0$ and the equation (3.28) is of the form

$$\begin{aligned} &\mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_A] + \mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_{A^c}] + c\lambda_1\mathbf{E}[\tilde{Z}_T(\tilde{\varphi} - \varphi)H\mathbf{1}_A] \\ &+ \lambda_1\mathbf{E}[l'((1 - \tilde{\varphi})H)(\tilde{\varphi} - \varphi)H\mathbf{1}_{A^c}] \\ &= (1 - c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_A] + \mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^c}] - \lambda_1\mathbf{E}[l'(H)\varphi H\mathbf{1}_{A^c}] = \lambda_2. \end{aligned} \quad (3.30)$$

The left side of (3.30) satisfies the following estimation

$$\begin{aligned} &(1 - c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_A] + \mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^c}] - \lambda_1\mathbf{E}[l'(H)\varphi H\mathbf{1}_{A^c}] \\ &\geq (1 - c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_A] + \mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^c}] - \lambda_1c\mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^c}] \\ &\geq (1 - c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_A + \tilde{Z}_T\varphi H\mathbf{1}_{A^c}]. \end{aligned}$$

If $\mathbf{E}[\tilde{Z}_T(\varphi - \tilde{\varphi})H\mathbf{1}_A + \tilde{Z}_T\varphi H\mathbf{1}_{A^c}] > 0$ then we take λ_1 such that $(1 - c\lambda_1) > 0$, in the opposite case, such that $(1 - c\lambda_1) < 0$. In both cases λ_2 given by (3.30) is nonnegative.

The second order condition for $\varepsilon = 0$ is

$$L''_\varepsilon(\varepsilon, \lambda_1, \lambda_2, \lambda_3) = \lambda_1\mathbf{E}[l''((1 - \tilde{\varphi})H) \cdot (\tilde{\varphi} - \varphi)^2 H^2] \geq 0,$$

and thus the solution of (3.27) is $\varepsilon = 0$. □

The Theorem 3.6 and the definitions of Ψ_1^l, Ψ_2^l lead us to the following result.

Theorem 3.7 *Let $c = c(v)$ be a solution of the equation*

$$\Psi_1^l(c) = v, \quad v \in [0, \mathbf{E}[l(H)]].$$

Then

$$\Phi_2^l(v) = \begin{cases} e^{-rT}\Psi_2^l(c) & \text{for } v \in [0, \mathbf{E}[l(H)]], \\ 0 & \text{for } v \geq \mathbf{E}[l(H)]. \end{cases}$$

4 Two dimensional model

In this section we determine explicit formulas for the functions Ψ_1^l, Ψ_2^l when $l(x) = x$ and $l(x) = \frac{x^p}{p}, p > 1$. In the latter case we use the notation $\Psi_1^p = \Psi_1^l, \Psi_2^p = \Psi_2^l$. We examine several examples of popular options.

For $l(x) = \frac{x^p}{p}$ we have $I(x) = x^{\frac{1}{p-1}}$ and in view of (3.24) the following holds

$$\Psi_1^p(c) = \frac{1}{p} \mathbf{E} \left[H^p \mathbf{1}_{A_c^c} \right] + \frac{1}{p} \mathbf{E} \left[(c\tilde{Z}_T)^{\frac{p}{p-1}} \mathbf{1}_{A_c} \right], \quad (4.31)$$

$$\Psi_2^p(c) = \tilde{\mathbf{E}} \left[\left(H - (c\tilde{Z}_T)^{\frac{1}{p-1}} \right) \mathbf{1}_{A_c} \right]. \quad (4.32)$$

where

$$A_c := \{c\tilde{Z}_T \leq H^{p-1}\}, \quad (4.33)$$

and A_c^c stands for the compliment of A_c .

Since our formulas are expressed in terms of integrals of normal densities, at the beginning we recall basic properties of the multidimensional normal distribution. They can be found in standard textbooks on probability theory or statistics, see for instance [7]. A random vector X taking values in \mathbb{R}^d has a multidimensional normal distribution if its density is of the form

$$f_X(x) = \frac{1}{(2\pi)^{\frac{d}{2}} (\det \Sigma)^{\frac{1}{2}}} \cdot e^{-\frac{1}{2}(x-m)^T \Sigma^{-1} (x-m)}, \quad x \in \mathbb{R}^d, \quad (4.34)$$

where $m \in \mathbb{R}^d$ is a mean of X and Σ is a symmetric positive definite $d \times d$ covariance matrix of X . The fact that X has a density (4.34) will be denoted by $X \sim N_d(m, \Sigma)$ or $\mathcal{L}(X) = N_d(m, \Sigma)$. If $d = 1$ then the subscript is omitted and $N(m, \sigma)$ denotes the normal distribution with mean m and variance σ . If $X \sim N_d(m, \Sigma)$ and A is a $k \times d$ matrix then,

$$AX \sim N_k(Am, A\Sigma A^T); \quad (4.35)$$

in particular if $a \in \mathbb{R}^d$ then

$$a^T X \sim N(a^T m, a^T \Sigma a). \quad (4.36)$$

Let X be a random vector taking values in \mathbb{R}^d and fix an integer $0 < k < d$. Let us divide X into two vectors $X^{(1)}$ and $X^{(2)}$ with lengths $k, d - k$ respectively, i.e.

$$X^{(1)} = (X_1, X_2, \dots, X_k)^T, \quad X^{(2)} = (X_{k+1}, X_{k+2}, \dots, X_d)^T.$$

Analogously, divide the mean vector m and the covariance matrix Σ

$$m = \begin{pmatrix} m^{(1)} \\ m^{(2)} \end{pmatrix}; \quad \Sigma = \begin{bmatrix} \Sigma^{(11)} & \Sigma^{(12)} \\ \Sigma^{(21)} & \Sigma^{(22)} \end{bmatrix},$$

so that $\mathbf{E}X^{(1)} = m^{(1)}, \mathbf{E}X^{(2)} = m^{(2)}, CovX^{(1)} = \Sigma^{(11)}, CovX^{(2)} = \Sigma^{(22)}, Cov(X^{(1)}, X^{(2)}) = \Sigma^{(12)} = \Sigma^{(21)^T}$. Denote by $\mathcal{L}(X^{(1)} | X^{(2)} = x^{(2)})$ the conditional distribution of $X^{(1)}$ given $X^{(2)} = x^{(2)} \in \mathbb{R}^{d-k}$. If $\Sigma^{(22)}$ is nonsingular then

$$\mathcal{L}(X^{(1)} | X^{(2)} = x^{(2)}) = N_k(m^{(1)}(x^{(2)}), \Sigma^{(11)}(x^{(2)})), \quad (4.37)$$

where

$$\begin{aligned} m^{(1)}(x^{(2)}) &= m^{(1)} + \Sigma^{(12)}\Sigma^{(22)^{-1}}(x^{(2)} - m^{(2)}), \\ \Sigma^{(11)}(x^{(2)}) &= \Sigma^{(11)} - \Sigma^{(12)}\Sigma^{(22)^{-1}}\Sigma^{(21)}. \end{aligned} \quad (4.38)$$

Actually the conditional variance $\Sigma^{(11)}(x^{(2)})$ does not depend on $x^{(2)}$ but we keep the notation for the sake of consistency. The conditional density will be denoted by $f_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$, where $x^{(1)} \in \mathbb{R}^k$. In particular if (X, Y) is a two dimensional normal vector with parameters

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}; \quad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix},$$

then

$$\mathcal{L}(X | Y = y) = N(m_1(y), \sigma_1(y)),$$

where

$$m_1(y) := m_1 + \frac{\sigma_{12}}{\sigma_{22}}(y - m_2), \quad \sigma_1(y) := \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}. \quad (4.39)$$

If X is a random vector then its distribution wrt. the measure \tilde{P} will be denoted by $\tilde{\mathcal{L}}(X)$ and its density by \tilde{f}_X . Analogously, $\tilde{f}_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$ stands for the conditional density with respect to the measure \tilde{P} .

Below we simplify the multidimensional notation to the case $d = 2$. The correlation matrix is of the form

$$Q = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

and thus we have

$$Q^{-1} = \frac{1}{\rho^2 - 1} \begin{bmatrix} -1 & \rho \\ \rho & -1 \end{bmatrix}, \quad Q^{-\frac{1}{2}} = \frac{1}{2} \begin{bmatrix} \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} \end{bmatrix}.$$

Hence the density of the martingale measure (2.1) can be written as

$$\tilde{Z}_T = e^{-A_1 W_T^1 - A_2 W_T^2 - \tilde{B}T} = e^{-A_1 \tilde{W}_T^1 - A_2 \tilde{W}_T^2 - \tilde{B}T}, \quad (4.40)$$

where

$$\begin{aligned} A_1 &:= \frac{1}{\rho^2 - 1} \left(-\frac{\alpha_1 - r}{\sigma_1} + \rho \frac{\alpha_2 - r}{\sigma_2} \right) \\ A_2 &:= \frac{1}{\rho^2 - 1} \left(\rho \frac{\alpha_1 - r}{\sigma_1} - \frac{\alpha_2 - r}{\sigma_2} \right) \\ B &:= \frac{1}{8} \left(\left(\left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} \right) \frac{\alpha_1 - r}{\sigma_1} + \left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} \right) \frac{\alpha_2 - r}{\sigma_2} \right)^2 \right. \\ &\quad \left. + \left(\left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} \right) \frac{\alpha_1 - r}{\sigma_1} + \left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} \right) \frac{\alpha_2 - r}{\sigma_2} \right)^2 \right) \\ \tilde{B} &:= B - A_1 \frac{\alpha_1 - r}{\sigma_1} - A_2 \frac{\alpha_2 - r}{\sigma_2}. \end{aligned}$$

In the following subsections we will use the universal constants: A_1, A_2, B, \tilde{B} defined in (4.40) as well as $a_1, a_2, b, \tilde{a}_1, \tilde{a}_2, \tilde{b}$ introduced below.

Fix numbers $K > 0, c \geq 0$. One can check the following

$$\{S_T^1 \geq K\} = \{W_T^1 \geq a_1\} = \{\widetilde{W}_T^1 \geq \tilde{a}_1\}, \quad (4.41)$$

$$\{S_T^2 \geq K\} = \{W_T^2 \geq a_2\} = \{\widetilde{W}_T^2 \geq \tilde{a}_2\}, \quad (4.42)$$

$$\{S_T^1 \geq S_T^2\} = \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\} = \{\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \geq \tilde{b}\}, \quad (4.43)$$

$$\{\tilde{Z}_T^{-1} \geq c\} = \{A_1 W_T^1 + A_2 W_T^2 \geq \ln c - BT\} = \{A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2 \geq \ln c - \tilde{B}T\}, \quad (4.44)$$

where

$$a_1 := \frac{1}{\sigma_1} \left(\ln \frac{K}{S_0^1} - (\alpha_1 - \frac{1}{2}\sigma_1^2)T \right), \quad \tilde{a}_1 := \frac{1}{\sigma_1} \left(\ln \frac{K}{S_0^1} - (r - \frac{1}{2}\sigma_1^2)T \right),$$

$$a_2 := \frac{1}{\sigma_2} \left(\ln \frac{K}{S_0^2} - (\alpha_2 - \frac{1}{2}\sigma_2^2)T \right), \quad \tilde{a}_2 := \frac{1}{\sigma_2} \left(\ln \frac{K}{S_0^2} - (r - \frac{1}{2}\sigma_2^2)T \right),$$

$$b := \ln \left(\frac{S_0^2}{S_0^1} \right) + (\alpha_2 - \alpha_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2))T, \quad \tilde{b} := \ln \left(\frac{S_0^2}{S_0^1} \right) - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)T.$$

In all the formulas appearing in the sequel it is understood that $\ln 0 = -\infty$ and Φ stands for the distribution function of $N(0, 1)$.

4.1 Digital option

Digital option is a contract with the payoff function of the form

$$H = K \cdot \mathbf{1}_{\{S_T^1 \geq S_T^2\}}, \quad \text{where } K > 0. \quad (4.45)$$

Let $(X, Y), (\tilde{X}, \tilde{Y})$ be random vectors defined by $X := \sigma_1 W_T^1 - \sigma_2 W_T^2, Y := A_1 W_T^1 + A_2 W_T^2, \tilde{X} := \sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2, \tilde{Y} := A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2$. They are normally distributed under P , resp. \tilde{P} and their parameters are given by (4.35).

Linear loss function

Using (4.40) and (4.43) we obtain

$$\Psi_1(c) = K \mathbf{E}(\mathbf{1}_{\{S_T^1 \geq S_T^2\}} \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}}) = KP(\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b, A_1 W_T^1 + A_2 W_T^2 \geq \ln c - BT),$$

and thus

$$\Psi_1(c) = K \int_b^{+\infty} \int_{\ln c - BT}^{+\infty} f_{X,Y}(x, y) dy dx.$$

Analogous computation yields

$$\Psi_2(c) = K \tilde{P}(\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \geq \tilde{b}, A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2 \geq \ln c - \tilde{B}T) = K \int_{\tilde{b}}^{+\infty} \int_{\ln c - \tilde{B}T}^{+\infty} f_{X,Y}(x, y) dy dx.$$

Power loss function

In view of (4.43) and (4.40) we have

$$\begin{aligned} A_c &:= \{c\tilde{Z}_T \leq H^{p-1}\} = \{c\tilde{Z}_T \leq K^{p-1}\mathbf{1}_{\{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\}}\} = \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b, c\tilde{Z}_T \leq K^{p-1}\} \\ &= \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b, A_1 W_T^1 + A_2 W_T^2 \geq \ln\left(\frac{K^{p-1}}{c}\right) - BT\} \end{aligned} \quad (4.46)$$

$$= \{\sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 \geq \tilde{b}, A_1 \tilde{W}_T^1 + A_2 \tilde{W}_T^2 \geq \ln\left(\frac{K^{p-1}}{c}\right) - \tilde{B}T\}, \quad (4.47)$$

and thus

$$\Psi_1^p(c) = \frac{1}{p}\mathbf{E}[K^p \mathbf{1}_{\{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\}} \mathbf{1}_{A_c^c}] + \frac{1}{p}c^{\frac{p}{p-1}}\mathbf{E}[\tilde{Z}_T^{\frac{p}{p-1}} \mathbf{1}_{A_c}],$$

$$\Psi_2^p(c) = \tilde{\mathbf{E}}[K \mathbf{1}_{\{\sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 \geq \tilde{b}\}} \mathbf{1}_{A_c}] - c^{\frac{1}{p-1}}\tilde{\mathbf{E}}[\tilde{Z}_T^{\frac{1}{p-1}} \mathbf{1}_{A_c}].$$

In view of (4.46) and (4.47) we have

$$\begin{aligned} \Psi_1^p(c) &= \frac{K^p}{p}P\left(\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b, A_1 W_T^1 + A_2 W_T^2 < \ln\left(\frac{K^{p-1}}{c}\right) - BT\right) + \frac{1}{p}c^{\frac{p}{p-1}}\mathbf{E}[\tilde{Z}_T^{\frac{p}{p-1}} \mathbf{1}_{A_c}] \\ &= \frac{K^p}{p} \int_b^{+\infty} \int_{-\infty}^{\ln\left(\frac{K^{p-1}}{c}\right) - BT} f_{X,Y}(x,y) dy dx + \frac{1}{p}c^{\frac{p}{p-1}} \int_b^{+\infty} \int_{\ln\left(\frac{K^{p-1}}{c}\right) - BT}^{+\infty} e^{-\frac{p(y+BT)}{p-1}} f_{X,Y}(x,y) dy dx, \end{aligned}$$

and

$$\begin{aligned} \Psi_2^p(c) &= K\tilde{P}(A_c) - c^{\frac{1}{p-1}}\tilde{\mathbf{E}}[e^{-A_1 \tilde{W}_T^1 - A_2 \tilde{W}_T^2 - \tilde{B}T} \mathbf{1}_{A_c}] \\ &= K \int_{\tilde{b}}^{+\infty} \int_{\ln\left(\frac{K^{p-1}}{c}\right) - \tilde{B}T}^{+\infty} \tilde{f}_{\tilde{X},\tilde{Y}}(x,y) dy dx - c^{\frac{1}{p-1}} \int_{\tilde{b}}^{+\infty} \int_{\ln\left(\frac{K^{p-1}}{c}\right) - \tilde{B}T}^{+\infty} e^{-\frac{y+\tilde{B}T}{p-1}} \tilde{f}_{\tilde{X},\tilde{Y}}(x,y) dy dx. \end{aligned}$$

4.2 Quantos

4.2.1 Quanto domestic

The contingent claim is of the form

$$H = S_T^2(S_T^1 - K)^+, \quad K > 0. \quad (4.48)$$

Linear loss function

Using (4.41) we obtain

$$\begin{aligned} \Psi_1(c) &= \mathbf{E}[S_T^2(S_T^1 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}}] = \mathbf{E}\left[S_T^2(S_T^1 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^1 > K\right]P(S_T^1 > K) \\ &= \mathbf{E}\left[S_T^2(S_T^1 - K) \mathbf{1}_{\{A_1 W_T^1 + A_2 W_T^2 \geq \ln c - BT\}} \mid W_T^1 > a_1\right]P(W_T^1 > a_1) \\ &= \int_{a_1}^{+\infty} \mathbf{E}\left[S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 W_T^2} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 W_T^1} - K) \mathbf{1}_{\{W_T^2 \geq \frac{\ln c - BT - A_1 W_T^1}{A_2}\}} \mid W_T^1 = x\right] f_{W_T^1}(x) dx \\ &= S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T} \int_{a_1}^{+\infty} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K) \int_{\frac{\ln c - BT - A_1 x}{A_2}}^{+\infty} e^{\sigma_2 y} f_{W_T^2 | W_T^1 = x}(y) dy f_{W_T^1}(x) dx, \end{aligned}$$

and

$$\begin{aligned}
\Psi_2(c) &= \tilde{\mathbf{E}}[S_T^2(S_T^1 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}}] = \tilde{\mathbf{E}}\left[S_T^2(S_T^1 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^1 > K\right] \tilde{P}(S_T^1 > K) \\
&= \tilde{\mathbf{E}}\left[S_T^2(S_T^1 - K) \mathbf{1}_{\{A_1 \tilde{W}_T^1 + A_2 \tilde{W}_T^2 \geq \ln c - \tilde{B}T\}} \mid \tilde{W}_T^1 > \tilde{a}_1\right] \tilde{P}(\tilde{W}_T^1 > \tilde{a}_1) \\
&= \int_{\tilde{a}_1}^{+\infty} \tilde{\mathbf{E}}\left[S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 \tilde{W}_T^2} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 \tilde{W}_T^1} - K) \mathbf{1}_{\{\tilde{W}_T^2 \geq \frac{\ln c - \tilde{B}T - A_1 \tilde{W}_T^1}{A_2}\}} \mid \tilde{W}_T^1 = x\right] \tilde{f}_{\tilde{W}_T^1}(x) dx \\
&= S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T} \int_{\tilde{a}_1}^{+\infty} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K) \int_{\frac{\ln c - \tilde{B}T - A_1 x}{A_2}}^{+\infty} e^{\sigma_2 y} \tilde{f}_{\tilde{W}_T^2 | \tilde{W}_T^1 = x}(y) dy \tilde{f}_{\tilde{W}_T^1}(x) dx.
\end{aligned}$$

Power loss function

The set (4.33) is of the form

$$\begin{aligned}
A_c &:= \{c\tilde{Z}_T \leq H^{p-1}\} = \left\{ \frac{\left(ce^{-A_1 W_T^1 - A_2 W_T^2 - BT}\right)^{\frac{1}{p-1}}}{S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 W_T^2}} \leq (S_T^1 - K)^+ \right\} \\
&= \left\{ \frac{c^{\frac{1}{p-1}}}{S_0^2} e^{-\frac{A_1}{p-1} W_T^1 - (\frac{A_2}{p-1} + \sigma_2) W_T^2 - (B + \alpha_2 - \frac{1}{2}\sigma_2^2)T} \leq S_T^1 - K, S_T^1 \geq K \right\}.
\end{aligned}$$

For simplicity we assume that $\frac{A_2}{p-1} + \sigma_2 > 0$. In the opposite case one has to modify the form of the set A_c and thus also the integration limits in the formulas below. We obtain

$$A_c = \{W_T^2 \geq w(W_T^1), W_T^1 \geq a_1\} = \{\tilde{W}_T^2 \geq \tilde{w}(\tilde{W}_T^1), \tilde{W}_T^1 \geq \tilde{a}_1\},$$

where

$$\begin{aligned}
w(x) &:= \frac{\frac{A_1}{p-1}x + \ln\left(\frac{S_0^2(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)}{c^{\frac{1}{p-1}}}\right) + (B + \alpha_2 - \frac{1}{2}\sigma_2^2)T}{-(\frac{A_2}{p-1} + \sigma_2)}, \\
\tilde{w}(x) &:= \frac{\frac{A_1}{p-1}x + \ln\left(\frac{S_0^2(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)}{c^{\frac{1}{p-1}}}\right) + (\tilde{B} + \alpha_2 - \frac{1}{2}\sigma_2^2)T}{-(\frac{A_2}{p-1} + \sigma_2)}.
\end{aligned}$$

In view of this above, (4.31), (4.32) and using conditional densities we obtain

$$\begin{aligned}
\Psi_1^p(c) &= \frac{(S_0^2)^p e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)pT}}{p} \left(\int_{a_1}^{+\infty} \int_{-\infty}^{+\infty} e^{p\sigma_2 y} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_2 x} - K)^p f_{W_T^2 | W_T^1 = x}(y) f_{W_T^1}(x) dy dx \right. \\
&\quad \left. - \int_{a_1}^{+\infty} \int_{w(x)}^{+\infty} e^{p\sigma_2 y} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_2 x} - K)^p f_{W_T^2 | W_T^1 = x}(y) f_{W_T^1}(x) dy dx \right) \\
&\quad + \frac{c^{\frac{p}{p-1}} e^{-\frac{BTp}{p-1}}}{p} \int_{a_1}^{+\infty} \int_{w(x)}^{+\infty} e^{-\left(\frac{A_1 p}{p-1} x + \frac{A_2 p}{p-1} y\right)} f_{W_T^2 | W_T^1 = x}(y) f_{W_T^1}(x) dy dx,
\end{aligned}$$

$$\begin{aligned}
\Psi_2^p(c) &= S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T} \int_{\tilde{a}_1}^{+\infty} \int_{\tilde{w}(x)}^{+\infty} e^{\sigma_2 y} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_2 x} - K) \tilde{f}_{\tilde{W}_T^2 | \tilde{W}_T^1 = x}(y) \tilde{f}_{\tilde{W}_T^1}(x) dy dx \\
&\quad - c^{\frac{1}{p-1}} e^{-\frac{BT}{p-1}} \int_{\tilde{a}_1}^{+\infty} \int_{\tilde{w}(x)}^{+\infty} e^{-\frac{1}{p-1}(A_1 x + A_2 y)} \tilde{f}_{\tilde{W}_T^2 | \tilde{W}_T^1 = x}(y) \tilde{f}_{\tilde{W}_T^1}(x) dy dx.
\end{aligned}$$

4.2.2 Quanto foreign

The payoff is of the form

$$H = \left(S_T^1 - \frac{K}{S_T^2} \right)^+, \quad K > 0.$$

Linear loss function

First let us notice that

$$\left\{ S_T^1 - \frac{K}{S_T^2} \geq 0 \right\} = \{ \sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d \} = \{ \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 \geq \tilde{d} \}, \quad (4.49)$$

where

$$d := \ln \frac{K}{S_0^1 S_0^2} - \left(\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \right) T, \quad \tilde{d} := \ln \frac{K}{S_0^1 S_0^2} - \left(2r - \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \right) T. \quad (4.50)$$

We have

$$\begin{aligned} \Psi_1(c) &= \mathbf{E} \left[\left(S_T^1 - \frac{K}{S_T^2} \right)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \right] \\ &= \mathbf{E} \left[\left(S_T^1 - \frac{K}{S_T^2} \right) \mathbf{1}_{\left\{ W_T^2 \geq \frac{\ln c - BT - A_1 W_T^1}{A_2} \right\}} \mid \sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d \right] P(\sigma_1 W_T^1 + \sigma_2 W_T^2 \geq d). \end{aligned}$$

Denoting $Z := \sigma_1 W_T^1 + \sigma_2 W_T^2$ and taking into account conditional distribution $\mathcal{L}(W_T^1, W_T^2 \mid Z)$ we obtain

$$\Psi_1(c) = \int_d^{+\infty} \int_{-\infty}^{+\infty} \int_{\frac{\ln c - BT - A_1 x}{A_2}}^{+\infty} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K S_0^2 e^{(-\alpha_2 + \frac{1}{2}\sigma_2^2)T - \sigma_2 y}) f_{(W_T^1, W_T^2) \mid Z=z}(x, y) dy dx f_Z(z) dz.$$

Using the same argument under the measure \tilde{P} with $\tilde{Z} := \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2$ yields

$$\begin{aligned} \Psi_2(c) &= \tilde{\mathbf{E}} \left[\left(S_T^1 - \frac{K}{S_T^2} \right) \mathbf{1}_{\left\{ \widetilde{W}_T^2 \geq \frac{\ln c - \tilde{B}T - A_1 \widetilde{W}_T^1}{A_2} \right\}} \mid \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 \geq \tilde{d} \right] \tilde{P}(\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 \geq \tilde{d}) \\ &= \int_{\tilde{d}}^{+\infty} \int_{-\infty}^{+\infty} \int_{\frac{\ln c - \tilde{B}T - A_1 x}{A_2}}^{+\infty} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K S_0^2 e^{(-r + \frac{1}{2}\sigma_2^2)T - \sigma_2 y}) \tilde{f}_{(\widetilde{W}_T^1, \widetilde{W}_T^2) \mid \tilde{Z}=z}(x, y) dy dx \tilde{f}_{\tilde{Z}}(z) dz. \end{aligned}$$

Power loss function

Using (4.49) one can check the following

$$\begin{aligned} A_c &:= \left\{ c \tilde{Z}_T \leq \left(\left(S_T^1 - \frac{K}{S_T^2} \right)^+ \right)^{p-1}, S_T^1 - \frac{K}{S_T^2} > 0 \right\} \\ &= \left\{ c \tilde{Z}_T \leq \left(\left(S_T^1 - \frac{K}{S_T^2} \right)^+ \right)^{p-1}, \sigma_1 W_T^1 + \sigma_2 W_T^2 > d \right\} \\ &= \left\{ \frac{A_1}{p-1} W_T^1 + \left(\frac{A_2}{p-1} - \sigma_2 \right) W_T^2 \geq v(\sigma_1 W_T^1 + \sigma_2 W_T^2), \sigma_1 W_T^1 + \sigma_2 W_T^2 > d \right\} \quad (4.51) \end{aligned}$$

$$= \left\{ \frac{A_1}{p-1} \widetilde{W}_T^1 + \left(\frac{A_2}{p-1} - \sigma_2 \right) \widetilde{W}_T^2 \geq \tilde{v}(\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2), \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 > \tilde{d} \right\}, \quad (4.52)$$

where d, \tilde{d} are given by (4.50) and

$$v(x) = \ln \left\{ \frac{S_0^1 S_0^2 e^{(\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))T + x} - K}{c^{\frac{1}{p-1}} S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2 - \frac{B}{p-1})T}} \right\},$$

$$\tilde{v}(x) = \ln \left\{ \frac{S_0^1 S_0^2 e^{(2r - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))T + x} - K}{c^{\frac{1}{p-1}} S_0^2 e^{(r - \frac{1}{2}\sigma_2^2 - \frac{B}{p-1})T}} \right\}.$$

To calculate Ψ_1^l, Ψ_2^l we use conditional distributions $\mathcal{L}(X | Y), \mathcal{L}(\tilde{X} | \tilde{Y})$, where $X := \frac{A_1}{p-1} W_T^1 + (\frac{A_2}{p-1} - \sigma_2) W_T^2, Y := \sigma_1 W_T^1 + \sigma_2 W_T^2, \tilde{X} := \frac{A_1}{p-1} \tilde{W}_T^1 + (\frac{A_2}{p-1} - \sigma_2) \tilde{W}_T^2, \tilde{Y} := \sigma_1 \tilde{W}_T^1 + \sigma_2 \tilde{W}_T^2$. Denote by k_1, k_2, k_3, k_4 constants satisfying $W_T^1 = k_1 X + k_2 Y, W_T^2 = k_3 X + k_4 Y, \tilde{W}_T^1 = k_1 \tilde{X} + k_2 \tilde{Y}, \tilde{W}_T^2 = k_3 \tilde{X} + k_4 \tilde{Y}$. Then we have

$$\begin{aligned} \Psi_1^p(c) &= \frac{1}{p} \int_d^\infty \int_{-\infty}^{v(y)} \left(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1(k_1 x + k_2 y)} - \frac{K}{S_0^1 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_1(k_3 x + k_4 y)}} \right)^p f_{X|Y=y}(x) f_Y(y) dx dy \\ &\quad + \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pBT}{p-1}} \int_d^{+\infty} \int_{v(y)}^{+\infty} e^{-\frac{pA_1}{p-1}(k_1 x + k_2 y) - \frac{pA_2}{p-1}(k_3 x + k_4 y)} f_{X|Y=y}(x) f_Y(y) dx dy, \end{aligned}$$

$$\begin{aligned} \Psi_2^p(c) &= \int_{\tilde{d}}^{+\infty} \int_{\tilde{v}(y)}^{+\infty} \left(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1(k_1 x + k_2 y)} - \frac{K}{S_0^1 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_1(k_3 x + k_4 y)}} \right) \tilde{f}_{\tilde{X}|\tilde{Y}=y}(x) \tilde{f}_{\tilde{Y}}(y) dx dy \\ &\quad - c^{\frac{1}{p-1}} e^{-\frac{BT}{p-1}} \int_{\tilde{d}}^{+\infty} \int_{\tilde{v}(y)}^{+\infty} e^{-\frac{A_1}{p-1}(k_1 x + k_2 y) - \frac{A_2}{p-1}(k_3 x + k_4 y)} \tilde{f}_{\tilde{X}|\tilde{Y}=y}(x) \tilde{f}_{\tilde{Y}}(y) dx dy. \end{aligned}$$

4.3 Outperformance option

The problem is studied for

$$H = (\max\{S_T^1, S_T^2\} - K)^+, \quad K > 0.$$

Linear loss function

By (4.41), (4.42) and (4.43) we get

$$\begin{aligned} \Psi_1(c) &= \mathbf{E} \left[(S_T^1 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^1 \geq K, S_T^1 \geq S_T^2 \right] P(S_T^1 \geq K, S_T^1 \geq S_T^2) \\ &\quad + \mathbf{E} \left[(S_T^2 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^2 \geq K, S_T^1 < S_T^2 \right] P(S_T^2 \geq K, S_T^1 < S_T^2) \\ &= \mathbf{E} \left[(S_T^1 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b \right] P(W_T^1 \geq a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b) \\ &\quad + \mathbf{E} \left[(S_T^2 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b \right] P(W_T^2 \geq a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 < b) \\ &= \int_{a_1}^{+\infty} \int_b^{+\infty} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K) \mathbf{1}_{\{A_1 x + A_2 \frac{\sigma_1 x - z}{\sigma_2} \geq \ln c - BT\}} f_{W_T^1, \sigma_1 W_T^1 - \sigma_2 W_T^2}(x, z) dz dx \\ &\quad + \int_{a_2}^{+\infty} \int_{-\infty}^b (S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K) \mathbf{1}_{\{A_1 \frac{z + \sigma_2 y}{\sigma_1} + A_2 y \geq \ln c - BT\}} f_{W_T^2, \sigma_1 W_T^1 - \sigma_2 W_T^2}(y, z) dz dy, \end{aligned}$$

$$\begin{aligned}
\Psi_2(c) &= \tilde{\mathbf{E}}\left[(S_T^1 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^1 \geq K, S_T^1 \geq S_T^2\right] \tilde{P}(S_T^1 \geq K, S_T^1 \geq S_T^2) \\
&+ \tilde{\mathbf{E}}\left[(S_T^2 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^2 \geq K, S_T^1 < S_T^2\right] \tilde{P}(S_T^2 \geq K, S_T^1 < S_T^2) \\
&= \tilde{\mathbf{E}}\left[(S_T^1 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid \tilde{W}_T^1 \geq \tilde{a}_1, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 \geq \tilde{b}\right] \tilde{P}(\tilde{W}_T^1 \geq \tilde{a}_1, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 \geq \tilde{b}) \\
&+ \tilde{\mathbf{E}}\left[(S_T^2 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid \tilde{W}_T^2 \geq \tilde{a}_2, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 < \tilde{b}\right] \tilde{P}(\tilde{W}_T^2 \geq \tilde{a}_2, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2 < \tilde{b}) \\
&= \int_{\tilde{a}_1}^{+\infty} \int_{\tilde{b}}^{+\infty} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 x} - K) \mathbf{1}_{\{A_1 x + A_2 \frac{\sigma_1 x - z}{\sigma_2} \geq \ln c - \tilde{B}T\}} \tilde{f}_{\tilde{W}_T^1, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2}(x, z) dz dx \\
&+ \int_{\tilde{a}_2}^{+\infty} \int_{-\infty}^{\tilde{b}} (S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 y} - K) \mathbf{1}_{\{A_1 \frac{z+\sigma_2 y}{\sigma_1} + A_2 y \geq \ln c - \tilde{B}T\}} \tilde{f}_{\tilde{W}_T^2, \sigma_1 \tilde{W}_T^1 - \sigma_2 \tilde{W}_T^2}(y, z) dz dy.
\end{aligned}$$

Power loss function

Taking into account (4.41), (4.42), (4.43) we can write

$$\begin{aligned}
A_c &= \{c\tilde{Z}_T \leq (S_T^1 \vee S_T^2 - K)^{p-1}, S_T^1 \vee S_T^2 - K > 0\} \\
&= \{c\tilde{Z}_T \leq (S_T^1 - K)^{p-1}, S_T^1 > K, S_T^1 \geq S_T^2\} \cup \{c\tilde{Z}_T \leq (S_T^2 - K)^{p-1}, S_T^2 > K, S_T^1 \leq S_T^2\}.
\end{aligned}$$

We consider the case when $A_1 > 0, A_2 > 0$:

$$\begin{aligned}
A_c &= \{W_T^2 \geq -\left(A_1 W_T^1 + BT + \ln\left(\frac{1}{c}(S_T^1 - K)^{p-1}\right)\right), W_T^1 > a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\} \\
&\cup \{W_T^1 \geq -\left(A_2 W_T^2 + BT + \ln\left(\frac{1}{c}(S_T^2 - K)^{p-1}\right)\right), W_T^2 > a_2, \sigma_1 W_T^1 - \sigma_2 W_T^2 \leq b\} \\
&= \left\{W_T^2 \geq v_1(W_T^1), W_T^1 > a_1, W_T^2 \leq \frac{\sigma_1 W_T^1 - b}{\sigma_2}\right\} \\
&\cup \left\{W_T^1 \geq v_2(W_T^2), W_T^2 > a_2, W_T^1 \leq \frac{\sigma_2 W_T^2 - b}{\sigma_1}\right\} \\
&= \left\{\tilde{W}_T^2 \geq \tilde{v}_1(\tilde{W}_T^1), \tilde{W}_T^1 > \tilde{a}_1, \tilde{W}_T^2 \leq \frac{\sigma_1 \tilde{W}_T^1 - \tilde{b}}{\sigma_2}\right\} \\
&\cup \left\{\tilde{W}_T^1 \geq \tilde{v}_2(\tilde{W}_T^2), \tilde{W}_T^2 > \tilde{a}_2, \tilde{W}_T^1 \leq \frac{\sigma_2 \tilde{W}_T^2 - \tilde{b}}{\sigma_1}\right\}, \tag{4.53}
\end{aligned}$$

where

$$\begin{aligned}
v_1(x) &= -\frac{1}{A_2} \left(A_1 x + BT + \ln\left(\frac{1}{c}(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)^p\right)\right), \\
v_2(x) &= -\frac{1}{A_1} \left(A_2 x + BT + \ln\left(\frac{1}{c}(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K)^p\right)\right), \\
\tilde{v}_1(x) &= -\frac{1}{A_2} \left(A_1 x + \tilde{B}T + \ln\left(\frac{1}{c}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)^p\right)\right), \\
\tilde{v}_2(x) &= -\frac{1}{A_1} \left(A_2 x + \tilde{B}T + \ln\left(\frac{1}{c}(S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K)^p\right)\right).
\end{aligned}$$

Using the representation (4.53) and accepting the convention that the integral over the empty set is zero, we obtain

$$\begin{aligned}
\Psi_1^p(c) &= \int_{a_1}^{+\infty} \int_{-\infty}^{v_1(x) \wedge \frac{\sigma_1 x - b}{\sigma_2}} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)^p f_{W_T^2 | W_T^1 = x}(y) dy f_{W_T^1}(x) dx \\
&+ \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pB}{p-1}} \int_{a_1}^{+\infty} \int_{v_1(x)}^{\frac{\sigma_1 x - b}{\sigma_2}} e^{-\frac{pA_1}{p-1}x - \frac{pA_2}{p-1}y} f_{W_T^2 | W_T^1 = x}(y) dy f_{W_T^1}(x) dx \\
&+ \int_{a_2}^{+\infty} \int_{-\infty}^{v_2(x) \wedge \frac{\sigma_2 x - b}{\sigma_1}} (S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K)^p f_{W_T^1 | W_T^2 = x}(y) dy f_{W_T^2}(x) dx \\
&+ \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pB}{p-1}} \int_{a_2}^{+\infty} \int_{v_2(x)}^{\frac{\sigma_2 x - b}{\sigma_1}} e^{-\frac{pA_1}{p-1}x - \frac{pA_2}{p-1}y} f_{W_T^1 | W_T^2 = x}(y) dy f_{W_T^2}(x) dx,
\end{aligned}$$

$$\begin{aligned}
\Psi_2^p(c) &= \int_{\tilde{a}_1}^{+\infty} \int_{\tilde{v}_1(x)}^{\frac{\sigma_1 x - \tilde{b}}{\sigma_2}} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K) \tilde{f}_{\tilde{W}_T^2 | \tilde{W}_T^1 = x}(y) dy \tilde{f}_{\tilde{W}_T^1}(x) dx \\
&- c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}}{p-1}T} \int_{\tilde{a}_1}^{+\infty} \int_{\tilde{v}_1(x)}^{\frac{\sigma_1 x - \tilde{b}}{\sigma_2}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y} \tilde{f}_{\tilde{W}_T^2 | \tilde{W}_T^1 = x}(y) dy \tilde{f}_{\tilde{W}_T^1}(x) dx \\
&+ \int_{\tilde{a}_2}^{+\infty} \int_{\tilde{v}_2(x)}^{\frac{\sigma_2 x - \tilde{b}}{\sigma_1}} (S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K) \tilde{f}_{\tilde{W}_T^1 | \tilde{W}_T^2 = x}(y) dy \tilde{f}_{\tilde{W}_T^2}(x) dx \\
&- c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}}{p-1}T} \int_{\tilde{a}_2}^{+\infty} \int_{\tilde{v}_2(x)}^{\frac{\sigma_1 x - \tilde{b}}{\sigma_1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y} \tilde{f}_{\tilde{W}_T^1 | \tilde{W}_T^2 = x}(y) dy \tilde{f}_{\tilde{W}_T^2}(x) dx.
\end{aligned}$$

4.4 Spread option

The payoff is of the form

$$H = (S_T^1 - S_T^2 - K)^+, \quad K > 0.$$

One can check the following

$$\{S_T^1 \geq S_T^2 + K\} = \{W_T^1 \geq d(W_T^2)\} = \{\tilde{W}_T^1 \geq \tilde{d}(\tilde{W}_T^2)\},$$

where

$$d(y) := \frac{1}{\sigma_1} \ln \left(\frac{S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K}{S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T}} \right), \quad \tilde{d}(y) := \frac{1}{\sigma_1} \ln \left(\frac{S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K}{S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T}} \right).$$

Linear loss function

We have

$$\begin{aligned}
\Psi_1(c) &= \mathbf{E} \left[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \right] = \int_{-\infty}^{+\infty} \mathbf{E} \left[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid W_T^2 = y \right] f_{W_T^2}(y) dy \\
&= \int_{-\infty}^{+\infty} \int_{d(y)}^{+\infty} \left(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \right) \mathbf{1}_{\{A_1 x + A_2 y \geq \ln c - BT\}} f_{W_T^1 | W_T^2 = y}(x) dx f_{W_T^2}(y) dy,
\end{aligned}$$

and

$$\begin{aligned}
\Psi_2(c) &= \tilde{\mathbf{E}} \left[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \right] = \int_{-\infty}^{+\infty} \tilde{\mathbf{E}} \left[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid \tilde{W}_T^2 = y \right] \tilde{f}_{\tilde{W}_T^2}(y) dy \\
&= \int_{-\infty}^{+\infty} \int_{\tilde{d}(y)}^{+\infty} \left(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \right) \mathbf{1}_{\{A_1 x + A_2 y \geq \ln c - \tilde{B}T\}} \tilde{f}_{\tilde{W}_T^1 | \tilde{W}_T^2 = y}(x) dx f_{\tilde{W}_T^2}(y) dy.
\end{aligned}$$

Power loss function

We have

$$\begin{aligned}
A_c &:= \{c\tilde{Z}_T \leq (S_T^1 - S_T^2 - K)^{p-1}, S_T^1 - S_T^2 - K > 0\} \\
&= \left\{ c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}W_T^1 - \frac{A_2}{p-1}W_T^2 - \frac{B}{p-1}T} \leq S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 W_T^1} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 W_T^2} - K, W_T^1 \geq d(W_T^2) \right\} \\
&= \left\{ W_T^1 \in \mathcal{A}(W_T^2) \right\} = \left\{ \tilde{W}_T^1 \in \tilde{\mathcal{A}}(\tilde{W}_T^2) \right\}, \tag{4.54}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}(y) &:= \{x : c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y - \frac{B}{p-1}T} \leq S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K, x \geq d(y)\}, \\
\tilde{\mathcal{A}}(y) &:= \{x : c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y - \frac{\tilde{B}}{p-1}T} \leq S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K, x \geq \tilde{d}(y)\}.
\end{aligned}$$

Let us notice that the set $A_c^c \cap \{H > 0\}$ is of the form

$$A_c^c \cap \{H > 0\} = \{W_T^1 \in \mathcal{B}(W_T^2)\}, \tag{4.55}$$

where

$$\mathcal{B}(y) := \{x : c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y - \frac{B}{p-1}T} > S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K, x \geq d(y)\}.$$

Taking into account (4.54) and (4.55) we obtain

$$\begin{aligned}
\Psi_1^p(c) &= \frac{1}{p} \int_{-\infty}^{+\infty} \int_{\mathcal{B}(y)} \left(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \right)^p f_{W_T^1|W_T^2=y}(x) dx f_{W_T^2}(y) dy \\
&\quad + \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pBT}{p-1}} \int_{-\infty}^{+\infty} \int_{\mathcal{A}(y)} \left(e^{-\frac{pA_1}{p-1}x - \frac{pA_2}{p-1}y} \right) f_{W_T^1|W_T^2=y}(x) dx f_{W_T^2}(y) dy, \\
\Psi_2^p(c) &= \int_{-\infty}^{+\infty} \int_{\tilde{\mathcal{A}}(y)} \left(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K \right) \tilde{f}_{\tilde{W}_T^1|\tilde{W}_T^2=y}(x) dx \tilde{f}_{\tilde{W}_T^2}(y) dy \\
&\quad + c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}T}{p-1}} \int_{-\infty}^{+\infty} \int_{\tilde{\mathcal{A}}(y)} \left(e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y} \right) \tilde{f}_{\tilde{W}_T^1|\tilde{W}_T^2=y}(x) dx \tilde{f}_{\tilde{W}_T^2}(y) dy.
\end{aligned}$$

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