Noname manuscript No.
(will be inserted by the editor)

Approximating European Options by Rebate Barrier Options

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Feb 11, 2011

Abstract When the underlying stock price is a strict local martingale process under an equivalent local martingale measure, Black-Scholes PDE associated with an European option may have multiple solutions. In this paper, we study an approximation for the smallest hedging price of such an European option. Our results show that a class of rebate barrier options can be used for this approximation, when its rebate and barrier are chosen appropriately. An asymptotic convergence rate is also achieved when the knocked-out barrier moves to infinity under suitable conditions.

Keywords European options · Financial bubbles · Local martingales · Truncation approximation · Convergence rate · Barrier options

Mathematics Subject Classification (2010) 60G44
JEL Classification G12 · G13

1 Introduction

We consider a single stock in the presence of the unique equivalent local martingale measure (ELMM) $\mathbb{P}$, under which the deflated price process follows

$$dX(s) = \sigma(X(s))dW(s), \;\; X(t) = x \geq 0, \tag{1.1}$$

where W is a standard Brownian motion with respect to a given probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_s : s \geq t\})$ satisfying usual conditions. For a contingent

The research of this author is supported in part by the Research Grants Council of Hong Kong No. CityU 104007.

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claim $f(X(T))$ with a fixed maturity $T > 0$, the smallest hedging price has the form of

$$V(x,t) = \mathbb{E}_{x,t}[f(X(T))] := \mathbb{E}[f(X^{x,t}(T))|\mathcal{F}_t]. \tag{1.2}$$

In the above, we suppress the superscripts (x,t) in $X^{x,t}$, and write $\mathbb{E}_{x,t}$ to indicate the expectation computed under these initial conditions.

Recently, [5] shows that the value function V of (1.2) is the $C^{2,1}(Q) \cap C(\overline{Q})$ solution of $BS(Q,f)$, where $BS(Q,f)$ refers to Black-Scholes equation

$$u_t + \frac{1}{2}\sigma^2(x)u_{xx} = 0 \text{ on } Q := \mathbb{R}^+ \times (0,T) \tag{1.3}$$

satisfying boundary-terminal condition

$$u(x,t) = f(x) \text{ on } \partial^* Q := [0,\infty) \times \{T\} \cup \{0\} \times (0,T), \tag{1.4}$$

However, next example taken from [2] shows that the value function V may not be the unique solution of $BS(Q,f)$ when the deflated price process X is a strict local martingale.

Example 1.1 (CEV model) Suppose the stock price follows a strict local martingale process $dX(s) = X^2 dW(s)$, with the initial $X(t) = x > 0$. Consider $V(x,t) = \mathbb{E}_{x,t}[X(T)]$. Then, V can be computed explicitly as

$$V(x,t) = x\Big(1 - 2\Phi\Big(-\frac{1}{x\sqrt{T-t}}\Big)\Big). \tag{1.5}$$

One can verify V satisfies $BS(Q,f)$. Another trivial solution is $u(x,t) = x$.

The difference $u - V$ of Example 1.1 is termed as a bubble in the literature, see [5], [7] and the references therein. Now, V is one of possibly multiple solutions of $BS(Q,f)$. A natural question is that how one can find a feasible numerical approximation of this value function V of (1.2). Similar question is also proposed by [6].

The next trivial example shows that the classical Monte Carlo method by Euler-Maruyama approximation does not lead to the desired value $V(x,t)$ of (1.5) of Example 1.1.

Example 1.2 Consider the strong Euler-Maruyama (EM) approximation to Example 1.1 is: with step size Δ

$$X^{\Delta}_{n+1} = X^{\Delta}_n + \sigma(X^{\Delta}_n)(W(n\Delta + \Delta) - W(n\Delta)), \quad X^{\Delta}_0 = x.$$

Let $X^{\Delta}(\cdot)$ be the piecewise constant interpolation of $\{X^{\Delta}_n : n \geq 0\}$, i.e.

$$X^{\Delta}(s) = X^{\Delta}_{[s/\Delta]}, \quad \forall s > 0. \tag{1.6}$$

Since $\{X^{\Delta}_n : n \geq 0\}$ is a martingale, the approximated value function leads to

$$\mathbb{E}_{x,0}[X^{\Delta}(T)] = \mathbb{E}_{x,0}[X^{\Delta}_{(T-t)/\Delta}] = x > V(x,0).$$

Our work in this paper is to find a feasible approximation to the smallest hedging price $V(x,t)$ of (1.2) in its domain Q of (1.3). It turns out that the value function of (1.2) can be obtained by a limit of a series of rebate option prices, which can be easily implemented by exactly the same EM method of Example 1.2.

To be more precise, we consider the following up-rebate: Suppose the knocked-out barrier is given by a positive constant β and the rebate function is a Borel measurable non-negative function g on $\overline{Q}$. Rebate option makes a payment $g(X(t),t)$ if either the asset price $X(t)$ of (1.1) reaches the barrier β or the time evolution reaches the maturity T. One of our results shows that, if the Rebate function g is chosen to satisfy growth conditions of Theorem 3.4, its rebate option price $V^{\beta,g}(x,t)$ of (3.1) converges to $V(x,t)$ as $\beta \to \infty$. Moreover, its error estimate is also provided in (3.10) in terms of growth factor of functions f and g, which may be used to choose a rebate function g a priori for better approximation.

In particular, taking g by

$$g(x,t) = f(x)\mathbf{1}_{\{(\mathbf{x},\mathbf{t})\in\partial^*\mathbf{Q}\}} \text{ on } \overline{\mathbf{Q}}, \tag{1.7}$$

the up-rebate becomes up-and-out barrier option, which is analogous to the truncation approximation of Black-Scholes PDE proposed by [4]. Theorem 3.4 also implies the convergence of the rebate option price with this specific choice of (1.7).

Thanks to the convergent result of Theorem 3.4, Example 3.5 shows that, if one can apply the natural EM approximation to an appropriate rebate option for a large barrier β and a small step size Δ, then the approximated up-rebate price $V_\Delta^{\beta,g}$ of (3.14) must be close to V.

On the other hand, if the rebate function g can be chosen to be continuous on $\overline{Q}$, then $V^{\beta,g}$ of (3.1) solves Black-Scholes equation $BS(Q_\beta, g)$ given by

$$u_t + \mathcal{L}u = 0 \text{ on } Q_\beta := (0,\beta)\times(0,T) \tag{1.8}$$

and Cauchy-Dirichlet data

$$u(x,t) = g(x,t) \text{ on } \partial^* Q_\beta := \{\beta,0\}\times(0,T)\cup[0,\beta]\times\{T\}. \tag{1.9}$$

uniquely in $C^{2,1}(Q_\beta)\cap C(\overline{Q}_\beta)$, see Lemma 3.7. Unique solvability provides alternative approximation to the above Monte Carlo methods: one can use any of numerical PDE methods, such as finite element method (FEM) or finite difference method (FDM).

However, if g is taken by a discontinuous function, for instance (1.7), one can not expect the solvability of $BS(Q_\beta,g)$ in $C^{2,1}(Q_\beta)\cap C(\overline{Q}_\beta)$. Moreover, it is well-known that singularity and discontinuity on boundary layer propogates the numerical errors quickly throughout its domain in using numerical PDE methods. This gives added difficulty if one may want to use numerical PDE approximation due to the discontinuity of the function g. To overcome this difficulty, we will provide a continuous rebate function g^β of (3.16) depending on the barrier β, and show its convergence and the unique solvability of $BS(Q_\beta,g)$

in $C^{2,1}(Q_\beta) \cap C(\overline{Q}_\beta)$, see Proposition 3.9. Note that Holder regularity of g^β does not decrease to zero as $\beta \to \infty$, but preserves Holder regularity of given payoff function f. This enables us to avoid the difficulty in using numerical PDE approximation with a singularity.

The rest of the paper is outlined as follows. We start with the precise problem formulation and some related preliminary results in Section 2. The main results, including convergence analysis and approximation with continuous Cauchy-Dichlet data, are included in Section 3. The last section is devoted to a brief summary of current and future studies.

2 Problem setup and some related preliminary results

We consider a single stock with price $X^{x,t}(\cdot)$ of (1.1). The desired value function $V(x,t)$ to be studied is the smallest hedging price of contingent claim $f(X^{x,t}(T))$, defined in (1.2). Throughout this paper, we use K as a generic constant, and impose the following two conditions on f and σ:

(A1) σ is locally Holder continuous with exponent $\frac{1}{2}$ satisfying $\sigma(x) > 0$ for all $x \in \mathbb{R}^+$, $\sigma(0) = 0$.
(A2) $f : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ is a $C_\gamma(\bar{\mathbb{R}}^+)$ a payoff function for some $\gamma \in [0,1]$.

In the above, $\mathbb{R}^+ = (0,\infty)$, $\bar{\mathbb{R}}^+ = \mathbb{R}^+ \cup \{0\}$, and $C_\gamma(A) = C(A) \cap D_\gamma(A)$, where $C(A)$ denotes the set of all continuous real functions on A, and $D_\gamma(A)$ denotes the set of all measurable functions $\varphi : A \to \bar{\mathbb{R}}^+$ satisfying growth condition

$$\varphi(x) \leq K(1 + |x|^\gamma), \ \forall x \in A. \tag{2.1}$$

By [8, 5.5.11], the assumption (A1) on σ ensures there exists a unique strong solution of (1.1) with absorbing state at zero.

To proceed, we need the next fundamental properties of the value function $V(x,t)$, which establishes the link with parabolic partial differential equation $BS(Q,f)$.

Proposition 2.1 *Assume (A1-A2). Then, value function V of* (1.2) *is*

1. *the smallest lower-bounded $C^{2,1}(Q) \cap C_\gamma(\overline{Q})$ solution of $BS(Q,f)$.*
2. *the unique $C^{2,1}(Q) \cap C(\overline{Q})$ solution of $BS(Q,f)$ if and only if σ satisfies*

$$\int_1^\infty \frac{x}{\sigma^2(x)} dx = \infty. \tag{2.2}$$

Proof Theorem 3.2 of [5] shows that V is a $C^{2,1}(Q) \cap C(\overline{Q})$ solution of $BS(Q,f)$. Applying supermartingale property of $X(T)$ and Jensen's inequality, the next derivation shows that $V \in C_\gamma(\overline{Q})$,

$$V(x,t) = \mathbb{E}_{x,t}[f(X(T))] \leq K(1 + \mathbb{E}_{x,t}[X^\gamma(T)]) \leq K(1 + x^\gamma).$$

For the necessary and sufficient condition on uniqueness, we refer the proof to [1]. It remains to show V is the smallest lower bounded solution. Sometimes, we use X to denote $X^{x,0}$ without ambiguity in this proof. Note that, by pathwise uniqueness of the solution to (1.1)

$$Y(t) \triangleq V(X^{x,0}(t), t) = \mathbb{E}[f(X^{X^{x,0}(t),t}(T))|\mathcal{F}_t] = \mathbb{E}[f(X^{x,0}(T))|\mathcal{F}_t]$$

is a martingale process. Suppose $\hat{V} \in C^{2,1}(Q) \cap C(\overline{Q})$ is an arbitrary lower bounded solution of $BS(Q, f)$, then Ito's formula applying to $\hat{Y}(t) \triangleq \hat{V}(X(t), t)$ leads to

$$\hat{Y}(t) = V(X(0), 0) + \int_0^t \hat{V}_x(X(s), s)\sigma(X(s))dW(s),$$

and $\hat{Y}(t)$ is a lower bounded local martingale, hence is a supermartingale. Therefore, we have

$$\hat{Y}(0) \geq \mathbb{E}[\hat{Y}(T)] = \mathbb{E}[f(X(T))] = Y(0)$$

and this implies

$$\hat{V}(x, 0) \geq V(x, 0).$$

We can similarly prove for $\hat{V}(x, t) \geq V(x, t)$ for all t.

Proposition 2.2 *Assuming (A1-A2), $BS(Q, f)$ only admits non-negative solution in the space of lower bounded $C^{2,1}(Q) \cap C(\overline{Q})$ functions.*

Proof Proposition 2.1 shows that V is the smallest lower-bounded solution of PDE. Since $V \geq 0$ by definition of (1.2), it implies any lower-bounded solution u satisfies $u \geq V \geq 0$.

In Example 1.1, we have seen that $BS(Q, f)$ of CEV model has multiple solutions. We continue this model to demonstrate Proposition 2.2, a solution smaller than V must be unbounded from below.

Example 2.3 By Proposition 2.1, the explicit solution $V \geq 0$ of (1.5) in CEV model smallest lower-bounded solution of $BS(Q, f)$. In fact one can find,

$$v(x, t) = x\Big(1 - \lambda\Phi\Big(-\frac{1}{x\sqrt{T-t}}\Big)\Big), \lambda > 2$$

is a smaller solution, i.e. $v \leq V$ in Q. However, v is not lower-bounded, i.e. $v(x, t) \to -\infty$ as $x \to \infty$.

Our goal is to find a feasible approximation $V(x, t)$ of (1.2) by using truncation method.

3 The truncation approximation with rebate options

In this section, we first give probabilistic definition of truncated value function $V^{\beta,g}$, which corresponds to *fair* price of the rebate option with knocked-out barrier β and rebate function g. We will also determine a class of rebate functions g which lead to the convergence $V^{\beta,g} \to V$ as $\beta \to \infty$. The convergence with g of (1.7) proposed by [4] is treated as a special case.

3.1 The definition of rebate barrier options: The truncated value function

Recall that the domain of the value function V is given by $\overline{Q}$ of (1.3), and its related truncated domain Q_β is given by (1.8). Let $g : \overline{Q} \to \bar{\mathbb{R}}^+$ be a measurable function. We introduce the truncated value function $V^{\beta,g}$ by

$$V^{\beta,g}(x,t) = \begin{cases} \mathbb{E}_{x,t}[g(X(\tau^\beta), \tau^\beta)], \ \forall (x,t) \in \overline{Q}_\beta, \\ 0 \qquad\qquad\qquad\qquad\qquad \text{Otherwise.} \end{cases} \tag{3.1}$$

where the stopping time τ^β (suppressing the initial condition (x,t)) is given by

$$\tau^{x,t,\beta} = \inf\{s > t : (X^{x,t}(s), s) \notin Q_\beta\}. \tag{3.2}$$

With the above setup, our goal is to find a suitable function g such that $V^{\beta,g} \to V$ as $\beta \to \infty$. Since $V^{\beta,g}(x,t) = g(x,t)$ on parabolic boundary $\partial^* Q_\beta$, we will restrict our search for g in the set of functions satisfying

$$g(x,t) = f(x), \ \forall (x,t) \in \partial^* Q. \tag{3.3}$$

To cover g of (1.7) in our analysis, we do not exclude possibly discontinuous function at the boundary.

The following question may be our *ultimate goal* of this work: What kind of rebate function g can result in the fastest convergence $V^{\beta,g} \to V$ among all possible g satisfying (3.3)? The answer is extremely simple, $g = V$ is best function due to the next lemma.

Lemma 3.1 *Assume (A1-A2). Then, $V(x,t) = V^{\beta,V}(x,t)$ for all $0 < x < \beta$.*

Proof $X^{x,t}$ is the unique strong solution of (1.1) due to (A1). Therefore, the conclusion follows from the following simple derivation using tower property and strong Markov property:

$$\begin{aligned} V(x,t) &= \mathbb{E}[f(X^{x,t}(T))|\mathcal{F}_t] \\ &= \mathbb{E}[f(X^{x,t}(T))|\mathcal{F}_{\tau^\beta}]|\mathcal{F}_t] \\ &= \mathbb{E}[\mathbb{E}[f(X^{X(\tau^\beta),\tau^\beta}(T))|\mathcal{F}_{\tau^\beta}]|\mathcal{F}_t] \\ &= \mathbb{E}[V(X(\tau^\beta),\tau^\beta)|\mathcal{F}_t] \\ &= V^{\beta,V}(x,t). \end{aligned}$$

Although, V is an unknown function and the above approximation by $V^{\beta,V}$ of Lemma 3.1 is apparently not implementable, Lemma 3.1 is placed in the above for later error estimates.

3.2 Convergence

Lemma 3.2 *Assume (A1-A2), and g satisfies* (3.3). $V^{\beta,g}$ *of* (3.1) *satisfies*

1. *For each $\beta > 0$, $V^{\beta,g_1} \le V^{\beta,g_2}$ whenever $g_1 \le g_2$.*

2. $\lim_{\beta\to\infty} V^{\beta,g}(x,t) \geq V(x,t)$. *In particular,* $\lim_{\beta\to\infty} V^{\beta,g}(x,t) = V(x,t)$ *if and only if*

$$\lim_{\beta\to\infty} \mathbb{E}_{x,t}\Big[g(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<\mathbf{T}\}}\Big] = \mathbf{0} \tag{3.4}$$

Proof Monotonicity in g follows directly from the definition of $V^{\beta,g}$ in the above. To show the the second statement, we start with the following observation: The solution $X := X^{t,x}$ of (1.1) does not explode almost surely by [8, 5.5.3], i.e.

$$\lim_{\beta\to\infty} \tau^\beta = T, \quad \text{a.s.}\mathbb{P} \tag{3.5}$$

Due to this fact with Monotone Convergence Theorem, we obtain following identities:

$$\lim_{\beta\to\infty} \mathbb{E}_{x,t}\Big[f(X(T))\mathbf{1}_{\{\tau^\beta=T\}}\Big] = \mathbb{E}_{x,t}\Big[\lim_{\beta\to\infty} f(X(T))\mathbf{1}_{\{\tau^\beta=T\}}\Big] = \mathbb{E}_{x,t}\Big[f(X(T))\Big] = V(x,t). \tag{3.6}$$

This results in

$$\begin{aligned}
&\lim_{\beta\to\infty} V^{\beta,g}(x,t) \\
&= \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(X(\tau^\beta),\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}] + \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(X(\tau^\beta),\tau^\beta)\mathbf{1}_{\{\tau^\beta=T\}}] \\
&= \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}] + \lim_{\beta\to\infty} \mathbb{E}_{x,t}[f(X(T))\mathbf{1}_{\{\tau^\beta=T\}}] \\
&= \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}] + V(x,t).
\end{aligned}$$

By rearranging the above identity, we have

$$V(x,t) = \lim_{\beta\to\infty} V^{\beta,g}(x,t) - \lim_{\beta\to\infty} \mathbb{E}_{x,t}[g(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}]. \tag{3.7}$$

Note that three terms in (3.7) are all non-negative. Hence, $\lim_{\beta\to\infty} V^{\beta,g}(x,t) \geq V(x,t)$ and equality holds if and only if (3.4) holds.

As mentioned in (3.5), the solution $X^{x,t}$ of (1.1) does not explode almost surely, and this implies rewritten as $\mathbb{P}(\tau^{x,t,\beta} < T) \to 0$ as $\beta \to \infty$. Then, how fast does this probability converge to zero? The next answer to this question may be useful to obtain the convergence rate of the truncated approximation.

Proposition 3.3 *Fix* $(x,t) \in Q$ *and assume (A1-A2). As* $\beta \to \infty$, *stopping time* $\tau^{x,t,\beta}$ *of* (3.2) *satisfies*

1. $\mathbb{P}\{\tau^{x,t,\beta} < T\} = O(1/\beta)$.
2. *Moreover,* $\mathbb{P}\{\tau^{x,t,\beta} < T\} = o(1/\beta)$ *if and only if* $\{X^{t,x}(s) : t \leq s \leq T\}$ *is a martingale .*

Proof By taking $g(x,t) = x$ in (3.7),

$$\lim_{\beta\to\infty} \mathbb{E}_{x,t}[X(\tau^\beta)] = \lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} + \mathbb{E}_{x,t}[X(T)].$$

For all $\beta > x$, since $\{X^{x,t}(\tau^\beta \wedge s) : s > t\}$ is a bounded local martingale, hence it is martingale. So, $\mathbb{E}_{x,t}[X(\tau^\beta)] = x$ for all $\beta > x$. Rearranging the above identity, we have

$$\lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} = x - \mathbb{E}_{x,t}[X(T)] \tag{3.8}$$

(3.8) implies

1. Since $\mathbb{E}_{x,t}[X(T)] \geq 0$, $\lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} \leq x < \infty$, which shows $\mathbb{P}\{\tau^{x,t,\beta} < T\} = O(1/\beta)$.
2. $\{X^{t,x}(s) : t \leq s \leq T\}$ is a martingale if and only if $x = \mathbb{E}_{x,t}[X(T)]$, if and only if $\mathbb{P}\{\tau^{x,t,\beta} < T\} = o(1/\beta)$.

To present the convergence result in the next, we need the following definition, for any $\varphi : \overline{Q} \to \mathbb{R}$

$$\overline{\varphi}(x) := \sup_{t\in[0,T)} \varphi(x,t). \tag{3.9}$$

Theorem 3.4 *Assume (A1-A2). Suppose $\overline{g}$ of (3.9) satisfies (3.3) and one of two following conditions:*

1. *$\overline{g}(x)$ is of sublinear growth, i.e. $\lim_{x\to\infty} x\overline{g}(x) = 0$;*
2. *$\overline{g}(x)$ is of linear growth, i.e. $\lim_{x\to\infty} x\overline{g}(x) < \infty$, and $X^{x,t}$ is a martingale.*

Then, we have the convergence

$$\lim_{\beta\to\infty} V^{\beta,g}(x,t) = V(x,t).$$

In addition, if $\overline{g} \in D_\eta(\mathbb{R}^+)$ with $\gamma \wedge \eta < 1$, then the convergence rate is the order of $1 - (\gamma \vee \eta)$ as $\beta \to \infty$, i.e.

$$|(V - V^{\beta,g})(x,t)| \leq K\beta^{-(1-(\gamma\vee\eta))}, \ \forall x < \beta. \tag{3.10}$$

Proof We first show its convergence, then obtain convergence rate with additional conditions.

1. Regarding its convergence, it is enough to verify (3.4) by Lemma 3.2. Note that

$$\lim_{\beta\to\infty} \mathbb{E}_{x,t}\Big[g(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}\Big] \leq \lim_{\beta\to\infty} \overline{g}(\beta)\mathbb{E}_{x,t}\Big[\mathbf{1}_{\{\tau^\beta<T\}}\Big] \leq \lim_{\beta\to\infty} \frac{\overline{g}(\beta)}{\beta} \lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\}.$$

   (a) If $\overline{g}$ is of sublinear growth, then $\lim_{\beta\to\infty} \frac{\overline{g}(\beta)}{\beta} = 0$. Hence, together with Proposition 3.3, (3.4) holds.
   (b) On the other hand, if $X^{t,x}$ is a martingale, then we have $\lim_{\beta\to\infty} \beta\mathbb{P}\{\tau^{x,t,\beta} < T\} = 0$ from Proposition 3.3, and (3.4) holds.

2. Since $V(x,t) = V^{\beta,V}(x,t)$ for all $\beta > x$ by Lemma 3.1, we have the following identity:

$$\begin{aligned}(V - V^{\beta,g})(x,t) &= (V^{\beta,V} - V^{\beta,g})(x,t) \\ &= \mathbb{E}[(V-g)(X(\tau^\beta),\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}] \\ &= \mathbb{E}[(V-g)(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}]\end{aligned} \tag{3.11}$$

Since $\overline{V} \in C_\gamma(\mathbb{R}^+)$ by Proposition 2.1 and $\overline{g} \in D_\gamma(\mathbb{R}^+)$, we have $\overline{V-g} \in D_{\gamma\vee\eta}(\mathbb{R}^+)$. Hence, write (3.11) by

$$|(V - V^{\beta,g})(x,t)| \leq |(\overline{V-g})(\beta)|\mathbb{E}_{x,t}[\mathbf{1}_{\{\tau^\beta<T\}}] \leq K\beta^{(\gamma\vee\eta)-1}. \tag{3.12}$$

Example 3.5 We have seen that the strong EM approximation converges to a wrong value in Example 1.2. Theorem 3.4 implies that, by taking

$$g(x,t) = x \cdot \mathbf{1}_{\{(x,t)\in\partial^* Q\}}, \tag{3.13}$$

the rebate option price is convergent to the smallest hedging price, i.e.

$$V^{\beta,g}(x,0) = \mathbb{E}_{x,0}[f(X(T))\mathbf{1}_{\{X(T)\leq\beta\}}] \to V(x,0) \quad \text{as } \beta\to\infty.$$

Note that $X^\Delta(T) \to X(T)$ almost surely. Dominated Convergence Theorem implies that

$$V_\Delta^{\beta,g}(x,0) = \mathbb{E}_{x,0}[f(X^\Delta(T))\mathbf{1}_{\{X^\Delta(T)\leq\beta\}}] \to V^{\beta,g}(x,0), \quad \text{as } \Delta\to 0. \tag{3.14}$$

As a result,

$$\lim_{\beta\to\infty}\lim_{\Delta\to 0} V_\Delta^{\beta,g}(x,0) = V(x,0).$$

Although the convergence can be obtained by the truncation methods given by g of (1.7) similar to Example 3.5, we can do better approximation by choosing different choice g, provided that the payoff f is bounded.

Example 3.6 Consider the stock price $X(t)$ given in CEV model of Example 1.1 and put type payoff $f(x) = (1-x)^+$ with maturity T. Now, we compare two approximations by taking different choices of rebate functions: first choice is taken as (3.13), and the other is simply taken as $g_1(x,t) = f(x)$. We denote the associated truncated values by $V^{\beta,g}$ and V^{β,g_1}. By Theorem 3.4, both approximation result in same order of error estimate $O(\beta^{-1})$. However, following arguments show that V^{β,g_1} has better error estimate than $V^{\beta,g}$, respectively.

Let the approximation errors be

$$e^\beta(x,t) = (V^{\beta,g} - V)(x,t),\ e_1^\beta(x,t) = (V^{\beta,g_1} - V)(x,t).$$

Rewriting (3.11), we have representation of two errors

$$e^\beta(x,t) = \mathbb{E}_{x,t}[V(\beta,\tau^\beta)\mathbf{1}_{\{\tau^\beta<T\}}]$$

and

$$e_1^\beta(x,t) = \mathbb{E}_{x,t}[(V(\beta,\tau^\beta) - f(\beta))\mathbf{1}_{\{\tau^\beta<T\}}]$$

On the other hand, one can easily verify that g_1 is a viscosity subsolution of $BS(Q_\beta, g_1)$, and V is the unique solution of $BS(Q_\beta, V)$. By comparison principle (see [3]), it follows that

$$V(x,t) \geq g_1(x,t) = f(x) \geq 0.$$

This leads to $0 \leq e_1^\beta(x,t) \leq e^\beta(x,t)$.

The other advantage is that, associated $BS(Q_\beta, g_1)$ has continuous Cauchy-Dirichlet boundary data, which facilitates use of any numerical PDE method.

3.3 Truncation with continuous Cauchy-Dirichlet data

The heuristic arguments of Feyman-Kac formula on $V^{\beta,g}$ lead to a PDE formulation of $BS(Q_\beta, g)$, defined in (1.8)-(1.9). If g is given by (1.7), then the boundary-terminal data of $BS(Q_\beta, g)$ is discontinuous at the corner (β, T). Therefore, one can not expect solvability of $BS(Q_\beta, g)$ in $C^{2,1}(Q_\beta) \cap C(\overline{Q}_\beta)$. In the following, Lemma 3.7 and Lemma 3.8 discusses solvability of $B(Q_\beta, g)$ for the continuous and discontinuous function g, respectively.

Lemma 3.7 *Assume (A1-A2). Provided that* $g \in C(\overline{Q})$*, then* $V^{\beta,g}$ *solves* $BS(Q_\beta, g)$ *uniquely in* $C^{2,1}(Q_\beta) \cap C(\overline{Q}_\beta)$.

Proof Fix $(x,t) \in Q_\beta$. Take $\alpha \in (0, x/2)$. Let $Q_\beta^\alpha = Q_\beta \cap (\overline{Q}_\alpha)^c$ be an open set. Also define $\tau^{\alpha,\beta} = \inf\{s > t : (X^{x,t}(s), s) \notin Q_\beta^\alpha\}$.

Due to the uniform ellipticity, $V^{\alpha,\beta,g}(x,t) = \mathbb{E}_{x,t}[g(X(\tau^{\alpha,\beta}), \tau^{\alpha,\beta})]$ solves $BS(Q_\beta^\alpha, g)$ uniquely in $C^{2,1}(Q_\beta^\alpha) \cap C(\overline{Q}_\beta^\alpha)$. Using the facts of the continuity of g and almost sure convergence $\tau^{\alpha,\beta} \to \tau^\beta$, together with dominated convergence theorem, one can check that

$$\lim_{\alpha\to 0} V^{\alpha,\beta,g}(x,t) = \mathbb{E}_{x,t}[\lim_{\alpha\to 0} g(X(\tau^{\alpha,\beta}, \tau^{\alpha,\beta})] = V^{\beta,g}(x,t).$$

On the other hand, let $d = \min\{\frac{x}{2}, t, T-t\}$, which must be less than the minimum distance of x to a point in the parabolic boundary $\partial^* Q_\beta^\alpha$. Consider a neighborhood of x given by $N_x = (x - \frac{d}{2}, x + \frac{d}{2}) \times (t - \frac{d}{2}, t + \frac{d}{2})$. By maximum principle (Theorem 2.11 of [10]), Shauder's estimate ([9]) , we have $|V^{\alpha,\beta,g}|_{2.5,N_x} \leq K_d$ uniformly in $0 < \alpha < x/2$. Hence, $V^{\beta,g} \in C^{2,1}(Q_\beta)$ solves $BS(Q_\beta, g)$. Uniqueness follows from maximum principle.

One can indeed generalize the regularity results of Lemma 3.7 to some discontinuous rebate functions g, which is applicable, for instance, digital barrier options and turbo warrants, etc.

Lemma 3.8 *Assume (A1-A2). If there exists* $\varphi_n \in C(\overline{Q})$ *satisfying growth condition* $\varphi_n(x,t) \leq K(1+|x|)$ *uniformly in* n*, such that* $\varphi_n \to g$ *pointwisely , then* $V^{\beta,g}$ *is in* $C^{2,1}(Q)$ *satisfying* (1.8).

Proof $V^{\beta,\varphi_n} \in C^{2,1}(Q_\beta) \cap C(\overline{Q}_\beta)$ solves $BS(Q_\beta, \varphi_n)$ uniquely by Lemma 3.7. Since $\sup_{Q_\beta} |V^{\beta,\varphi_n}(x,t)| < K(1+\beta)$ by maximum principle (Theorem 2.11 of [10]), Shauder estimate ([9]) implies

$$|V^{\beta,\varphi_n}|_{2.5,Q_\beta(d)} \le K_d |V^{\beta,\varphi_n}|_{0,Q_\beta} \le K_d, \tag{3.15}$$

where $Q_\beta(d) := \{(x,t) \in Q_\beta : dist((x,t); \partial^* Q_\beta) \ge d\}$. Hence, there exists a convergent subsequence in $C^{2,1}(Q_\beta(d))$, which limit denoted by u^∞. Then, u^∞ solves (1.8) in $C^{2,1}(Q_\beta)$. It remains to show that $V^{\beta,g} = u^\infty$ on Q_β. In fact, this can be shown using dominated convergence theorem, for $(x,t) \in Q_\beta$,

$$\begin{aligned} V^{\beta,g}(x,t) &= \mathbb{E}[g(X(\tau^\beta), \tau^\beta)] \\ &= \mathbb{E}[\lim_{n\to\infty} \varphi_n(X(\tau^\beta), \tau^\beta)] \\ &= \lim_{n\to\infty} \mathbb{E}[\varphi_n(X(\tau^\beta), \tau^\beta)] \\ &= \lim_{n\to\infty} V^{\beta,\varphi_n}(x,t) = u^\infty. \end{aligned}$$

Theorem 3.4 together with Lemma 3.8 implies that the value function V can be approximated by a series of smooth function $V^{\beta,g}$ with g of (1.7) as $\beta \to \infty$. In fact, $V^{\beta,g}$ can be computed by Euler-Maruyama methods as of Example 3.5. An alternative approach is to compute $V^{\beta,g}$ by solving $BS(Q_\beta, g)$ numerically, either by finite element method (FEM) or finite difference method (FDM). It is well-known that, when solving PDE numerically, the discontinuity or singularity of Cauchy-Dirichlet data would generate quick error propagation throughout the domain. This leads to another question: *can we find truncation approximation with continuous terminal-boundary data (the rebate function)?*

The answer is positive. In this below, we will construct one such an approximation. Let the rebate function be

$$g^\beta(x,t) = f(x)\mathbf{1}_{\{x \le \beta/2\}} + \frac{2f(x)(\beta - x)}{\beta}\mathbf{1}_{\{\beta/2 < x \le \beta\}} \text{ on } (x,t) \in \overline{Q}. \tag{3.16}$$

Observe that $g^\beta \in C(\overline{Q})$ depends on β, while the choice of g in Section 3.1 is invariant of β. Moreover, g^β violates (3.3), instead it satisfies

$$\lim_{\beta\to\infty} g^\beta(x,t) = f(x), \text{ on } \partial^* Q. \tag{3.17}$$

Recall the definition of V^{β,g^β} of (3.1). Now, we have the following convergence results.

Proposition 3.9 *Assume (A1-A2), and g^β is given by* (3.16)*. Then, V^{β,g^β} of* (3.1) *is unique $C^{2,1}(Q_\beta) \cap C(\overline{Q}_\beta)$ solution of $BS(Q_\beta, g^\beta)$, and $\lim_{\beta\to 0} V^{\beta,g^\beta}(x,t) \to V(x,t)$ for all $(x,t) \in Q$. In addition, if $\gamma < 1$ in (A2), then the convergence rate is $|V^{\beta,g^\beta} - V|(x,t) \le K\beta^{-1+\gamma}$.*

Proof Unique solvability follows from Lemma 3.7. Fix $(x,t) \in Q$. Then, for any $\beta > 2x$, we have inequality by its definition (3.1)

$$V^{\beta/2,g}(x,t) \le V^{\beta,g^\beta}(x,t) \le V^{\beta,g}(x,t) \tag{3.18}$$

where g is given by (1.7). Taking $\lim_{\beta\to\infty}$ in the above inequality and using Theorem 3.4, the convergence result follows. The rate of the convergence is the combined result of (3.18) and (3.10).

4 Further remarks

This paper studies an approximation to the smallest hedging price of European option using rebate options. From mathematical point of view, this work concerns on the approximation of the value function V of (1.2) by truncating the domain Q and imposing suitable Cauchy-Dirichlet data g.

The main result on the convergence Theorem 3.4 provides that, if the function g is chosen to satisfy sublinear growth in x uniformly in $t \in [0, T)$, then the truncated value $V^{\beta,g}$ converges to V. This enables practitioners to adopt EM methods on big enough truncated domain Q_β to get a close value of V, as demonstrated in Example 3.5.

On the other hand, to adopt numerical PDE techniques, continuous Cauchy-Dirichlet data is desired to get a good approximation. However, if the payoff f is given as of linear growth, g is taken as sublinear growth in x for the purpose of the convergence by Theorem 3.4, then it's not possible to have a $C(\overline{Q})$ function g satisfying (3.3), which basically requires the deferred rebate function $g(\cdot, T)$ agrees with payoff $f(\cdot)$ of European option. Alternatively, we provide a continuous function g^β in (3.16), which asymptotically agrees the payoff $f(\cdot)$ at T.

One question is that, if there is a rebate function $g \in C(\overline{Q})$ satisfying (3.3) and being independent of barrier β, such that $V^{\beta,g} \to V$ when f is of linear growth and X is strict local martingale (for instance Example 1.1)? The answer is positive. If one is lucky enough to take $g = V$, then $V^{\beta,g}$ is obviously convergent to V. But, V is unknown, and $g = V$ is not practical. Then, is there a feasible choice of continuous g satisfying (3.3) such that $V^{\beta,g} \to V$? This may be one of our future directions of this study.

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