Constructing Exactly Solvable Pseudo-hermitian Many-particle Quantum Systems by Isospectral Deformation

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Abstract

A class of non-Dirac-hermitian many-particle quantum systems admitting entirely real spectra and unitary time-evolution is presented. These quantum models are isospectral with Dirac-hermitian systems and are exactly solvable. The general method involves a realization of the basic canonical commutation relations defining the quantum system in terms of operators those are hermitian with respect to a pre-determined positive definite metric in the Hilbert space. Appropriate combinations of these operators result in a large number of pseudo-hermitian quantum systems admitting entirely real spectra and unitary time evolution. Examples of a pseudo-hermitian rational Calogero model and XXZ spin-chain are considered.

1 Introduction

The (non-)hermiticity of an operator crucially depends on the choice of the metric(or inner-product) in the Hilbert space, which has been taken as an identity operator in the standard treatment of quantum mechanics. The question of necessity of such a choice in formulating quantum physics is as old as the subject itself. A renewed interest[1, 2, 3] has been generated over the last decade in addressing the same question in a systematic manner. The current understanding is that a quantum system with unbroken combined Parity(P) and Time-reversal(T) symmetry admits entirely real spectra even though the system may be non-Dirac-hermitian. A consistent quantum description including reality of the entire spectra and unitary time-evolution of the non-Dirac-hermitian system is possible with the choice of a new inner-product[1]. An alternative description of non-Dirac-hermitian quantum systems admitting entirely real spectra is in terms of pseudo-hermitian operator[2, 3]. The existence of a positive-definite metric in the Hilbert space is again crucial in this formalism for showing reality of the entire spectra as well as unitary time evolution.

A large number of non-Dirac-hermitian quantum systems admitting entirely real spectra have been found. A few prototype examples from a vast list of such systems are contained in Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The real difficulty in giving a complete description of these systems lies in finding the exact positive-definite metric in the Hilbert space. It is worth emphasizing here that no expectation values of physical observables or correlation functions can be calculated without the knowledge of the metric in the Hilbert space. This makes a non-Dirac-hermitian quantum system incomplete, even though the complete energy spectrum and the associated eigenfunctions may be known explicitly. There are not many non-Dirac-hermitian quantum systems for which the metric is known exactly and explicitly, the scenario being even worse for such system with several degrees of freedom.

The purpose of this contribution is to present a class of exactly solvable pseudo-hermitian many-particle quantum systems with a pre-determined metric in the Hilbert space. The motivation behind considering a pre-determined metric lies in the fact that it can be used to construct non-Dirac-hermitian quantum systems with a complete description, simply by deforming known Dirac-hermitian quantum systems. Although the non-Dirac-hermitian Hamiltonian constructed in this way is isospectral with the corresponding Dirac-hermitian Hamiltonian, the difference may appear in the description of different correlation functions of these two quantum systems[7].

The motivation behind constructing such non-Dirac-hermitian Hamiltonian by isospectral deformation is the following. First of all, quantum systems constructed in this way may serve as prototype examples for testing different ideas and methods related to the subject. This is important particularly in the context of many-particle systems, where the number of exactly solvable models with an explicit knowledge of the metric in the Hilbert space are very few and validity of approximate and/or numerical methods are required to be checked before applications. Secondly, a significant number of non-Dirac-hermitian quantum systems are known to admit entirely real spectra for which the origin of the reality of the spectra is not obvious. It is to be seen whether the reality of the spectra of some of these models could be related to certain pseudo-hermitian quantum system or not. In fact, the asymmetric XXZ spin-chain[12], which is relevant in the context of two species reaction-diffusion processes and Kardar-Parisi-Zhang-type growth phenomenon, is shown to be pseudo-hermitian following the general approach[4] prescribed in this article. Finally, construction of physically realizable quantum systems is always desirable.

The general method involves a realization of the basic canonical commutation relations defining the quantum system in terms of operators those are hermitian with respect to a pre-determined positive definite metric η_+ in the Hilbert space. Consequently, any Hamiltonian that is constructed using appropriate combination of these operators is hermitian with respect to η_+. However, in general, the same Hamiltonian may not be Dirac-hermitian, thereby giving rise to a pseudo-hermitian Hamiltonian. A pseudo-hermitian quantum system constructed this way may or may not be exactly solvable. The examples considered in this article include an exactly solvable non-Dirac-hermitian Calogero Model and an XXZ spin-chain. Both of these models are pseudo-hermitian and isospectral with known Dirac-hermitian model. Many other interesting quantum systems following from this construction are described in Ref. [4].

2 Preliminaries and Examples

The Hilbert space that is endowed with the standard inner product $\langle ., . \rangle$ is denoted as $\mathcal{H}_D$. The subscript D indicates that the Dirac-hermiticity condition is used in this Hilbert space. On the other hand, the Hilbert space that is endowed with the positive-definite metric η_+ and the modified inner product,

$$\langle\langle ., . \rangle\rangle_{\eta_+} := \langle ., \eta_+ . \rangle. \tag{1}$$

is denoted as $\mathcal{H}_{\eta_+}$. Corresponding to a hermitian operator $\hat{\mathcal{B}}$ in the Hilbert space $\mathcal{H}_D$, a hermitian operator $\hat{B}$ in the Hilbert space $\mathcal{H}_{\eta_+}$ can be defined as[2],

$$\hat{B} = \rho^{-1} \hat{\mathcal{B}} \rho, \quad \rho := \sqrt{\eta_+}. \tag{2}$$

An interesting consequence of Eq. (2) is that a set of operators $\hat{\mathcal{B}}_i$ obey the same canonical commutation relations as those satisfied by the corresponding set of operators $\hat{B}_i$ and the vice verse. The relation (2) is important for identifying observables in $\mathcal{H}_{\eta_+}$ and also crucial for the discussion that follows.

2.1 Pseudo-hermitian Rational Calogero Model

The rational Calogero model[13] is one of the most celebrated examples of exactly solvable many-particle quantum systems. This model[13] and its variants[14] are relevant to the study of a diverse branches of contemporary physics. Calogero-Sutherland-type models have been constructed previously[9, 10, 11] within the context of PT-symmetric quantum systems. A new class of pseudo-hermitian quantum system involving rational Calogero model is presented below.

A Dirac-non-hermitian rational A_{N+1} Calogero model may be introduced as follows:

$$\begin{aligned}
H &= -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\lambda(\lambda-1)\sum_{i\neq j} X_{ij}^{-2} + \frac{1}{2}\sum_{i=1}^{N} x_i^2,\\
X_{12} &= (x_1-x_2)\,cosh\phi + i\,(x_1+x_2)\,sinh\phi,\\
X_{1j} &= x_1 cosh\phi + i x_2 sinh\phi - x_j, \quad j>2,\\
X_{2j} &= -i x_1 sinh\phi + x_2 cosh\phi - x_j \quad j>2,\\
X_{ij} &= x_i - x_j, \quad (i,j)>2.
\end{aligned} \tag{3}$$

The parameters λ, ϕ appearing in H are real. The coordinates x_i and their conjugate momenta p_i are hermitian in $\mathcal{H}_D$. Unlike the standard rational Calogero model, the two-body inverse-square interaction term is neither invariant under translation nor singular for $x_1 = x_i, i > 1$ and $x_2 = x_i, i > 2$. However, the Hamiltonian is invariant under a combined PT operation with the P and T transformations defined as,

$$P: x_1 \to x_2, \quad x_2 \to x_1, \quad x_i \to x_i \;\; \forall\, i>2; \;\; T: i \to -i. \tag{4}$$

The operation of P may be identified as a permutation of the particles '1' and

'2' in one dimension. Alternatively, with the interpretation of H as describing a single-particle system in N dimensions, P is a valid parity transformation in the N dimensional space. It may be recalled at this point that the transformation,

$$x_1 \rightarrow x_1 cos\theta + x_2 sin\theta,$$
$$x_2 \rightarrow x_1 sin\theta - x_2 cos\theta, \quad 0 \leq \theta \leq 2\pi, \tag{5}$$

corresponds to parity transformation in two dimensions with the familiar forms $x_1 \rightarrow x_1, x_2 \rightarrow -x_2$ or $x_1 \rightarrow -x_1, x_2 \rightarrow x_2$ reproduced by $\theta = 0$ and $\theta = \pi$, respectively. The parity transformation corresponding to $\theta = \frac{\pi}{2}$, i.e. $x_1 \rightarrow x_2$ and $x_2 \rightarrow x_1$, has been embedded in the N dimensional space for introducing P in Eq. (4) and the choice of θ is unique for the Calogero model considered in this article.

The claim is that the non-Dirac-hermitian H is isospectral with the standard rational Calogero model. The reality of the entire spectra of H could be attributed to an underlying pseudo-hermiticity. A positive-definite metric η_+ in the Hilbert space $\mathcal{H}_{\eta_+}$ may be considered:

$$\eta_+ := e^{-2\gamma \mathcal{L}_{12}}, \quad \mathcal{L}_{12} = x_1 p_2 - x_2 p_1, \quad \gamma \in R. \tag{6}$$

The coordinates x_i and the momenta p_i are not hermitian in $\mathcal{H}_{\eta_+}$. A new set of canonical conjugate operators those are hermitian in the Hilbert space $\mathcal{H}_{\eta_+}$ may be introduced by using the relation (2) as follows:

$$X_1 = x_1 \ cosh\phi + i x_2 \ sinh\phi,$$
$$X_2 = -i x_1 \ sinh\phi + x_2 \ cosh\phi, \quad X_i = x_i \ for \ i > 2$$
$$P_1 = p_1 \ cosh\phi + i p_2 \ sinh\phi,$$
$$P_2 = -i p_1 \ sinh\phi + p_2 \ cosh\phi, \ P_i = p_i \ for \ i > 2. \tag{7}$$

It may be noted that $L_{12} = X_1 P_2 - X_2 P_1 = \mathcal{L}_{12}$ is hermitian both in $\mathcal{H}_D$ and $\mathcal{H}_{\eta_+}$. This ensures that η_+ defined in Eq. (6) is positive-definite.

The Hamiltonian H can be re-written in terms of (X_i, P_i) as,

$$H = -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial X_i^2} + \frac{1}{2}\lambda(\lambda-1)\sum_{i\neq j} X_{ij}^{-2} + \frac{1}{2}\sum_{i=1}^{N} X_i^2, \qquad (8)$$

which implies hermiticity of H in $\mathcal{H}_{\eta_+}$. The non-Dirac-hermitian H can also be mapped to Dirac-hermitian Hamiltonian h through a similarity transformation,

$$\begin{aligned} h &:= \left(e^{-\gamma\mathcal{L}_{12}}\right)\ H\ \left(e^{\gamma\mathcal{L}_{12}}\right) \\ &= -\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}\lambda(\lambda-1)\sum_{i\neq j} x_{ij}^{-2} + \frac{1}{2}\sum_{i=1}^{N} x_i^2. \qquad (9) \end{aligned}$$

Thus, the Hamiltonian H is isospectral with the standard rational Calogero model h. The eigenfunctions ψ_{η_+} of H are related to the eigenfunctions ψ_D of h through the relation, $\psi_{\eta_+} = e^{\gamma\mathcal{L}_{12}}\ \psi_D$. The wave-functions ψ_{η_+} constitute a complete set of orthonormal states in $\mathcal{H}_{\eta_+}$.

A few comments are in order.

(i) The rational Calogero model in its original formulation[13] has been first solved for a definite ordering of the particles and then extended it to the whole of the configuration space in a continuous fashion by using the underlying permutation symmetry. New states have been found[15] for h by considering more general boundary conditions and including singular points/lines in the configuration space. Thus, with the use of these generalized boundary conditions, the fact that H is non-singular for $x_1 = x_i, i > 1$ and $x_2 = x_i, i > 2$ has no special significance. However, if singular points/lines are not included in the configuration space, careful analysis of the eigenvalue problem of H is required.

(ii) There are $\frac{N(N-1)}{2}$ numbers of angular-momentum operators in the N-dimensional hyper-spherical coordinate system. More general metric involving these angular momentum operators can be constructed with new non-Dirac-hermitian Calogero models admitting entirely real spectra and unitary time-evolution.

(iii) The construction can be trivially generalized to rational Calogero models corresponding to other root systems.

2.2 Pseudo-hermitian XXZ Spin-chain

It is a well known fact that non-hermitian quantum spin chains correspond to two-dimensional classical systems with positive Boltzmann weights. Examples of non-Dirac-hermitian spin chains are also abundant in the literature. The list includes XY and XXZ spin chain Hamiltonian with Dzyaloshinsky-Moriya interaction[16], the integrable chiral Potts model[17, 18], asymmetric XXZ spin chains[12] and quantum ising spin chain in one dimension[19]. Within the context of $\mathcal{PT}$-symmetric theory, non-hermitian spin chains have been studied in Refs. [4, 7, 8].

A non-Dirac-hermitian XXZ spin-chain in an external complex magnetic field may be introduced as follows:

$$\begin{aligned} H_A &= \sum_{i=1}^{N-1} [\Gamma \left(e^{w_i - w_{i+1}} \mathcal{S}_i^+ \mathcal{S}_{i+1}^- + e^{-(w_i - w_{i+1})} \mathcal{S}_i^- \mathcal{S}_{i+1}^+ \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z \\ &+ \left(A_i coshw_i - i B_i sinhw_i \right) \mathcal{S}_i^x + \left(B_i coshw_i + i A_i sinhw_i \right) \mathcal{S}_i^y \\ &+ C_i \mathcal{S}_i^z], \end{aligned} \tag{10}$$

where $\mathcal{S}_i^{\pm} := \mathcal{S}_i^x \pm i \mathcal{S}_i^y$, $\{\Gamma, \Delta, A_i, B_i, C_i, w_i\} \in R$ and $S_i^{x,y,z}$ are hermitian in $\mathcal{H}_D$. The non-hermitian interaction in H_A without the external magnetic field may be interpreted as arising due to imaginary vector potential. It may be noted that such imaginary gauge potentials are also relevant in the context of metal-insulator transitions or depinnning of flux lines from extended defects in type-II superconductors[20]. A Hamiltonian resembling the random-hopping model of Ref. [20] can be obtained from H_A by using a hard-core boson representation and mapping it to a non-hermitian quadratic form of bosonic operators with nearest-neighbour interactions[4]. The inclusion of the complex magnetic field is justified, since it may shed light on the nature of ordinary second order phase transitions as in the case of popular Yang-Lee model[21]. As discussed below, H_A reduces to the asymmetric XXZ model[12] that arises in the context of two species reaction-diffusion processes and Kardar-Parisi-Zhang-type growth phenomenon.

The parity and the time-reversal transformations involving spin-operators are defined as,

$$P : \vec{S}_i \rightarrow \vec{S}_i, \quad T : \vec{S}_i \rightarrow -\vec{S}_i, \quad i \rightarrow -i. \tag{11}$$

It may be noted that spin being an axial vector does not change sign under the parity operation. The Hamiltonian H_A is not invariant under the combined operation of PT. However, the Hamiltonian H_A with $A_i = 0 = B_i \ \forall \ i$ is invariant under an anti-linear $\mathcal{PT}$ transformation, where $\mathcal{T} : i \rightarrow -i$ and $\mathcal{P}$ is defined as a discrete symmetry in the spin-space with its actions on the spin operators as follows:

$$\begin{aligned}
S_i^x &\rightarrow \bar{S}_i^x = S_i^x cos\theta + S_i^y sin\theta \\
S_i^y &\rightarrow \bar{S}_i^y = S_i^x sin\theta - S_i^y cos\theta \\
S_i^z &\rightarrow \bar{S}_i^z = S_i^z, \quad 0 \leq \theta \leq 2\pi.
\end{aligned} \tag{12}$$

The discrete transformation $\mathcal{P}$ is similar to a non-standard parity transformation in three dimensions involving the position co-ordinates. The $\mathcal{PT}$ symmetry can be promoted to be the symmetry of H_A with non-vanishing A_i and B_i for a fixed θ, provided these parameters are related to each other through the relations:

$$\frac{B_i}{A_i} = tan\frac{\theta}{2}, \quad \forall \ i. \tag{13}$$

The Hamiltonian H_A may be invariant under a more general anti-linear transformation for arbitrary A_i and B_i which is not known at this point.

The Hamiltonian H_A is hermitian in $\mathcal{H}_{\eta_+}$ with the metric:

$$\eta_+ := \prod_{i=1}^{N} e^{-2\gamma_i S_i^z}. \tag{14}$$

A set of spin-operators $T_{x,y,z}$ may be introduced which are hermitian in $\mathcal{H}_{\eta_+}$:

$$\begin{aligned} T_i^x &:= coshw_i \ \mathcal{S}_i^x + isinhw_i \ \mathcal{S}_i^y \\ T_i^y &:= -isinhw_i \ \mathcal{S}_i^x + coshw_i \ \mathcal{S}_i^y \\ T_i^z &:= \mathcal{S}_i^z. \end{aligned} \tag{15}$$

Consequently, H_A can be re-written as,

$$H_A = \sum_{i=1}^{N-1} \left[\Gamma \left(T_i^+ T_{i+1}^- + T_i^- T_{i+1}^+ \right) + \Delta T_i^z T_{i+1}^z + A_i T_i^x + B_i T_i^y + C_i T_i^z \right], \tag{16}$$

showing its hermiticity in $\mathcal{H}_{\eta_+}$, where $T_i^{\pm} := T_i^x \pm i T_i^y$. The Hamiltonian H_A can be mapped to a Dirac-hermitian Hamiltonian,

$$\begin{aligned} h &:= (\eta_+^{\frac{1}{2}} H_A \eta_+^{-\frac{1}{2}}) \\ &= \sum_{i=1}^{N-1} \left[\Gamma \left(\mathcal{S}_i^x \mathcal{S}_{i+1}^x + \mathcal{S}_i^y \mathcal{S}_{i+1}^y \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z + A_i \mathcal{S}_i^x + B_i \mathcal{S}_i^y + C_i \mathcal{S}_i^z \right], \end{aligned} \tag{17}$$

implying that both H_A and h have entirely real spectra.

A few comments are in order at this point.

(i) A typical choice for w_k as $w_k = w - (k-1)\phi$ leads to a site-independent global phase factor $e^{\pm\phi}$ in lieu of $e^{\pm(w_i - w_{i+1})}$ and H_A reduces to asymmetric XXZ Hamiltonian that has been studied in the literature[12] in the context of two species reaction-diffusion processes and Kardar-Parisi-Zhang-type growth phenomenon. Although the transformation that maps non-hermitian asymmetric XXZ Hamiltonian to a hermitian Hamiltonian is known in the literature[12], the realization of pseudo-hermiticity is new. Thus, with the discovery of the pseudo-hermiticity of H_A, it may be used to describe unitary time evolution in $\mathcal{H}_{\eta_+}$.

(ii) With the choice of $w_i \equiv w \ \forall \ i$ in Eq. (10), a symmetric XXZ spin-chain Hamiltonian in an external complex magnetic field may be constructed,

$$\begin{aligned} H_S &= \sum_{i=1}^{N-1} [\Gamma \left(\mathcal{S}_i^x \mathcal{S}_{i+1}^x + \mathcal{S}_i^y \mathcal{S}_{i+1}^y \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z + \left(A_i coshw - iB_i sinhw \right) \mathcal{S}_i^x \\ &+ \left(B_i coshw + iA_i sinhw \right) \mathcal{S}_i^y + C_i \mathcal{S}_i^z], \end{aligned} \tag{18}$$

which is non-Dirac-hermitian, but, hermitian in $\mathcal{H}_{\eta_+}$. The equivalent Dirac-hermitian Hamiltonian $h := (\eta_+^{\frac{1}{2}} H_S \eta_+^{-\frac{1}{2}})$ is still given by Eq. (17).

The Hamiltonian h has several integrable limits. Consequently, H_A and H_S are also integrable in these limits with entirely real spectra and unitary time-evolution. For example, h reduces to a transverse-field Ising model for $\Gamma = B_i = C_i = 0, A_i = A \ \forall \ i$ and both h and H_S have been studied in some detail[7] for this limiting case. For $\Delta = 0, A_i = 0, B_i = 0 \ \forall \ i$, h reduces to an XX model in a transverse magnetic field and is exactly solvable[22, 23]. Although H_S is hermitian in $\mathcal{H}_D$ for this choice of the parameters, H_A is non-hermitian. Thus, the non-hermitian H_A is exactly solvable and has an equivalent description in terms of a hermitian XX model in an external magnetic field. For the following choice of the parameters,

$$\begin{aligned} &\Gamma = 1, \Delta = coshq, C_1 = -C_N = -sinhq, \\ &A_i = B_i = 0 \ \forall \ i; C_i = 0, i = 2, 3, \ldots, N-1, \end{aligned} \tag{19}$$

$h - \Delta$ reduces to an $SU_q(2)$ invariant[24] integrable[25] spin-chain Hamiltonian. The XXZ spin-chain with Sl_2 loop symmetry[26] may also be obtained as a limiting case. The corresponding non-hermitian Hamiltonian H_A is also integrable and allows an unitary description.

(iii) A more general $\mathcal{PT}$-symmetric Hamiltonian may be introduced which contains H_A as a special case. In particular,

$$\tilde{H}_A = \sum_{i=1}^{N-1} \left[\Gamma \left(\gamma_{i,i+1} \mathcal{S}_i^+ \mathcal{S}_{i+1}^- + \delta_{i,i+1} \mathcal{S}_i^- \mathcal{S}_{i+1}^+ \right) + \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z \right]$$

$$+ \sum_{i=1}^{N-1} \left[\left(\alpha_i^R + i\alpha_i^I \right) \mathcal{S}_i^x + \left(\beta_i^R + i\beta_i^I \right) \mathcal{S}_i^y + C_i \mathcal{S}_i^z \right], \tag{20}$$

is invariant under the $\mathcal{PT}$ transformation provided that, for a fixed θ, the real parameters $\alpha_i^R, \alpha_i^I, \beta_i^R, \beta_i^I$ satisfy the equations,

$$\frac{\beta_i^R}{\alpha_i^R} = -\frac{\alpha_i^R}{\beta_i^I} = tan(\frac{\theta}{2}) \ \ \forall \ i. \tag{21}$$

It may be noted that there are no restrictions on the real parameters $\gamma_{i,i+1}$ and $\delta_{i,i+1}$ in order to ensure the $\mathcal{PT}$ symmetry. Further, $\alpha_i^{R,I}$ and $\beta_i^{R,I}$ are independent of the parameters $\gamma_{i,i+1}$ and $\delta_{i,i+1}$ in $\tilde{H}_A$. The Hamiltonian H_A is reproduced with the choice of the parameters:

$$\begin{aligned}
&\gamma_{i,i+1} = e^{w_i - w_{i+1}}, \ \ \delta_{i,i+1} = e^{-(w_i - w_{i+1})} \\
&\alpha_i^R = A_i coshw_i, \ \alpha_i^I = -B_i sinhw_i, \\
&\beta_i^R = B_i coshw_i, \ \beta_i^I = A_i sinhw_i.
\end{aligned} \tag{22}$$

The region in the parameter-space of $\tilde{H}_A$ for which $\mathcal{PT}$ is unbroken must contain the region defined by the above equations. A concrete investigation to find the region of unbroken $\mathcal{PT}$ symmetry for $\tilde{H}_A$ is desirable.

(iv) The general prescription given in this article may be used to construct models of non-Dirac-hermitian spin-chain with long-range interaction. For example, the non-Dirac-hermitian spin-chain Hamiltonian,

$$H = \pm \sum_{i<j} \frac{\vec{T}_i \cdot \vec{T}_j}{2sin^2 \frac{\pi}{N}(i-j)}, \tag{23}$$

is isospectral with the celebrated Haldane-Shastry[27] model. The equivalent hermitian Hamiltonian in $\mathcal{H}_D$ may be obtained as,

$$h := \rho H \rho^{-1} = \pm \sum_{i<j} \frac{\vec{\mathcal{S}}_i \cdot \vec{\mathcal{S}}_j}{2sin^2 \frac{\pi}{N}(i-j)}. \tag{24}$$

It may be noted that, in general, eigenstates of h and H are different. However, with proper identification of physical observables in $\mathcal{H}_{\eta_+}$ through Eq. (2), different correlation functions of the quantum systems governed by H and h are identical.

3 Conclusions

A class of non-Dirac-hermitian many-particle quantum systems admitting entirely real spectra has been presented. The time-evolution of these systems is guaranteed to be unitary with the modified inner-product in the Hilbert space involving the pre-determined metric. These quantum systems are isospectral with known Dirac-hermitian quantum systems and are exactly solvable. In fact, several previously studied non-Dirac-hermitian quantum systems involving transverse ising-chain[7], asymmetric XXZ spin-chain[4, 12] belong to this class of pseudo-hermitian quantum system. New exactly solvable interesting pseudo-hermitian many-particle quantum systems involving rational Calogero model, XXZ spin-chain and Haldane-Shastry spin-chain have also been constructed. Many other physically relevant pseudo-hermitian quantum system involving Swanson model[5], Dicke model, Lipkin model, quadratic boson/fermion form etc. are described in Ref. [4].

The general approach that has been followed in the construction of these quantum systems is the following. The basic canonical commutation relations defining these systems have been realized in terms of non-Dirac-hermitian operators which are hermitian with respect to the modified inner product in the Hilbert space involving the pre-determined metric. Consequently, appropriate combinations of these operators result in a very large number of pseudo-hermitian quantum systems. The construction in this article is purely mathematical. Physical realizations of such models are desirable.

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