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A Roe-type Riemann solver based on the spectral decomposition of the equations of Relativistic Magnetohydrodynamics

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Abstract. In a recent paper (Antón et al. 2010) we have derived sets of right and left eigenvectors of the Jacobians of the relativistic MHD equations, which are regular and span a complete basis in any physical state including degenerate ones. We present a summary of the main steps followed in the above derivation and the numerical experiments carried out with the linearized (Roe-type) Riemann solver we have developed, and some note on the (non-)convex character of the relativistic MHD equations.

1. Introduction

Relativistic flows in association with intense gravitational and magnetic fields are commonly linked up to extremely energetic phenomena in the Universe, viz. pulsar winds, anomalous X-ray pulsars, soft gamma-ray repeaters, gamma-ray bursts, relativistic jets in active galactic nuclei, etc. The necessity to model the aforementioned astrophysical scenarios in the framework of relativistic MHD (RMHD), together with the fast increase in computing power, is pushing towards the development of more efficient numerical algorithms. In the last years, considerable progress has been achieved in numerical special RMHD (SRMHD), by extending the existing high-resolution shock-capturing (HRSC) methods of special relativistic hydrodynamics (e.g., Martí & Müller 2003). In the so called Godunov-type methods, an important subsample of HRSC methods, numerical fluxes are evaluated through the exact or approximate solution of the (local) Riemann problem. Despite the fact that such an exact solution in SRMHD is known (Romero et al. 2005; Giacomazzo & Rezzolla 2006), approximate algorithms are usually preferred because of their larger numerical efficiency. Several authors (see, e.g., Antón et al. 2010, and references therein) have developed independent *Roe-type* algorithms based on linearized Riemann solvers relying on the characteristic structure of the RMHD equations.

The purpose of the present paper is twofold. On one hand, the objective is to present a *regular* set of right and left eigenvectors of the flux vector Jacobian matrices of the RMHD equations, and span a complete basis in *any* physical state, including degenerate states. On the other hand, wish to evaluate numerically the performance of a RMHD Riemann solver based on the aforementioned spectral decomposition. Both the theoretical analysis and the numerical applications presented in this paper are based

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on the work developed by Antón et al. (2010), where we have characterized thoroughly all the degeneracies of RMHD in terms of the components of the magnetic field normal and tangential to the wavefront in the fluid rest frame. Our numerical method deviates in several aspects from previous works based on linearized Riemann solver approaches (Komissarov 1999; Balsara 2001; Koldoba et al. 2002). First, numerical fluxes are computed from the spectral decomposition in conserved variables. Second, we present explicit expressions also for the left eigenvectors. Third, and most important, we have extended classical MHD strategy (Brio & Wu 1988) to relativistic flows, giving sets of right and left eigenvectors which are well defined through degenerate states. Based on the full wave decomposition (FWD) provided by the renormalized set of eigenvectors in conserved variables, we have also developed a linearized (Roe-type) Riemann solver.

Extensive testing against one- and two-dimensional standard numerical problems allows us to conclude that our solver is very robust. When compared with a family of simpler solvers that do not require the knowledge of the full characteristic structure of the equations in the computation of the numerical fluxes, our solver turns out to be less diffusive than HLL and HLLC, and comparable in accuracy to the HLLD solver. The amount of operations needed by the FWD solver makes it less efficient computationally than those of the HLL family in one-dimensional problems. However its relative efficiency increases in multidimensional simulations.

2. The equations of ideal relativistic magnetohydrodynamics

The equations of ideal RMHD correspond to the conservation of rest-mass and energy-momentum, and the Maxwell equations. In the following, the standard Einstein sum convention is assumed. Greek indices will run from 0 to 3 (or from t to z) while Roman run from 1 to 3 (or from x to z). We use units in which the speed of light is $c = 1$ and the $(4\pi)^{1/2}$ factor is absorbed in the definition of the magnetic field. Specializing for a flat space-time and Cartesian coordinates, these equations can be written as a system of conservation laws, which reads

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x^i} = 0, \tag{1}$$

where the state vector, $\mathbf{U}$ (vector of *conserved variables*), and the fluxes, $\mathbf{F}^i$ ($i = 1, 2, 3$ or $i = x, y, z$), are the following column vectors,

$$\mathbf{U} = (D, S^j, \tau, B^k)^T \tag{2}$$

$$\mathbf{F}^i = (Dv^i, S^j v^i + p^* \delta^{ij} - b^j B^i/W, \tau v^i + p^* v^i - b^0 B^i/W, v^i B^k - v^k B^i)^T \tag{3}$$

where the superscript $^{\mathrm{T}}$ stands for the transposition.

In the preceding equations, D, S^j and τ stand, respectively, for the rest-mass density, the momentum density of the magnetized fluid in the j-direction and its total energy density as measured in the laboratory (i.e., Eulerian) frame,

$$D = \rho W, \qquad S^j = \rho h^* W^2 v^j - b^0 b^j, \qquad \tau = \rho h^* W^2 - p^* - (b^0)^2 - D. \tag{4}$$

where ρ is the proper rest-mass density, $h^* = 1 + \epsilon + p/\rho + b^2/\rho$ is the specific enthalpy including the contribution from the magnetic field (b^2 stands for $b^\mu b_\mu$), ϵ is the specific internal energy, p the thermal pressure, and $p^* = p + b^2/2$ the total pressure. The four-vectors representing the fluid velocity, u^μ, and the magnetic field measured in the fluid rest frame, b^μ, and there is an equation of state relating the thermodynamic variables, p, ρ and ϵ, $p = p(\rho, \epsilon)$. All the discussion will be valid for a general equation of state but results will be shown for an ideal gas, for which $p = (\gamma - 1)\rho\epsilon$, where γ is the adiabatic exponent. Quantities v^i stand for the components of the fluid velocity trivector as measured in the laboratory frame; they are related with the components of the fluid four-velocity according to the following expression $u^\mu = W(1, v^x, v^y, v^z)$, where W is the flow Lorentz factor, $W^2 = 1/(1 - v^i v_i)$.

The following fundamental relations hold between the components of the magnetic field four-vector in the comoving frame and the three vector components B^i measured in the laboratory frame,

$$b^0 = W\,\mathbf{B}\cdot\mathbf{v} \quad , \quad b^i = \frac{B^i}{W} + b^0 v^i \tag{5}$$

$\mathbf{v}$ and $\mathbf{B}$ being, respectively, the tri-vectors (v^x, v^y, v^z) and (B^x, B^y, B^z).

$$b^2 = \frac{\mathbf{B}^2}{W^2} + (\mathbf{B}\cdot\mathbf{v})^2 \; . \tag{6}$$

The preceding system must be complemented with the usual divergence constraint

$$\frac{\partial B^i}{\partial x^i} = 0 \; , \tag{7}$$

which should be fulfilled at all times.

Fluxes $\mathbf{F}^i$ $(i = x, y, z)$ are functions of the conserved variables, $\mathbf{U}$, although for the RMHD this dependence, in general, can not be expressed explicitly. It is therefore necessary to introduce another set of variables, the so-called *primitive variables*, derived from the conserved ones, in terms of which the fluxes can be computed explicitly. We have used the following set of primitive variables

$$\mathbf{V} = (\rho, p, v^x, v^y, v^z, B^x, B^y, B^z)^T . \tag{8}$$

3. Characteristic structure of the RMHD equations

The hyperbolicity of the equations of RMHD including the derivation of wavespeeds and the corresponding eigenvectors, and the analysis of various degeneracies has been reviewed by Anile (1989), in a covariant framework, using a set of variables of dimension 10, the so-called *covariant variables* (Anile's variables, in the next):

$$\tilde{\mathbf{U}} = (u^\mu, b^\mu, p, s)^T , \tag{9}$$

where s is the specific entropy.

In terms of variables $\tilde{\mathbf{U}}$, the system of RMHD equations can be written as a quasi-linear system of the form

$$\mathcal{A}^\mu \tilde{\mathbf{U}}_{;\mu} = 0, \tag{10}$$

where the subscript $;\mu$ stands for the covariant derivative, and four 10×10 Jacobian matrices $\mathcal{A}^\mu$ can be found in Anile's book. It is important to remark that the 10 covariant variables we have used to write the system of equations are not independent, since they are related by the constraints

$$u^\alpha u_\alpha = -1 \quad , \quad b^\alpha u_\alpha = 0 \quad , \quad \partial_\alpha(u^\alpha b^0 - u^0 b^\alpha) = 0 \, , \tag{11}$$

The latter condition, is a covariant representation of the divergence constraint (Eq. 7).

3.1. Wavespeeds and degeneracies

The system of (ideal) RMHD equations have the same seven wavespeeds as in classical MHD: the entropic, Alfvén, slow magnetosonic, and fast magnetosonic waves. They can be ordered as follows

$$\lambda_f^- \leq \lambda_a^- \leq \lambda_s^- \leq \lambda_e \leq \lambda_s^+ \leq \lambda_a^+ \leq \lambda_f^+, \tag{12}$$

where the subscripts e, a, s and f stand for *entropic*, *Alfvén*, *slow magnetosonic* and *fast magnetosonic* respectively, and the superscript $-$ or $+$ refer to the lower or higher value of each pair. Unlike classical MHD, it is however not possible, in general, to obtain simple expressions for the magnetosonic speeds since they are given by the solutions of a quartic equation.

As in the case of classical MHD, degeneracies are encountered for waves propagating perpendicular to the magnetic field direction (Type I) and for waves propagating along the magnetic field direction (Type II). Finally, a particular subcase of Type II degeneracy appears when the sound speed is equal to c_a.

For Type I degeneracy, the two Alfvén waves, the entropic wave and the two slow magnetosonic waves propagate at the same speed ($\lambda_a^- = \lambda_s^- = \lambda_e = \lambda_s^+ = \lambda_a^+$). For Type II degeneracy, an Alfvén wave and a magnetosonic wave (slow or fast) of the same class propagate at the same speed ($\lambda_f^- = \lambda_a^-$ or $\lambda_a^- = \lambda_s^-$ or $\lambda_s^+ = \lambda_a^+$ or $\lambda_a^+ = \lambda_f^+$). In the special Type II′ subcase, an Alfvén wave and both the slow and fast magnetosonic waves of the same class propagate at the same speed ($\lambda_f^- = \lambda_a^- = \lambda_s^-$ or $\lambda_s^+ = \lambda_a^+ = \lambda_f^+$).

3.2. Renormalized right eigenvectors

As it is well known in classical MHD, Alfvén and magnetosonic eigenvectors have a pathological behaviour at degeneracies, since they become zero or linearly dependent and they do not form a basis. In Antón et al. (2010), we have derived a new set of renormalized Alfvén and magnetosonic eigenvectors for RMHD. Our renormalized Alfvén right eigenvectors (following Brio & Wu 1988 methodology) are a linear combination of the ones proposed by Komissarov (1999), for the Type II degeneracy case. However, contrary to the Komissarov's choice, our expressions are free of pathologies not only in the Type II degeneracy but also in the Type I degeneracy case. Our derivation of the renormalized magnetosonic right eigenvectors is algebraically more cumbersome and reader interested is addressed to Antón et al. (2010). The final result of this analysis allows to have a complete set of right eigenvectors linearly independent for all possible states. Following the same procedure we have used to renormalize the right eigenvectors, we have derived (see Antón et al. 2010) left eigenvectors well behaved for degenerate states.

4. A Full Wave Decomposition Riemann Solver in RMHD (FWD)

Let us summarize the main steps allowing us to derive a FWD Riemann Solver in RMHD:

- Let $r_{\widetilde{\mathbf{U}}}$ be a generic right eigenvector derived in terms of Anile's variables (9).
- The corresponding eigenvector in terms of the primitive variables (8) is derived according to: $r_{\mathbf{V}} = \left(\partial_{\widetilde{\mathbf{U}}}\mathbf{V}\right) r_{\widetilde{\mathbf{U}}}$.
- Finally, the corresponding vector in terms of the conserved variables (2) is obtained from $\mathbf{R} \equiv r_{\mathbf{U}} = (\partial_{\mathbf{V}}\mathbf{U})\, r_{\mathbf{V}}$.
- Analogously, for the left eigenvectors. This procedure, which starts with renormalized eigenvectors, allows one to get the full spectral decomposition of the Jacobian matrices (associated to the fluxes), and free of pathologies in the degeneracies.
- We use a linearized (Roe's type) Riemann solver:
  $\widehat{\mathbf{f}}_{j\pm\frac{1}{2}} = \frac{1}{2}\left(\mathbf{f}(\mathbf{u}^L_{j\pm\frac{1}{2}}) + \mathbf{f}(\mathbf{u}^R_{j\pm\frac{1}{2}}) - \sum_{\alpha=1}^{p} \mid \widetilde{\lambda}_\alpha \mid \Delta\widetilde{\omega}_\alpha \widetilde{r}_\alpha\right)$
  where $\mathbf{u}^L$, $\mathbf{u}^R$, are the left and right reconstructed variables; $\Delta\omega$, is the jump of characteristic variables.

Our FWD linearized Riemann solver has been exhaustively tested (Antón et al. 2010).

5. A note on RMHD convexity

For the sake of conciseness let us remind some definitions. A characteristic field C_α $(\alpha = 1, 2, ..., d)$ (d is the number of equations) satisfying

$$C_\alpha : \quad \frac{dx}{dt} = \lambda_\alpha \quad (\alpha = 1, 2, ..., d) \tag{13}$$

is said to be *genuinely nonlinear* or *linearly degenerate* if, respectively,

$$\vec{\nabla}_{\mathbf{u}}\, \lambda_\alpha \cdot \mathbf{r}_\alpha \neq 0\,, \tag{14}$$

$$\vec{\nabla}_{\mathbf{u}}\, \lambda_\alpha \cdot \mathbf{r}_\alpha = 0 \tag{15}$$

where the operator $\vec{\nabla}_{\mathbf{u}}$ acts on the space of conserved variables.

In a convex system, all the characteristic fields are genuinely non-linear or linearly degenerate. Non-convexity is associated to those states for which the condition (14) is not fulfiled.

For the system of equations governing relativistic (ideal) flows it can be shown that the convexity is strongly dependent on the second derivatives of pressure (or the first derivatives of the sound speed). In the following, we analize when genuinely nonlinear fields become linearly degenerate, by examining the products $\mathcal{P}_\pm := \vec{\nabla}_{\mathbf{w}}\, \lambda_\pm(\mathbf{w}) \cdot \mathbf{r}_\pm(\mathbf{w})$. After some algebra, we find

$$\mathcal{P}_\pm = \pm\, T(a, b, c_s) \left(\frac{\partial c_s}{\partial \rho} + \frac{p}{\rho^2}\,\frac{\partial c_s}{\partial \epsilon} + \frac{c_s}{\rho}\,(1 - c_s^2)\right) \tag{16}$$

being

$$T(a,b,c_s) \;=\; (a\; c_s \;\pm\; \delta^{1/2})^{-2}\; (1-a^2)^2\; \delta^{-1/2}\; W^2 \tag{17}$$

where a and b stand for, respectively, the spatial components of the velocity field in the x-direction and the tangential one. The quantity δ is defined by $\delta = W^2(1 - a^2 - b^2 c_s^2)$.

¿From the above relations (16,17) it turns out that the loss of convexity is closely related with the properties of the equation of state (second term in Eq. 16). The first and second thermodinamical derivatives of pressure play a fundamental role regarding with this issue (that was noticed by Menikoff & Plohr (1989), for equations of state having phase transitions). Furthermore, we realize that

i) In the purely one-dimensional case ($b = 0$), non-convexity only appears in the ultra-relativistic regime ($a \rightarrow 1$):

$$T(a,0,c_s) = (1 \pm a\, c_s)^{-2}\, W^{-2} \;\;;\;\; a \rightarrow 1 \Longrightarrow T(a,0,c_s) \longrightarrow 0\,,\, O(W^{-2}) \tag{18}$$

ii) Likewise, if $a = 0$, non-convexity arises in the ultrarelativistic regime ($b \rightarrow 1$):

$$T(0,b,c_s) = \Delta^{-3/2}\; W^2 \;;\;\; b \rightarrow 1 \Longrightarrow T(0,b,c_s) \longrightarrow 0\,,\, O(W^{-1}) \tag{19}$$

Brio & Wu (1988) noted that the equations of classical MHD are non-convex at the degenerate states (magnetosonic waves change from genuinely non-linear to linearly degenerate) . We have faced on the problem of non-convex character of RMHD, and preliminar results allows one to conclude that the degenerate states are, as in the classical MHD, non-convex, being the magnetosonic fields the ones changing their character. We refer the reader to Antón (2008) (appendix G), where an analysis of the characteristic fields of RMHD in terms of Anile's covariant variables is presented. Much more theoretical work is necessary in order to asses all the richness of other possible non-convex states in RMHD. The previous analysis in special relativistic hydrodynamics serves us as a road-map to the full characterization of RMHD pathological behaviours (we remind the reader that the non-convex character of both the classical and relativistic MHD equations is source of several pathologies, as the development of the so-called compound waves).

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