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Effective Action and Phase Transitions in Thermal Yang-Mills Theory on Spheres

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Abstract: We study the covariantly constant Savvidy-type chromomagnetic vacuum in finite-temperature Yang-Mills theory on the four-dimensional curved spacetime. Motivated by the fact that a positive spatial curvature acts as an effective gluon mass we consider the compact Euclidean spacetime $S^1 \times S^1 \times S^2$, with the radius of the first circle determined by the temperature $a_1 = (2\pi T)^{-1}$. We show that covariantly constant Yang-Mills fields on S^2 cannot be arbitrary but are rather a collection of monopole-antimonopole pairs. We compute the heat kernels of all relevant operators exactly and show that the gluon operator on such a background has negative modes for any compact semi-simple gauge group. We compute the infrared regularized effective action and apply the result for the computation of the entropy and the heat capacity of the quark-gluon gas. We compute the heat capacity for the gauge group $SU(2N)$ for a field configuration of N monopole-antimonopole pairs. We show that in the high-temperature limit the heat capacity is well defined in the infrared limit and exhibits a typical behavior of second-order phase transition $\sim (T - T_c)^{-3/2}$ with the critical temperature $T_c = (2\pi a)^{-1}$, where a is the radius of the 2-sphere S^2.

Keywords: Confinement, Nonperturbative Effects, QCD

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1 Introduction

Despite the tremendous success of quantum chromodynamics (QCD) in describing the phenomenology of strong interactions of elementary particles at high energies, a deep understanding of the physics at low energies is still lacking. At high energies, non-Abelian gauge theory is asymptotically free, and as a result, perturbation theory is an adequate tool. However, at low energies, the interaction becomes strong, and perturbation theory fails. It has been suggested that this failure is directly linked to the phenomenon of confinement in QCD, which is a well-known experimental fact. However, the precise nature of a non-pertubative mechanism ensuring confinement is still not well understood. In field-theoretic terms, this means that the vacuum of QCD at low energies has a far more complicated structure than the trivial perturbative one.

A model of a non-perturbative vacuum for an $SU(2)$ gauge theory was put forward in 1977 by Savvidy [16]. He proposed an explicit ansatz for the vacuum gauge fields in form of a constant chromomagnetic field, or more precisely, a gauge field with covariantly constant field strength in flat four-dimensional Minkowski space-time with only one nonvanishing color component. Savvidy showed that, due to quantum fluctuations of the gauge fields, the energy of such a field configuration is below the perturbative vacuum level, which leads to infrared instability of the perturbative vacuum under creation of a constant chromomagnetic field. Further investigations [13, 14] showed that the Savvidy vacuum itself is unstable too, meaning that the physical nonperturbative vacuum has an even more complicated structure. It has been suggested that the real vacuum is likely to have a small domain structure with random constant chromomagnetic fields (spaghetti vacuum).

In our papers [3, 4] we extended Savvidy's investigation by considering more complicated gauge groups and flat spacetimes of dimension higher than four. We showed that for an arbitrary compact simple gauge group in dimensions higher than four there exist more general nontrivial field configurations with several color and space-time components that turn out to be stable. In [4] we proposed an explicit example of such background field configurations.

In the present paper we propose a new mechanism to stabilize the Savvidy vacuum in four dimensions. The main idea of this approach is that a *positive space curvature could provide an effective mass term* for the gauge fields on the chromomagnetic vacuum, thus, making the vacuum stable. To simplify the calculations, we consider the space-times with compact space slices with the product structure $S^1 \times S^2$. To also study the finite-temperature effects we consider Euclidean spacetimes of the form $S^1 \times S^1 \times S^2$.

However, as we show below, topological considerations on the sphere constrain the magnetic field to be of the same order of magnitude as the space curvature, thus negating the stabilization effect of the curvature term. Moreover, even under these constraints, an interesting second-order phase transition occurs at a critical temperature near the inverse radius of the sphere.

2 Yang-Mills Theory

Let (M, g) be an n-dimensional pseudo-Riemannian orientable spin manifold without boundary with a globally hyperbolic metric g. We denote the local coordinates on M by x^μ, with Greek indices running over $0, 1, \dots, n-1$. We denote the frame indices by the low case Latin indices from the beginning of the alphabet, which also run over $0, 1, \dots, n-1$. The frame indices should not be confused with the group indices introduced below that are enclosed in parenthesis. We use Einstein summation convention and sum over repeated indices. The coordinate indices are raised and lowered by the metric tensor $g_{\mu\nu}$ and the frame indices are raised and lowered by the Minkowski metric, η_{ab}.

We choose a local Lorentz frame on the tangent bundle TM, $e_a = e_a{}^\mu \partial_\mu$, and the dual frame on the cotangent bundle T^*M, $\sigma^a = \sigma^a{}_\mu\, dx^\mu$. We denote the corresponding spin connection 1-form by $\omega^a{}_b = \omega^a{}_{b\mu}\, dx^\mu$ and its curvature 2-form by $\Theta^a{}_b = \frac{1}{2}\Theta^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu$ so that $R_{\alpha\nu} = e_a{}^\mu e^b{}_\alpha \Theta^a{}_{b\mu\nu}$ is the Ricci tensor, and $R = g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature. Let $\mathcal{T}$ be a spin-tensor bundle realizing a representation T of the spin group, $\mathrm{Spin}(1, n-1)$, with generators Σ_{ab}. The spin connection induces a connection ∇^T on the bundle $\mathcal{T}$ with the curvature $\mathcal{R}_{\mu\nu} = \frac{1}{2}\Theta^{ab}{}_{\mu\nu} T(\Sigma_{ab})$. Recall that the generators of the spin group in the vector representation and the spinor representation are

$$T_1(\Sigma_{ab})^c{}_d = 2\delta^c{}_{[a}\eta_{b]d}\,, \qquad T_{\mathrm{spin}}(\Sigma_{ab}) = \frac{1}{2}\gamma_{ab}\,, \tag{2.1}$$

where η_{bd} is the Minkowski metric and $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$. Here and everywhere below the square brackets denote the antisymmetrization over indices included.

Let G be a m-dimensional compact simple Lie group and $\mathfrak{g}$ be its Lie algebra. We use lower case gothic letters to denote Lie algebras, for example, $\mathfrak{spin}(1, n-1)$. We denote the group indices, which run over $1, 2, \dots, m$,, by the low case Latin letters from the middle of the alphabet. Let $C^i{}_{jk}$ be the structure constants of G in a given basis. The real $m \times m$ matrices C_i defined by $(C_i)^j{}_k = C^j{}_{ik}$ form a basis in the Lie algebra $\mathfrak{g}$ and define the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathbb{R}^m)$ of the algebra $\mathfrak{g}$ by endomorphisms of the vector space $\mathbb{R}^m$. Of course, this also defines the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathbb{R}^m)$ of the group G into the automorphism group of the vector space $\mathbb{R}^m$. In the following we will identify the algebra $\mathfrak{g}$ with its adjoint representation. The Cartan-Killing metric γ_{ij} on the Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{tr}\, C_i C_j = C^k{}_{il} C^l{}_{jk} = -2\gamma_{ij}\,, \tag{2.2}$$

the precise form depends, of course, on the choice of the basis. We will determine it later. For compact semi-simple groups it is positive definite. We will use it to raise and lower the group indices.

We consider the principal fiber bundle over the manifold M with the structure group G and the typical fiber G. Let $\rho_W : G \to \mathrm{Aut}(W)$ be an irreducible representation of the group G into the automorphism group of an N-dimensional (real or complex) vector space W. Sometimes, we will denote the representation ρ_W by the vector space W; this should not cause any confusion. For example, we will denote the generators of this representation by $W(C_i)$. Let $\mathcal{W}$ be the associated vector bundle with the structure group G and the typical fiber W. Then for any spin-tensor bundle $\mathcal{T}$ (realizing a representation T of the spin group, $\mathrm{Spin}(1, n-1)$) the vector bundle $\mathcal{V} = \mathcal{W} \otimes \mathcal{T}$ is a twisted spin-tensor bundle realizing the representation $V = W \otimes T$. The sections of the bundle $\mathcal{V}$ are represented locally by (real or complex) N-tuples of spin-tensors.

Let $\mathcal{A}_\mu = A^i{}_\mu C_i$ and $\mathcal{A} = \mathcal{A}_\mu dx^\mu$ be the Yang-Mills connection 1-form taking values in the Lie algebra $\mathfrak{g}$ and $\mathcal{F}_{\mu\nu} = \mathcal{F}^i{}_{\mu\nu} C_i$ and $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}\, dx^\mu \wedge dx^\nu = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ be its curvature 2-form. It is worth remembering that in a non-trivial bundle, there are several overlapping coordinate patches; the connection one-form is not globally well-defined, in general, but is rather described by a collection of its representations in each patch, which are related in overlapping patches by gauge transformations.

Let ∇^V be the total connection on the twisted spin-tensor bundle $\mathcal{V}$. The curvature of the total connection on the twisted spin-tensor bundle is equal to $\mathbb{I}_W \otimes T(\mathcal{R}_{\mu\nu}) + W(\mathcal{F}_{\mu\nu}) \otimes \mathbb{I}_T$. We will usually omit the identity matrices for the sake of simplicity of notation. The covariant Laplacian $\Delta_{T\otimes W} = g^{\mu\nu}\nabla^V_\mu\nabla^V_\nu$ acting on sections of the twisted spin-tensor bundle $\mathcal{V}$ has the form

$$\Delta_{T\otimes W} = g^{-1/2}\left[\partial_\mu + \frac{1}{2}\omega^{ab}{}_\mu T(\Sigma_{ab}) + W(\mathcal{A}_\mu)\right] g^{1/2} g^{\mu\nu}\left[\partial_\nu + \frac{1}{2}\omega^{cd}{}_\nu T(\Sigma_{cd}) + W(\mathcal{A}_\nu)\right]. \tag{2.3}$$

We will use twisted Lie derivatives defined as follows. Suppose that there is a faithful representation $\rho_X : \mathrm{Spin}(1, n-1) \to \mathrm{Aut}(W)$ of the spin group in the same vector space W with generators $X(\Sigma_{ab})$. Then the matrices

$$G_{ab} = \mathbb{I}_W \otimes T(\Sigma_{ab}) - X(\Sigma_{ab}) \otimes \mathbb{I}_T \tag{2.4}$$

are the generators of the twisted representation $\rho_{W\otimes T} : \mathrm{Spin}(1, n-1) \to \mathrm{Aut}(V)$ of the spin group. Let ξ be a Killing vector field. The twisted Lie derivative of sections of the vector bundle $\mathcal{V}$ along ξ is defined by

$$\begin{aligned} \mathcal{L}_\xi &= \xi^\mu \nabla^V_\mu - \frac{1}{2}\xi_{[a;b]}G^{ab} \\ &= \xi^\mu\partial_\mu + \frac{1}{2}\left[\xi^\mu\omega_{ab\mu} - \xi_{[a;b]}\right] T(\Sigma^{ab}) + \xi^\mu A^i{}_\mu W(C_i) + \frac{1}{2}\xi_{[a;b]}X(\Sigma^{ab}). \end{aligned} \tag{2.5}$$

The action of the Yang-Mills theory in curved spacetime is constructed as follows. We consider two associated vector bundles $\mathcal{W}_{\rm spin}$ and $\mathcal{W}_0$ of dimensions $N_{\rm spin}$ and N_0 respectively realizing some irreducible representations, $W_{\rm spin}$ and W_0, of the gauge group. Usually, the representation $W_{\rm spin}$ realized by the spinor fields is taken to be the fundamental (or defining) representation of the gauge group. The scalar fields are just sections of the bundle $\mathcal{W}_0$, whereas the spinor fields are sections of the spinor bundle twisted by $\mathcal{W}_{\rm spin}$. Then the classical action of the model is the functional

$$S = -\int\limits_M dx\, g^{1/2}\left\{\frac{1}{2e^2}|\mathcal{F}|^2 + \left\langle\psi, [\gamma^\mu\nabla_\mu + M]\psi\right\rangle_{W_{\rm spin}} + \frac{1}{2}\left\langle\nabla^\mu\varphi, \nabla_\mu\varphi\right\rangle_{W_0} + V(\varphi)\right\}, \tag{2.6}$$

where $g = \det g_{\mu\nu}$, $|\mathcal{F}|^2 = -\frac{1}{4}\mathrm{tr}_{Ad}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, e is the Yang-Mills coupling constant, $V(\varphi)$ is a potential for scalar fields (such that $V(0) = V'(0) = 0$) and M is the spinor mass matrix.

3 Effective Action

In this paper we will not quantize gravity, assuming the gravitational field to be classical and simply ignoring all quantum-gravitational effects. The energy scale of our primary interest will be well below the Planckian scale, so that this assumption is reasonable in any theory of quantum gravity.

The effective action is expressed in terms of functional determinants of differential operators (see, for example, [9, 10]). Such determinants can be defined, strictly speaking, only for elliptic operators on compact manifolds by making use of a regularization procedure, for example, zeta-function regularization. For hyperbolic operators and for non-compact manifolds such determinants do not have a direct mathematical meaning.

That is why we will make the further assumption that the background is static, that is, there is a global time-like Killing vector field ∂_t. Moreover, we will assume that the spacetime has the simple structure of a product manifold $M = \mathbb{R} \times \Sigma$ and that all background fields are static and do not have time-like components.

In the case of static background one can make an analytic continuation to a purely imaginary time $t \to i\tau$, with a positive-definite Riemannian metric. Moreover, we can go even further and compactify the Euclidean time by replacing $\mathbb{R}$ by a circle S^1 of radius a_1, that is, by restricting the range, $0 \le \tau \le \beta$, where $\beta = 2\pi a_1$ is the circumference of the circle S^1 and requiring all fields to be periodic in the Euclidean time τ with period β. The "Euclidean" space-time $M = S^1 \times \Sigma$ is then a compact manifold and Lorentzian spin group $\mathrm{Spin}(1, n-1)$ becomes the Euclidean spin group $\mathrm{Spin}(n)$, which is compact. This corresponds to a statistical ensemble at a finite temperature $T = 1/\beta$. In the limit of infinite radius $\beta \to \infty$ we recover the zero-temperature theory.

We will consider a background in which there are no matter fields and Yang-Mills fields and gravitational field are covariantly constant (parallel), that is,

$$\nabla_\mu R_{\rho\sigma\alpha\beta} = 0\,, \qquad \nabla_\mu \mathcal{F}_{\alpha\beta} = 0\,. \tag{3.1}$$

More precisely, we will study the case when $\Sigma = S^1 \times S^2$, that is, $M = S^1 \times S^1 \times S^2$.

There exists a minimal gauge such that all differential operators involved are second-order differential operators of Laplace type

$$L = -\Delta + Q \tag{3.2}$$

with some endomorphism Q. The precise nature of the Laplacian $\Delta = g^{\mu\nu}\nabla_\mu\nabla_\nu$ depends, of course, on the fields it is acting upon.

Let L^{YM}_{ghost} be the Laplacian acting on scalar fields in adjoint representation of the gauge group defined by

$$L^{YM}_{\text{ghost}} = -\Delta_{T_0 \otimes Ad}\,. \tag{3.3}$$

Let Q_{vect} be the endomorphism acting on vectors in adjoint representation of the gauge group defined by

$$(Q_{\text{vect}}\varphi)^a = (R^a{}_b \mathbb{I}_{Ad} - 2\mathcal{F}^a{}_b)\,\varphi^b\,. \tag{3.4}$$

Let L_{vect} be an operator acting on vector fields in adjoint representation of the gauge group defined by

$$L_{\text{vect}} = -\Delta_{T_1 \otimes Ad} + Q_{\text{vect}}\,. \tag{3.5}$$

We will suppose that the spinor mass matrix M commutes with the Dirac operator $\gamma^\mu\nabla_\mu$. Let Q_{spin} be the endomorphism acting on spinor fields in the representation W_{spin} of the gauge group defined by

$$Q_{\text{spin}} = \frac{1}{4} R \mathbb{I}_{T_{\text{spin}}} \otimes \mathbb{I}_{W_{\text{spin}}} - \frac{1}{2}\gamma^{ab} W_{\text{spin}}(\mathcal{F}_{ab}) + M^2\,. \tag{3.6}$$

Let L_{spin} be the differential operator acting on spinor fields in the representation W_{spin} of the gauge group defined by

$$L_{\text{spin}} = -\Delta_{T_{\text{spin}} \otimes W_{\text{spin}}} + Q_{\text{spin}}\,. \tag{3.7}$$

Let L_0 be a differential operator acting on scalar fields in some representation W_0 of the gauge group defined by

$$L_0 = -\Delta_{T_0\otimes W_0} + Q_0\,, \tag{3.8}$$

where Q_0 is a matrix defined by

$$\frac{d^2}{d\varepsilon^2}V(\varepsilon h)\Big|_{\varepsilon=0} = \langle h, Q_0 h\rangle_{W_0}\;. \tag{3.9}$$

The most important observation that should be made at this point is that the positive curvature acts as a mass (or positive potential) term in both the Yang-Mills operator and the spinor operator. While the magnetic field reduces the eigenvalues of the Yang-Mills operator the positive Ricci tensor increases them. Roughly speaking, *it is the balance of these two terms that determines whether or not the Yang-Mills operator is positive* (so that the vacuum is stable).

In the minimal gauge the one-loop effective action is given by [3, 4]

$$\Gamma = S + \hbar\,(\Gamma_{(1)YM} + \Gamma_{(1)\mathrm{mat}}) + O(\hbar^2)\,, \tag{3.10}$$

where

$$\Gamma_{(1)YM} = \frac{1}{2}\,\log\mathrm{Det}\,L_{\mathrm{vect}} - \log\mathrm{Det}\,L^{YM}_{\mathrm{ghost}}\,, \tag{3.11}$$

and

$$\Gamma_{(1)\mathrm{mat}} = \frac{1}{2}\,\log\mathrm{Det}\,L_0 - \frac{1}{2}\,\log\mathrm{Det}\,L_{\mathrm{spin}}\,, \tag{3.12}$$

are the contributions of the Yang-Mills field, and matter fields respectively, and Det is the functional determinant. The Planck constant is introduced here just for illustrative purposes. Henceforth, we set $\hbar = 1$.

The operators introduced above are second-order elliptic partial differential operators on compact manifold. The spectrum of these operators depends, of course, on the background fields. Elliptic operators on compact manifolds can only have a finite number of negative eigenvalues. The negative eigenvalues indicate instability of the vacuum at low energies.

That is why, to study the infrared behavior of the system, one has to introduce an infrared regularization and take it off at the very end as in [4]. For example, one could introduce a sufficiently large mass parameter z so that all operators are positive, which is equivalent to replacing the operators L by $L + z$, and study the dependence of the infrared regularized effective action on z. If there are no infrared divergences then there is a well defined limit $z \to 0$. In the case of non–trivial low-energy behavior there appears an imaginary part of the effective action or some infrared logarithmic singularities. One should stress that although the ultraviolet regularization is a rather formal method, the infrared regularization parameter can take, in principle, a *direct physical meaning*, something like Λ_{QCD}.

We will assume that this has been done, that is, we will add a mass parameter to the Yang-Mills and the ghost operators so that all operators are positive. The determinants of *positive* elliptic operators can be regularized by the zeta-function regularization method which can be summarized by [1, 5, 9]

$$\log\mathrm{Det}\,(L + z) = -\zeta'(0)\,, \tag{3.13}$$

where $\zeta'(0) = \frac{\partial}{\partial s}\zeta(s)\Big|_{s=0}$ and

$$\zeta(s) = \sum_{k=1}^{\infty} d_k \left(\frac{\lambda_k + z}{\mu^2}\right)^{-s} \tag{3.14}$$

where $\lambda_k = \lambda_k(L)$ are the eigenvalues of the operator L and d_k are their multiplicities. The analytic continuation of the zeta function gives a meromorphic function of s, which is analytic at $s = 0$; therefore, the determinant (3.13) is a well defined invariant.

We can also express the zeta-function in terms of the heat trace of the operator L defined by

$$\operatorname{Tr}\exp(-tL) = \sum_{k=1}^{\infty} d_k e^{-t\lambda_k} = \int_M dx\, g^{1/2} \operatorname{tr} U_L^{\rm diag}(t)\,, \tag{3.15}$$

where $U_L(t; x, x') = \exp(-tL)\delta(x, x')$ is the heat kernel of the operator L and $U_L^{\rm diag}(t; x) = U_L(t; x, x)$ is the heat kernel diagonal. Then the zeta-function is related to the heat trace by the Mellin transform

$$\zeta(s) = \frac{\mu^{2s}}{\Gamma(s)} \int\limits_0^{\infty} dt\, t^{s-1} e^{-tz} \operatorname{Tr}\exp(-tL)\,, \tag{3.16}$$

where μ is a renormalization parameter introduced to preserve dimensions.

We introduce a useful function $\Theta_L(t)$ as follows

$$\Theta_L(t) = (4\pi t)^{n/2} \operatorname{Tr}\exp(-tL)\,. \tag{3.17}$$

It is well-known that as $t \to 0$

$$\Theta_L(t) \sim \sum_{k=0}^{\infty} B_k t^k\,, \tag{3.18}$$

where $B_k = B_k(L)$ are some spectral invariants of the operator L. The first three coefficients have the form [1, 5, 10]

$$B_0 = \int_M dx g^{1/2} \operatorname{tr} \mathbb{I}\,, \tag{3.19}$$

$$B_1 = \int_M dx g^{1/2} \operatorname{tr}\left(\frac{1}{6} R\mathbb{I} - Q\right), \tag{3.20}$$

$$\begin{aligned} B_2 = \int_M dx g^{1/2} \operatorname{tr} \Bigg\{ & \frac{1}{2}\left(\frac{1}{6}R\mathbb{I} - Q\right)^2 + \frac{1}{180} R_{abcd}R^{abcd}\mathbb{I} - \frac{1}{180} R_{ab}R^{ab}\mathbb{I} \\ & + \frac{1}{12}\left[\mathcal{R}_{ab} + \mathcal{F}_{ab}\right]\left[\mathcal{R}^{ab} + \mathcal{F}^{ab}\right]\Bigg\}\,. \end{aligned} \tag{3.21}$$

It will be convenient to represent the heat trace as $t \to 0$ as follows

$$\Theta_L(t) \sim e^{-t\lambda} \sum_{k=0}^{\infty} A_k(\lambda) t^k\,, \tag{3.22}$$

where λ is a new arbitrary parameter (that should not be confused with z) and

$$A_k(\lambda) = \sum_{j=0}^{k} \frac{1}{j!} \lambda^j B_{k-j} \,. \tag{3.23}$$

The analytic continuation of the zeta function to $s = 0$ can be obtained by integration by parts. Then the zeta-regularized determinant can be expressed directly in terms of an integral of the heat trace. For even n we obtain

$$\log \mathrm{Det}_{\mu}(L + z) = \frac{(4\pi)^{-n/2}}{\Gamma\left(1 + \frac{n}{2}\right)} \int_0^\infty dt \,\left[\log(\mu^2 t) + \Psi\left(1 + \frac{n}{2}\right)\right] \left(\frac{\partial}{\partial t}\right)^{1+\frac{n}{2}} \left[e^{-tz}\Theta_L(t)\right] \,, \tag{3.24}$$

where $\Psi(s) = \Gamma'(s)/\Gamma(s)$ is the logarithmic derivative of the Gamma function.

Let us consider the case of four dimensions, $n = 4$, in more detail. Then

$$\log \mathrm{Det}_{\mu}(L + z) = \frac{1}{2}(4\pi)^{-2} \int_0^\infty dt \left[\log(\mu^2 t) + \frac{3}{2} - \mathbb{C}\right] \left(\frac{\partial}{\partial t}\right)^3 \left[e^{-tz}\Theta_L(t)\right] \,, \tag{3.25}$$

where $\mathbb{C} \approx 0.58...$ is the Euler constant. It is not difficult to find the dependence of the determinant on the renormalization parameter μ

$$\log \mathrm{Det}_{\mu}(L + z) = -(4\pi)^{-2} \log \frac{\mu^2}{\lambda} \, A_2(-z) + \log \mathrm{Det}_{\sqrt{\lambda}}(L + z) \,. \tag{3.26}$$

Now, let

$$\Theta_L^{\mathrm{ren}}(t) \,=\, \Theta_L(t) - e^{-t\lambda} \left[A_0 + A_1(\lambda)t + A_2(\lambda)t^2\right] \,, \tag{3.27}$$

The renormalized heat trace has the following asymptotics: as $t \to 0$

$$\Theta_L^{\mathrm{ren}}(t) = A_3(\lambda)t^3 + O(t^4) \,, \tag{3.28}$$

and as $t \to \infty$

$$\Theta_L^{\mathrm{ren}}(t) = -e^{-t\lambda} \left[A_2(\lambda)t^2 + A_1(\lambda)t + A_0(\lambda)\right] + (4\pi)^2 d_1 t^2 e^{-t\lambda_1} + O(e^{-t\lambda_2}) \,. \tag{3.29}$$

Thus, we can consider the integral

$$\log \mathrm{Det}_{\mathrm{ren}}(L + z) \,=\, -(4\pi)^{-2} \int_0^\infty \frac{dt}{t^3} e^{-tz} \,\Theta_L^{\mathrm{ren}}(t) \,, \tag{3.30}$$

which is well defined since it converges both at 0 and ∞.

One can compute the dependence of the renormalized determinant on λ exactly. First, we show that

$$\lambda \frac{\partial}{\partial \lambda} \log \mathrm{Det}_{\mathrm{ren}}(L + z) = -(4\pi)^{-2} \left(A_2(-z) + \lambda A_1(-z) + \frac{1}{2}\lambda^2 A_0\right) \,. \tag{3.31}$$

Then, by integrating this equation we get

$$\log \mathrm{Det}_{\,\mathrm{ren}}(L+z) \;=\; -(4\pi)^{-2}\left[A_2(-z)\log\frac{\lambda}{\lambda_0}+\lambda A_1(-z)+\frac{1}{4}\lambda^2 A_0\right]+\mathrm{const}\;, \tag{3.32}$$

where λ_0 is some constant. Notice that in the limit when $\lambda \to 0$ there is an infrared divergence

$$\log \mathrm{Det}_{\,\mathrm{ren}}(L+z) = -(4\pi)^{-2}A_2(-z)\log\frac{\lambda}{\lambda_0}+O(1)\,. \tag{3.33}$$

Now, by integrating by parts one can show that

$$\log \mathrm{Det}_{\,\mu}(L+z) = \log \mathrm{Det}_{\,\mathrm{ren}}(L+z)-(4\pi)^{-2}\log\frac{\mu^2}{\lambda}\;A_2(-z)+c_0\lambda^2A_0+c_1\lambda A_1(-z)+c_2A_2(-z)\,. \tag{3.34}$$

Here c_0, c_1 and c_2 are some numerical constants dependent on the regularization scheme, in particular, they can be set to zero without loss of generality.

By using this regularization of functional determinants we obtain the effective action in the form

$$\Gamma_{(1)YM} \;=\; -\frac{1}{2}(4\pi)^{-2}\left\{\beta_{YM}\log\frac{\mu^2}{\lambda}+\int_0^\infty\frac{dt}{t^3}\;e^{-tz}\Theta^{\mathrm{ren}}_{YM}(t)\right\}\;, \tag{3.35}$$

$$\Gamma_{(1)\mathrm{mat}} \;=\; -\frac{1}{2}(4\pi)^{-2}\left\{\beta_{\mathrm{mat}}\log\frac{\mu^2}{\lambda}+\int_0^\infty\frac{dt}{t^3}\;\Theta^{\mathrm{ren}}_{\mathrm{mat}}(t)\right\}\;, \tag{3.36}$$

where

$$\Theta^{\mathrm{ren}}_{YM}(t) \;=\; \Theta^{\mathrm{ren}}_{L_{\mathrm{vect}}}(t)-2\Theta^{\mathrm{ren}}_{L^{YM}_{\mathrm{ghost}}}(t)\,, \tag{3.37}$$

$$\Theta^{\mathrm{ren}}_{\mathrm{mat}}(t) \;=\; \Theta^{\mathrm{ren}}_{L_0}(t)-\Theta^{\mathrm{ren}}_{L_{\mathrm{spin}}}(t)\,, \tag{3.38}$$

and

$$\begin{aligned}\beta_{YM} \;=\; & B_2(L_{\mathrm{vect}})-zB_1(L_{\mathrm{vect}})+\frac{z^2}{2}B_0(L_{\mathrm{vect}})\\ & -2B_2(L^{YM}_{\mathrm{ghost}})+2zB_1(L_{\mathrm{ghost}})-z^2B_0(L_{\mathrm{ghost}})\,,\end{aligned} \tag{3.39}$$

$$\beta_{\mathrm{mat}} \;=\; B_2(L_0)-B_2(L_{\mathrm{spin}})\,. \tag{3.40}$$

The main idea of the renormalization group is based on the realization that the total effective action, $\Gamma = S + \hbar\Gamma_{(1)} + \cdots$, should not depend on the arbitrary renormalization parameter μ. This means that in renormalizable field theories the coupling constants in the classical action should depend on μ in such a way to exactly compensate the dependence of the one-loop effective action on μ, that is,

$$\mu\frac{\partial}{\partial\mu}S \;=\; -\mu\frac{\partial}{\partial\mu}\Gamma_{(1)} = (4\pi)^{-2}\left(\beta_{YM}+\beta_{\mathrm{mat}}\right)\;. \tag{3.41}$$

This means that the classical action should have terms of the same type as those in the coefficients β_{YM} and $\beta_{\rm mat}$.

As we discussed above, all relevant operators are of Laplace type $L = -\Delta + Q$. For a covariantly constant background the endomorphism Q is covariantly constant, and, therefore, commutes with the Laplacian. Therefore, the heat semigroup of the operator L is determined by the heat semigroup of the Laplacian

$$\exp(-tL) = \exp(-tQ)\exp(t\Delta)\,, \tag{3.42}$$

so, the heat kernel diagonal of the operator L has the form

$$U^{\rm diag}_L(t) = \exp(-tQ)U^{\rm diag}(t)\,, \tag{3.43}$$

where $U^{\rm diag}(t)$ denotes the heat kernel diagonal of the pure Laplacian acting on a vector bundle $\mathcal{V}$.

By using this property we can express the heat kernel diagonals of the operators introduced above in terms of the heat kernel of the corresponding Laplacians,

$$U^{\rm diag}_{L_{\rm vect}}(t) \;=\; \exp\left(-tQ_{\rm vect}\right) U^{\rm diag}_{T_1\otimes Ad}(t)\,, \tag{3.44}$$
$$U^{\rm diag}_{L^{YM}_{\rm ghost}}(t) \;=\; U^{\rm diag}_{T_0\otimes Ad}(t)\,, \tag{3.45}$$
$$U^{\rm diag}_{L_0}(t) \;=\; \exp\left(-tQ_0\right) U^{\rm diag}_{T_0\otimes W_0}(t)\,, \tag{3.46}$$
$$U^{\rm diag}_{L_{\rm spin}}(t) \;=\; \exp\left(-tQ_{\rm spin}\right) U^{\rm diag}_{T_{\rm spin}\otimes W_{\rm spin}}(t)\,. \tag{3.47}$$

This means that

$$\Theta_{YM}(t) \;=\; {\rm vol}\,(M)(4\pi t)^2 {\rm tr}_{\,Ad}\Big[{\rm tr}_{\,T_1} \exp(-tQ_{\rm vect})U^{\rm diag}_{T_1\otimes Ad}(t) - 2U^{\rm diag}_{T_0\otimes Ad}(t)\Big]\,, \tag{3.48}$$

$$\begin{aligned}\Theta_{\rm mat}(t) \;=\; {\rm vol}\,(M)(4\pi t)^2\Big\{&{\rm tr}_{\,W_0} \exp(-tQ_0)U^{\rm diag}_{T_0\otimes W_0}(t)\\ &-{\rm tr}_{\,W_{\rm spin}}{\rm tr}_{\,T_{\rm spin}} \exp(-tQ_{\rm spin})U^{\rm diag}_{T_{\rm spin}\otimes W_{\rm spin}}(t)\Big\}\,.\end{aligned} \tag{3.49}$$

The renormalized functions $\Theta^{\rm ren}_{YM}(t)$ and $\Theta^{\rm ren}_{\rm mat}(t)$ are obtained from this by finding the smallest eigenvalue and then subtracting some terms according to the prescription (3.27). Thus, we need to compute the heat kernel diagonals for the Laplacians only.

4 Geometry of the Sphere S^2

In this section we follow mainly our paper [7].

4.1 Metric

We cover the sphere S^2 (of radius a) by two coordinate patches: one patch covering the South pole and another patch covering the North pole. We will use the spherical coordinates (r,φ), which range over $0 \le r \le a\pi$ and $0 \le \varphi \le 2\pi$. The South coordinate patch is the neighborhood of the South pole

$r = 0$, whereas the North coordinate patch is the neighborhood of the North pole $r = a\pi$. The volume of S^2 is, of course, $\mathrm{vol}\,(S^2) = 4\pi a^2$.

The metric in spherical coordinates is

$$ds^2 = dr^2 + a^2 \sin^2(r/a) d\varphi^2 . \tag{4.1}$$

We choose an orthonormal basis of 1-forms

$$\sigma^1 = \cos\varphi\, dr - a \sin(r/a) \sin\varphi\, d\varphi , \tag{4.2}$$

$$\sigma^2 = \sin\varphi\, dr + a \sin(r/a) \cos\varphi\, d\varphi . \tag{4.3}$$

Then the spin connection one-form is

$$\omega_{ab} = \varepsilon_{ab}[1 - \cos(r/a)]\, d\varphi , \tag{4.4}$$

the Riemann curvature is

$$R_{abcd} = \frac{1}{a^2}\varepsilon_{ab}\varepsilon_{cd} = \frac{1}{a^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) , \tag{4.5}$$

and the Ricci tensor and the scalar curvature are

$$R_{ab} = \frac{1}{a^2}\delta_{ab} , \qquad R = \frac{2}{a^2} . \tag{4.6}$$

4.2 Connection

In two dimensions the Yang-Mills connection with a covariantly constant curvature is necessarily Abelian and can always be chosen to be proportional to the spin connection, that is,

$$\mathcal{A} = -X[1 - \cos(r/a)]\, d\varphi , \tag{4.7}$$

where $X = \mathrm{ad}(\Sigma)$ is an $m \times m$ real anti-symmetric matrix. The Yang-Mills curvature 2-form is

$$\mathcal{F} = -X\frac{1}{a} \sin(r/a)\, dr \wedge d\varphi , \tag{4.8}$$

in components, $\mathcal{F}_{cd} = \varepsilon_{cd} H$, where $H = -X/a^2$. The connection has to be redefined (via a gauge transformation) in the North coordinate patch. By defining

$$\mathcal{A}' = X[1 + \cos(r/a)]\, d\varphi , \tag{4.9}$$

we obtain the connection that is well defined globally. Now, we have that

$$\mathcal{A}' - \mathcal{A} = dUU^{-1} , \tag{4.10}$$

where $U = \exp(2X\varphi)$ is an element of the group G.

Since U should be periodic in φ, the matrix X should satisfy the condition

$$\exp(4\pi X) = \mathbb{I} , \tag{4.11}$$

which means that its eigenvalues must be either zero or imaginary half-integers. Only in this case the vector bundle is globally defined.

Since X is anti-symmetric, its non-zero eigenvalues must appear in pairs. That is, the spectrum of the matrix X must be

$$\mathrm{Spec}(X) = \left\{ \underbrace{0,\dots,0}_{r}, i\frac{n_1}{2}, -i\frac{n_1}{2}, \dots, i\frac{n_p}{2}, -i\frac{n_p}{2} \right\}, \tag{4.12}$$

where $r = m - 2p$, and n_j, $j = 1, \dots, p$, are some non-zero integers.

Such construction for each n is nothing but the Hopf complex line bundle $\mathcal{H}_n$ over S^2, which is equal to a tensor product of n Hopf bundles $\mathcal{H}_1$ (or dual Hopf bundles for $n < 0$). This is sometimes called the topological quantization condition. The deep fundamental reason for this condition is the Chern theorem [11], which in particular says that the Chern form

$$\frac{i}{2\pi}\int_{S^2} \mathcal{F} = -2iX \tag{4.13}$$

has integer eigenvalues. Note that the matrix X is the generator of a representation $\rho_X : \mathrm{Spin}(2) \to \mathrm{Aut}(\mathbb{R}^m)$ of the double cover of the group $SO(2)$ (which we, by definition, still call the spin group Spin(2) in two dimensions).

It is important to understand that the numbers n_j are not independent. Because X lies in the adjoint representation of the compact semi-simple Lie algebra $\mathfrak{g}$, these eigenvalues are determined by the roots of the algebra $\mathfrak{g}$. We will discuss this in the next section.

4.3 Roots

In this subsection we follow [12]. Since X is a real antisymmetric matrix it can be diagonalized, and thus lies in the Cartan subalgebra. Let r be the rank of the group G, which is equal to the dimension of the Cartan subalgebra. The generators C_j, $j = 1, \dots, r$, of the Cartan subalgebra in adjoint representation are $m \times m$ diagonal matrices of the form

$$C_j = \mathrm{diag}\left(\underbrace{0,\dots,0}_{r}, i\alpha_{1j}, -i\alpha_{1j}, \dots, i\alpha_{pj}, -i\alpha_{pj}, \right), \tag{4.14}$$

where $\alpha_k = (\alpha_{kj})$, $k = 1, \dots, p$, are some covectors in $\mathbb{R}^r$ called the positive roots of the Lie algebra $\mathfrak{g}$, p is the number of positive roots related to the rank by $r = m - 2p$. Then the matrix X has the form

$$X = \sum_{i=1}^{r} x^j C_j = \mathrm{diag}\left(\underbrace{0,\dots,0}_{r}, i\alpha_1(x), -i\alpha_1(x), \dots, i\alpha_p(x), -i\alpha_p(x), \right), \tag{4.15}$$

where $x = (x^j)$, $j = 1, \dots, r$, is a vector in $\mathbb{R}^r$, and $\alpha_k(x) = \sum_{i=1}^{r} \alpha_{kj} x^j$ is the canonical value of the covector α_k on the vector x in $\mathbb{R}^r$. The vector x must be such that the value of each root on it is a half-integer, that is,

$$\alpha_k(x) = \frac{n_k}{2}, \qquad n_k \in \mathbb{Z}, \qquad k = 1, \dots, p. \tag{4.16}$$

Thus, the number of (possibly) non-zero eigenvalues is equal to the number p of positive roots of the algebra, and the eigenvalues themselves are equal to the values of the roots on the vector x.

The roots α_k and the vector x are vectors in $\mathbb{R}^r$. However, it is convenient to consider this space $\mathbb{R}^r$ as the hyperplane in $\mathbb{R}^{r+1}$ orthogonal to the vector $b = \sum_{i=1}^{r+1} \hat{e}_i$, where $\hat{e}_i$, $i = 1, \dots, r+1$, is the canonical orthonormal basis in $\mathbb{R}^{r+1}$. Then the roots α_k and the vector x can be represented in terms of the basis e_i of $\mathbb{R}^{r+1}$ as $x = \sum_{i=1}^{r+1} \hat{x}^i \hat{e}_i$, and $\alpha_k = \sum_{i=1}^{r+1} \hat{\alpha}_{ki} \hat{e}_i$, $k = 1, \dots, p$. Notice that not all of the coordinates $\hat{x}^i$ are independent. Since these vectors lie in the hyperplane orthogonal to the vector b, the sum of all coordinates of these vectors should be equal to zero, that is, $\sum_{i=1}^{r+1} \hat{x}^i = 0$, and $\sum_{i=1}^{r+1} \hat{\alpha}_{ki} = 0$. Then the values of the roots on the vector x are

$$\alpha_k(x) = \sum_{i=1}^{r+1} \hat{\alpha}_{ki} \hat{x}^i = \sum_{i=1}^{r} (\hat{\alpha}_{ki} - \hat{\alpha}_{k,r+1}) \, \hat{x}^i \, . \tag{4.17}$$

For all classical Lie algebras A_n, B_n, C_n and D_n the coordinates of the roots are integers (up to a uniform normalization factor), that is, $\hat{\alpha}_{ki} = \lambda \beta_{ki}$, $k = 1, \dots, p$; $i = 1, \dots, r+1$, where β_{ki} are integers and λ is a normalization factor. The normalization constant can be determined from the chosen Cartan-Killing metric (2.2) by requiring

$$\lambda^2 \sum_{k=1}^{p} \beta_{ki} \beta_{kj} = \gamma_{ij}. \tag{4.18}$$

This can also be viewed as the definition of the metric γ_{ij}. The constant λ is then just a uniform factor that can be set to $\lambda = 1$.

Therefore, $\alpha_k(x)$ will be half-integer if $\hat{x}^i = \frac{1}{2} k_i$, where k_i, $i = 1, \dots, r$, are arbitrary integers, that is,

$$\alpha_k(x) = \frac{1}{2} \sum_{i=1}^{r} (\beta_{ki} - \beta_{k,r+1}) \, k_i \, . \tag{4.19}$$

To be specific let us consider the group $G = SU(N)$. The algebra $\mathfrak{su}(N)$ is isomorphic to the classical algebra A_{N-1}. The dimension and the rank of $SU(N)$ are

$$m = \dim SU(N) = N^2 - 1, \qquad r = \operatorname{rank} SU(N) = N - 1 \, . \tag{4.20}$$

The positive roots are labeled by two integers $1 \le i < j \le N$, and have the form $\alpha_{ij} = \hat{e}_i - \hat{e}_j$. The number of positive roots is

$$p = \frac{m - r}{2} = \frac{N(N-1)}{2} \, . \tag{4.21}$$

The Cartan-Killing metric is $\gamma_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$. Then

$$\alpha_{ij}(x) = \hat{x}^i - \hat{x}^j = \frac{1}{2}(k_i - k_j) \, . \tag{4.22}$$

Therefore, in this case the integers determining the eigenvalues of the matrix X are also labeled by two indices

$$n_{ij} = k_i - k_j \, , \qquad 1 \le i < j \le N \, . \tag{4.23}$$

We will call these integers monopole numbers for $n_{ij} > 0$ (or antimonopole numbers for $n_{ij} < 0$). Here k_i, $i = 1, \ldots, N$, are arbitrary integers whose sum is equal to zero. In particular, some or all of them can be equal to zero. Note, however, that it is impossible to have only one non-zero number k_i. Therefore, either they are all equal to zero, or there are at least two non-zero integers k_i. Another important observation is that for any choice of non-zero integers k_i some of the integers n_{ij} will have absolute value greater or equal to 2. In other words, it is impossible to have $n_{ij} = 0, \pm 1$ for all i, j (except, of course, the trivial case when all integers $k_i = 0$). This observation has profound implications for the stability of the chromomagnetic vacuum studied in this paper.

For the classical groups the situation is similar. Let, as above, k_i, $i = 1, \ldots, N$, be an arbitrary collection of N integers whose sum is equal to zero. Then for the group D_{N-1} the possible monopole numbers are

$$n_{ij} = \pm k_i \pm k_j \,. \tag{4.24}$$

For the group B_{N-1} there are two possible combinations

$$n_{ij} = \pm k_i \pm k_j \qquad \text{and} \qquad n_i = \pm k_i \,, \tag{4.25}$$

and for the group C_{N-1} the possible combinations are

$$n_{ij} = \pm k_i \pm k_j \qquad \text{and} \qquad n_i = \pm 2k_i \,. \tag{4.26}$$

It is not difficult to see that in all these cases there is no choice of non-zero integers k_i such that the only monopole numbers are $0, \pm 1$. There will be necessarily monopole numbers with absolute value greater or equal to 2.

Thus, *for any of the compact simple classical groups if the matrix X has at least one non-zero eigenvalue, then it will have at least one eigenvalue with absolute value greater or equal to* 2.

4.4 Weights

Now let us consider an irreducible representation $\mathfrak{g} \to \mathrm{End}(W)$ of the Lie algebra $\mathfrak{g}$ in a N-dimensional complex vector space W. The generators of the Cartan subalgebra in this representation, $W(C_i)$, $i = 1, \ldots, r$, are $N \times N$ complex diagonal matrices of the form

$$W(C_j) = \mathrm{diag}\left(i\nu_{1j}, \ldots, i\nu_{Nj}\right) , \tag{4.27}$$

where $\nu_k = (\nu_{kj})$, $k = 1, \ldots, N$, are some covectors in $\mathbb{R}^r$ called the weights of the representation W. Contrary to roots, the weights can be degenerate, that is, have multiplicity greater than 1, and be equal to zero with some multiplicity too. Then the matrix $W(X)$ has the form

$$W(X) = \sum_{i=1}^{r} x^j W(C_j) = \mathrm{diag}\left(i\nu_1(x), \ldots, i\nu_N(x)\right), \tag{4.28}$$

where $x = (x^j)$, $j = 1, \ldots, r$, is a vector in $\mathbb{R}^r$, and $\nu_k(x) = \sum_{i=1}^r \nu_{kj} x^j$ is the canonical value of the covector α_k on the vector x in $\mathbb{R}^r$. The vector x must be such that the value of each weight on it is a half-integer, that is,

$$\nu_k(x) = \frac{m_k}{2} \,, \qquad m_k \in \mathbb{Z} \,, \qquad k = 1, \ldots, N \,. \tag{4.29}$$

The weights lie in the same space as the roots. So, we can represent them by $\nu_k = \sum_{i=1}^{r+1} \hat{\nu}_{ki}\hat{e}_i$, $k = 1, \ldots, N$, where $\sum_{i=1}^{r+1} \hat{\nu}_{ki} = 0$. Then the values of the weights on the vector x are

$$\nu_k(x) = \sum_{i=1}^{r+1} \hat{\nu}_{ki}\hat{x}^i = \sum_{i=1}^{r} (\hat{\nu}_{ki} - \hat{\nu}_{k,r+1})\,\hat{x}^i = \frac{1}{2}\sum_{i=1}^{r} (\hat{\nu}_{ki} - \hat{\nu}_{k,r+1})\,k_i\,, \tag{4.30}$$

where k_i, $i = 1, \ldots, r$, are arbitrary integers.

To be specific let us consider the *fundamental* (defining) representation of the algebra $\mathfrak{g} = \mathfrak{su}(N)$ by $N \times N$ complex traceless anti-Hermitian matrices. The Cartan subalgebra is generated by diagonal matrices. Therefore, the generators $W(C_i)$, $i = 1, \ldots, N-1$, of the Cartan subalgebra must have N imaginary eigenvalues whose sum is equal to zero. Then the matrix $W(X)$ must have the form

$$W(X) = \sum_{i=1}^{r} x^j W(C_j) = \operatorname{diag}\left(i\frac{k_1}{2}, \ldots, i\frac{k_N}{2}\right), \tag{4.31}$$

where k_i, $i = 1, \ldots, N$, are N integers whose sum is equal to zero, These are exactly the integers that define the values of the roots $\alpha_{ij}(x) = (k_i - k_j)/2$ for the adjoint representation of the group $SU(N)$.

4.5 Isometries

It is well-known that $S^2 = SO(3)/SO(2)$, so that $SO(3)$ is the isometry group and $SO(2)$ is the isotropy (or holonomy) group of S^2. The Killing vectors, ξ_A, of S^2 have the form [7]

$$\xi_1 = \cos\varphi\,\partial_r - \frac{1}{a}\cot(r/a)\sin\varphi\,\partial_\varphi\,, \tag{4.32}$$

$$\xi_2 = \sin\varphi\,\partial_r + \frac{1}{a}\cot(r/a)\cos\varphi\,\partial_\varphi\,, \tag{4.33}$$

$$\xi_3 = \partial_\varphi\,. \tag{4.34}$$

One can check that the Killing vector fields form a representation of the isometry algebra, $SO(3)$, [7]

$$[\xi_1, \xi_2] = -\frac{1}{a^2}\xi_3\,, \qquad [\xi_3, \xi_1] = -\xi_2 \qquad [\xi_2, \xi_3] = -\xi_1\,. \tag{4.35}$$

The Cartan metric of the group $SO(3)$ has the form

$$(\gamma_{AB}) = \operatorname{diag}(1, 1, a^2)\,, \qquad (\gamma^{AB}) = \operatorname{diag}\left(1, 1, \frac{1}{a^2}\right). \tag{4.36}$$

Therefore, the Casimir operator, $\Delta = \gamma^{AB}\xi_A\xi_B$, of the Lie algebra of the group $SO(3)$ is nothing but the scalar Laplacian

$$\Delta = \partial_r^2 + \frac{1}{a}\cot(r/a)\partial_r + \frac{1}{a^2\sin^2(r/a)}\partial_\varphi^2\,. \tag{4.37}$$

Now, suppose the group G has the spin group Spin(2) as a subgroup and let $\alpha : \mathrm{Spin}(2) \to G$ be the corresponding embedding. Since we also have a representation $\rho_W : G \to \mathrm{Aut}(W)$ of the gauge group G in the vector space W, this defines a new representation of the spin group $\rho_X = \rho_W \circ \alpha :$

$\mathrm{Spin}(2) \to \mathrm{Aut}(W)$. Let Σ be the generator of the spin group Spin(2). Then $X(\Sigma)$ is the generator of the spin group Spin(2) in the representation ρ_X and

$$G = \mathbb{I}_W \otimes T(\Sigma) - X(\Sigma) \otimes \mathbb{I}_T \tag{4.38}$$

is the generator of the twisted representation $X \otimes T$ of the spin group Spin(2). This generator should not be confused with the gauge group denoted by the same symbol.

The twisted Lie derivatives $\mathcal{L}_A = \mathcal{L}_{\xi_A}$ along Killing vectors ξ_A of sections of the vector bundle $\mathcal{V}$ are [7]

$$\begin{aligned}
\mathcal{L}_1 &= \cos\varphi\,\partial_r - \frac{1}{a}\sin\varphi\cot(r/a)\,\partial_\varphi + \sin\varphi\frac{1-\cos(r/a)}{a\sin(r/a)}G\,, && (4.39)\\
\mathcal{L}_2 &= \sin\varphi\,\partial_r + \frac{1}{a}\cos\varphi\cot(r/a)\,\partial_\varphi - \cos\varphi\frac{1-\cos(r/a)}{a\sin(r/a)}G\,, && (4.40)\\
\mathcal{L}_3 &= \partial_\varphi + G\,. && (4.41)
\end{aligned}$$

One can show that these operators form a representation of the isometry algebra $\mathfrak{so}(3)$ [7]

$$[\mathcal{L}_1, \mathcal{L}_2] = -\frac{1}{a^2}\mathcal{L}_3\,, \qquad [\mathcal{L}_3, \mathcal{L}_1] = -\mathcal{L}_2 \qquad [\mathcal{L}_2, \mathcal{L}_3] = -\mathcal{L}_1\,. \tag{4.42}$$

The generalized Laplacian is expressed in terms of the Casimir operators of the isometry group $SO(3)$ and the holonomy group $SO(2)$

$$\Delta = \gamma^{AB}\mathcal{L}_A\mathcal{L}_B - \frac{1}{a^2}G^2 \tag{4.43}$$

and is equal to [7]

$$\Delta = \partial_r^2 + \frac{1}{a}\cot(r/a)\partial_r + \frac{1}{a^2\sin^2(r/a)}\left(\partial_\varphi + [1-\cos(r/a)]G\right)^2\,. \tag{4.44}$$

We will be interested in the limit as $a \to \infty$ (when the curvature of the sphere vanishes, so, formally, $S^2 \to \mathbb{R}^2$) and $X \to \infty$, so that the magnetic field $H = -X(\Sigma)/a^2$ remains constant. In this limit the generator G becomes $G \to a^2 H \otimes \mathbb{I}_T$. It is worth noting that this limit is only defined locally in the South coordinate patch. Therefore, the global bundle structure is destroyed in this limit. Thus, in the following we will be working in the South coordinate patch (for small r) in S^2, which approaches $\mathbb{R}^2$ as $a \to \infty$.

Notice that as $a \to \infty$ the Killing vectors of the sphere S^2 become the Killing vectors of the Euclidean space $\mathbb{R}^2$ and the Laplacian becomes exactly the Laplacian with a constant magnetic field H in the Euclidean plane $\mathbb{R}^2$.

5 Heat Traces

We will employ the algebraic methods for calculation of the heat kernel developed in [6–8]. To compute the heat trace directly we need to compute the spectrum of the Laplacian, its eigenvalues and their multiplicities. However, we can also compute the heat trace as an integral of the heat kernel diagonal. Since on the sphere the heat kernel diagonal is constant the heat kernel is simply proportional to the fiber trace of the heat kernel diagonal,

$$\mathrm{Tr}\ \exp(-tL) = \mathrm{vol}\,(M)\mathrm{tr}\ \exp(-tQ)U^{\mathrm{diag}}(t)\,. \tag{5.1}$$

That is why, we need to compute the heat kernel diagonal of the Laplacian.

Next, we note that on the product manifold $S^1 \times S^1 \times S^2$ the Laplacian splits naturally

$$\Delta_{S^1\times S^1\times S^2} = \Delta_{S^1\times S^1} + \Delta_{S^2}\,, \tag{5.2}$$

and, therefore, the heat kernel factorizes

$$U^{\mathrm{diag}}_{S^1\times S^1\times S^2}(t) = U^{\mathrm{diag}}_{S^1\times S^1}(t)U^{\mathrm{diag}}_{S^2}(t)\,. \tag{5.3}$$

Therefore,

$$\mathrm{Tr}\ \exp(-tL) = \mathrm{vol}\,(M)\mathrm{tr}\ \exp(-tQ)U^{\mathrm{diag}}_{S^1\times S^1}(t)U^{\mathrm{diag}}_{S^2}(t)\,. \tag{5.4}$$

That is why, we need to compute the heat kernel diagonals of the Laplacian on S^1 and S^2 only.

5.1 Heat Kernel on $\mathbb{R}^2$ and $S^1 \times S^1$ without Magnetic Field

The Laplacian on $\mathbb{R}$ is $\Delta = \partial_x^2$. The heat kernel diagonal of such an operator is easily computed by Fourier transform

$$U^{\mathrm{diag}}_{\mathbb{R}}(t) = \frac{1}{\sqrt{4\pi t}}\,. \tag{5.5}$$

For $\mathbb{R}^2$ we obviously have the product

$$U^{\mathrm{diag}}_{\mathbb{R}^2}(t) = \frac{1}{4\pi t}\,. \tag{5.6}$$

On the circle S^1 of radius a the heat kernel can be computed by Fourier expansion. The spectrum of the operator $(-\Delta)$ is

$$\lambda_l(\Delta) = \frac{l^2}{a^2}\,, \tag{5.7}$$

where $l = 0, 1, 2, \dots$, with multiplicities $d_0 = 1$ and $d_l = 2$ for $l = 1, 2, \dots$. Then the heat kernel diagonal on S^1 is

$$U^{\mathrm{diag}}_{S^1}(t) = \frac{1}{\sqrt{4\pi t}}\Omega\left(\frac{t}{a^2}\right)\,, \tag{5.8}$$

where

$$\Omega(t) = \sqrt{\frac{t}{\pi}}\left\{1 + 2\sum_{l=1}^{\infty} e^{-tl^2}\right\} = \sqrt{\frac{t}{\pi}}\,\theta_3\left(0, e^{-t}\right)\,, \tag{5.9}$$

and $\theta_3(v, q)$ is the third Jacobi theta function. Therefore, for the product $S^1 \times S^1$ with radii a_1, a_2 the heat kernel diagonal is

$$U^{\mathrm{diag}}_{S^1\times S^1}(t) = \frac{1}{4\pi t}\Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right)\,. \tag{5.10}$$

An important property of $\Omega(t)$ is the Poisson duality formula, which gives a nontrivial relation

$$\Omega(t) = \sqrt{\frac{t}{\pi}}\,\Omega\left(\frac{\pi^2}{t}\right) = \theta_3\left(0, e^{-\pi^2/t}\right). \tag{5.11}$$

By differentiating the equation (5.11) we get

$$\Omega'(t) = \frac{2\pi^2}{t^2}\sum_{l=1}^{\infty} l^2 \exp\left(-\frac{\pi^2}{t}l^2\right), \tag{5.12}$$

which immediately shows that Ω is an increasing function.

We will need the asymptotics of this function. It is not difficult to see that as $t \to 0$

$$\Omega(t) = 1 + 2\exp\left(-\frac{\pi^2}{t}\right) + O\left(e^{-4\pi^2/t}\right), \tag{5.13}$$

$$\Omega'(t) = 2\pi^2 t^{-2}\exp\left(-\frac{\pi^2}{t}\right) + O\left(e^{-4\pi^2/t}\right), \tag{5.14}$$

$$\Omega''(t) = 2\pi^2 t^{-4}\left(\pi^2 - 2t\right)\exp\left(-\frac{\pi^2}{t}\right) + O\left(e^{-4\pi^2/t}\right), \tag{5.15}$$

and as $t \to \infty$, we have

$$\Omega(t) = \frac{1}{\sqrt{\pi}}\left[t^{1/2} + 2t^{1/2}e^{-t} + O\left(e^{-4t}\right)\right], \tag{5.16}$$

$$\Omega'(t) = \frac{1}{\sqrt{\pi}}\left[\frac{1}{2}t^{-1/2} + \left(t^{-1/2} - 2t^{1/2}\right)e^{-t} + O\left(e^{-4t}\right)\right], \tag{5.17}$$

$$\Omega''(t) = \frac{1}{\sqrt{\pi}}\left[-\frac{1}{4}t^{-3/2} + \left(2t^{1/2} - 2t^{-1/2} - \frac{1}{2}t^{-3/2}\right)e^{-t} + O\left(e^{-4t}\right)\right]. \tag{5.18}$$

Notice that although, in general, as $t \to \infty$ the derivatives of the function $\Omega(t)$ decrease only as powers of t, a particular combination of the derivatives, which we will need later, is exponentially decreasing as $t \to \infty$,

$$\Omega'(t) + 2t\Omega''(t) = \frac{2}{\sqrt{\pi}}t^{1/2}\left(2t-3\right)e^{-t} + O\left(e^{-4t}\right). \tag{5.19}$$

This will have a major impact on the calculation of the heat capacity of the quark-gluon gas.

5.2 Heat Kernel on $\mathbb{R}^2$ with Magnetic Field

Let us now compute the heat kernel diagonal of the Laplacian with a constant magnetic field on $\mathbb{R}^2$. In the flat space limit the Laplacian on $\mathbb{R}^2$ in polar coordinates with a constant magnetic field can be written as

$$\Delta = \nabla_1^2 + \nabla_2^2, \tag{5.20}$$

where

$$\nabla_1 = \cos\varphi\,\partial_r - \frac{1}{r}\sin\varphi\,\partial_\varphi - \frac{r}{2}\sin\varphi H, \tag{5.21}$$

$$\nabla_2 = \sin\varphi\,\partial_r + \frac{1}{r}\cos\varphi\,\partial_\varphi + \frac{r}{2}\cos\varphi H. \tag{5.22}$$

The heat kernel can be computed as follows. First, by observing that the covariant derivatives form the Heisenberg algebra

$$[\nabla_1, \nabla_2] = H, \qquad [\nabla_1, H] = [\nabla_2, H] = 0\,, \tag{5.23}$$

one can prove that the heat semigroup can be expressed in the form [2]

$$\exp(t\Delta) \;=\; \frac{1}{4\pi}\frac{H}{\sin(tH)}\int\limits_{\mathbb{R}^2} dq\;\exp\left(-\frac{1}{4}H\cot(tH)|q|^2 + \langle q, \nabla\rangle\right), \tag{5.24}$$

where $|q|^2 = (q^1)^2 + (q^2)^2$ and $\langle q, \nabla\rangle = q^1\nabla_1 + q^2\nabla_2$. Further, it is not difficult to show that

$$\left[\exp\langle q, \nabla\rangle\,\delta(x, x')\right]^{\rm diag} = \delta(q)\,, \tag{5.25}$$

so that the integral over q becomes trivial and we immediately obtain the heat kernel diagonal [2]

$$U^{\rm diag}_{\mathbb{R}^2}(t) = \frac{1}{4\pi}\frac{H}{\sin(tH)}\,. \tag{5.26}$$

Of course, since $\mathbb{R}^2$ is non-compact, the spectrum of the Laplacian Δ is degenerate, that is, even if the eigenvalues are discrete their multiplicities are infinite. Therefore, the heat trace in the limit $a \to \infty$ is infinite. However, the heat kernel diagonal is still well defined. Therefore, on the sphere S^2 in the limit as $a \to \infty$ and fixed $H = -X/a^2$ the heat kernel diagonal locally must have the following limit

$$\lim_{a\to\infty} U^{\rm diag}_{S^2}(t) = \frac{1}{4\pi}\frac{H}{\sin(tH)}\,. \tag{5.27}$$

Recall that H is a real anti-symmetric matrix with purely imaginary eigenvalues, so that this heat kernel diagonal is well defined.

5.3 Heat Trace on S^2

5.3.1 Algebraic Approach

Let (q^1, q^2, ω) be the canonical coordinates on the isometry group $SO(3)$. Let C be the contour of integration in the complex plane of ω defined by

$$C = \frac{1}{2}(C_+ + C_-)\,. \tag{5.28}$$

By using the isometry algebra and the representation of the Laplacian in terms of the twisted Lie derivatives one can show that the heat semigroup $\exp(t\Delta)$ can be represented in form of an integral over the isometry group [7]

$$\begin{aligned}\exp(t\Delta) \;=\;& \frac{1}{4\pi t}\exp\left\{\left(\frac{1}{4} - G^2\right)\frac{t}{a^2}\right\}\int\limits_C \frac{d\omega}{\sqrt{4\pi t/a^2}}\int\limits_{\mathbb{R}^2} dq\;\exp\left\{-\frac{|q|^2}{4t} - \frac{a^2}{4t}\omega^2\right\}\\ &\times\frac{\sin[\omega/2]}{\omega/2}\exp\left[\langle q, \mathcal{L}\rangle + \omega\mathcal{L}_3\right]\,.\end{aligned} \tag{5.29}$$

where $|q|^2 = (q^1)^2 + (q^2)^2$, $\langle q, \mathcal{L} \rangle = q^1 \mathcal{L}_1 + q^2 \mathcal{L}_2$. We will see later that one can take here either C_+ or C_-, which gives identical results.

By using the isometry algebra one can show that [7]

$$\exp\left[\langle q, \mathcal{L} \rangle + \omega \mathcal{L}_3\right] \delta(x, x')\Big|_{x=x'} = \left(\frac{\omega/2}{\sin[\omega/2]} \right)^2 \exp(\omega G)\, \delta(q)\,. \tag{5.30}$$

Substituting this into (5.29) we obtain the heat kernel diagonal in the form

$$U^{\rm diag}(t) = \frac{1}{4\pi t} \exp\left[\frac{t}{a^2} \left(\frac{1}{4} - G^2 \right) \right] \int\limits_C \frac{a d\omega}{\sqrt{4\pi t}} \exp\left\{ -\frac{a^2 \omega^2}{4t} + G\omega \right\} \frac{\omega/2}{\sin[\omega/2]}\,. \tag{5.31}$$

This can be written in the form

$$U^{\rm diag}(t) = \frac{1}{4\pi t} \exp\left[\frac{t}{a^2} \left(\frac{1}{4} - G^2 \right) \right] \Psi\left(\frac{t}{a^2}; -2iG \right), \tag{5.32}$$

where $\Psi(t; n)$, with an integer n, is a function defined by

$$\Psi(t; n) = \int\limits_C \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{ -\frac{\omega^2}{4t} + in\omega/2 \right\} \frac{\omega/2}{\sin[\omega/2]}, \tag{5.33}$$

Recall that G has imaginary half-integer eigenvalues.

5.3.2 Properties of the Function $\Psi(t; n)$

Let

$$\Psi_\pm(t; n) = \int\limits_{C_\pm} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{ -\frac{\omega^2}{4t} + in\omega/2 \right\} \frac{\omega/2}{\sin[\omega/2]}\,, \tag{5.34}$$

so that

$$\Psi(t; n) = \frac{1}{2} \left[\Psi_+(t; n) + \Psi_-(t; n) \right]\,. \tag{5.35}$$

By deforming the contour of integration one can show that

$$\Psi_-(t; n) = \Psi_+(t; n) + i[R_-(t; n) + R_+(t; n)]\,, \tag{5.36}$$

where

$$R_+(t; n) = \left(\frac{\pi}{t} \right)^{1/2} \sum_{k=1}^{\infty} (-1)^{k(1+n)} 2\pi k \exp\left\{ -\frac{\pi^2 k^2}{t} \right\}\,, \tag{5.37}$$

$$R_-(t; n) = \left(\frac{\pi}{t} \right)^{1/2} \sum_{k=-1}^{-\infty} (-1)^{k(1+n)} 2\pi k \exp\left\{ -\frac{\pi^2 k^2}{t} \right\}\,. \tag{5.38}$$

Obviously,

$$R_-(t;n) = -R_+(t;n)\,. \tag{5.39}$$

Therefore,

$$\Psi(t;n) = \Psi_+(t;n) = \Psi_-(t;n)\,, \tag{5.40}$$

and

$$\Psi(t;n) = \fint_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi t}}\ \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} \cos(n\omega/2)\,, \tag{5.41}$$

where $\fint$ denotes the Cauchy principal value of the integral. This can also be written as

$$\begin{aligned}\Psi(t;n) = \sum_{k=-\infty}^{\infty} (-1)^{k(1+n)} \fint_0^{2\pi} \frac{d\omega}{\sqrt{4\pi t}}\ \exp\left\{-\frac{1}{4t}(\omega+2\pi k)^2\right\} \\ \times \frac{(\omega+2\pi k)/2}{\sin[\omega/2]} \cos(n\omega/2)\,,\end{aligned} \tag{5.42}$$

which is nothing but the sum over closed geodesics of S^2.

We can also rewrite the function $\Psi(t;n)$ in the form

$$\Psi(t;n) = \fint_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}}\ \exp\left\{-\frac{\omega^2}{4}\right\} \frac{\omega\sqrt{t}/2}{\sin\left[\omega\sqrt{t}/2\right]} \cos\left[n\omega\sqrt{t}/2\right]\,, \tag{5.43}$$

which allows the analytic continuation in the complex plane of t with a cut along the negative real axis (so that $t = |t|e^{i\theta}$ with $|\theta| < \pi$),

$$\Psi(t;n) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}}\ \exp\left\{-\frac{\omega^2}{4}\right\} \frac{\omega\sqrt{-t}/2}{\sinh\left[\omega\sqrt{-t}/2\right]} \cosh\left[n\omega\sqrt{-t}/2\right]\,. \tag{5.44}$$

Note also that the function $\Psi(t;n)$ is an even function of n, that is,

$$\Psi(t;n) = \Psi(t;-n) = \Psi(t;|n|)\,, \tag{5.45}$$

so that it depends only on the absolute value $|n|$. In terms of the matrix G this means that $\Psi(t;-2iG)$ depends only on the absolute value $|G|$ of the matrix G, which can be defined as the positive square root of the real symmetric matrix $|G| = \sqrt{-G^2}$. Recall that G has purely imaginary half-integer eigenvalues, therefore, $|G|$ has positive half-integer eigenvalues.

We can also find the dual representation of the function $\Psi(t;n)$ as follows. As we have seen above one can use either contour, C_+ or C_- for the calculation of the function $\Psi(t;n)$. We use the series the series

$$\frac{\omega/2}{\sin[\omega/2]} = -i\omega \sum_{l=0}^{\infty} \exp\left\{i\left(l+\frac{1}{2}\right)\omega\right\} \tag{5.46}$$

for $\mathrm{Im}\,\omega > 0$ and

$$\frac{\omega/2}{\sin[\omega/2]} = i\omega \sum_{l=0}^{\infty} \exp\left\{-i\left(l+\frac{1}{2}\right)\omega\right\} \tag{5.47}$$

for $\mathrm{Im}\,\omega < 0$; they both converge absolutely.

Now we notice that

$$\int\limits_{C_+-C_-} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} \sin[n\omega/2] = 0\,. \tag{5.48}$$

Indeed, the integrand is analytic on the real axis and, therefore, this integral vanishes. Therefore, we can add it to the original integral to get

$$\begin{aligned}
\Psi(t;n) \;=\; & \frac{1}{2} \int\limits_{C_++C_-} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} \cos[n\omega/2] \\
& + \frac{i}{2} \int\limits_{C_+-C_-} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} \sin[|n|\omega/2] \\
= \; & \frac{1}{2} \int\limits_{C_+} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t} + i|n|\omega/2\right\} \frac{\omega/2}{\sin[\omega/2]} \\
& + \frac{1}{2} \int\limits_{C_-} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t} - i|n|\omega/2\right\} \frac{\omega/2}{\sin[\omega/2]}\,.
\end{aligned} \tag{5.49}$$

Now, by using the expansion above we get

$$\begin{aligned}
\Psi(t;n) \;=\; & \frac{1}{2} \int\limits_{C_+} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t} + i|n|\omega/2\right\} \sum_{l=0}^{\infty} (-i\omega) \exp\left\{i\left(l+\frac{1}{2}\right)\omega\right\} \\
& + \frac{1}{2} \int\limits_{C_-} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t} - i|n|\omega/2\right\} \sum_{l=0}^{\infty} i\omega \exp\left\{-i\left(l+\frac{1}{2}\right)\omega\right\}\,.
\end{aligned} \tag{5.50}$$

Finally, by using the equation

$$\begin{aligned}
& \mp \int\limits_{C_\pm} \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t} \pm i\left(l + \frac{1+|n|}{2}\right)\omega\right\} i\omega \\
& \qquad = 2t\left(l + \frac{1+|n|}{2}\right) \exp\left\{-t\left(l+\frac{1+|n|}{2}\right)^2\right\}\,,
\end{aligned} \tag{5.51}$$

we obtain the spectral representation

$$\Psi(t;n) = t \sum_{l=0}^{\infty} 2\left(l + \frac{1+|n|}{2}\right) \exp\left\{-t\left(l+\frac{1+|n|}{2}\right)^2\right\}\,. \tag{5.52}$$

By using this equation we see that all functions $\Psi(t;n)$ for different n can be reduced to just two functions, $\Psi(t;0)$ and $\Psi(t;1)$, namely, for $n \geq 0$

$$\Psi(t;2n) = \Psi(t;0) - t\sum_{l=0}^{n-1}(2l+1)\exp\left\{-t\left(l+\frac{1}{2}\right)^2\right\}, \tag{5.53}$$

$$\Psi(t;2n+1) = \Psi(t;1) - 2t\sum_{l=1}^{n} l e^{-tl^2}. \tag{5.54}$$

Also, it is not difficult to see that the function $\Psi(t;0)$ can be expressed in terms of the function $\Psi(t;1)$ as follows

$$\Psi(t;0) = 2\Psi\left(\frac{t}{4};1\right) - \Psi(t;1). \tag{5.55}$$

This property can be obtained directly from the integral representation (5.41).

One can also show that the function $\Psi(t;n)$ has the following dual integral representation

$$\Psi(t;n) = t\int_\Gamma d\nu\ \tan(\pi\nu) i\left(\nu+\frac{|n|}{2}\right)\exp\left[-t\left(\nu+\frac{|n|}{2}\right)^2\right], \tag{5.56}$$

where Γ is a contour that goes counterclockwise from $(\infty + i\varepsilon)$ to $(\frac{1}{2} + i\varepsilon)$, then around the point $\frac{1}{2}$ between 0 and $\frac{1}{2}$ on the real axis to the point $(\frac{1}{2} - i\varepsilon)$, and finally to $(\infty - i\varepsilon)$.

5.3.3 Heat Trace and the Spectrum

By using the spectral representation of the function $\Psi(t;n)$ we can write the heat kernel diagonal of the Laplacian in the form

$$\begin{aligned} U^{\text{diag}}(t) &= \frac{1}{4\pi a^2}\sum_{l=0}^{\infty} 2\left(l+\frac{1}{2}+|G|\right)\exp\left\{-\frac{t}{a^2}\left[\left(l+\frac{1}{2}+|G|\right)^2-\frac{1}{4}+G^2\right]\right\} \\ &= \frac{1}{4\pi a^2}\sum_{l=0}^{\infty} d_l\,(2|G|)\exp\left\{-\frac{t}{a^2}\lambda_l\,(2|G|)\right\}, \end{aligned} \tag{5.57}$$

where

$$\lambda_l(n) = l(l+n+1)+\frac{n}{2}, \tag{5.58}$$

$$d_l(n) = 2l+n+1. \tag{5.59}$$

Then the heat trace of the Laplace type operator $L = -\Delta + Q$ is

$$\text{Tr}\exp(-tL) = \text{tr}\sum_{l=0}^{\infty} d_l\,(2|G|)\exp\left\{-\frac{t}{a^2}\left[\lambda_l\,(2|G|)+a^2Q\right]\right\}. \tag{5.60}$$

Recall that G is a generator of $SO(2)$ in some N-dimensional representation. We can diagonalize as follows

$$G = \text{diag}\left(i\frac{m_1}{2},\dots,i\frac{m_N}{2}\right), \tag{5.61}$$

where $m_1, \ldots, m_N$ are some integers. Since the matrix Q commutes with the matrix G, it can be diagonalized simultaneously with G, that is,

$$Q = \mathrm{diag}\,(Q_1, \ldots, Q_N)\,. \tag{5.62}$$

By using this decomposition the heat trace takes the form

$$\mathrm{Tr}\,\exp(-tL) \;=\; \sum_{j=1}^{N}\sum_{l=0}^{\infty} d_l(|m_j|)\exp\left(-\frac{t}{a^2}\lambda_{l,j}\right), \tag{5.63}$$

where

$$\lambda_{l,j} \;=\; \lambda_l(|m_j|) + a^2 Q_j\,, \qquad j = 1, \ldots, N,\,. \tag{5.64}$$

Therefore, the eigenvalues of the operator L are $\frac{1}{a^2}\lambda_{l,j}$, $j = 1, \ldots, N$; $l = 0, 1, 2, \ldots$, with multiplicities $d_l(|m_j|)$. The smallest eigenvalue of the operator L is

$$\lambda_{\min}(L) = \frac{1}{a^2}\min_{1\le j\le N}\left\{\lambda_0(|m_j|) + Q_j\right\} = \min_{1\le j\le N}\left\{\frac{1}{a^2}\frac{|m_j|}{2} + Q_j\right\}. \tag{5.65}$$

In terms of the function $\Psi(t;n)$ the heat trace can be written as

$$\mathrm{Tr}\,\exp(-tL) \;=\; \frac{a^2}{t}\sum_{j=1}^{N}\exp\left[\frac{t}{a^2}\left(\frac{(1+m_j^2)}{4} - a^2 Q_j\right)\right]\Psi\left(\frac{t}{a^2}; m_j\right). \tag{5.66}$$

5.3.4 Asymptotics

The asymptotics of the function $\Psi(t;n)$ as $t \to \infty$ can be directly obtained from the eigenvalue representation; it is given by the lowest eigenvalue

$$\Psi(t;n) \sim (1+|n|)\,t\exp\left\{-t\frac{(1+|n|)^2}{4}\right\} + (3+|n|)\,t\exp\left\{-t\frac{(3+|n|)^2}{4}\right\} + \cdots. \tag{5.67}$$

To compute the asymptotics as $t \to 0$ we use the rescaled representation (5.43). This form is particularly useful to compute the short-time asymptotics as $t \to 0$; we just expand the exponent in the powers of t and compute the Gaussian integrals over ω. The asymptotics as $t \to 0$ are

$$\Psi(t;n) \sim \sum_{k=0}^{\infty} c_k(n)t^k, \tag{5.68}$$

where

$$c_0(n) \;=\; 1, \tag{5.69}$$

$$c_1(n) \;=\; \frac{1}{12} - \frac{n^2}{4}, \tag{5.70}$$

$$c_2(n) \;=\; \frac{7}{480} - \frac{n^2}{16} + \frac{n^4}{32}, \tag{5.71}$$

$$c_3(n) \;=\; \frac{31}{8064} - \frac{7n^2}{384} + \frac{5n^4}{384} - \frac{n^6}{384}. \tag{5.72}$$

Therefore, the heat kernel diagonal asymptotics as $t \to 0$ are

$$U^{\rm diag}(t) \sim (4\pi t)^{-1} \sum_{k=0}^{\infty} a_k \left(\frac{t}{a}\right)^k , \tag{5.73}$$

where

$$a_0 = 1, \qquad a_1 = \frac{1}{3}, \qquad a_2 = \frac{1}{15} + \frac{1}{6}G^2 . \tag{5.74}$$

This coincides with the well-known general heat kernel coefficients (3.21).

Moreover, this integral can be used to compute the asymptotics of the heat kernel diagonal as $a \to \infty$. It amounts to just shifting the contour of integration in (5.31) by $\omega \mapsto \omega + 2Gt/a^2$ to get

$$U^{\rm diag}(t) = \frac{1}{4\pi t} \exp\left(\frac{t}{4a^2}\right) \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}} \exp\left(-\frac{\omega^2}{4}\right) \frac{\omega\sqrt{t}/(2a) + Gt/a^2}{\sin\left[\omega\sqrt{t}/(2a) + Gt/a^2\right]} \tag{5.75}$$

Next, let $G = T + a^2 H$, where $T = T(\Sigma)$ is a generator of the group $SO(2)$ in the representation T and $H = -X(\Sigma)/a^2$ is the generator of the group $SO(2)$ in the representation X taking values in the adjoint representation of the Lie algebra of the gauge group. Then

$$U^{\rm diag}(t) = \frac{1}{4\pi t} \exp\left(\frac{t}{4a^2}\right) \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{4\pi}} \exp\left(-\frac{\omega^2}{4}\right) \frac{tH + \omega\sqrt{t}/(2a) + tT/a^2}{\sin\left[tH + \omega\sqrt{t}/(2a) + tT/a^2\right]} . \tag{5.76}$$

Now, the asymptotics as $a \to \infty$ can be computed by expanding the integrand in a Taylor series in inverse powers of a and computing the Gaussian integrals over ω. This gives the correct leading asymptotics that coincides with the Euclidean space limit (with the magnetic field H)

$$\lim_{a\to\infty} U^{\rm diag}(t) = \frac{1}{4\pi} \frac{H}{\sin(tH)} . \tag{5.77}$$

Of course, such asymptotics describe the behavior of the heat kernel for large a and fixed t. The behavior of the heat kernel for large t is sensitive to the value of a, it is described by the lowest eigenvalue of the Laplacian, which is different on the sphere S^2 and on the Euclidean space $\mathbb{R}^2$.

5.3.5 Spectral Approach

As an alternative, let us compute the spectrum of the Laplacian directly. The Laplacian (4.44) is a differential operator acting on the space $L^2([0, a\pi] \times [0, 2\pi], a\sin(r/a)dr\, d\varphi)$. We will find it useful to introduce a new variable $x = \cos(r/a)$. Then the Laplacian acts on the space $L^2([-1, 1] \times [0, 2\pi], a^2 dx\, d\varphi)$ and takes the form

$$\Delta = \frac{1}{a^2}\left\{(1 - x^2)\partial_x^2 - 2x\partial_x + \frac{1}{1 - x^2}\left[\partial_\varphi + (1 - x)G\right]^2\right\} . \tag{5.78}$$

Let α, β be *positive* Hermitian operators defined by

$$\alpha = \left|\partial_\varphi\right| , \qquad \beta = \left|\partial_\varphi + 2G\right| , \tag{5.79}$$

and

$$\rho(x) = (1-x)^\alpha (1+x)^\beta . \tag{5.80}$$

We intertwine the Laplacian as follows

$$\begin{aligned} \tilde{\Delta} &= \rho^{-1/2} \Delta \rho^{1/2} \\ &= -\frac{1}{a^2}\left(K_{\alpha,\beta} + \frac{1}{4}(\alpha+\beta+1)^2 - \frac{1}{4} + G^2\right) , \end{aligned} \tag{5.81}$$

where

$$K_{\alpha,\beta} = -(1-x^2)\partial_x^2 - [\beta - \alpha - (\alpha+\beta+2)x]\,\partial_x . \tag{5.82}$$

Then the operator $\tilde{\Delta}$ has the same spectrum and, therefore, the same heat trace as the original Laplacian. The operator $K_{\alpha,\beta}$ is an elliptic self-adjoint operator on $L^2([-1,1]\times[0,2\pi],\rho)$ with the weight function $\rho(x)$. It is well-known [15] that the operator $K_{\alpha,\beta}$ (with positive real $\alpha, \beta > 0$) has a discrete spectrum with simple eigenvalues

$$\lambda_l(K_{\alpha,\beta}) = l(l+\alpha+\beta+1) , \tag{5.83}$$

where $l = 0, 1, \ldots$, with eigenfunctions proportional to the Jacobi polynomials $P_l^{(\alpha,\beta)}(x)$.

This immediately enables us to compute the heat trace of a Laplace type operator $L = -\Delta + Q$ as follows

$$\mathrm{Tr}\ \exp(-tL) = \sum_{l=0}^\infty \mathrm{Tr}\ \exp\left\{-\frac{t}{a^2}\left(\left(l + \frac{1}{2}(\alpha+\beta+1)\right)^2 + G^2 - \frac{1}{4} + a^2 Q\right)\right\} , \tag{5.84}$$

where the trace Tr in the last equation is the combined trace over the operator ∂_φ and the fiber trace over the matrix G. Since the eigenvalues of the operator ∂_φ are purely imaginary integers, im, with $m \in \mathbb{Z}$, this can finally be written as

$$\begin{aligned} \mathrm{Tr}\ \exp(-tL) &= \mathrm{tr}\ \exp\left\{-\frac{t}{a^2}\left(G^2 - \frac{1}{4} + a^2 Q\right)\right\} \\ &\times \sum_{l=0}^\infty \sum_{m=-\infty}^\infty \exp\left\{-\frac{t}{a^2}\left[l + \frac{1}{2}(|m| + |m - 2iG| + 1)\right]^2\right\} , \end{aligned} \tag{5.85}$$

where the trace tr is now over the fiber indices only.

Now, one can show that this sum is equal to the previous result (5.57), (5.60), obtained by a completely different algebraic approach.

6 Effective Action

6.1 Yang-Mills Effective Action

The magnitude of the magnetic field in our scenario can be written as $|\mathcal{F}|^2 = \frac{1}{4}K/a^4$, where $K = \sum_{i=1}^{p} n_i^2$, with p being the number of positive roots of the gauge algebra, and, therefore, the classical (Euclidean) action on the covariantly constant background $S^1 \times S^1 \times S^2$ is

$$S = 16\pi^3\sigma\frac{1}{xy}\,, \tag{6.1}$$

where $\sigma = K/8e^2$,

$$x = \frac{a}{a_1}, \qquad y = \frac{a}{a_2}\,, \tag{6.2}$$

and a_1 and a_2 are the radii of the two circles. The total effective action is given by the sum of this classical part and the one-loop effective action (3.10).

For the Yang-Mills ghost operator, the endomorphism Q and the generator G are

$$Q^{YM}_{\rm ghost} = 0, \qquad G^{YM}_{\rm ghost} = -X\,, \tag{6.3}$$

where X is the generator of the spin group Spin(2) taking values in the adjoint representation of the Lie group G. The volume of the manifold $M = S^1 \times S^1 \times S^2$ is $\mathrm{vol}\,(M) = 16\pi^3 a^2 a_1 a_2$. By taking into account the contribution of the torus $S^1 \times S^1$, the heat trace of the Yang-Mills ghost operator is

$$\Theta^{YM}_{\rm ghost}(t) = 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right)\mathrm{tr}_{\,Ad}\exp\left[\frac{t}{a^2}\left(\frac{1}{4} - X^2\right)\right]\Psi\left(\frac{t}{a^2}; 2iX\right). \tag{6.4}$$

For the Yang-Mills operator $L_{\rm vect}$ the endomorphism $Q_{\rm vect}$ is given by

$$a^2 Q_{\rm vect} = P + 2EX\,, \tag{6.5}$$

where P and E are 4×4 real matrices of the form

$$P^a{}_b = \delta^a{}_2\delta_{2b} + \delta^a{}_3\delta_{3b}\,, \qquad E^a{}_b = \delta^a{}_2\delta_{3b} - \delta^a{}_3\delta_{2b}\,. \tag{6.6}$$

Notice that P is a 2-dimensional projection, that is, $P^2 = P$, and $E^2 = -P$. The generator $G_{\rm vect}$ is

$$G_{\rm vect} = E - X\,. \tag{6.7}$$

Therefore,

$$G^2_{\rm vect} + a^2 Q_{\rm vect} = X^2. \tag{6.8}$$

So, again, by adding the corresponding factors for the torus $S^1 \times S^1$ we get the heat kernel diagonal of the Yang-Mills operator

$$\Theta_{\rm vect}(t) = 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right)\mathrm{tr}_{\,Ad}\exp\left[\frac{t}{a^2}\left(\frac{1}{4} - X^2\right)\right]\mathrm{tr}_{\,T_1}\Psi\left(\frac{t}{a^2}; 2i(X-E)\right). \tag{6.9}$$

Thus, the total Yang-Mills heat trace is

$$\Theta_{YM}(t) = 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right) W_{YM}\left(\frac{t}{a^2}\right), \tag{6.10}$$

where

$$\begin{aligned} W_{YM}(t) = & \int_C \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} \\ & \times \mathrm{tr}_{Ad}\left\{\exp\left[t\left(\frac{1}{4} - X^2\right) - X\omega\right][\mathrm{tr}_{T_1} \exp(E\omega) - 2]\right\}. \end{aligned} \tag{6.11}$$

The trace over the vector representation can be easily computed. By using

$$\exp(E\omega) = \mathbb{I} - P + P\cos\omega + E\sin\omega, \tag{6.12}$$

we get

$$\mathrm{tr}_{T_1} \exp(E\omega) = 2 + 2\cos\omega. \tag{6.13}$$

Thus, the function $W_{YM}(t)$ is

$$W_{YM}(t) = \int_C \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} 2\cos\omega \, \mathrm{tr}_{Ad} \exp\left[t\left(\frac{1}{4} - X^2\right) - X\omega\right]. \tag{6.14}$$

Recall that the eigenvalues of the matrix X are imaginary half-integers

$$\mathrm{Spec}(X) = \left\{\underbrace{0,\dots,0}_{r}, i\frac{n_1}{2}, -i\frac{n_1}{2}, \dots, i\frac{n_p}{2}, -i\frac{n_p}{2}, \right\}, \tag{6.15}$$

where p is the number of positive roots, r is the rank of the group, and $n_1, \dots, n_p$ are some integers that we call monopole numbers.

Therefore, the trace over the adjoint representation can be also computed to take more explicit form

$$\begin{aligned} W_{YM}(t) = & \sum_{j=1}^{p} \exp\left[\frac{t(1+n_j^2)}{4}\right] \int_C \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} 4\cos\left[n_j\omega/2\right]\cos\omega \\ & + r\exp\left(\frac{t}{4}\right) \int_C \frac{d\omega}{\sqrt{4\pi t}} \exp\left\{-\frac{\omega^2}{4t}\right\} \frac{\omega/2}{\sin[\omega/2]} 2\cos\omega, \end{aligned} \tag{6.16}$$

which can be written in terms of the function $\Psi(t;n)$

$$W_{YM}(t) = 2\sum_{j=1}^{p} \exp\left[\frac{t(1+n_j^2)}{4}\right] \left\{\Psi(t; 2+|n_j|) + \Psi(t; 2-|n_j|)\right\} + 2re^{t/4}\Psi(t;2). \tag{6.17}$$

The ultraviolet (high-energy) properties of the effective action depend of the asymptotics of the function W_{YM} as $t \to 0$. By using the asymptotics of the function $\Psi(t; n)$ we obtain

$$W_{YM}(t) \sim 2m + C_1 t + C_2 t^2 + \cdots , \tag{6.18}$$

where

$$C_1 = -\frac{19}{6} p - \frac{4}{3} r - \frac{4}{3} \sum_{j=1}^{p} n_j^2 , \tag{6.19}$$

$$C_2 = \frac{4}{15} p + \frac{19}{30} r + \frac{11}{6} \sum_{j=1}^{p} n_j^2 . \tag{6.20}$$

By using this result, we obtain easily

$$\beta_{YM} = 16\pi^3 \frac{1}{xy} \left(C_2 - za^2 C_1 + z^2 a^4 m \right) . \tag{6.21}$$

Now, by subtracting enough terms of the Taylor expansion at $t = 0$ we obtain the renormalized heat trace

$$\Theta^{\rm ren}_{YM}(a^2 t) = 16\pi^3 \frac{a^4}{xy} \left\{ \Omega\left(x^2 t\right) \Omega\left(y^2 t\right) W_{YM}(t) - e^{-ta^2\lambda} R_{YM}(t; a^2\lambda) \right\} , \tag{6.22}$$

where

$$R_{YM}(t; a^2\lambda) = 2m + \left(C_1 + 2ma^2\lambda \right) t + \left(C_2 + C_1 a^2 \lambda + ma^4 \lambda^2 \right) t^2 . \tag{6.23}$$

Then the one-loop Yang-Mills effective action is

$$\Gamma_{(1)\,YM}(x, y, \mu, a^2 z) = -\frac{\pi}{2} \frac{1}{xy} \left\{ \left(C_2 - za^2 C_1 + z^2 a^4 m \right) \log \frac{\mu^2}{\lambda} + \Phi_{YM}\left(x, y; a^2\lambda, a^2 z \right) \right\} , \tag{6.24}$$

where

$$\Phi_{YM}(x, y; a^2\lambda, a^2 z) = \int_0^\infty \frac{dt}{t^3} e^{-ta^2 z} \left\{ \Omega\left(x^2 t\right) \Omega\left(y^2 t\right) W_{YM}(t) - e^{-ta^2\lambda} R_{YM}(t; a^2\lambda) \right\} . \tag{6.25}$$

The convergence properties of this integral as $t \to \infty$ (infrared region, low energies) crucially depends on the asymptotics of the function $W_{YM}(t)$ as $t \to \infty$. Let

$$\lambda_0(n) = 1 - |n| + \left| 1 - \frac{|n|}{2} \right| = \begin{cases} 2 & \text{for } n = 0, \\ \frac{1}{2} & \text{for } |n| = 1, \\ -\frac{|n|}{2} & \text{for } |n| \geq 2 . \end{cases} \tag{6.26}$$

Let

$$\lambda_{\min} = \min_{1 \leq j \leq p} \lambda_0(n_j) \tag{6.27}$$

and

$$c = \sum_{\substack{1\le j\le p \\ \lambda_0(n_j)=\lambda_{\rm min}}} \Big(1 + |2 - |n_j||\Big)\,, \tag{6.28}$$

where the summation goes over all j such that $\lambda_0(n_j) = \lambda_{\rm min}$. Then as $t \to \infty$

$$W_{YM}(t) \sim 2tce^{-t\lambda_{\rm min}} + 6rte^{-2t} + \cdots\,. \tag{6.29}$$

Thus we see that for $|n_j| \ge 2$ the lowest eigenvalue is negative, which leads to the exponential growth of the function $W_{YM}(t)$ as $t \to \infty$ and, as a result, to the infrared divergence of the integral for the zeta-function and the effective action and to the instability of the chromomagnetic vacuum. The only stable configurations are those in which all monopole numbers are equal to 0 or ± 1. Recall that the flat space limit $a \to \infty$ is recovered as $X = -a^2 H \to \infty$, keeping H constant. That is, in the limit $a \to \infty$ the matrix $|X| \to \infty$ and, of course, the monopole numbers $|n_j| \to \infty$ (which can be interpreted physically as condensation of magnetic monopoles) leading to the instability. This is nothing but the instability of the Savvidy chromomagnetic vacuum in the flat Euclidean space.

However, as we have shown above since the monopole numbers n_j are not independent but are rather determined by the roots of the Lie algebra $\mathfrak{g}$ it is impossible to have all monopole numbers bounded between -1 and 1. Therefore, the lowest eigenvalue is necessarily negative and is equal to

$$\lambda_{\rm min} = -\frac{1}{2} n_{\rm max}\,, \tag{6.30}$$

where $n_{\rm max} = \max_{1\le j\le p} |n_j|$ is the largest monopole number. Since $n_{\rm max} \ge 2$, then the constant c is now

$$c = (n_{\rm max} - 1)q\,, \tag{6.31}$$

where q is the multiplicity of the largest monopole number.

Let us consider the group $SU(N)$ for simplicity. Then the monopole numbers are $n_{ij} = k_i - k_j$, where k_i are some integers whose sum is equal to zero. For $N = 2$ we have just two non-zero integers, k_1 and k_2, whose sum is equal to zero. Then there is one non-zero monopole number

$$n_1 = 2k_1\,. \tag{6.32}$$

Therefore,

$$n_{\rm max} = 2|k_1| \ge 2 \qquad \text{and} \qquad \lambda_{\rm min} = -|k_1| \le -1. \tag{6.33}$$

For $N = 3$ there are three non-zero integers k_1, k_2, k_3 whose sum is equal to zero. Then there are three non-zero monopole numbers

$$n_1 = k_1 - k_2\,, \qquad n_2 = 2k_1 + k_2\,, \qquad n_3 = k_1 + 2k_2\,. \tag{6.34}$$

Then, if $k_1 = k_2$,

$$n_{\rm max} = 3|k_1| \ge 3\,. \tag{6.35}$$

If $k_1 > k_2 > 0$, then

$$n_{\rm max} = 2k_1 + k_2 \ge 3\,. \tag{6.36}$$

If $k_2 < 0 < 2|k_2| \leq k_1$, then

$$n_{\max} = 2k_1 - |k_2| \geq 3\,. \tag{6.37}$$

If $k_2 < 0 < k_1$ and $|k_2| \leq 2k_1 \leq 4|k_2|$, then

$$n_{\max} = k_1 + |k_2| \geq 3\,. \tag{6.38}$$

Finally, if If $k_2 < 0 < k_1$ and $2k_1 \leq |k_2|$, then

$$n_{\max} = 2|k_2| - k_1 \geq 3\,. \tag{6.39}$$

In this case

$$\lambda_{\min} \leq -\frac{3}{2}\,. \tag{6.40}$$

More generally, one can show that for the group $SU(N)$ we have $n_{\max} \geq 2$ for even N and $n_{\max} \geq 3$ for odd N. The equality is achieved by choosing the non-zero integers $k_1, \dots, k_N$, (whose sum is equal to zero) such that there are as many as possible (monopole-antimonopole) pairs $(+1, -1)$. Then for even N, the largest difference $|k_i - k_j|$ is equal to 2. For odd N, we will have as many pairs $(+1, -1)$ as possible and a triple $(2, -1, -1)$. Then the largest difference $|k_i - k_j|$ is equal to 3. Thus, the bottom eigenvalue for the group $SU(N)$ is less or equal to (-1) for even N and less or equal to $(-3/2)$ for odd N.

Let us compute this function for the group $SU(2N)$ so that $m = 4N^2 - 1$, $r = 2N - 1$ and $p = N(2N - 1)$, in the simple case when the numbers k_j are chosen as N pairs $(+1, -1)$, that is, $k_1 = \cdots = k_N = 1$ and $k_{N+1} = \cdots = k_{2N} = -1$. Then there are only N^2 nonzero monopole numbers which are equal to 2 and the rest are equal to zero. Then

$$\begin{aligned} W_{YM}^{SU(2N)}(t) &= 2N^2 e^{5t/4}\left\{\Psi(t;4) + \Psi(t;0)\right\} + 2(2N^2 - 1)e^{t/4}\Psi(t;2) \\ &= t\left[2N^2 e^t + 6N^2 e^{-t} + 6(2N^2 - 1)e^{-2t}\right] \\ &\quad + \left[4N^2 e^{5t/4} + 2(2N^2 - 1)e^{t/4}\right]\Psi(t;4)\,. \end{aligned} \tag{6.41}$$

The asymptotics of this function as $t \to \infty$ are

$$W_{YM}^{SU(2N)}(t) = 2N^2 te^t + O(te^{-t})\,. \tag{6.42}$$

A few remarks are in order. First of all, the effective action does not depend on the parameter λ; it can be chosen arbitrarily, for example, $\lambda = 1/a^2$. Second, for a sufficiently large $\mathrm{Re}\, z$ the function $\Phi_{YM}(x, y; a^2\lambda, a^2 z)$, as a function of z, is analytic. Since the function $W_{YM}(t)$ grows exponentially as $t \to \infty$, the function Φ_{YM} has a branching singularity at

$$z_0 = -\frac{\lambda_{\min}}{a^2} = \frac{1}{2a^2} n_{\max}\,. \tag{6.43}$$

6.2 Effective Action of Matter Fields

For the scalar operator the potential term is Q_0, which is determined by the second derivative of the scalar potential $V(\phi)$, and the generator G_0 is

$$G_0 = -X_0\,, \tag{6.44}$$

where X_0 is the generator of the spin group Spin(2) taking values in the representation W_0 of the gauge group. Therefore, the heat trace of the operator L_0 is

$$\Theta_0(t) = 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right)\mathrm{tr}_{\,W_0}\exp\left[-\frac{t}{a^2}\left(X_0^2 + a^2 Q_0 - \frac{1}{4}\right)\right]\Psi\left(\frac{t}{a^2}; -2iX_0\right). \tag{6.45}$$

For the spinor operator the generator $G_{\rm spin}$ is now

$$G_{\rm spin} = \frac{1}{2}\gamma - X_{\rm spin}\,, \tag{6.46}$$

where $\gamma = \gamma_3\gamma_4$, with γ_a being the Dirac matrices, so that $\frac{1}{2}\gamma$ is the generator of the spinor representation; of course, $\gamma^2 = -\mathbb{I}$, and $X_{\rm spin} = X_{\rm spin}(\Sigma_{34})$ is the generator of the spin group Spin(2) taking values in the representation $W_{\rm spin}$ of the gauge group. The endomorphism $Q_{\rm spin}$ is

$$Q_{\rm spin} = \frac{1}{2a^2}\mathbb{I} + \frac{1}{a^2}\gamma X_{\rm spin} + M^2\,. \tag{6.47}$$

Then

$$G^2_{\rm spin} + a^2 Q_{\rm spin} = \frac{1}{4}\mathbb{I} + X^2_{\rm spin} + a^2 M^2\,. \tag{6.48}$$

Therefore, the heat trace for the operator $L_{\rm spin}$ is

$$\begin{aligned}\Theta_{\rm spin}(t) = {} & 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right)\mathrm{tr}_{\,W_{\rm spin}}\mathrm{tr}_{\,T_{\rm spin}}\exp\left[-\frac{t}{a^2}\left(X^2_{\rm spin} + a^2 M^2\right)\right] \\ & \times\Psi\left(\frac{t}{a^2}; i(2X_{\rm spin} - \gamma)\right).\end{aligned} \tag{6.49}$$

Thus the total heat trace for the matter fields reads

$$\Theta_{\rm mat}(t) = 16\pi^3 a^2 a_1 a_2 \Omega\left(\frac{t}{a_1^2}\right)\Omega\left(\frac{t}{a_2^2}\right) W_{\rm mat}\left(\frac{t}{a^2}\right), \tag{6.50}$$

where

$$\begin{aligned}W_{\rm mat}(t) = {} & \int_C \frac{d\omega}{\sqrt{4\pi t}}\;\exp\left\{-\frac{\omega^2}{4t}\right\}\frac{\omega/2}{\sin[\omega/2]} \\ & \times\Bigg\{\mathrm{tr}_{\,W_0}\exp\left[-t\left(X_0^2 + a^2 Q_0 - \frac{1}{4}\right) - X_0\omega\right] \\ & -\mathrm{tr}_{\,W_{\rm spin}}\mathrm{tr}_{\,T_{\rm spin}}\exp\left[-t\left(X^2_{\rm spin} + a^2 M^2\right) + \left(\frac{1}{2}\gamma - X_{\rm spin}\right)\omega\right]\Bigg\}.\end{aligned} \tag{6.51}$$

To proceed further, we assume that the matrix M does not transform with respect to the spinor representation (it does not have spinor indices, but only the group indices). Then by using

$$\exp\left[\frac{1}{2}\gamma\omega\right] = \cos(\omega/2)\mathbb{I} + \gamma\sin(\omega/2)\,, \tag{6.52}$$

the trace over spinor representation is easily computed

$$\mathrm{tr}_{\,T_{\rm spin}}\exp\left[\frac{1}{2}\gamma\omega\right] = 4\cos(\omega/2)\,. \tag{6.53}$$

Thus, we get

$$\begin{aligned} W_{\rm mat}(t) \;=\; & \int\limits_C \frac{d\omega}{\sqrt{4\pi t}}\;\exp\left\{-\frac{\omega^2}{4t}\right\}\frac{\omega/2}{\sin\left[\omega/2\right]} \\ & \times\left\{\mathrm{tr}_{\,W_0}\exp\left[-t\left(X_0^2 + a^2Q_0 - \frac{1}{4}\right) - X_0\omega\right]\right. \\ & \left. -4\mathrm{tr}_{\,W_{\rm spin}}\exp\left[-t\left(X_{\rm spin}^2 + a^2M^2\right) - X_{\rm spin}\omega\right]\cos(\omega/2)\right\}. \end{aligned} \tag{6.54}$$

The eigenvalues of the matrices X_0 and $X_{\rm spin}$ must be imaginary half-integers

$$\mathrm{Spec}(X_0) = \left\{i\frac{m_1}{2},\ldots,i\frac{m_{N_0}}{2},\right\}, \tag{6.55}$$

$$\mathrm{Spec}(X_{\rm spin}) = \left\{i\frac{k_1}{2},\ldots,i\frac{k_{N_{\rm spin}}}{2},\right\}, \tag{6.56}$$

where m_i and k_j are some integers determined by the weights of the representations realized by spinors and scalars.

For simplicity, we assume that the symmetry is not broken so that all scalar fields have the same mass m_0 and all spinor fields have mass $m_{\rm spin}$. Then the potential terms Q_0 and M are proportional to the identity, that is,

$$Q_0 = m_0^2\mathbb{I}\,, \qquad M = m_{\rm spin}\mathbb{I}\,. \tag{6.57}$$

In this case

$$\begin{aligned} W_{\rm mat}(t) \;=\; & \exp\left[\left(\frac{1}{4} - a^2m_0^2\right)t\right]\sum_{j=1}^{N_0} e^{tm_j^2/4}\Psi(t;m_j) \\ & -2e^{-ta^2m_{\rm spin}^2}\sum_{j=1}^{N_{\rm spin}} e^{tk_j^2/4}\left[\Psi(t;1-k_j) + \Psi(t;1+k_j)\right]. \end{aligned} \tag{6.58}$$

The leading asymptotics of this function as $t\to\infty$ is

$$W_{\rm mat}(t) \sim r_0te^{-ta^2m_0^2} - 2\sum_{j=1}^{N_{\rm spin}}|k_j|\;te^{-ta^2m_{\rm spin}^2}\,, \tag{6.59}$$

where r_0 is the number of zeros among the scalar weights m_j.

This indicates infrared instability for massless fields, when $m_0 = m_{\rm spin} = 0$. It is interesting to note that the infrared instability of massless spinor fields is not caused by the zero numbers k_j; in fact, that contribution is proportional to $\Psi(t;1)$ and has a nice exponentially decreasing behavior at $t \to \infty$. This instability is intrinsic and cannot be avoided if there are non-zero numbers k_j. This simply means that massless Dirac operator with a constant non-zero magnetic field always has zero modes on S^2. We can even compute the multiplicity of the zero eigenvalue, that is, the dimension of the kernel of massless Dirac operator,

$$\dim \operatorname{Ker} L_{\rm spin}\Big|_{m_{\rm spin}=0} = 2\sum_{j=1}^{N_{\rm spin}} |k_j| = 4 \operatorname{tr}_{W_{\rm spin}} |X_{\rm spin}| \,. \tag{6.60}$$

Let us compute the contribution of spinors in the fundamental representation of the group $SU(2N)$ when the numbers $k_1, \dots, k_{2N}$ are chosen as $k_1 = \cdots = k_N = 1$ and $k_{N+1} = \cdots = k_{2N} = -1$, so that $N_{\rm spin} = 2N$. Then

$$\begin{aligned} W^{SU(2N)}_{\rm spin}(t) &= -4N \exp\left[\left(\frac{1}{4} - a^2 m^2_{\rm spin}\right)t\right] [\Psi(t;2) + \Psi(t;0)] \\ &= -4Nte^{-a^2 m^2_{\rm spin} t} - 8N \exp\left[\left(\frac{1}{4} - a^2 m^2_{\rm spin}\right)t\right] \Psi(t;2)\,. \end{aligned} \tag{6.61}$$

Now, the heat trace for matter fields can be renormalized in the same fashion as we have done for Yang-Mills fields. Then we get the effective action for matter fields exactly in the same form as for the Yang-Mills fields with the function $\Phi_{\rm mat}(x, y, a^2\lambda, a^2 z)$ that is expressed in terms of the function $W_{\rm mat}(t)$ exactly in the same way as for the Yang-Mills fields.

7 Thermodynamics of Yang-Mills Theory

In this section we investigate the entropy and the heat capacity of the gluon gas. The temperature is related to the radius of the first circle by $T = \frac{1}{2\pi a_1}$. The volume of the space is expressed in terms of the radius of the second circle and the radius of the sphere $V = 8\pi^2 a_2 a^2$.

For a canonical statistical ensemble with fixed T and V the free energy $F = E - TS$ is a function of T, V defined by

$$F = T\Gamma = \frac{1}{2\pi}\frac{\Gamma}{a_1}\,. \tag{7.1}$$

Then the entropy is defined by

$$S = -\frac{\partial}{\partial T} F = a_1^2 \frac{\partial}{\partial a_1}\frac{\Gamma}{a_1} \tag{7.2}$$

and the heat capacity at constant volume is

$$C_v = T\frac{\partial}{\partial T} S = -a_1^2 \left(a_1 \frac{\partial^2}{\partial a_1^2}\frac{\Gamma}{a_1} + 2\frac{\partial}{\partial a_1}\frac{\Gamma}{a_1}\right)\,. \tag{7.3}$$

By using our results for the effective action we obtain

$$\frac{\Gamma}{a_1} = 16\pi^3\sigma\frac{a_2}{a^2} - \frac{\pi}{2}\frac{a_2}{a^2}\left\{b\log\frac{\mu^2}{\lambda} + \Phi\left(\frac{a}{a_1}, \frac{a}{a_2}; a^2\lambda, a^2 z\right)\right\}, \tag{7.4}$$

where the function Φ is given by the sum of the corresponding functions for the Yang-Mills fields and the matter fields, $\Phi = \Phi_{YM} + \Phi_{\rm mat}$, and

$$b = \frac{1}{16\pi^3}\frac{a^2}{a_1 a_2}\left(\beta_{YM} + \beta_{\rm mat}\right). \tag{7.5}$$

It is easy to see that neither the classical part nor the coefficient b depend on the temperature. Therefore, the entropy and the heat capacity do not depend neither on the classical term nor on the renormalization parameter μ. Recall also that the parameter λ is completely arbitrary; so we can set it equal to $\lambda = 1/a^2$. Thus the entropy and the heat capacity of the quark-gluon gas are given by the derivatives of the functions Φ

$$S = \frac{\pi}{2y}\Phi_x(x, y, 1, a^2 z), \tag{7.6}$$

$$C_v = \frac{\pi}{2}\frac{x}{y}\Phi_{xx}(x, y, 1, a^2 z) = \frac{\pi}{2}\frac{a_2}{a_1}\Phi_{xx}(x, y, 1, a^2 z). \tag{7.7}$$

The entropy and the heat capacity per unit volume are

$$\frac{S}{V} = \frac{1}{16\pi a^3}\Phi_x(x, y, 1, a^2 z), \tag{7.8}$$

$$\frac{C_v}{V} = \frac{1}{16\pi a_1 a^2}\Phi_{xx}(x, y, 1, a^2 z). \tag{7.9}$$

Now, by differentiating the function Φ with respect to x we get

$$\Phi_x(x, y; 1, a^2 z) = 2x\int_0^\infty \frac{dt}{t^2}e^{-ta^2 z}\Omega'\left(x^2 t\right)\Omega\left(y^2 t\right)W(t), \tag{7.10}$$

$$\begin{aligned}\Phi_{xx}(x, y; 1, a^2 z) &= 2\int_0^\infty \frac{dt}{t^2}e^{-ta^2 z}\left\{\Omega'\left(x^2 t\right) + 2x^2 t\Omega''\left(x^2 t\right)\right\}\Omega\left(y^2 t\right)W(t) \\ &= 2x^2\int_0^\infty \frac{dt}{t^2}e^{-ta^2 z/x^2}\left\{\Omega'(t) + 2t\Omega''(t)\right\}\Omega\left(\frac{y^2}{x^2}t\right)W\left(\frac{t}{x^2}\right),\end{aligned} \tag{7.11}$$

where

$$W(t) = W_{YM}(t) + W_{\rm mat}(t). \tag{7.12}$$

Now by using the asymptotics of the function Ω we easily see that the integrals (7.10) and (7.11) converge at $t \to 0$. Moreover, for large z these integrals also converge as $t \to \infty$. Since the function W_{YM} increases exponentially at infinity, the function Φ has a singularity at a finite positive value of z. However, because of the asymptotics of the function $\Omega'(t) + 2t\Omega''(t)$ as $t \to \infty$, (5.19), we immediately see that the integral (7.11) for the heat capacity may converge even in the infrared limit $z \to 0$, due to the presence of an extra exponential factor e^{-t}. Namely, for the groups $SU(2N)$ there are monopole configurations such that the function $W_{YM}(t/x^2)$ increases as e^{t/x^2}, and therefore, the integral would converge for sufficiently large x.

That is why we investigate this case in more detail. We set $z = 0$ (and $y = 0$, for simplicity) to get

$$\Phi_{xx}(x, 0; 1, 0) = 2x^2 \int_0^\infty \frac{dt}{t^2} \left\{\Omega'(t) + 2t\Omega''(t)\right\} W\left(\frac{t}{x^2}\right) . \tag{7.13}$$

We consider the group $SU(2N)$ when the function W is given by (6.42) and (6.61) for the spinors. We decompose it according to

$$W\left(\frac{t}{x^2}\right) = 2N^2 \frac{t}{x^2} e^{t/x^2} - 4Ns\frac{t}{x^2} + V\left(\frac{t}{x^2}\right) , \tag{7.14}$$

where $s = 0$ for massive spinors and $s = 1$ for massless spinors, when $m_{\rm spin} = 0$, and the function V is exponentially decreasing as $t \to \infty$.

We now split the integral (7.13) into three parts accordingly

$$\Phi_{xx}(x, 0; 1, 0) = I_1 + I_2 - 8Ns\nu_1 , \tag{7.15}$$

where

$$\nu_1 = \int_0^\infty \frac{dt}{t} \left\{\Omega'(t) + 2t\Omega''(t)\right\} , \tag{7.16}$$

and

$$I_1 = 4N^2 \int_0^\infty \frac{dt}{t} e^{t/x^2} \left\{\Omega'(t) + 2t\Omega''(t)\right\} , \tag{7.17}$$

$$I_2 = 2x^2 \int_0^\infty \frac{dt}{t^2} \left\{\Omega'(t) + 2t\Omega''(t)\right\} V\left(\frac{t}{x^2}\right) . \tag{7.18}$$

Recall that $x = a/a_1 = 2\pi aT$; so $x \to \infty$ is the high-temperature limit and $x \to 0$ is the limit of zero temperature.

We consider the high-temperature limit first. As $x \to \infty$ the second integral is

$$I_2 \sim 4(4N^2 - 1)\nu_2 x^2 , \tag{7.19}$$

where

$$\nu_2 = \int_0^\infty \frac{dt}{t^2} \left\{\Omega'(t) + 2t\Omega''(t)\right\} . \tag{7.20}$$

For the first integral in the limit $x \to \infty$ we get

$$I_1 \sim 4N^2 \nu_1 . \tag{7.21}$$

Thus, as $T \to \infty$ we obtain

$$\frac{C_v}{V} \sim 2\pi^2 (4N^2 - 1)\nu_2 T^3 + \cdots , \tag{7.22}$$

which is similar to the black body photon gas.

The first integral exhibits much more interesting behavior due to the presence of the growing exponential factor e^{t/x^2}. It converges at $t \to 0$ for any x. However, at $t \to \infty$ its convergence depends on the critical temperature. Recall that the other part of the integrand has an exponential factor e^{-t}. Therefore, the integral converges in the infrared domain $t \to \infty$ if $x > 1$ and diverges otherwise. This defines the *critical temperature*

$$T_c = \frac{1}{2\pi a}\,. \tag{7.23}$$

At the critical temperature $T \approx T_c$ the second integral is simply

$$I_2 = 2\int_0^\infty \frac{dt}{t^2}\left\{\Omega'(t) + 2t\Omega''(t)\right\} V(t)\,. \tag{7.24}$$

However, the first integral is nonanalytic near T_c, namely, by using the asymptotics (5.19), we get as $x \to 1^+$

$$I_1 \sim 8N^2\left(1 - \frac{1}{x^2}\right)^{-3/2} \sim \frac{4N^2}{\sqrt{2}}(x-1)^{-3/2}\,. \tag{7.25}$$

Therefore, the heat capacity near the critical temperature, $T \to T_c^+$, is

$$\frac{C_v}{V} \sim \frac{N^2}{4\pi\sqrt{2}a^3}\left(\frac{T - T_c}{T_c}\right)^{-3/2}\,. \tag{7.26}$$

This indicates the second-order phase transition with the critical exponent $\alpha = 3/2$.

8 Conclusion

The primary goal of this paper was to study of the Yang-Mills vacuum in the low-energy (long-distance) limit. The Savvidy model of such a vacuum with constant chromomagnetic field in Minkowski spacetime suffers from a well-known instability, which exhibits itself in negative eigenvalues of the gluon operator. We noticed that a positive spatial curvature of the spacetime manifold acts as an effective mass term and, therefore, can stabilize the Savvidy vacuum. That is why we considered the case of a compact spacetime manifold of the form $S^1 \times S^1 \times S^2$ with a covariantly constant chromomagnetic Yang-Mills field on the sphere S^2. On the sphere, such a configuration is of monopole type parametrized by a collection of half-integers (monopole numbers). Such a configuration has a well-defined (Savvidy type) flat space limit with constant chromomagnetic field when the radius a of the 2-sphere S^2, as well as the monopole numbers, n_j, go to infinity, $n_j, a \to \infty$, such that the ratio n_j/a^2 (which defines the magnetic field) remains constant. This limit can be interpreted physically as the condensation of monopoles.

We computed exactly the spectra and the trace of the heat kernels of all relevant operators, which enabled us to compute exactly the one-loop effective action. We have found that the gluon operator does not have negative eigenvalues only when the monopole numbers are between -1 and 1. That is, any monopole number n_j, with $|n_j| \geq 2$, leads to an instability of the chromomagnetic vacuum. This confirms once again that the flat space limit with constant chromomagnetic field is unstable since it is created by infinitely large monopole numbers, formally $|n_j| \to \infty$. We showed that for any compact simple gauge group there are always monopole numbers with absolute value greater or equal to 2, which means that *there is no stable constant chromomagnetic configuration also in curved space* (at least in the model $S^1 \times S^1 \times S^2$).

We also studied the thermal properties of Yang-Mills theory, in particular, we computed the entropy and the heat capacity of the quark-gluon gas. We have found that the heat capacity is well defined even in the infrared limit and computed the high-temperature asymptotics of the heat capacity. Moreover, in a particular model $SU(2N)$ we found that the heat capacity has a typical branching singularity $\sim (T - T_c)^{-3/2}$ at a finite critical temperature $T_c = 1/(2\pi a)$ indicating the second-order phase transition.

We conclude that to stabilize the chromomagnetic vacuum at lower energies one should consider non-constant magnetic fields on non-compact spaces. Constant magnetic fields on compact symmetric spaces are too rigid, they are completely determined by the spin connection and are of the same order as the space curvature. This makes it impossible that the gluon operator with the potential term $R^a{}_b - 2\mathcal{F}^a{}_b$ is strictly positive. What one needs is a large Ricci tensor and a small independent magnetic field to make this work.

It is also interesting to study the Yang-Mills vacuum on the Einstein model $S^1 \times S^3$. We intend to carry out such a study in future work. Another interesting question to pose is whether the Yang-Mills vacuum is stable when gravity is treated as a dynamical field. The technical calculations are not that difficult, but then, of course, we immediately face the non-renormalizability of general relativity, so we have to consider higher-derivative gravity instead.

It is hard to imagine that this model can be directly relevant in hadron physics in the study of the confinement because of the completely different energy scales dictated by the gravitational constant and the cosmological constant. However, it can be relevant in the study of the structure of the quark-gluon plasma in the early Universe.

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