

Dirac Operator on Complex Manifolds and Supersymmetric Quantum Mechanics

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ABSTRACT

We explore a new simple $\mathcal{N} = 2$ SQM model describing the motion over complex manifolds in external gauge fields. The nilpotent supercharge Q of the model can be interpreted as a (twisted) exterior holomorphic derivative, such that the model realizes the twisted Dolbeault complex. The sum $Q + \bar{Q}$ can be interpreted as the Dirac operator: the standard Dirac operator if the manifold is Kähler and a certain “truncated” Dirac operator for a generic complex manifold. Focusing on the Kähler case, we give new simple physical proofs of the two mathematical facts: (i) the equivalence of the twisted Dirac and twisted Dolbeault complexes and (ii) the Atiyah-Singer theorem.

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1 Introduction

Complexes are the algebraic objects associated with smooth manifolds and studied in differential geometry [1]. The most known is the *de Rham* complex involving the exterior derivative d and the Hermitian-conjugate operator $d^\dagger = *d*$ acting on the space of p -forms. The operators d and d^\dagger are nilpotent, while their anticommutator $\{d, d^\dagger\}$ coincides with the covariant Laplacian acting on the forms. The other important complexes are the *Dolbeault* complex, which is defined on complex manifolds and involves holomorphic exterior derivative ∂ and its hermitian conjugate ∂^\dagger , and the *Dirac* complex associated with the Dirac operator. The complexes may be *twisted* by adding background Abelian or non-Abelian gauge fields¹.

An important characteristics of all these complexes are their indices. The index of an elliptic operator \mathcal{O} can be defined if the whole Hilbert space of objects (states) where it acts can be divided into two subspaces (call them \mathcal{H}_L and \mathcal{H}_R) and there are symmetry operators (commuting with \mathcal{O}) that transform a state from \mathcal{H}_L into a state from \mathcal{H}_R and a state from \mathcal{H}_R into a state from \mathcal{H}_L .

In such case, one can always define nilpotent projections: an operator that brings a state from \mathcal{H}_L into a state from \mathcal{H}_R and annihilates any state from \mathcal{H}_R , and its Hermitian conjugate: the operator bringing a state from \mathcal{H}_R into \mathcal{H}_L and annihilating the states from \mathcal{H}_L . The anticommutator of these nilpotent projections is a symmetry operator too. In the simplest case, it coincides with \mathcal{O} . Then all eigenstates of \mathcal{O} with nonzero eigenvalues are double degenerate (take an eigenstate from \mathcal{H}_L and act upon it by a symmetry operator). It is not true for zero eigenvalues. The index is then defined as the difference between the number of states in the kernel of \mathcal{O} belonging to \mathcal{H}_L and such a number for \mathcal{H}_R .

For example, for the de Rham complex, $\mathcal{O} = -\Delta_{\text{cov}}$, and the Hilbert space of all relevant forms can be divided into the subsets of even and odd forms. The relevant symmetry operators are d and d^\dagger . For the Dirac complex², $\mathcal{O} = -\mathcal{D}^2$, the Hilbert space of all spinors can be subdivided into the left-handed spinors and the right-handed ones. The symmetry operators are \mathcal{D} and $\mathcal{D}\gamma^{D+1}$ (the index of \mathcal{O} coinciding with the index of \mathcal{D} can be defined only for even-dimensional manifolds where γ^{D+1} , a multidimensional generalization of γ^5 , can be defined).

The indices have beautiful integral representations. Consider, e.g., the 2-dim Dirac operator in an external Abelian field on the plane. Its index coincides with the magnetic flux,

$$I_{\mathcal{D}} = \frac{1}{2\pi} \int B d^2x . \quad (1.1)$$

The integral representations for all indices were systematically derived by Atiyah and Singer [2]. In their derivation, they used the so called *heat kernel* method [3] based on the semiclassical (small β) expansion of the matrix element $\langle x | \Gamma e^{-\beta \mathcal{O}} | x \rangle$, where Γ

¹ There are two parallel terminological systems: physical and mathematical. For example, what a physicist calls *Abelian gauge field* is called *connection on a line bundle* by a mathematician. We will mostly use the physical terminology.

²The sign is chosen so that the operator \mathcal{O} is positive-definite with $\mathcal{D} = \mathcal{D}_A \gamma^A$ and Hermitian γ^A .

is the grading operator distinguishing between \mathcal{H}_L and \mathcal{H}_R , such that $\Gamma\Psi = \Psi$ when $\Psi \in \mathcal{H}_L$ and $\Gamma\Psi = -\Psi$ when $\Psi \in \mathcal{H}_R$.

An interesting, from the physical viewpoint, modification of this method is based on the observation that the indices of elliptic (Euclidean) operators are associated via the level crossing picture with the *anomalies* of certain Minkowski space currents [4]. For example, the index (1.1) is associated with the anomalous divergence of the 2-dim axial current $J_\mu = \bar{\psi}\gamma_\mu\gamma^5\psi$,

$$\partial_\mu J_\mu = \frac{1}{4\pi}\epsilon_{\alpha\beta}F_{\alpha\beta} . \quad (1.2)$$

One of the ways to derive (1.2) is to regularize the current by the Schwinger splitting $J_\mu \rightarrow J_\mu(\epsilon) = \bar{\psi}(x+\epsilon)\gamma_\mu\gamma^5\psi(x-\epsilon)$ and calculate then the Euclidean fermion propagator in external Abelian field $\langle\psi(x-\epsilon)\bar{\psi}(x+\epsilon)\rangle_A$ using the Fock-Schwinger gauge technique [5].

For the index (1.1), the heat kernel calculation is rather explicit, but it is much more intricate in more complicated cases of mathematical and physical interest.

Back in 1981, Witten noticed [6] that this set of mathematical problems has a beautiful physical interpretation: the operators \mathcal{O} can be viewed of as Hamiltonians of certain supersymmetric quantum mechanics (SQM) systems, while the nilpotent projections discussed above are interpreted as supercharges. The index of \mathcal{O} coincides then with the Witten index of the corresponding SQM system,

$$I = \text{Tr}\{\Gamma e^{-\beta H}\} = \text{Tr}\{(-1)^F e^{-\beta H}\} , \quad (1.3)$$

where β is a parameter having the meaning of inverse temperature and F is an operator that commutes with H and has even eigenvalues for the states from \mathcal{H}_L and odd eigenvalues for the states from \mathcal{H}_R . Physically, F is interpreted as the fermion number. Due to degeneracy between the excited states in \mathcal{H}_L and \mathcal{H}_R , the index does not depend on β .

Now, the r.h.s. of (1.3) has a functional integral representation. For small β , this functional integral can be evaluated by semiclassical methods. As a result, the Atiyah-Singer integral theorems are reproduced. This program was carried out in [7].

In our paper, we concentrate on *complex* manifolds and construct, using the superfield formalism, the $\mathcal{N} = 2$ SQM model³ which, in our opinion, is most appropriate for calculating the relevant indices. Its classical $\mathcal{N} = 2$ superfield Lagrangian is a particular case of the general Lagrangian given in [8]. In the Kähler case, this SQM model is reduced to the model considered in [7], whereas in a generic complex case its Lagrangian is different. Also, in the Kähler case, our approach differs from the approach in Ref. [7] by the choice of supercharges. Instead of the supercharges \mathcal{P} and $\mathcal{P}\gamma^{D+1}$ that realize the supersymmetry algebra for any even-dimensional manifold, we use the (Hermitian) supercharges

$$i\mathcal{P} = i\gamma^A\mathcal{D}_A, \quad S = I^A_B\gamma^B\mathcal{D}_A, \quad (1.4)$$

³Following the convention adopted now by the most practitioners of SQM, \mathcal{N} counts the number of *real* supercharges. Thus, the minimal interesting case where supersymmetry (double degeneracy of all excited levels) is present in the spectrum of the Hamiltonian corresponds to $\mathcal{N} = 2$.

where I_B^A is the complex structure matrix. The existence of the supercharge S (such that $S^2 = H$ and $\{S, \mathcal{D}\} = 0$) is specific for Kähler manifolds [9]. The supercharges (1.4) are naturally obtained in our superfield framework as a real and imaginary part of a certain complex nilpotent supercharge.

After fixing the complex geometry notations in Sect. 2, we present our model in Sect. 3. In Sect. 4, we show how this $\mathcal{N} = 2$ SQM model can, in the Kähler case, be completed to the extended $\mathcal{N} = 4$ SQM model. In Sect. 5, we give a geometric interpretation (1.4) for the Nöther supercharges derived in Sect. 3. We also observe that the nilpotent supercharge $\mathcal{D} + iS$ can be interpreted as the (twisted) operator of the holomorphic exterior derivative. This allows one to prove, in a rather manifest way, the known mathematical fact: *for Kähler manifolds, the twisted Dirac complex and the twisted Dolbeault complex are equivalent*. Sect. 6 is devoted to the functional integral derivation of the Atiyah-Singer theorem. The derivation is similar in spirit to the derivation in Ref. [7], but we do it in a much more detailed way (almost no details of the calculation were given in [7]), focusing on the Kähler case.

2 Complex geometry

Let us start with recalling some mathematical facts on complex geometry adapted for immediate use in the next section where the relevant SQM system will be introduced.

We assume the manifold to be even-dimensional of dimension $D = 2n$ and described by complex coordinates $z^N = (z^j, \bar{z}^{\bar{k}})$. The metric is assumed to have the Hermitian form $ds^2 = 2h_{j\bar{k}}dz^j d\bar{z}^{\bar{k}}$. In other words⁴,

$$g_{MN} = \begin{pmatrix} 0 & h_{j\bar{k}} \\ h_{k\bar{j}} & 0 \end{pmatrix}. \quad (2.1)$$

The covariant derivative that we will use in the following is defined as

$$\nabla \psi^M = \dot{\psi}^M + z^N \hat{\Gamma}_{NL}^M \psi^L. \quad (2.2)$$

It involves generically the *affine connections*

$$\hat{\Gamma}_{NK}^M = \Gamma_{NK}^M + \frac{1}{2}g^{ML}C_{LNK}, \quad (2.3)$$

where Γ_{NK}^M are the standard Christoffel symbols for the metric g_{MN} and C_{LNK} is the totally antisymmetric torsion tensor. In the following, we will stick to a *special* form of the torsion tensor with the nonvanishing components

$$C_{j\bar{k}\bar{l}} = \partial_k h_{j\bar{l}} - \partial_j h_{k\bar{l}}, \quad C_{\bar{j}k\bar{l}} = (C_{j\bar{k}\bar{l}})^* = \partial_{\bar{k}} h_{l\bar{j}} - \partial_{\bar{j}} h_{l\bar{k}} \quad (2.4)$$

⁴Such a manifold is not necessarily *complex* in the precise mathematical sense. The genuine complex manifold is required to be divisible in several maps such that the metric has the form (2.1) in each map *and* the coordinates $(z^j, \bar{z}^{\bar{k}})$ in such different overlapping maps are expressed through each other by means of holomorphic functions. Thus, S^4 (in contrast to S^2) is not a complex manifold even though its metric can be represented as in Eq.(2.1) in both the northern and the southern hemispheres. Sill notice that the requirement for the metric to be representable locally in the form (2.1) is nontrivial and singles out some subset of even-dimensional manifolds.

(and those obtained from them by the cyclic permutation of indices). The torsion (2.4) has a non-zero curl, $\partial_{[M}C_{NPQ]} \neq 0$ ⁵. Non-vanishing components of $\Gamma_{M,NP}$ and $\hat{\Gamma}_{M,NP}$ are

$$\begin{aligned}\Gamma_{\bar{m},np} &= \frac{1}{2}(\partial_n h_{p\bar{m}} + \partial_p h_{n\bar{m}}), & \Gamma_{m,n\bar{p}} &= \Gamma_{m,\bar{p}n} = \frac{1}{2}C_{mn\bar{p}}, & (\text{and c.c.}), \\ \hat{\Gamma}_{\bar{m},np} &= \partial_p h_{n\bar{m}}, & \hat{\Gamma}_{m,n\bar{p}} &= C_{mn\bar{p}}, & \hat{\Gamma}_{m,\bar{p}n} &= 0, & (\text{and c.c.}).\end{aligned}\quad (2.5)$$

Let us introduce complex vielbeins as

$$e_k^a e_{\bar{i}}^{\bar{a}} = h_{k\bar{i}}, \quad e_a^k e_{\bar{a}}^{\bar{i}} = h^{\bar{i}k}, \quad h^{\bar{i}k} h_{k\bar{j}} = \delta_{\bar{j}}^{\bar{i}}, \quad h_{k\bar{j}} g^{\bar{j}l} = \delta_k^l, \quad (2.6)$$

$$e_k^a e_a^j = \delta_k^j, \quad e_a^k e_k^b = \delta_a^b, \quad e_{\bar{k}}^{\bar{a}} e_{\bar{a}}^{\bar{j}} = \delta_{\bar{k}}^{\bar{j}}, \quad e_{\bar{a}}^{\bar{k}} e_{\bar{k}}^{\bar{b}} = \delta_{\bar{a}}^{\bar{b}}. \quad (2.7)$$

The nonzero components of the standard spin connections

$$\Omega_{M,AB} = e_{AN}(\partial_M e_B^N + \Gamma_{MK}^N e_B^K)$$

are

$$\Omega_{j,\bar{b}a} = -\Omega_{j,a\bar{b}} = e_p^b(\partial_j e_a^p + \Gamma_{jk}^p e_a^k), \quad \Omega_{j,\bar{a}b} = \frac{1}{2}e_{\bar{a}}^{\bar{s}} e_{\bar{b}}^{\bar{k}} C_{j\bar{s}\bar{k}} \quad (2.8)$$

and complex conjugated $\bar{\Omega}_{\bar{j},b\bar{a}}$, $\bar{\Omega}_{\bar{j},ab}$, with $C_{j\bar{k}\bar{s}}$ defined in (2.4).

When the torsion is present, one can define a generalized spin connection related to the generalized affine connection $\hat{\Gamma}_{ML}^N$:

$$\hat{\Omega}_{M,AB} = e_{AN}(\partial_M e_B^N + \hat{\Gamma}_{MK}^N e_B^K) = \Omega_{M,AB} + \frac{1}{2}e_A^K e_B^L C_{MLK}. \quad (2.9)$$

The nonzero components of $\hat{\Omega}$ are

$$\begin{aligned}\hat{\Omega}_{i,\bar{b}a} &= -\hat{\Omega}_{i,a\bar{b}} = e_k^b \partial_i e_a^k + e_{\bar{b}}^{\bar{l}} e_a^k \hat{\Gamma}_{\bar{l},ik} = e_a^k \partial_k e_i^b - e_a^k \partial_i e_k^b + e_a^k e_{\bar{b}}^{\bar{l}} e_i^c \partial_k e_{\bar{l}}^{\bar{c}}, \\ \hat{\Omega}_{\bar{i},b\bar{a}} &= (\hat{\Omega}_{i,\bar{b}a})^*,\end{aligned}\quad (2.10)$$

while the components $\hat{\Omega}_{j,\bar{a}b}$ and $\hat{\Omega}_{\bar{j},ab}$ vanish.

The vielbeins and the generalized spin connection satisfy the Maurer-Cartan structure equation

$$de_A + \hat{\Omega}_{AB} \wedge e_B = T_A, \quad (2.11)$$

where

$$e_A = e_{AM} dx^M, \quad \hat{\Omega}_{AB} = \hat{\Omega}_{M,AB} dx^M, \quad T_A = \frac{1}{2}e_A^M C_{MNP} dx^N \wedge dx^P.$$

The Maurer-Cartan equation for the standard torsion-free spin connection is

$$de_A + \Omega_{AB} \wedge e_B = 0. \quad (2.12)$$

⁵This is in contrast to Ref. [10] where the Atiyah-Singer theorem on manifolds involving extra curl-free torsion was considered.

The equations (2.11) and (2.12) are equivalent, as can be checked using the relation (2.9). For the Hermitian metric (2.1), with the torsion defined in (2.4), these equations imply the identity

$$\partial_{[k} e_{l]}^a - e_{\bar{a}}^{\bar{i}} e_{[k}^d \partial_{l]} e_{\bar{i}}^{\bar{d}} = \frac{1}{2} e_{\bar{a}}^{\bar{j}} C_{lk\bar{j}} \quad (\text{and c.c.}) . \quad (2.13)$$

For Kähler manifolds, the metric (2.1) is derived from the Kähler potential,

$$h_{j\bar{k}}(z, \bar{z}) = \partial_j \partial_{\bar{k}} K(z, \bar{z}) . \quad (2.14)$$

In this case

$$\partial_{\bar{l}} h_{j\bar{k}} - \partial_{\bar{k}} h_{j\bar{l}} = \partial_k h_{j\bar{l}} - \partial_j h_{k\bar{l}} = 0 , \quad (2.15)$$

and, as a result,

$$C_{MNK} = 0 \Rightarrow \hat{\Gamma}_{NK}^M = \Gamma_{NK}^M . \quad (2.16)$$

The only nonvanishing components of $\Gamma_{M,NP}$ in the Kähler case are:

$$\Gamma_{\bar{m},np} = \partial_n h_{p\bar{m}} , \quad \Gamma_{m,\bar{n}\bar{p}} = (\Gamma_{\bar{m},np})^* = \partial_{\bar{n}} h_{m\bar{p}} . \quad (2.17)$$

The expressions for the non-vanishing components of the spin connections are also greatly simplified,

$$\hat{\Omega}_{j,\bar{b}a} = \Omega_{j,\bar{b}a} = e_{\bar{b}}^{\bar{k}} \partial_j e_{\bar{k}}^{\bar{a}} \stackrel{\text{def}}{=} \omega_{j,\bar{b}a} , \quad \hat{\bar{\Omega}}_{\bar{j},b\bar{a}} = \bar{\Omega}_{\bar{j},b\bar{a}} = e_b^k \partial_{\bar{j}} e_k^a \stackrel{\text{def}}{=} \bar{\omega}_{\bar{j},b\bar{a}} . \quad (2.18)$$

In the Kähler case, the structure equation (2.13) acquires the form

$$\partial_{[k} e_{l]}^a - e_{\bar{a}}^{\bar{i}} e_{[k}^d \partial_{l]} e_{\bar{i}}^{\bar{d}} = 0 \quad (\text{and c.c.}) , \quad (2.19)$$

which is now a non-trivial constraint on the vielbeins $e_l^a, e_{\bar{l}}^{\bar{a}}$ (equivalent to the constraint (2.15) on the metric). The only non-vanishing components of the Kähler Riemann tensor are

$$\begin{aligned} R_{j\bar{k},l\bar{t}} &= \partial_j \partial_{\bar{k}} h_{l\bar{t}} - h_{p\bar{s}} \Gamma_{jl}^p \Gamma_{\bar{k}\bar{t}}^{\bar{s}} = \partial_j \partial_{\bar{k}} h_{l\bar{t}} - h^{\bar{s}n} (\partial_j h_{l\bar{s}}) (\partial_{\bar{k}} h_{n\bar{t}}) \\ &= e_l^a e_{\bar{t}}^{\bar{b}} (\partial_j \bar{\omega}_{\bar{k},a\bar{b}} + \partial_{\bar{k}} \omega_{j,\bar{b}a} + \omega_{j,\bar{b}d} \bar{\omega}_{\bar{k},a\bar{d}} - \omega_{j,\bar{d}a} \bar{\omega}_{\bar{k},d\bar{b}}) . \end{aligned} \quad (2.20)$$

Finally, note the useful generic relations:

$$\Omega_{i,\bar{a}b} = \omega_{i,\bar{a}b} + \frac{1}{2} e_b^l e_{\bar{a}}^{\bar{t}} C_{il\bar{t}} , \quad \hat{\Omega}_{i,\bar{a}b} = \omega_{i,\bar{a}b} + e_b^l e_{\bar{a}}^{\bar{t}} C_{il\bar{t}} , \quad (2.21)$$

where the expressions $\omega_{i,\bar{a}b}$ coincide by form with those defined in (2.18). Note that the objects $\omega_{i,\bar{a}b}$ can be given a geometric interpretation even in a non-Kähler case. They coincide with the appropriate components of a generalized spin connection associated with $\tilde{\Gamma}_{NK}^M = \Gamma_{NK}^M - \frac{1}{2} g^{MT} C_{TNK}$.

3 $\mathcal{N} = 2$ SQM model

We formulate a general complex $\mathcal{N} = 2, d = 1$ SQM sigma model in terms of $2n$ mutually conjugated chiral and anti-chiral superfields $Z^j(t_L, \theta), \bar{Z}^{\bar{j}}(t_R, \bar{\theta})$ ($j, \bar{j} = 1, \dots, n$),

$$\bar{D}Z^j(t_L, \theta) = D\bar{Z}^{\bar{j}}(t_R, \bar{\theta}) = 0, \quad (3.1)$$

where

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_t, \bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\partial_t, \quad \{D, \bar{D}\} = 2i\partial_t, \quad t_L = t - i\theta\bar{\theta}, t_R = t + i\theta\bar{\theta}. \quad (3.2)$$

The basic superfields have the following component expansion

$$Z^j = z^j + \sqrt{2}\theta\psi^j - i\theta\bar{\theta}\dot{z}^j, \quad \bar{Z}^{\bar{j}} = \bar{z}^{\bar{j}} - \sqrt{2}\bar{\theta}\bar{\psi}^{\bar{j}} + i\theta\bar{\theta}\dot{\bar{z}}^{\bar{j}}. \quad (3.3)$$

The $\mathcal{N} = 2$ transformation properties of the component fields are as follows:

$$\begin{aligned} \delta z^j &= -\sqrt{2}\epsilon\psi^j, & \delta\psi^j &= \sqrt{2}i\bar{\epsilon}\dot{z}^j, \\ \delta\bar{z}^{\bar{j}} &= \sqrt{2}\bar{\epsilon}\bar{\psi}^{\bar{j}}, & \delta\bar{\psi}^{\bar{j}} &= -\sqrt{2}i\epsilon\dot{\bar{z}}^{\bar{j}}. \end{aligned} \quad (3.4)$$

The superfield action we start with reads

$$\begin{aligned} S &= \int dt d^2\theta (\mathcal{L}_\sigma + \mathcal{L}_{gauge}), \\ \mathcal{L}_\sigma &= -\frac{1}{4}h_{i\bar{j}}(Z, \bar{Z}) DZ^i \bar{D}\bar{Z}^{\bar{j}}, \quad \mathcal{L}_{gauge} = W(Z, \bar{Z}). \end{aligned} \quad (3.5)$$

Here, $h_{i\bar{j}}(Z, \bar{Z})$ and $W(Z, \bar{Z})$ are unconstrained functions of the superfields. In general [8], one can add to \mathcal{L} the terms

$$\sim \mathcal{B}_{ik}(Z, \bar{Z}) DZ^i DZ^k + c.c.. \quad (3.6)$$

These additional terms do not change the target space metric in the component action and affect only fermionic terms (introducing some extra non-zero components of the torsion). For this reason, we shall not consider them in what follows.

The Kähler case corresponds to the choice

$$h_{j\bar{k}}(Z, \bar{Z}) = \partial_j\partial_{\bar{k}}K(Z, \bar{Z}), \quad (3.7)$$

where the Kähler potential $K(Z, \bar{Z})$ is an arbitrary real function of the superfields ⁶.

The component form of the full action is

$$\begin{aligned} S &\equiv \int dt (L_\sigma + L_{gauge}) = \int dt \left\{ h_{j\bar{k}} \left[\dot{z}^j \dot{\bar{z}}^{\bar{k}} + \frac{i}{2} (\psi^j \dot{\bar{\psi}}^{\bar{k}} - \dot{\psi}^j \bar{\psi}^{\bar{k}}) \right] \right. \\ &\quad - \frac{i}{2} \left[(2\partial_j h_{t\bar{k}} - \partial_t h_{j\bar{k}}) \dot{z}^t - (2\partial_{\bar{k}} h_{j\bar{t}} - \partial_{\bar{t}} h_{j\bar{k}}) \dot{\bar{z}}^{\bar{t}} \right] \psi^j \bar{\psi}^{\bar{k}} + (\partial_t \partial_{\bar{t}} h_{j\bar{k}}) \psi^t \psi^j \bar{\psi}^{\bar{t}} \bar{\psi}^{\bar{k}} \\ &\quad \left. + 2\partial_j \partial_{\bar{k}} W \psi^j \bar{\psi}^{\bar{k}} - i \left(\partial_j W \dot{z}^j - \partial_{\bar{j}} W \dot{\bar{z}}^{\bar{j}} \right) \right\}. \end{aligned} \quad (3.8)$$

⁶For CP^1 and with the restriction $K = W$, this SQM model was earlier considered at the classical (component and superfield) and quantum levels in Refs.[11].

The geometric meaning of the different terms in the ‘‘sigma-model’’ part L_σ of the Lagrangian in (3.8) can be clarified, if rewriting it in the following form

$$L_\sigma = \frac{1}{2} \left[g_{MN} \dot{z}^M \dot{z}^N + ig_{MN} \psi^M \nabla \psi^N - \frac{1}{3!} \partial_P C_{MNT} \psi^P \psi^M \psi^N \psi^T \right], \quad (3.9)$$

where the metric g_{MN} is written in (2.1), the covariant derivative $\nabla \psi^N$ was defined in (2.2) and the torsion tensor C_{MNT} in (2.4).

The Lagrangian L_{gauge} (the last line in (3.8)) describes the interactions with the Abelian gauge field $A_M = (-i\partial_j W, i\partial_{\bar{j}} W) \equiv I_M^N \partial_N W$, the double derivative $F_{j\bar{k}} = -F_{\bar{k}j} = 2i\partial_j \partial_{\bar{k}} W$ being the magnetic field strength. This Lagrangian can also be rewritten in a form analogous to (3.9)

$$L_{gauge} = A_M \dot{Z}^M - \frac{i}{2} F_{MN} \psi^M \psi^N. \quad (3.10)$$

The prepotential $W(Z, \bar{Z})$ is an arbitrary function. A particularly clever choice is $W \propto \ln \det h$ (see Eq.(5.1) below). The corresponding bundle $(-i\partial_j W, i\partial_{\bar{j}} W)$ is called *canonical* by mathematicians.

The fermion variables $\psi^j, \bar{\psi}^{\bar{k}}$ are not canonically conjugated, their Poisson bracket being equal to $-ih^{\bar{k}j}$. It is convenient to introduce the canonically conjugated fermionic fields with the tangent space indices,

$$\psi^a = e_j^a \psi^j, \quad \bar{\psi}^{\bar{b}} = e_{\bar{k}}^{\bar{b}} \bar{\psi}^{\bar{k}}, \quad (3.11)$$

such that all the variables have the canonical Poisson brackets,

$$\{z^j, P_k\}_{PB} = \delta_k^j, \quad \{\bar{z}^{\bar{j}}, P_{\bar{k}}\}_{PB} = \delta_{\bar{k}}^{\bar{j}}, \quad \{\psi^a, \bar{\psi}^{\bar{b}}\}_{PB} = -i\delta^{a\bar{b}}. \quad (3.12)$$

Then, using the invariance of the Lagrangian in (3.8) under the transformations (3.4) (modulo a total time derivative), it is easy to compute the corresponding canonical supercharges

$$\begin{aligned} Q &= \sqrt{2} \left[\Pi_k e_a^k \psi^a - i\psi^b \psi^d \bar{\psi}^{\bar{a}} (e_b^k \partial_{[k} e_{l]}^a e_d^l) \right], \\ \bar{Q} &= \sqrt{2} \left[\bar{\Pi}_{\bar{k}} e_{\bar{a}}^{\bar{k}} \bar{\psi}^{\bar{a}} - i\bar{\psi}^{\bar{c}} \bar{\psi}^{\bar{a}} \psi^d (e_{\bar{c}}^{\bar{k}} \partial_{[\bar{k}} e_{\bar{l}}^{\bar{d}} e_{\bar{a}}^{\bar{l}}]) \right], \end{aligned} \quad (3.13)$$

where

$$\Pi_k = P_k + i \partial_k W, \quad \bar{\Pi}_{\bar{k}} = P_{\bar{k}} - i \partial_{\bar{k}} W, \quad (3.14)$$

and $P_k, P_{\bar{k}}$ are the canonical momenta.

Using the definitions (2.8), (2.4) and the relations (2.13), (2.21), these supercharges can be brought in a more suggestive geometric form

$$Q = \sqrt{2} \left[\Pi_k - i\bar{\psi}^{\bar{a}} \psi^b \Omega_{k,\bar{a}b} \right] e_c^k \psi^c, \quad \bar{Q} = \sqrt{2} e_{\bar{c}}^{\bar{k}} \bar{\psi}^{\bar{c}} \left[\bar{\Pi}_{\bar{k}} + i\bar{\psi}^{\bar{a}} \psi^d \bar{\Omega}_{\bar{k},d\bar{a}} \right]. \quad (3.15)$$

It should be pointed out that the 3-fermionic terms in these supercharges contain the spin connections $\Omega_{k,\bar{a}b}, \bar{\Omega}_{\bar{k},d\bar{a}}$ corresponding to the standard symmetric Christoffel symbols Γ_{MK}^N and defined by the relations (2.8), (2.21).

Using (3.12), it is easy to find

$$\{Q, \bar{Q}\}_{PB} = -2iH_{cl}, \quad \{Q, Q\}_{PB} = \{\bar{Q}, \bar{Q}\}_{PB} = [Q, H_{cl}]_{PB} = [\bar{Q}, H_{cl}]_{PB} = 0. \quad (3.16)$$

The canonical classical Hamiltonian H_{cl} can be represented in the following compact form:

$$\begin{aligned} H_{cl} = & h^{\bar{k}j} \left(\Pi_j - i\hat{\Omega}_{j,\bar{b}a} \bar{\psi}^{\bar{b}} \psi^a \right) \left(\bar{\Pi}_{\bar{k}} + i\hat{\Omega}_{\bar{k},c\bar{d}} \bar{\psi}^{\bar{d}} \psi^c \right) \\ & - 2e_a^j e_b^{\bar{k}} \partial_j \partial_{\bar{k}} W \psi^a \bar{\psi}^{\bar{b}} - e_a^t e_c^j e_b^{\bar{l}} e_d^{\bar{k}} (\partial_t \partial_{\bar{l}} h_{j\bar{k}}) \psi^a \psi^c \bar{\psi}^{\bar{b}} \bar{\psi}^{\bar{d}}. \end{aligned} \quad (3.17)$$

It is interesting to note that, in the generic case, the spin connections entering the supercharges (3.15) and the classical hamiltonian (3.17) do not coincide with each other: they differ by a term proportional to the torsion.

In the Kähler case, when the torsion is vanishing, the situation simplifies. Both the supercharges and the Hamiltonian are expressed through the same connections (2.18). In addition, the last four-fermionic term in (3.17) vanishes.

We thus obtain

$$Q^{\text{Kähl}} = \sqrt{2} [\Pi_k - i\bar{\psi}^{\bar{a}} \psi^b \omega_{k,\bar{a}b}] e_c^k \psi^c, \quad \bar{Q}^{\text{Kähl}} = \sqrt{2} e_{\bar{c}}^{\bar{k}} \bar{\psi}^{\bar{c}} [\bar{\Pi}_{\bar{k}} + i\bar{\psi}^{\bar{a}} \psi^d \bar{\omega}_{\bar{k},d\bar{a}}] \quad (3.18)$$

and

$$H_{cl}^{\text{Kähl}} = h^{\bar{k}j} \left(\Pi_j - i\omega_{j,\bar{b}a} \bar{\psi}^{\bar{b}} \psi^a \right) \left(\bar{\Pi}_{\bar{k}} + i\bar{\omega}_{\bar{k},c\bar{d}} \bar{\psi}^{\bar{d}} \psi^c \right) - 2e_a^j e_b^{\bar{k}} \partial_j \partial_{\bar{k}} W \psi^a \bar{\psi}^{\bar{b}}. \quad (3.19)$$

The expression for the Lagrangian also simplifies a lot in the Kähler case. The four-fermionic term in (3.8), (3.9) vanishes. The remaining terms in L_σ can be presented as

$$L_\sigma^{\text{Kähl}} = h_{j\bar{k}} \dot{z}^j \dot{\bar{z}}^{\bar{k}} + \frac{i}{2} (\psi^a \dot{\bar{\psi}}^{\bar{a}} - \dot{\psi}^a \bar{\psi}^{\bar{a}}) + i \left(\dot{\bar{z}}^{\bar{k}} \bar{\omega}_{\bar{k},ab} - \dot{z}^k \omega_{k,\bar{b}a} \right) \psi^a \bar{\psi}^{\bar{b}}. \quad (3.20)$$

Let us now turn to quantum theory. The Poisson brackets (3.12) are replaced by the (anti)commutators:

$$[z^j, P_k] = i\delta_k^j, \quad [\bar{z}^{\bar{j}}, P_{\bar{k}}] = i\delta_{\bar{k}}^{\bar{j}}, \quad \{\psi^a, \bar{\psi}^{\bar{b}}\} = \delta^{a\bar{b}}. \quad (3.21)$$

As is well known, there exist, generically, many different quantum theories corresponding to a given classical one, due to ordering ambiguities. To make a selection, we require that the supersymmetry algebra (3.16) remains intact at the quantum level and that Q_{qu} and \bar{Q}_{qu} are Hermitian conjugate to each other. As was noticed in [12], these two requirements can be simultaneously fulfilled only provided that the classical expressions for the *supercharges* are Weyl-ordered in the quantum case. The correct expression for the quantum Hamiltonian is obtained as the anticommutator of the Weyl-ordered Q_{qu} and \bar{Q}_{qu} . Note that this correct quantum Hamiltonian *does not* coincide with the operator obtained through Weyl-ordering of the classical Hamiltonian defined by the relations (3.16).

We thus obtain

$$\begin{aligned} Q^{\text{flat}} &= \frac{1}{\sqrt{2}} [\{\Pi_k, e_a^k\} \psi^a + i\{\psi^b \psi^d, \bar{\psi}^{\bar{a}}\} e_b^k \Omega_{k,\bar{a}d}], \\ \bar{Q}^{\text{flat}} &= \frac{1}{\sqrt{2}} [\{\bar{\Pi}_{\bar{k}}, e_{\bar{a}}^{\bar{k}}\} \bar{\psi}^{\bar{a}} + i\{\bar{\psi}^{\bar{c}} \bar{\psi}^{\bar{a}}, \psi^d\} e_{\bar{c}}^{\bar{k}} \bar{\Omega}_{\bar{k},d\bar{a}}]. \end{aligned} \quad (3.22)$$

These Weyl-ordered supercharges were dubbed “flat” because they act on the wave functions normalized by the condition [12]

$$\int \prod_k dz^k d\bar{z}^{\bar{k}} \prod_a d\psi^a d\bar{\psi}^{\bar{a}} \exp\{\bar{\psi}^{\bar{a}} \psi^a\} \bar{\Psi}(z^k, \bar{z}^{\bar{k}}, \bar{\psi}^{\bar{a}}) \Psi(\bar{z}^{\bar{k}}, z^k, \psi^a) = 1 \quad (3.23)$$

with the flat Hilbert space measure. In particular, it is straightforward to see that the Weyl-ordered supercharges Q and \bar{Q} are Hermitian-conjugate to each other with respect to such flat inner product⁷.

It is more natural, however, to deal with the covariant supercharges $Q^{\text{cov}}, \bar{Q}^{\text{cov}}$ which act on the Hilbert space in which the inner product is defined with the covariant integration measure

$$\int \prod_k dz^k d\bar{z}^{\bar{k}} \det h \prod_a d\psi^a d\bar{\psi}^{\bar{a}} \exp\{\bar{\psi}^{\bar{a}} \psi^a\} \bar{\Psi}(z^k, \bar{z}^{\bar{k}}, \bar{\psi}^{\bar{a}}) \Psi(\bar{z}^{\bar{k}}, z^k, \psi^a) = 1 \quad (3.24)$$

(note that $\det h \propto \sqrt{\det g}$). They are related to the flat supercharges by a similarity transformation

$$(Q^{\text{cov}}, \bar{Q}^{\text{cov}}) = (\det h)^{-1/2} (Q^{\text{flat}}, \bar{Q}^{\text{flat}}) (\det h)^{1/2}, \quad (3.25)$$

which yields the expressions:

$$\begin{aligned} Q^{\text{cov}} &= \sqrt{2} \psi^c e_c^k \left[\Pi_k - \frac{i}{2} \partial_k (\ln \det \bar{e}) + i \psi^b \bar{\psi}^{\bar{a}} \Omega_{k,\bar{a}b} \right] \\ \bar{Q}^{\text{cov}} &= \sqrt{2} \bar{\psi}^{\bar{c}} e_{\bar{c}}^{\bar{k}} \left[\bar{\Pi}_{\bar{k}} - \frac{i}{2} \partial_{\bar{k}} (\ln \det e) + i \bar{\psi}^{\bar{b}} \psi^a \bar{\Omega}_{\bar{k},a\bar{b}} \right]. \end{aligned} \quad (3.26)$$

Here,

$$\Pi_k = -i \left(\frac{\partial}{\partial z^k} - \partial_k W \right), \quad \bar{\Pi}_{\bar{k}} = -i \left(\frac{\partial}{\partial \bar{z}^{\bar{k}}} + \partial_{\bar{k}} W \right). \quad (3.27)$$

The supercharges obey the relations of the $\mathcal{N} = 2, d = 1$ Poincaré superalgebra

$$\{Q^{\text{cov}}, \bar{Q}^{\text{cov}}\} = 2H_{qu}^{\text{cov}}, \quad \{Q^{\text{cov}}, Q^{\text{cov}}\} = \{\bar{Q}^{\text{cov}}, \bar{Q}^{\text{cov}}\} = [Q^{\text{cov}}, H_{qu}^{\text{cov}}] = [\bar{Q}^{\text{cov}}, H_{qu}^{\text{cov}}] = 0.$$

The expression for the quantum Hamiltonian H_{qu}^{cov} can be obtained in two ways:
(i) By directly calculating the anticommutator of quantum supercharges (3.26) or

⁷The same concerns the fermion operators ψ^a and $\bar{\psi}^{\bar{a}} = \frac{\partial}{\partial \psi^a}$. They are Hermitian-conjugated with a particular Berezin integration measure in (3.23), (3.24) involving the factor $\exp\{\bar{\psi}^{\bar{a}} \psi^a\}$.

(ii) by Weyl-ordering the Grönewold-Moyal bracket [13] of the classical supercharges (3.15) and performing then the similarity transformation, like in (3.25).

We obtain

$$H_{qu}^{\text{cov}} = -\frac{1}{2}\Delta^{\text{cov}} + \frac{1}{8}\left(R - \frac{1}{2}h^{\bar{k}j}h^{\bar{l}t}h^{\bar{i}n}C_{jt\bar{i}}C_{\bar{k}\bar{l}n}\right) - 2\langle\psi^a\bar{\psi}^b\rangle e_a^k e_b^{\bar{l}}\partial_k\partial_{\bar{l}}W - \langle\psi^a\psi^c\bar{\psi}^b\bar{\psi}^d\rangle e_a^t e_c^j e_b^{\bar{l}} e_d^{\bar{k}}(\partial_t\partial_{\bar{l}}h_{j\bar{k}}). \quad (3.28)$$

Here, $\langle\dots\rangle$ denotes the Weyl-ordered products of fermions, R is the standard scalar curvature of the metric $h_{j\bar{k}}$, and Δ^{cov} is the covariant Laplacian calculated with the “hatted” affine connections in (2.5) and including also the (hatted) spin connections,

$$-\Delta^{\text{cov}} = h^{\bar{k}j}\left(\mathcal{P}_j\bar{\mathcal{P}}_{\bar{k}} + i\hat{\Gamma}_{j\bar{k}}^{\bar{q}}\bar{\mathcal{P}}_{\bar{q}} + \bar{\mathcal{P}}_{\bar{k}}\mathcal{P}_j + i\hat{\Gamma}_{\bar{k}j}^s\mathcal{P}_s\right), \quad (3.29)$$

where $\mathcal{P}_j = \Pi_j + i\hat{\Omega}_{j,\bar{b}a}\langle\psi^a\bar{\psi}^b\rangle$ and $\bar{\mathcal{P}}_{\bar{k}} = \bar{\Pi}_{\bar{k}} - i\hat{\Omega}_{\bar{k},ab}\langle\psi^a\bar{\psi}^b\rangle$. Note that the scalar curvature R is related to its “hatted” counterpart \hat{R} associated with the non-symmetric affine connection $\hat{\Gamma}$ by the simple formula

$$\hat{R} = R - \frac{1}{4}C_{MNP}C^{MNP} = R - \frac{3}{2}h^{\bar{k}j}h^{\bar{l}t}h^{\bar{i}n}C_{jt\bar{i}}C_{\bar{k}\bar{l}n}.$$

In the Kähler case, the expression for the quantum Hamiltonian greatly simplifies:

$$H_{\text{Kähler}}^{\text{cov}} = -\frac{1}{2}\Delta^{\text{cov}} + \frac{R}{8} - 2\langle\psi^a\bar{\psi}^b\rangle e_a^k e_b^{\bar{l}}\partial_k\partial_{\bar{l}}W, \quad (3.30)$$

where now $-\Delta^{\text{cov}} = h^{\bar{k}j}(\mathcal{P}_j\bar{\mathcal{P}}_{\bar{k}} + \bar{\mathcal{P}}_{\bar{k}}\mathcal{P}_j)$ and $\hat{\Omega}, \hat{\bar{\Omega}}$ are reduced to $\Omega, \bar{\Omega} = \omega, \bar{\omega}$ according to the relations (2.18).

An important remark is to the point here. The Lagrangian (3.20) can also be expressed through real variables,

$$L = \frac{g_{MN}}{2}[\dot{z}^M\dot{z}^N + i\psi^M\nabla\psi^N] = \frac{1}{2}\left[g_{MN}\dot{z}^M\dot{z}^N + i\psi^A(\dot{\psi}^A + \Omega_{M,AB}\dot{z}^M\psi^B)\right]. \quad (3.31)$$

This Lagrangian is well-known [7]. It can be (and was) also considered for a generic (not necessarily complex) manifold. In a generic case, it is manifestly invariant only under $\mathcal{N} = 1$ supersymmetry transformations (with a real Grassmann parameter). The corresponding Nöther supercharge is

$$Q = \psi^A e_A^M \left[P_M - \frac{i}{2}\Omega_{M,BC}\psi^B\psi^C \right]. \quad (3.32)$$

The covariant quantum supercharge (obtained by Weyl-ordering of the classical supercharge and taking a correct account of the measure factor as in (3.25)) is given by the *same* expression, where now $\{\psi^A, \psi^B\} = \delta^{AB}$. It can be interpreted as the Dirac operator. By construction, it is Hermitian with the Hilbert space metric including the factor $\sqrt{\det g}$.

The corresponding quantum Hamiltonian [3] coincides, up to a proper similarity transformation, with (3.30) rewritten in real notations.

As was mentioned in the Introduction (and we will return to the discussion of this issue in Sect. 6), for an even-dimensional manifold, the second real supercharge

$$\tilde{Q} = 2^{D/2} Q \prod_{A=1}^D \psi^A \quad (3.33)$$

associated with $\mathcal{D}\gamma^{D+1}$ can also be defined. However, for $D \geq 4$, this second supercharge has nothing to do with the supercharges (3.26).

To recapitulate:

- For any even-dimensional manifold, the system (3.31) admits two real quantum supercharges (3.32) and (3.33).
- For any manifold with Hermitian metric (2.1), a generically *different* system (3.8) involves a *different* pair of supercharges (3.26). We will show in Sect. 5 that these supercharges can be interpreted as an exterior holomorphic derivative and its complex conjugate.
- We will also show in Sect. 5 that, in the *Kähler* case, the real and imaginary parts of the supercharge Q in (3.26) can be interpreted as the Dirac operator \mathcal{D} and the operator S defined in Eq. (1.4).
- Thus, for Kähler manifolds, when the Lagrangians (3.8) and (3.31) coincide, two different $\mathcal{N} = 2$ supersymmetry structures $(\mathcal{D}^2; \mathcal{D}, \mathcal{D}\gamma^{D+1})$ and $(\mathcal{D}^2; \mathcal{D}, S)$ are possible. Note that this does not imply an extended $\mathcal{N} = 4$ supersymmetry because the anticommutator $\{S, \mathcal{D}\gamma^{D+1}\}$ does not vanish.
- Note, however, that, for *hyper-Kähler* manifolds where three different complex structures are present, one can construct three different new supercharges

$$S^{(f)} = iI_B^{A(f)} \gamma^B \mathcal{D}_A, \quad f = 1, 2, 3, \quad (3.34)$$

such that the Lagrangian (3.20), (3.31) enjoys an $\mathcal{N} = 4$ supersymmetry [9].

- There exists also an $\mathcal{N} = 4$ *completion* of the system (3.20) for any Kähler manifold, as will be discussed in Sect. 4.

3.1 Examples

Here we consider two examples of SQM on complex manifolds.

1. $\mathbb{C}\mathbb{P}^n$ model. This is a Kähler manifold, so the torsion (2.4) vanishes and many formulas look simpler. The corresponding Kähler potential is

$$K = \ln(1 + z\bar{z}), \quad z\bar{z} \equiv z^j \bar{z}^{\bar{j}}. \quad (3.35)$$

We choose

$$W = -\frac{c_0}{2} K = -\frac{c_0}{2} \ln(1 + z\bar{z}). \quad (3.36)$$

The metric is given by the well known Fubini-Study expressions:

$$\begin{aligned} h_{j\bar{k}} &= \partial_j \partial_{\bar{k}} \ln(1 + z\bar{z}) = \frac{1}{1 + z\bar{z}} \left(\delta_{j\bar{k}} - \frac{z^k \bar{z}^{\bar{j}}}{1 + z\bar{z}} \right), \\ h^{\bar{k}j} &= (1 + z\bar{z}) \left(\delta^{\bar{k}j} + z^j \bar{z}^{\bar{k}} \right). \end{aligned} \quad (3.37)$$

Note the specific for \mathbb{CP}^n relation

$$K = -\frac{1}{n+1} \ln \det h. \quad (3.38)$$

We choose the vielbeins in the form [9]:

$$\begin{aligned} e_l^a &= \frac{1}{\sqrt{1 + z\bar{z}}} \left(\delta_l^a - \frac{z^a \bar{z}^{\bar{l}}}{\sqrt{1 + z\bar{z}}(1 + \sqrt{1 + z\bar{z}})} \right), \\ e_b^l &= \sqrt{1 + z\bar{z}} \left(\delta_b^l + \frac{z^l \bar{z}^{\bar{b}}}{1 + \sqrt{1 + z\bar{z}}} \right), \\ e_{\bar{a}}^{\bar{l}} &= \sqrt{1 + z\bar{z}} \left(\delta_{\bar{a}}^{\bar{l}} + \frac{z^a \bar{z}^{\bar{l}}}{1 + \sqrt{1 + z\bar{z}}} \right), \\ e_{\bar{l}}^{\bar{a}} &= \frac{1}{\sqrt{1 + z\bar{z}}} \left(\delta_{\bar{l}}^{\bar{a}} - \frac{z^l \bar{z}^{\bar{a}}}{\sqrt{1 + z\bar{z}}(1 + \sqrt{1 + z\bar{z}})} \right). \end{aligned} \quad (3.39)$$

The supercharges (3.26) in this special case look as follows

$$Q^{\text{cov}} = \sqrt{2} \psi^c e_c^k \left[\tilde{\Pi}_k + i \psi^b \bar{\psi}^{\bar{a}} \omega_{k,\bar{a}b} \right], \quad \bar{Q}^{\text{cov}} = \sqrt{2} \bar{\psi}^{\bar{c}} e_{\bar{c}}^{\bar{k}} \left[\tilde{\Pi}_{\bar{k}} + i \bar{\psi}^{\bar{a}} \psi^b \bar{\omega}_{\bar{k},b\bar{a}} \right], \quad (3.40)$$

where

$$\begin{aligned} \tilde{\Pi}_k &= \frac{1}{i} \left[\frac{\partial}{\partial z^k} + \frac{1}{2} \left(c_0 - \frac{n+1}{2} \right) \frac{\bar{z}^{\bar{k}}}{1 + z\bar{z}} \right], \\ \hat{\Pi}_{\bar{k}} &= \frac{1}{i} \left[\frac{\partial}{\partial \bar{z}^{\bar{k}}} - \frac{1}{2} \left(c_0 + \frac{n+1}{2} \right) \frac{z^k}{1 + z\bar{z}} \right] \end{aligned} \quad (3.41)$$

and $\omega_{k,\bar{a}b}$, $\bar{\omega}_{\bar{k},b\bar{a}}$ were defined in Eq.(2.18).

2. S^4 model. As a second example, we consider a 4-dimensional conformally flat manifold with the metric

$$ds^2 = \frac{2 dz^j d\bar{z}^{\bar{j}}}{f^2}, \quad j = 1, 2. \quad (3.42)$$

When $f = 1 + z\bar{z}$, this is the metric of S^4 . Under a natural choice of vielbeins, $\det e = \det \bar{e} = 1/f^2$ and the non-zero components of the spin connection $\Omega_{k,\bar{a}b}$ are

$$\Omega_{1,\bar{1}1} = \Omega_{2,\bar{2}1} = -\partial_1 \ln f, \quad \Omega_{1,\bar{1}2} = \Omega_{2,\bar{2}2} = -\partial_2 \ln f. \quad (3.43)$$

This is not a Kähler manifold. Taking the general expression (3.26) for the supercharges, we derive for $W = 0$,

$$\mathcal{Q} = -i\sqrt{2}\psi^a[f\partial_a - (\partial_a f)] - i\sqrt{2}\psi^1\psi^2[(\partial_2 f)\bar{\psi}^1 - (\partial_1 f)\bar{\psi}^2]. \quad (3.44)$$

An $\mathcal{N} = 4$ SQM model describing the motion over any conformally flat 4-dimensional manifold with the metric (3.42) with or without background gauge field was constructed in [14] based on the action given in [15] (see also [16]). In the case when the gauge field is absent, the *flat* (in the Hilbert space sense, as discussed above) supercharges have the form

$$Q_\alpha = (\sigma_\mu \bar{\psi})_\alpha P_\mu - i(\partial_\mu f)\psi_\gamma \bar{\psi}^\gamma (\sigma_\mu \bar{\psi})^\alpha, \quad \bar{Q}^\alpha = (\psi \sigma_\mu^\dagger)^\alpha P_\mu + i(\partial_\mu f)(\psi \sigma_\mu^\dagger)^\alpha \psi_\gamma \bar{\psi}^\gamma, \quad (3.45)$$

where $\sigma_\mu = (i, \vec{\sigma})$, $\sigma_\mu^\dagger = (-i, \vec{\sigma})$.

It is straightforward to see that, after performing the similarity transformation (3.25), the supercharge (3.44) coincides with \bar{Q}^1 in (3.45) under the identification

$$z^1 = \frac{x_3 + ix_4}{\sqrt{2}}, \quad z^2 = \frac{x_1 - ix_2}{\sqrt{2}},$$

or with \bar{Q}^2 , under the identification

$$z^1 = \frac{x_1 + ix_2}{\sqrt{2}}, \quad z^2 = \frac{ix_4 - x_3}{\sqrt{2}}.$$

These two possibilities reflect the presence of two different $\mathcal{N} = 2$ Poincaré superalgebras in the $\mathcal{N} = 4$ superalgebra.

4 Completion to Kähler $\mathcal{N} = 4$ SQM model

Our starting point is the $\mathcal{N} = 2$ SQM model with the superfield Lagrangian \mathcal{L}_σ in (3.5) involving the Kähler metric (3.7). We do not add the gauge part \mathcal{L}_{gauge} . So we choose

$$h_{j\bar{k}}(Z, \bar{Z}) = \partial_j \partial_{\bar{k}} K(Z, \bar{Z}), \quad W = 0 \quad (4.1)$$

in (3.5). The corresponding component Lagrangian was written in (3.20).

Using the chirality properties of $Z^j, \bar{Z}^{\bar{k}}$ and the algebra of $\mathcal{N} = 2$ spinor derivatives, it will be convenient to rewrite the corresponding superfield Lagrangian in the following three equivalent (they coincide up to a total time derivative) forms:

$$\begin{aligned} \mathcal{L}^K &= -\frac{1}{4} \partial_j \partial_{\bar{k}} K(Z, \bar{Z}) DZ^j \bar{D}\bar{Z}^{\bar{k}} \simeq -\frac{i}{2} \dot{Z}^j \partial_j K \\ &\simeq \frac{i}{2} \dot{\bar{Z}}^{\bar{k}} \partial_{\bar{k}} K \simeq \frac{i}{4} \left(\dot{\bar{Z}}^{\bar{k}} \partial_{\bar{k}} K - \dot{Z}^j \partial_j K \right). \end{aligned} \quad (4.2)$$

Now consider an extended Lagrangian

$$\tilde{\mathcal{L}}^K = \mathcal{L}^K + \frac{1}{4} h_{j\bar{k}} \Phi^j \bar{\Phi}^{\bar{k}}, \quad (4.3)$$

where Φ^j , $\bar{\Phi}^{\bar{k}}$ are chiral and anti-chiral fermionic $\mathcal{N} = 2$ (0+1)-dimensional superfields, $\bar{D}\Phi^j = D\bar{\Phi}^{\bar{k}} = 0$. It is straightforward to check that (4.3) is invariant, modulo a total derivative, under the following extra $\mathcal{N} = 2$ supersymmetry transformations:

$$\delta Z^j = -\zeta \Phi^j, \quad \delta \bar{Z}^{\bar{k}} = \bar{\zeta} \bar{\Phi}^{\bar{k}}, \quad \delta \Phi^j = 2i \bar{\zeta} \dot{Z}^j, \quad \delta \bar{\Phi}^{\bar{k}} = -2i \zeta \dot{\bar{Z}}^{\bar{k}}. \quad (4.4)$$

These variations form the same algebra with respect to Lie brackets as the variations (3.4) corresponding to the manifest world-line $\mathcal{N} = 2$ supersymmetry. Thus, they extend the latter to off-shell (0+1)-dimensional $\mathcal{N} = 4$ supersymmetry.

The superfields Φ^j , $\bar{\Phi}^{\bar{k}}$ have the following θ expansions

$$\Phi^j = \sqrt{2} \chi^j + \theta d^j - i\sqrt{2} \theta \bar{\theta} \dot{\chi}^j, \quad \bar{\Phi}^{\bar{k}} = \sqrt{2} \bar{\chi}^{\bar{k}} + \bar{\theta} \bar{d}^{\bar{k}} + i\sqrt{2} \theta \bar{\theta} \dot{\bar{\chi}}^{\bar{k}}. \quad (4.5)$$

We observe that they contain no new bosonic fields of physical dimension, only the auxiliary bosonic fields $d^j, \bar{d}^{\bar{k}}$ as well as the extra physical fermionic fields $\chi^j, \bar{\chi}^{\bar{k}}$. Thus, in this model we deal with n off-shell $\mathcal{N} = 4$ supermultiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$, the subsequent numerals standing, respectively, for the numbers of the physical bosonic, physical fermionic and auxiliary bosonic fields⁸. The manifest $\mathcal{N} = 2$ supersymmetry acts on the component fields in (4.5) as

$$\delta \chi^j = -\frac{1}{\sqrt{2}} \epsilon d^j, \quad \delta d^j = 2\sqrt{2} i \bar{\epsilon} \dot{\chi}^j, \quad \delta \bar{\chi}^{\bar{k}} = -\frac{1}{\sqrt{2}} \bar{\epsilon} \bar{d}^{\bar{k}}, \quad \delta \bar{d}^{\bar{k}} = 2\sqrt{2} i \epsilon \dot{\bar{\chi}}^{\bar{k}}. \quad (4.6)$$

The second supersymmetry transformations (4.4) has the following realization in components:

$$\begin{aligned} \delta z^j &= -\sqrt{2} \zeta \chi^j, \quad \delta \psi^j = \frac{1}{\sqrt{2}} \zeta d^j, \quad \delta \chi^j = \sqrt{2} i \bar{\zeta} \dot{z}^j, \quad \delta d^j = -2\sqrt{2} i \bar{\zeta} \dot{\psi}^j, \\ \delta \bar{z}^{\bar{k}} &= \sqrt{2} \bar{\zeta} \bar{\chi}^{\bar{k}}, \quad \delta \bar{\psi}^{\bar{k}} = \frac{1}{\sqrt{2}} \bar{\zeta} \bar{d}^{\bar{k}}, \quad \delta \bar{\chi}^{\bar{k}} = -\sqrt{2} i \zeta \dot{\bar{z}}^{\bar{k}}, \quad \delta \bar{d}^{\bar{k}} = -2\sqrt{2} i \zeta \dot{\bar{\psi}}^{\bar{k}}. \end{aligned} \quad (4.7)$$

After going to the component fields in the action corresponding to the modified superfield Lagrangian (4.3) and eliminating the auxiliary fields $d^j, \bar{d}^{\bar{k}}$ by their equations of motion,

$$d^j = 2h^{\bar{p}j} \partial_l h_{t\bar{p}} \chi^t \psi^l, \quad \bar{d}^{\bar{k}} = 2h^{\bar{k}p} \partial_{\bar{l}} h_{p\bar{j}} \bar{\psi}^{\bar{l}} \bar{\chi}^{\bar{j}}, \quad (4.8)$$

the contribution of the second term in (4.3) to the total component Lagrangian reads:

$$\Delta L = \frac{i}{2} h_{j\bar{k}} \left(\chi^j \nabla \bar{\chi}^{\bar{k}} - \nabla \chi^j \bar{\chi}^{\bar{k}} \right) + R_{j\bar{k}l\bar{p}} \psi^j \bar{\psi}^{\bar{k}} \chi^l \bar{\chi}^{\bar{p}}. \quad (4.9)$$

⁸In this notation, the $\mathcal{N} = 2$ multiplets corresponding to the superfields Z^i and Φ^i can be denoted as $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ and $(\mathbf{0}, \mathbf{2}, \mathbf{2})$.

Here,

$$\nabla \bar{\chi}^{\bar{k}} = \dot{\bar{\chi}}^{\bar{k}} + \dot{z}^{\bar{p}} \Gamma_{\bar{p}\bar{j}}^{\bar{k}} \bar{\chi}^{\bar{j}}, \quad \nabla \chi^j = \dot{\chi}^j + \dot{z}^l \Gamma_{lp}^j \chi^p, \quad (4.10)$$

and $R_{j\bar{k}l\bar{p}}$ is the Riemann tensor for the Kähler metric defined in (2.20). Its appearance in the Lagrangian is an important new feature of the $\mathcal{N} = 4$ case compared to Eq.(3.20).

The total $\mathcal{N} = 4$ supersymmetric component Lagrangian can be concisely written as

$$L = h_{j\bar{k}} \left[\dot{z}^j \dot{\bar{z}}^{\bar{k}} + \frac{i}{2} \left(\psi^j \nabla \bar{\psi}^{\bar{k}} + \chi^j \nabla \bar{\chi}^{\bar{k}} - \nabla \psi^j \bar{\psi}^{\bar{k}} - \nabla \chi^j \bar{\chi}^{\bar{k}} \right) \right] + R_{j\bar{k}l\bar{p}} \psi^j \bar{\psi}^{\bar{k}} \chi^l \bar{\chi}^{\bar{p}}. \quad (4.11)$$

The $\mathcal{N} = 4$ supersymmetry closes on shell, since we have eliminated the auxiliary fields $d^j, \bar{d}^{\bar{k}}$.

The Lagrangian (4.11) is well known. It coincides with the Lagrangian obtained by deleting spatial derivatives in the (1+1)-dimensional $\mathcal{N} = 2$ σ -model Lagrangian [17] and discussed, e.g., in [18, 12] (there, fermionic fields ψ^j, χ^j were combined into a SU(2) doublet). We refer the reader to [12] for the expressions for the classical and quantum supercharges, the Hamiltonian, etc.

It is worth also recalling that the Lagrangian (4.11) coincides with the generic SQM sigma-model Lagrangian involving D supermultiplets $(\mathbf{1}, \mathbf{2}, \mathbf{1})$ [20],

$$L = g_{MN} \left(\frac{1}{2} \dot{z}^M \dot{z}^N + i \bar{\psi}^M \nabla \psi^M \right) + \frac{1}{2} R_{MNPQ} \bar{\psi}^M \psi^N \bar{\psi}^P \psi^Q. \quad (4.12)$$

For a generic metric, the latter Lagrangian enjoys only $\mathcal{N} = 2$ supersymmetry, but in the Kähler case, a second pair of supercharges can be found. Note also that, when an external gauge field is present, there is no such second pair. A related almost equivalent statement is that no $\mathcal{N} = 4$ completion based on the linear chiral $\mathcal{N} = 4, d = 1$ multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ is possible for the theory (3.5) with $W \neq 0$. Note that such a completion becomes possible, if extending the $\mathcal{N} = 2$ chiral multiplets $(\mathbf{2}, \mathbf{2}, \mathbf{0})$ to *nonlinear* versions of the $\mathcal{N} = 4$ multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ or $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ [19].

5 Quantum supercharges and geometry

Let us assume that $\det \bar{e} = \det e = \sqrt{\det h}^9$ and choose

$$W = \frac{c_0}{2(n+1)} \ln \det h. \quad (5.1)$$

Then the general supercharges (3.26) take the form (3.40), (3.41) where we should replace $\omega_{k,\bar{a}b} \rightarrow \Omega_{k,\bar{a}b}$ and

$$\begin{aligned} \frac{\bar{z}^k}{1+z\bar{z}} &\rightarrow -\frac{1}{n+1} \partial_k (\ln \det h), \\ \frac{z^k}{1+z\bar{z}} &\rightarrow -\frac{1}{n+1} \partial_{\bar{k}} (\ln \det h). \end{aligned} \quad (5.2)$$

⁹Such a choice amounts to fixing a gauge with respect to the local frame U(1) transformations of the vielbeins.

We see that there are special values $c_0 = \pm(n+1)/2$ where either $\tilde{\Pi}_k$ or $\bar{\tilde{\Pi}}_k$ coincide with the usual holomorphic or antiholomorphic derivatives. Consider first the case $c_0 = (n+1)/2$. It is not difficult to check that the action of Q^{cov} on the wave functions

$$\Psi(z^k, \bar{z}^k; \psi^a) = A^{(0)}(z^k, \bar{z}^k) + \psi^a A_a^{(1)}(z^k, \bar{z}^k) + \dots + \psi^{a_1} \dots \psi^{a_n} A_{[a_1 \dots a_n]}^{(n)}(z^k, \bar{z}^k) \quad (5.3)$$

is isomorphic to the action of the exterior holomorphic derivative ∂ on the set of $n+1$ holomorphic (p,0)-forms (the term $\propto \Omega$ in Q^{cov} cancels out the term coming from differentiation of the vielbeins in virtue of the structure equation (2.12)). The Hermitian-conjugate operator \bar{Q}^{cov} is then isomorphic to ∂^\dagger . In other words, in this case the supercharges (3.40) realize the standard untwisted (i.e. involving no additional gauge field) Dolbeault complex.

Likewise, in the case $c_0 = -(n+1)/2$, the action of the operator \bar{Q}^{cov} on *anti*-holomorphic wave functions $\Psi(z^k, \bar{z}^k; \bar{\psi}^a)$ is isomorphic to the action of the operator $\bar{\partial}$ on antiholomorphic (0,p)-forms, the operator Q^{cov} playing the role of $\bar{\partial}^\dagger$. Thus, in this case we are dealing with the anti-holomorphic untwisted Dolbeault complex.

For any other value of c_0 , an extra Abelian gauge field is present in the framework of both the holomorphic and antiholomorphic Dolbeault interpretations, i.e.

$$A_k = \frac{i}{4} \left(1 - \frac{2c_0}{n+1} \right) \partial_k \ln \det h \quad (5.4)$$

in the holomorphic case and

$$A_{\bar{k}} = \frac{i}{4} \left(1 + \frac{2c_0}{n+1} \right) \partial_{\bar{k}} \ln \det h \quad (5.5)$$

in the antiholomorphic case. We face what is called twisted Dolbeault complex.

Until now we dealt with the general (non-Kähler) $\mathcal{N} = 2$ SQM model, the only restriction was the relation (5.1). If the manifold is Kähler, the supercharges admit *another* even more interesting geometric interpretation: when $c_0 = 0$, the *sum* $Q^{\text{cov}} + \bar{Q}^{\text{cov}}$ can be interpreted as the untwisted Dirac operator. When $c_0 \neq 0$, an extra Abelian gauge field is present.

Indeed, the standard untwisted Dirac operator in the real notations is [cf. (3.32)]

$$\mathcal{D} = \gamma^A e_A^M \left(\partial_M + \frac{1}{4} \Omega_{M,BC} \gamma^B \gamma^C \right) \equiv \gamma^A \mathcal{D}_A. \quad (5.6)$$

When splitting $M \equiv (k, \bar{k})$, $A \equiv (a, \bar{a})$ and introducing $\sqrt{2}\psi^a \equiv \gamma^a$, $\sqrt{2}\bar{\psi}^{\bar{a}} \equiv \gamma^{\bar{a}}$, one can be convinced that, for Kähler manifolds, one can represent

$$\mathcal{D} = \mathcal{D}^{\text{Hol}} - (\mathcal{D}^{\text{Hol}})^\dagger, \quad (5.7)$$

where

$$\mathcal{D}^{\text{Hol}} = \sqrt{2}\psi^b e_b^k \left[\partial_k + \frac{1}{2} \omega_{k,\bar{a}d} (\bar{\psi}^{\bar{a}} \psi^d - \psi^d \bar{\psi}^{\bar{a}}) \right] \quad (5.8)$$

and

$$(\mathcal{P}^{\text{Hol}})^\dagger = -\sqrt{2}\bar{\psi}^{\bar{b}}e_{\bar{b}}^{\bar{k}} \left[\partial_{\bar{k}} + \frac{1}{2}\bar{\omega}_{\bar{k},a\bar{a}}(\psi^a\bar{\psi}^{\bar{d}} - \bar{\psi}^{\bar{d}}\psi^a) \right]. \quad (5.9)$$

These operators coincide, up to the factor i , with the supercharges (3.26), (3.27) in which one chooses $\Omega = \omega$ and $W = 0$:

$$\mathcal{P}^{\text{Hol}} = iQ^{\text{cov}}, \quad (\mathcal{P}^{\text{Hol}})^\dagger = -i\bar{Q}^{\text{cov}}, \quad (C_{k\bar{i}\bar{l}} = c_0 = 0). \quad (5.10)$$

For $c_0 \neq 0$, an additional Abelian gauge field is present, and we are facing the twisted Dirac operator in this case. Note that the definition of “twisting” or “untwisting” is different in the interpretations in terms of Dolbeault and Dirac complexes. E.g., the choice $c_0 = 0$ corresponds to an *untwisted* Dirac complex, but to the *twisted* Dolbeault complex (as is seen from (5.4), (5.5)).

The operator \mathcal{P} is anti-Hermitian. Consider now the *real* part of \mathcal{P}^{Hol} ,

$$S = \mathcal{P}^{\text{Hol}} + (\mathcal{P}^{\text{Hol}})^\dagger. \quad (5.11)$$

One can be convinced that instead of $\sqrt{2}\psi^A\mathcal{D}_A \equiv \gamma^A\mathcal{D}_A$ (with the imaginary \equiv anti-Hermitian part of \mathcal{P}^{Hol}), we obtain the structure

$$S = \gamma^B I_B^A \mathcal{D}_A, \quad (5.12)$$

where $I_B^A, I^2 = -1$, is the complex structure: $I = \text{diag}(i\sigma_2, \dots, i\sigma_2)$ under the natural conventions. The pair of supercharge (1.4) is thus reproduced. We emphasize again that the existence of the supercharge S is specific for Kähler manifolds.

A by-product of this analysis is a physical proof of the purely mathematical fact: for Kähler manifolds, the twisted Dirac complex is equivalent to the twisted Dolbeault complex, bearing in mind that the twisting (the adding of Abelian gauge fields) in the Dirac complex and in the Dolbeault complex is different. This fact is known to mathematicians, see e.g. the Propositions 1.4.23 and 1.4.25 in the book [21].

If the manifold is not Kähler, the decomposition (5.7) - (5.9) is no longer valid. Besides the connections $\Omega_{k,\bar{a}\bar{d}}, \bar{\Omega}_{\bar{k},a\bar{d}}$, there are also the nonzero connections $\Omega_{k,\bar{a}\bar{b}}$ and $\bar{\Omega}_{\bar{k},ab}$ [see Eq.(2.8)]. The sum $Q + \bar{Q}$ does not thus coincide with the full Dirac operator, but represents a certain “truncated” Dirac operator, where the terms involving the connections $\Omega_{k,\bar{a}\bar{b}}$ and $\bar{\Omega}_{\bar{k},ab}$ are suppressed, thus breaking the full local tangent space group $O(2n)$ down to $U(n)$. For the moment, it is unclear to us whether this truncated operator is related to the non-truncated one, and if so, then how. A further analysis of this interesting question is necessary.

6 Index

The Euclidean path integral representation for the index (1.3) of our system is

$$I = \int \prod_{j\tau} \frac{d\pi_j(\tau)d\bar{\pi}_{\bar{j}}(\tau)dz^j(\tau)d\bar{z}^{\bar{j}}(\tau)}{(2\pi)^2} \prod_{a\tau} d\psi^a(\tau)d\bar{\psi}^{\bar{a}}(\tau) \times \exp \left\{ \int_0^\beta \left[i\pi_j \dot{z}^j + i\bar{\pi}_{\bar{j}} \dot{\bar{z}}^{\bar{j}} + \dot{\bar{\psi}}^{\bar{a}} \psi^a - H(\pi_j, \bar{\pi}_{\bar{j}}, z^j, \bar{z}^{\bar{j}}; \bar{\psi}^{\bar{a}}, \psi^a) \right] \right\}, \quad (6.1)$$

where both bosonic *and* fermionic variables satisfy the periodic boundary conditions, $z^j(\beta) = z^j(0)$, etc. Expand all the variables in the Fourier series,

$$z^j(\tau) = z^{j(0)} + \sum_{m \neq 0} z^{j(m)} e^{2\pi i m \tau / \beta} \quad \bar{z}^{\bar{j}}(\tau) = \bar{z}^{\bar{j}(0)} + \sum_{m \neq 0} \bar{z}^{\bar{j}(m)} e^{-2\pi i m \tau / \beta} , \quad (6.2)$$

and similarly for $\pi_j(\tau)$, $\bar{\pi}_{\bar{j}}(\tau)$ and $\psi^a(\tau)$, $\bar{\psi}^{\bar{a}}(\tau)$. If β is small, we *seemingly* (see below) can neglect the nonzero modes in the expansion, neglect thereby the terms with time derivatives in (6.1), and rewrite (6.1) as an *ordinary* integral [22]:

$$I = \int \prod_j \frac{d\pi_j^{(0)} d\bar{\pi}_{\bar{j}}^{(0)} dz^{j(0)} d\bar{z}^{\bar{j}(0)}}{(2\pi)^2} \prod_a d\psi^{a(0)} d\bar{\psi}^{\bar{a}(0)} \times \exp \left\{ -\beta H(\pi_j^{(0)}, \bar{\pi}_{\bar{j}}^{(0)}, z^{j(0)}, \bar{z}^{\bar{j}(0)}; \psi^{a(0)}, \bar{\psi}^{\bar{a}(0)}) \right\} . \quad (6.3)$$

The functional integral is reduced to the ordinary one in the semiclassical limit $\beta \rightarrow 0$. However, the index (1.3) does not depend on β , and the estimate (6.3) should be true for any β .

Substituting here the Hamiltonian (3.17) with the choice (5.1) (remember that, for Kähler manifolds, the last term in (3.17) vanishes), we can easily integrate over $\prod_j d\pi_j d\bar{\pi}_{\bar{j}}$ and over $\prod_a d\psi^a d\bar{\psi}^{\bar{a}}$ to obtain

$$I = \left(\frac{1}{2\pi} \right)^n \int \prod_j dz^j d\bar{z}^{\bar{j}} \det \|h_{j\bar{k}}\| \det \|i\mathcal{F}_{a\bar{b}}\| , \quad (6.4)$$

where $i\mathcal{F}_{a\bar{b}} = -2e_a^j e_{\bar{b}}^{\bar{k}} \partial_j \partial_{\bar{k}} W$ is related to the 2-form describing the magnetic field strength. In the simplest $\mathbb{C}\mathbb{P}^n$ case under the choice (5.1), we have $i\mathcal{F}_{a\bar{b}} = c_0 \delta_{a\bar{b}}$ leading to

$$I = \left(\frac{c_0}{2\pi} \right)^n \int \prod_j dz^j d\bar{z}^{\bar{j}} (\det h) . \quad (6.5)$$

The calculation with the Fubini-Study metric (3.37), i.e. with $\det h = \frac{1}{(1+z\bar{z})^{n+1}}$, gives

$$I_{\mathbb{C}\mathbb{P}^n} \stackrel{?}{=} \frac{(c_0)^n}{n!} . \quad (6.6)$$

This result looks suspicious. Indeed, to make it integer (the index should be integer for the Dirac operator to make sense: only in this case the manifold admits *spin structure*), c_0 should depend on n in an odd way.

Actually, the estimate (6.6) is *wrong*. The correct estimate for any (not necessarily Kähler) even-dimensional manifold reads [2, 7]

$$I = \int e^{\mathcal{F}/2\pi} \det^{-1/2} \left[\frac{\sin \frac{\mathcal{R}}{4\pi}}{\frac{\mathcal{R}}{4\pi}} \right] , \quad (6.7)$$

where \mathcal{F} is the field strength 2-form and \mathcal{R} is the matrix 2-form associated with the Riemann curvature,

$$\mathcal{F} = \frac{1}{2} F_{MN} dx^M \wedge dx^N, \quad \mathcal{R}^{AB} = \frac{1}{2} R^{AB}{}_{MN} dx^M \wedge dx^N, \quad (6.8)$$

A, B being the tangent space indices.

The precise meaning of the representation (6.7) is that the volume integral in its r.h.s. projects out only the forms of the maximal rank D from the Taylor expansion of the integrand. Thus, for 4-dimensional manifolds, the index is represented as the sum of two terms,

$$I_{d=4} = \frac{1}{8\pi^2} \int \mathcal{F} \wedge \mathcal{F} + \frac{1}{192\pi^2} \int \text{Tr}\{\mathcal{R} \wedge \mathcal{R}\}. \quad (6.9)$$

The topological invariants in the r.h.s. are known as the second Chern class c_2 and the Hirzebruch signature τ (the latter enters with the coefficient $-1/8$). For higher dimensions, the index is a sum of many different invariants.

It is convenient to represent the determinant factor in (6.7) as

$$\det^{-1/2}[\dots] = \prod_{\alpha=1}^n \frac{\lambda_\alpha/(4\pi)}{\sinh(\lambda_\alpha/(4\pi))}, \quad (6.10)$$

where λ_α are the eigenvalues of the antisymmetric matrix \mathcal{R}^{AB} . This can be derived by diagonalizing,

$$\mathcal{R} \longrightarrow (i\sigma_2\lambda_1, \dots, i\sigma_2\lambda_n),$$

and noting that, for any *even* function $f(\mathcal{R})$,

$$\det^{-1/2} f(\mathcal{R}) = \prod_{\alpha=1}^n \frac{1}{f(i\lambda_\alpha)}.$$

The estimate (6.6) for $\mathbb{C}\mathbb{P}^n$ would be reproduced, if ignoring this curvature-dependent determinant factor in (6.7). When including this factor, we obtain instead

$$I_{\mathbb{C}\mathbb{P}^n} = \binom{c_0 + (n-1)/2}{n}, \quad (6.11)$$

where c_0 must be integer for odd n and half-integer for even n . The index is given by Eq.(6.11) when $c_0 \geq (n+1)/2$. For negative $c_0 \leq -(n+1)/2$, it is given by

$$I_{\mathbb{C}\mathbb{P}^n}(c_0 < 0) = (-1)^n \binom{|c_0| + (n-1)/2}{n}. \quad (6.12)$$

The index vanishes for $|c_0| < (n+1)/2$ ¹⁰.

¹⁰Note in passing that the index (6.11) is closely related to the Witten index in 3d supersymmetric Yang-Mills-Chern-Simons theory [23]. See [24] for detailed discussion.

The result (6.11) for the index in $\mathbb{C}\mathbb{P}^n$ can also be derived directly, simply by counting the number of independent ground states [25, 9], i.e. the number of the normalized (with the measure (3.24), in which $\det h = 1/(1 + z\bar{z})^{n+1}$) solutions to the equations

$$Q^{cov}\Psi_0 = \bar{Q}^{cov}\Psi_0 = 0, \quad (6.13)$$

with Q^{cov} and \bar{Q}^{cov} defined in (3.40), (3.41). Choosing, e.g., the holomorphic representation (5.3) for the wave functions, we find that, in the sector of *zero* fermionic charge, the equation $\bar{Q}^{cov}\Psi_0 = 0$ is satisfied identically, while the equation $Q^{cov}\Psi_0 = 0$ implies

$$\partial_k \Psi_0 = -\frac{s\bar{z}^k}{1 + z\bar{z}} \Psi_0, \quad (6.14)$$

with

$$2s = c_0 - \frac{n+1}{2}. \quad (6.15)$$

We see that the normalized solutions exist only at $s \geq 0$. They have the form

$$\Psi_0 = \Psi(z, \bar{z}) = (1 + z\bar{z})^{-s} \Phi(\bar{z}), \quad (6.16)$$

where $\Phi(\bar{z})$ is a polynomial of $\bar{z}^{\bar{j}}$ of the rank not higher than $2s$. Then the number of independent ground states is given by the binomial coefficient

$$\frac{(n+2s)!}{n!(2s)!} = \frac{(c_0 + \frac{n-1}{2})!}{n!(c_0 - \frac{n+1}{2})!}, \quad (6.17)$$

which exactly coincides with Eq. (6.11). For negative c_0 , the vacuum states are present in the sector of fermion charge $F = n$, hence the factor $(-1)^n$ in Eq. (6.12).

What was wrong then in the calculation having led to (6.6)? The answer is that the recipe [22] that allowed us to replace the functional integral (6.1) by the ordinary integral (6.3) and that works well for many SQM and supersymmetric field theory systems *fails* in this case. To obtain the correct estimate for the index, one should take into account the nonzero Fourier modes in the expansion (6.2) and integrate them over in the *Gaussian* (see below) approximation. This integral gives exactly the determinant factor in (6.7).

To perform the actual calculation ¹¹, we assume β to be small, impose periodic boundary conditions, subdivide the interval $(0, \beta)$ into a large number N of integration points and integrate first over $\prod_{j\tau} \frac{d\pi_j(\tau)d\bar{\pi}_{\bar{j}}(\tau)}{2\pi}$ to obtain

$$\begin{aligned} I &= \int \prod_{\tau} \det h(\bar{z}^{\bar{j}}(\tau), z^j(\tau)) \prod_j \frac{d\bar{z}^{\bar{j}}(\tau)dz^j(\tau)}{2\pi(\beta/N)} \prod_a d\psi^a(\tau)d\bar{\psi}^{\bar{a}}(\tau) \\ &\quad \times \exp \left\{ - \int_0^{\beta} L_E(\tau) d\tau \right\}, \end{aligned} \quad (6.18)$$

¹¹ It is rather similar in spirit to the calculation of the functional integral for SQM describing the complex $(\mathcal{P}^2; \mathcal{P}, \mathcal{P}\gamma^5)$ [7]. However, because almost no details of the calculation were presented in [7], we decided to describe these details at some length here concentrating on the Kähler case.

with

$$L_E = h_{j\bar{k}} \dot{z}^j \dot{\bar{z}}^{\bar{k}} + \frac{1}{2} (\psi^a \dot{\bar{\psi}}^{\bar{a}} - \dot{\psi}^a \bar{\psi}^a) + \left(\dot{\bar{z}}^{\bar{k}} \omega_{\bar{k},a\bar{b}} - \dot{z}^k \omega_{k,\bar{b}a} \right) \psi^a \bar{\psi}^{\bar{b}}. \quad (6.19)$$

The product \prod_{τ} in (6.18) runs over N discrete points $\tau_r = \beta r/N$, $r = 0, \dots, N-1$. For simplicity, we suppressed the gauge part that was already successfully handled earlier by the Cecotti-Girardello method. It is the determinant factor depending only on the Riemannian manifold geometry that is of interest for us now.

Substitute now the expansion (6.2) into (6.19). If the number of points N is large, but finite, we have also to keep the number of Fourier modes finite, so that the sum in (6.2) runs over $m = -M, \dots, 0, \dots, M$, where $N = 2M + 1$. To calculate the functional integral in the Gaussian approximation¹² (we will justify the validity of this approximation later), we keep only quadratic (in $\bar{z}_m^j, z_m^j, \bar{\psi}_m^{\bar{a}}$ and ψ_m^a) terms and do the τ -integral. The quadratic part of the Lagrangian gives

$$\begin{aligned} \int_0^\beta L_E^{(2)} d\tau &= -i\beta \sum'_m \Omega_m \psi_m^a \bar{\psi}_m^{\bar{a}} - i\beta \sum'_m \Omega_m \left[\psi^{a(0)} \omega_{k,\bar{b}a}^{(0)} \bar{\psi}_m^{\bar{b}} z_m^k - \bar{\psi}^{\bar{b}(0)} \psi_m^a \bar{z}_m^{\bar{k}} \omega_{\bar{k},a\bar{b}}^{(0)} \right] \\ &+ \beta \sum'_m z_m^j \bar{z}_m^{\bar{k}} \left[\Omega_m^2 h_{j\bar{k}}^{(0)} - i\Omega_m \psi^{a(0)} \bar{\psi}^{\bar{b}(0)} (\partial_j \omega_{\bar{k},a\bar{b}}^{(0)} + \partial_{\bar{k}} \omega_{j,\bar{b}a}^{(0)}) \right] \\ &- i\beta \sum'_m \Omega_m \left[\psi^{a(0)} \omega_{\bar{k},a\bar{b}}^{(0)} \bar{z}_m^{\bar{k}} \bar{\psi}_{-m}^{\bar{b}} - \bar{\psi}^{\bar{a}(0)} z_m^k \psi_{-m}^b \omega_{k,\bar{a}b}^{(0)} \right] \\ &- i\beta \sum'_m \Omega_m \psi^{a(0)} \bar{\psi}^{\bar{b}(0)} \left[\partial_k \omega_{j,\bar{b}a}^{(0)} z_m^j z_{-m}^k + \partial_{\bar{k}} \omega_{j,\bar{a}b}^{(0)} \bar{z}_m^{\bar{j}} \bar{z}_{-m}^{\bar{k}} \right], \quad (6.20) \end{aligned}$$

with $\Omega_m = 2\pi m/\beta$. The sum \sum'_m runs over all nonzero modes. When writing this, we assumed $m \ll M$. If $m \sim M$, one is not allowed to approximate the finite differences in the Euclidean action entering the finite-number-of-point approximation (6.18) of the path integral by time derivatives. An accurate analysis displays that the only change one should implement for large m is to substitute

$$\Omega_m \longrightarrow \tilde{\Omega}_m = -i \frac{N}{\beta} (1 - e^{-2\pi i m/N}) \quad (6.21)$$

in Eq.(6.20). However, as we will see later, this replacement affects only the overall coefficient in the functional integral that is fixed separately, while the nontrivial dependence of the integrand on the metric is determined by the contribution of only first few Fourier modes.

Thus, we keep for the moment $\Omega_m = 2\pi m/\beta$ and diagonalize the sum in (6.20) by the substitution

$$\begin{aligned} \psi_m^b &\Rightarrow \psi_m^b + \psi^{a(0)} [\bar{z}_{-m}^{\bar{k}} \omega_{\bar{k},a\bar{b}}^{(0)} - z_m^k \omega_{k,\bar{b}a}^{(0)}], \\ \bar{\psi}_m^{\bar{b}} &\Rightarrow \bar{\psi}_m^{\bar{b}} + \bar{\psi}^{\bar{a}(0)} [z_{-m}^k \omega_{k,\bar{a}b}^{(0)} - \bar{z}_m^{\bar{k}} \omega_{\bar{k},\bar{b}a}^{(0)}]. \quad (6.22) \end{aligned}$$

¹²Incidentally, the result (6.18) can also be reproduced by trading the variables $\pi_j(\tau_r)$ and $\bar{\pi}_{\bar{j}}(\tau_r)$ for their Fourier modes and performing then the Gaussian integral over $\prod'_m d\pi_j^{(m)} d\bar{\pi}_{\bar{j}}^{(m)}$. In this case, the factor N^N seen in (6.18) appears as the Jacobian of the variable change (6.2).

It brings (6.20) to the simple form

$$\int_0^\beta L_E^{(2)} d\tau = \sum'_m A_{m\bar{a}\bar{b}} \psi_m^a \bar{\psi}_m^{\bar{b}} + \sum'_m D_{mj\bar{k}} z_m^j \bar{z}_m^{\bar{k}}, \quad (6.23)$$

where

$$A_{m\bar{a}\bar{b}} = -i\beta \Omega_m \delta_{\bar{a}\bar{b}}, \quad D_{mj\bar{k}} = \beta \left[\Omega_m^2 h_{j\bar{k}}^{(0)} - i\Omega_m R_{j\bar{k},\bar{a}\bar{b}} \psi^{a(0)} \bar{\psi}^{\bar{b}(0)} \right] \quad (6.24)$$

and $R_{j\bar{k},\bar{a}\bar{b}}$ is the Riemann tensor defined in (2.20). In the process of passing from (6.20) to (6.23) we used the property $\Omega_{-m} = -\Omega_m$ and also the identity

$$\partial_{[l\omega_k],\bar{b}a} - \omega_{[l,\bar{d}a}\omega_k],\bar{b}d} = 0 \quad (\text{and c.c.}) \quad (6.25)$$

(the l.h.s. of Eq.(6.25) is none other than the component $R_{l\bar{k}\bar{b}a}$ of the Riemann tensor that vanishes for Kähler manifolds).

Note that the matrix of the partial derivatives corresponding to the substitution (6.22) is triangle and so has a unit superdeterminant. The super-Jacobian for the variable change (6.2) is also equal to unity, because the bosonic and fermion determinants cancel each other. The functional integral over non-zero modes is then given by a product of a large (in the continuous limit, infinite) number of finite-dimensional determinants, which can be symbolically written as

$$\begin{aligned} \text{grav. factor} &= \mu \prod'_m \prod_j \frac{dz_m^j d\bar{z}_m^{\bar{j}}}{2\pi} \prod_a d\psi_m^a d\bar{\psi}_m^{\bar{a}} \exp \{ -A_m \psi_m \bar{\psi}_m - D_m z_m \bar{z}_m \} \\ &= \mu \prod'_m \det \|A_m\| \cdot \det^{-1} \|D_m\|, \end{aligned} \quad (6.26)$$

where

$$\mu = \left(\det \|h_{i\bar{k}}^{(0)}\| \right)^N \prod_{m=1}^M \Omega_m^{2n} \quad (6.27)$$

is the appropriate measure. The factor $\left(\det \|h_{i\bar{k}}^{(0)}\| \right)^N$ in (6.27) comes from the factor $\prod_{r=0}^{N-1} \det \|h_{i\bar{k}}(\tau_r)\|$ in (6.18), where the dependence of $z^j(\tau)$, $\bar{z}^{\bar{j}}(\tau)$ on higher Fourier harmonics has been suppressed. This suppression can be justified by noticing that the characteristic values of z_m in the integral $\prod_{mj} dz_m^j d\bar{z}_m^{\bar{j}} \exp\{-\int_0^\beta L_E^{(2)} d\tau\}$ are $z_m \sim 1/\Omega_m \sqrt{\beta} \sim \sqrt{\beta}$, which is small at small β . The dimensional factor β^{-2Mn} in (6.27) comes from the factor $\beta^{-nN} = \beta^{-2nM} \times \beta^{-n}$ in (6.18) (the factor β^{-n} having been borrowed to be displayed in the constant mode integral (6.3) after performing the integration over momenta). To derive from (6.18) the correct numerical factor in the measure, notice that the coefficient N^N present in (6.18) can be represented as

$$N^N = \prod_{m=1}^M (\beta \tilde{\Omega}_m) (\beta \tilde{\Omega}_{-m}), \quad (6.28)$$

which follows in turn from the known identity ¹³

$$\prod_{r=1}^{N-1} (1 - w^r) = N, \quad \text{if } w = e^{2\pi i/N}. \quad (6.29)$$

Then, bearing in mind that only first few values m are relevant (see below), we can replace $\tilde{\Omega}_m \rightarrow \Omega_m$, which yields (6.27). It is much easier, of course, to fix the factor in (6.27) from the condition that the r.h.s. of Eq.(6.26) is equal to 1 in the flat case $h_{j\bar{k}} = \delta_{j\bar{k}}$.

The calculation gives

$$\begin{aligned} \text{grav. factor} &= \left(\det \|h_{i\bar{k}}^{(0)}\| \right)^N \prod'_m \frac{\Omega_m^{2n}}{\det \|h_{i\bar{k}}^{(0)}\| \det \|\Omega_m^2 \delta_j^q - i\Omega_m R_j^q\|} \\ &= \det \|h_{i\bar{k}}^{(0)}\| \prod_{m=1}^{\infty} \frac{\Omega_m^{2n}}{\det \|\Omega_m^2 \delta_j^q + R_j^s R_s^q\|}, \end{aligned} \quad (6.30)$$

where

$$R_j^q = g^{q\bar{k}} R_{j\bar{k}l\bar{p}} \psi^{l(0)} \bar{\psi}^{\bar{p}(0)} \quad (6.31)$$

and we took into account the relation $N = 2M + 1$ and sent $M \rightarrow \infty$ afterwards. We see that only *one* power of the determinant $\det h^{(0)}$ is left.

The infinite m -product in (6.30) can be done by writing the determinant as the product of the eigenvalues and using the identity

$$\prod_{m=1}^{\infty} \frac{(2\pi m)^2}{(2\pi m)^2 + a^2} = \frac{a}{2 \sinh(a/2)}. \quad (6.32)$$

For $a \sim 1$, only few first values of m are essential in this product, and it justifies as promised the assumption $m \ll M$ under which Eq.(6.20) was derived.

We finally obtain

$$I^{\text{pure gravity}} = \frac{1}{(2\pi\beta)^n} \int \prod_{j=1}^n d\bar{z}^{\bar{j}} dz^j d\psi^j d\bar{\psi}^{\bar{j}} \det \frac{\beta R/2}{\sinh(\beta R/2)}, \quad (6.33)$$

where we suppressed the superscripts ⁽⁰⁾ and passed back to the integration over the fermionic zero modes with the world indices $\psi^j, \bar{\psi}^{\bar{j}}$ (this absorbs the remaining factor $\det h$ in (6.30)). Multiplying the integrand by $\exp\{-i\beta \mathcal{F}_{j\bar{k}} \psi^j \bar{\psi}^{\bar{k}}\}$ and doing the fermion integral, we arrive at (6.10) and hence to (6.7) ¹⁴. It is clear now why, in

¹³Consider $P(x) = x^N - 1 = \prod_{r=0}^{N-1} (x - w^r)$ and calculate $P'(1)$.

¹⁴ To establish the exact correspondence, one has to keep in mind that the skew-symmetric matrix \mathcal{R} defined in (6.8) is represented in the Kähler case as

$$\mathcal{R}^{AB} = \begin{pmatrix} 0 & -R^{a\bar{b}} \\ R^{b\bar{a}} & 0 \end{pmatrix}, \quad (6.34)$$

where

$$R^{a\bar{b}} = e_i^a e_{\bar{k}}^{\bar{b}} R^{i\bar{k}}_{j\bar{l}} dz^j \wedge d\bar{z}^{\bar{l}}. \quad (6.35)$$

this particular case, we had to insert the 1-loop gravitational factor in the tree-level integral (6.3) for the index. Formally, the factor (6.30) tends to 1 for small β and, naively, the corrections involving β and its higher powers can be neglected. We see, however, that each factor β in the expansion is multiplied by a bi-fermion structure $\sim \psi\bar{\psi}$, as is also the case for the expansion of the integrand in (6.3). For the fermion integral not to vanish, we have to pick up the terms $\sim \beta^n(\psi\bar{\psi})^n$ in the expansion of both the factor $\exp\{-i\beta\mathcal{F}\psi\bar{\psi}\}$ inside the tree-level integral and of the 1-loop factor (6.30) — they come on equal footing.

On the other hand, the possible semiclassical corrections involving more powers of β than those coming from $\psi\bar{\psi}$ are not relevant (cf. a remark in the paragraph after (6.27)). This justifies neglecting two-loop and higher-loop effects in the functional integral (6.1).

7 Final comments

In the previous Section, we have constructed the proof of the Atiyah-Singer theorem based on the analysis of our SQM model for the Kähler manifolds and for Abelian gauge fields. The same method can be and was used, however, to prove it for any even-dimensional manifold. To this end, one should consider the system defined by the Lagrangian (3.31) accompanied by the external gauge field Lagrangian (3.10).

As was discussed above, in the generic case, the $\mathcal{N} = 2$ supersymmetry algebra is realized not by the supercharges (1.4), but by the supercharges \mathcal{P} and $\mathcal{P}\gamma^{D+1}$. The Witten index (1.3) of this model still coincides with the Atiyah-Singer index of \mathcal{P} . One can be easily convinced in it by introducing the holomorphic variables $\chi^1 = \psi^{1-i2}/\sqrt{2}$, $\chi^2 = \psi^{3-i4}/\sqrt{2}$, etc, and noting that

$$\gamma^{D+1} \equiv (2i)^{D/2} \prod_{A=1}^D \psi^A = \prod_{a=1}^{D/2} (\bar{\chi}^a \chi^a - \chi^a \bar{\chi}^a) \equiv (-1)^F . \quad (7.1)$$

Then we have to expand the Euclidean version of the Lagrangians (3.31) and (3.10) into the modes and to perform basically the same calculation as described above. It gives the same answer (6.7). Exploring somewhat more complicated SQM systems, this method can be generalized to non-Abelian gauge fields too.

Our model (3.9) coincides with the model (3.31) in the Kähler case, but, for a generic complex (or “quasi-complex”, like S^4 ,) manifold, its Lagrangian is different. As was explained in Sect. 5, the corresponding supercharges in all cases realize the twisted Dolbeault complex. It would be interesting to generalize the functional integral calculation of the index, which was done in the previous section for the Kähler manifolds, for the generic Lagrangians (3.9), (3.10). One should reproduce in this way the known integral representation for the index of the Dolbeault complex.

We want also to mention here that there is another way to evaluate the curvature-dependent corrections to the naive leading order semiclassical result (6.6). One can proceed in the framework of the Hamiltonian formalism and notice that the index is given by the phase space integral of the Weyl symbol of the operator $e^{-\beta H}$. The

point is that, generically, $[e^{-\beta H}]_W$ differs from $e^{-\beta H_W}$, there appear corrections which involve higher powers of β . The simplest correction of this type for a generic SQM system with the phase space variables $(p_j, q_j; \bar{\psi}_a, \psi_a)$ is expressed as [26]

$$[e^{-\beta H}]_W = e^{-\beta H_W} (1 + \delta + O(\beta^4)) \quad (7.2)$$

with

$$\begin{aligned} \delta(p_j, q_j; \psi_a, \bar{\psi}_a) &= \frac{\beta^2}{48} \left[\frac{\partial^2}{\partial \Psi_a \partial \bar{\psi}_a} - \frac{\partial^2}{\partial \psi_a \partial \bar{\Psi}_a} + i \left(\frac{\partial^2}{\partial q_j \partial P_j} - \frac{\partial^2}{\partial Q_j \partial p_j} \right) \right]^2 \\ &\times H(p_j, q_j; \bar{\psi}_a, \psi_a) H(P_j, Q_j; \bar{\Psi}_a, \Psi_a) |_{P=p, Q=q; \bar{\Psi}=\bar{\psi}, \Psi=\psi}. \end{aligned} \quad (7.3)$$

In most cases, this correction is suppressed at small β and so is irrelevant. However, in our case, it gives a relevant β -independent contribution,

$$\Delta_\delta I = -\frac{1}{96\pi^2} \int \prod_j dz^j d\bar{z}^{\bar{j}} g^{k\bar{l}} g^{p\bar{t}} \epsilon^{ms} \epsilon^{\bar{n}\bar{q}} R_{k\bar{t}m\bar{n}} R_{p\bar{l}s\bar{q}} = -\frac{\tau}{8}, \quad (7.4)$$

where τ is the Hirzebruch signature. This coincides with the second term in (6.9).

We see that the Lagrangian method is much more convenient than the Hamiltonian one: the one-loop correction manifestly seen within the Lagrangian method corresponds to a complicated series in β on the Hamiltonian side. To find a relevant $\propto \beta^4$ term in the expansion (7.2) is already a pretty difficult task.

Finally, it is worth mentioning that there are also other cases when the index cannot be expressed as the simple phase space integral (6.3).

First of all, this concerns the systems with the continuous spectrum, like super-conformal quantum mechanics [27] or super-Yang-Mills quantum mechanics, where the integrals like (6.3) give meaningless fractional values [28]. In these cases, due to the absence of the gap, such integrals cannot be “focused” on zero energy normalized states, but are “contaminated” by the states from continuum.

The systems with continuous spectrum are widely known and discussed in the literature. There is, however, *another* interesting class of systems, the SQM systems related to Abelian [29] and non-Abelian [30] chiral supersymmetric 4D gauge theories. In the latter case, the spectrum seems to be discrete, the index is well-defined, and still the integral (6.3) gives a fractional value. It would be rather interesting to see whether this “anomaly” can be cured by taking into account the 1-loop determinant in the spirit of (6.7).

There is also a problem in the index calculation for “symplectic” supersymmetric $\mathcal{N} = 4$ σ -models with bosonic part describing the motion over a 3D conformally flat manifold [31]. For example, for S^3 , the index is equal to 2, while the integral (6.3) gives a meaningless irrational number. One of us has shown in [26] that the corrections to (6.3) *are* present in this case and that they are of the same order as the tree-level contribution. It would be interesting to try to sum up all such corrections by the Lagrangian functional integral method.

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