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Parent formulation at the Lagrangian level

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ABSTRACT. The recently proposed first-order parent formalism at the level of equations of motion is specialized to the case of Lagrangian systems. It is shown that for diffeomorphism-invariant theories the parent formulation takes the form of an AKSZ-type sigma model. The proposed formulation can be also seen as a Lagrangian version of the BV-BRST extension of the Vasiliev unfolded approach. We also discuss its possible interpretation as a multidimensional generalization of the Hamiltonian BFV–BRST formalism. The general construction is illustrated by examples of mechanics, relativistic particle, Yang–Mills theory, and gravity.

1 Introduction

The Batalin–Vilkovisky (BV) formalism [1, 2] allows reformulating nearly any gauge system as a universal BV theory that has an elegant and unique form irrespective of the particular structure of the starting point system. In so doing all the information about the Lagrangian, gauge transformations, Noether identities and higher structures of the gauge algebra are encoded in the BV master action. This is achieved by introducing ghost fields and antifields in such a way that the entire field-antifield space acquires an odd Poisson bracket (the antibracket). All the compatibility conditions like gauge invariance of the action, reducibility relation and so on are then encoded in the master equation which is merely equivalent to requiring the BRST transformation to be nilpotent.

All the ingredients of the BV formalism can be naturally seen as geometric objects defined on an abstract manifold and the BV formalism makes perfect sense in the purely geometrical setting. In the context of local gauge field theory the manifold in question has an extra structure: it is the space of suitable maps (field histories) between the space-time and the target-space manifolds. Moreover, all the ingredients such as the Lagrangian, gauge generators, structure functions and so on are required to involve space-time derivatives of finite order. In the BV formalism the locality is usually taken into account [3, 4, 5, 5] by approximating the space of field histories by the respective jet bundle (see e.g. [6, 7, 8, 9] for a review on jet bundle approach). More technically, the formalism involves the total de Rham differential along with the BRST differential so that the naive BRST complex becomes a part of the appropriate bicomplex.

Although the jet space extension of the BV formalism has proved extremely useful in studying, e.g., renormalization, anomalies, and consistent deformations [3, 5, 10] (see [11] for a review) it is not completely satisfactory because the jet space approximation can be too restrictive. For instance, the boundary dynamics is not captured in a straightforward way. In addition, the jet space structures such as, e.g., generalized connections and curvatures of [12, 13, 14] do not have a direct dynamical meaning and are not manifestly realized in the formulation.

An interesting alternative to the jet space description of gauge theories is the unfolded formalism [15, 16] developed in the context of higher spin gauge theories. In this approach on-shell independent derivatives of fields are treated as new independent fields and the equations of motion are represented as a free differential algebra (FDA) [17]. The latter structure also underlies somewhat related approaches to supergravity [18, 19]. It is within the unfolded framework that the interacting theory of higher spin fields on the AdS space has been derived [20, 21, 22]. The unfolded approach is also a powerful tool in studying gauge field theories invariant under one or another space-time symmetry algebras [23, 24].

At the level of equations of motion the relation between the BV formalism and the unfolded approach was established in [25] (see also [26, 27]) for linear systems and in [28] in the general case by constructing the so-called parent formulation such that both the BV and the unfolded formulation can be arrived at via straightforward reductions. The parent formulation itself or some of its extensions can be considered as a new formulation generalizing and unifying both the BV and the unfolded formulation at the level of equations of motion. Moreover, it is the parent formulation that gives a systematic way to construct (and proves the existence of) the unfolded form of a given theory.

In this paper we specialize the parent formulation to the case of Lagrangian systems giving a parent extension of the BV formalism. In particular, we identify the precise set of fields and antifields, prescribe the antibracket and construct the master action satisfying the classical master equation. We show that for diffeomorphism-invariant theories the parent formulation is a sigma model of Alexandrov–Kontsevich–Schwartz–Zaboronsky (AKSZ) type [29] (see also [30, 31, 32, 33, 34, 35, 36, 37, 38, 39] for further developments and applications of AKSZ-type sigma models) for which the target space is the BV jet space of the starting point system while the starting point Lagrangian plays the role of a potential.

2 Parent Lagrangian

2.1 Preliminaries

Suppose we are given a regular local Lagrangian gauge field theory. Within the BV formalism the theory is defined by the master action $S[\psi,\psi^*]$, where ψ^A,ψ^*_A are fields and antifields. The space of fields and antifields carry an integer ghost degree $\mathrm{gh}(\cdot)$ such that fields of the theory are those ψ^A with $\mathrm{gh}(\psi^A)=0$ while the remaining ψ^A-s are ghost fields, ghosts for ghosts, and so on, and carry positive ghost degrees. The master action S carries vanishing ghost degree and satisfies the master equation

$$\big(S,S\big)=0\,, \tag{2.1}$$

with respect to the antibracket defined by

$$\big(\psi^A(x),\psi^*_A(x')\big)=\delta^A_B\delta^{(n)}(x-x')\,,\quad \big(\psi^A(x),\psi^B(x')\big)=\big(\psi^*_A(x),\psi^*_B(x')\big)=0\,, \tag{2.2}$$

where x^μ, $\mu=1,\ldots,n$ denote space-time coordinates. The ghost numbers and the Grassmann parities of the antifields are determined by those of the fields through $\mathrm{gh}(\psi^*_A)=-1-\mathrm{gh}(\psi^A)$ and $|\psi^*_A|=|\psi^A|+1\bmod 2$ so that the antibracket is Grassmann odd and carries ghost degree 1.

We restrict ourselves to the case of theories with closed algebra. For such theories $S[\psi,\psi^*]$ can be chosen at most linear in antifields. More precisely, S can be taken as

$$S=\int d^nxL_0[\psi]+\int d^nx\,\psi^*_A(\gamma\psi^A)\,, \tag{2.3}$$

where γ is a gauge part of the complete BRST differential s and $L_0[\psi]$ is the Lagrangian. In our case, γ is nilpotent and enters the complete BRST differential $s=\big(\cdot,S\big)$ as $s=\delta+\gamma$. Here, δ is the Koszul–Tate term implementing the equations of motion determined by L_0 and their reducibility relations. Note that in general γ is nilpotent only modulo equations of motion and $s=\delta+\gamma+\ldots$, where dots refer to terms originating from the terms in S of the second and higher orders in ψ^*_A.

We first recall the construction [28] of the parent theory at the level of the equations of motion. In the present context it is convenient to concentrate on the gauge structure encoded in γ and temporarily disregard the actual equations of motion implemented through δ and the antifields ψ^*_A. This corresponds to the off–shell truncation of the parent formulation in [28]. The extended set of fields (including ghost fields etc.) is given by $\psi^A_{(\lambda)[\nu]}$, where (λ) denotes a symmetric multi-index and $[\nu]$ an antisymmetric one. Introducing bosonic variables y^λ and fermionic variables θ^ν, all the fields can be packed into the generating function

$$\widetilde{\psi}^A(x,y,\theta)=\sum_{k,l\geqslant 0}\frac{1}{k!l!}\,\theta^{\nu_l}\ldots\theta^{\nu_1}y^{\lambda_k}\ldots y^{\lambda_1}\psi^A_{\lambda_1\ldots\lambda_k|\nu_1\ldots\nu_l}(x)\equiv\theta^{(\nu)}y^{(\lambda)}\psi^A_{(\lambda)[\nu]}(x)\,. \tag{2.4}$$

The ghost degrees of the component fields are determined by the ghost degree of ψ^A if one prescribes $\mathrm{gh}(y^\lambda) = 0$ and $\mathrm{gh}(\theta^\nu) = 1$. In what follows we also use the condensed notation ψ^α for all the fields so that α stands for $A, (\mu), [\nu]$ and ranges over an infinite but countable set. The lowest component $\psi^A_{()[]}$ is identified with ψ^A. Fields $\psi^A_{(\lambda)[\nu]}$ are refereed to as θ and y-derivatives (or descendants) of ψ^A.

We need to introduce some useful operations on the space of fields of the parent theory. Given a differential operator $\mathcal{O}$ on the space of y, θ and x we associate a functional vector field $\mathcal{O}^F$ on the space of fields $\psi^A_{(\lambda)[\nu]}(x)$ according to (see [28] for more details)

$$\mathcal{O}^F(\widetilde{\psi}^A) = (-1)^{|A||\mathcal{O}|}\mathcal{O}\widetilde{\psi}^A\,, \tag{2.5}$$

where $\mathcal{O}^F$ is assumed to act from the right. Here, $\mathcal{O}$ acts on y, θ, x while $\mathcal{O}^F$ acts on the space of fields $\psi^A_{(\lambda)[\nu]}(x)$. Relation (2.5) is compatible with the commutator in the sense that $([\mathcal{O}_1, \mathcal{O}_2])^F = [\mathcal{O}_1^F, \mathcal{O}_2^F]$. To fit with the usual conventions for the master action (see, e.g., [40]) we have exchanged the left and right action with respect to [28]. Using (2.5) one defines $d^F, \sigma^F, \frac{\partial^F}{\partial y^\mu}, \frac{\partial^F}{\partial \theta^\mu}$ associated to $\sigma = \theta^\mu \frac{\partial}{\partial y^\mu}$, $d = \theta^\mu \frac{\partial}{\partial x^\mu}$, $\frac{\partial}{\partial y^\mu}$, and $\frac{\partial}{\partial \theta^\mu}$. In what follows we need some explicit relations:

$$\begin{gathered}\frac{\partial^F}{\partial\theta^\nu}\psi^A = (-1)^{|A|}\psi^A_{()\nu}\,, \quad \frac{\partial^F}{\partial y^\nu}\psi^A = \psi^A_{\nu[]}\,, \quad d^F\psi^A_{(\lambda)[]} = \sigma^F\psi^A_{(\lambda)[]} = 0\,, \\ d^F\psi^A_{()\nu} = (-1)^{|A|}\partial_\nu\psi^A\,, \quad \sigma^F\psi^A_{()\nu} = (-1)^{|A|}\psi^A_{\nu[]}\,.\end{gathered} \tag{2.6}$$

We often employ the language of jet spaces (see, e.g., [7, 6]) and hence replace the space of field histories $\psi^\alpha(x)$ by the respective jet space with coordinates x^μ, ψ^α, and all x-derivatives $\psi^\alpha_{(\mu)}$. We also use ∂_μ to denote the total derivative:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} + \psi^\alpha_\mu \frac{\partial}{\partial \psi^\alpha} + \psi^\alpha_{\mu\mu_1}\frac{\partial}{\partial \psi^\alpha_{\mu_1}} + \dots\,. \tag{2.7}$$

Functional vector fields defined by (2.5) can be also seen as vector fields on the jet space.

The gauge part γ of the BRST differential can then be naturally seen as acting on the space with coordinates $x^\mu, \psi^\alpha_{(\mu)}$. This is achieved as follows: for ψ^A one defines $\bar{\gamma}\psi^A = \gamma\psi^A$, where the derivatives $\partial_{(\mu)}\psi^A$ in the HRS are replaced by $\psi^A_{(\mu)[]}$. The action of $\bar{\gamma}$ on coordinates $\psi^A_{(\lambda)[]}$ is uniquely determined by requiring $[\frac{\partial}{\partial x^\mu} + \frac{\partial^F}{\partial y^\mu}, \bar{\gamma}] = 0$. Finally the action on θ-derivatives $\psi^A_{(\lambda)[\nu]}$ and x-derivatives of all the fields is obtained by the usual prolongation $[\partial_\mu, \bar{\gamma}] = [\frac{\partial^F}{\partial\theta^\nu}, \bar{\gamma}] = 0$.

Finally, the BRST differential of the parent theory is given by [28]

$$\gamma^P = d^F - \sigma^F + \bar{\gamma}\,. \tag{2.8}$$

It was shown in [28] that the parent formulation is equivalent to the starting point one via elimination of generalized auxiliary fields (see Section **2.3** for the definition and [41, 25] for details on this notion of equivalence).

2.2 Parent master action

To simplify the exposition, we assume for the moment that the starting point Lagrangian $L_0[\psi]$ is strictly gauge invariant so that $\gamma L_0 = 0$. The general case where L_0 is gauge invariant modulo a total derivative is considered next.

Associated to each field ψ^α we introduce an antifield Λ_α or in components $\Lambda_A^{(\mu)[\nu]}$ and postulate the usual antibracket, ghost number and Grassmann parity assignments:

$$\begin{gathered} \big(\psi^\alpha(x), \Lambda_\beta(x')\big)^P = \delta^\alpha_\beta \delta^{(n)}(x-x'), \\ \mathrm{gh}(\Lambda_\alpha) = -\mathrm{gh}(\psi_\alpha) - 1, \quad |\Lambda_\alpha| = |\psi_\alpha| + 1 \bmod 2\,. \end{gathered} \tag{2.9}$$

Consider then the following functional

$$S^P = \int d^n x\, \left(\Lambda_\alpha(d^F - \sigma^F + \bar{\gamma})\psi^\alpha + L_0(\psi^A_{(\lambda)[\,]}, x)\right)\,, \tag{2.10}$$

where $L_0(\psi^A_{(\lambda)[\,]}, x)$ is the starting point Lagrangian in which derivatives $\partial_{(\mu)}\psi^A$ are replaced with $\psi^A_{(\mu)[\,]}$. Because space-time derivatives enter only through d^F this action is a first-order one.

Proposition 2.1. *S^P satisfies the master equation along with the usual ghost number and Grassmann parity assignments*

$$\big(S^P, S^P\big) = 0\,, \qquad \mathrm{gh}(S^P) = 0\,, \quad |S^P| = 0\,, \tag{2.11}$$

and hence can be considered a BV master action of a gauge field theory.

Proof. It is useful to work in terms of integrands (understood modulo total derivatives). Let $K = \Lambda_\alpha(d^F - \sigma^F + \bar{\gamma})\psi^\alpha$ and L_0 be the integrands of respectively the first and the second terms in (2.10). The equation $\big(K, K\big)^P = 0$ is just a consequence of the nilpotency of the vector field $d^F - \sigma^F + \bar{\gamma}$. $\big(L_0, L_0\big)^P = 0$ is obvious because L_0 is independent of the antifields. Finally, nonvanishing contributions to $\big(L_0, K\big)^P$ can only originate from terms in K involving $\Lambda_A^{(\mu)[0]}$. But $(d^F - \sigma^F)\psi^A_{(\mu)[0]} = 0$ so that $\big(L_0, K\big)^P = \big(L_0, \Lambda_A^{(\mu)[0]}\bar{\gamma}\psi^A_{(\mu)[0]}\big)^P = 0$ as a consequence of $\gamma L_0 = 0$. $\square$

The number of fields entering master action (2.10) is infinite. This complicates the analysis and makes the interpretation of (2.10) ambiguous. Fortunately, it turns out that the action can be consistently truncated to the one involving only finitely many fields and finitely many terms. To see this, we consider the degree $N_{\partial_y} + N_{\partial_\theta} - K\mathrm{gh}_{\mathrm{T}}$, called truncation degree, where

$$N_{\partial_y} = \sum_{l \geqslant 0} l\psi^A_{\lambda_1\ldots\lambda_l[\nu]} \frac{\partial}{\partial\psi^A_{\lambda_1\ldots\lambda_l[\nu]}}\,, \qquad N_{\partial_\theta} = \sum_{l \geqslant 0} l\psi^A_{(\lambda)\nu_1\ldots\nu_l} \frac{\partial}{\partial\psi^A_{(\lambda)\nu_1\ldots\nu_l}}\,, \tag{2.12}$$

and gh_{T} is the target space ghost degree defined through $\mathrm{gh}_{\mathrm{T}}(\psi^A_{(\lambda)[\nu]}) = \mathrm{gh}(\psi^A)$ and K denotes the maximal degree of a term in $\bar{\gamma}$ that is homogeneous with respect to N_{∂_y} (or, equivalently, the maximal degree in space time derivatives of the starting point differential γ). It follows that $d^F - \sigma^F + \bar{\gamma}$ doesn't increase the truncation degree. By assigning to Λ_α conjugate to ψ^α the same truncation degree as to ψ^α and putting all the fields of a degree higher than a sufficiently high[1] integer M to zero, one then ends up with a consistent master action. In particular, the truncated master action still satisfies the master equation.

This observation gives master action (2.10) the following interpretation: S^P is to be understood as a usual master action involving a finite number of fields. However, it is useful not to fix the truncation bound and work as if all necessary fields were present. Here and below we assume that γ and L_0 involve derivatives up to a finite order and the ghost degree of fields ψ^A is also finite. In particular, this is necessary for the above truncation to exist.

In what follows we refer to the local gauge field theory determined by S^P (or its generalizations considered below) as the parent formulation. According to the principles of the BV formalism the fields of the parent formulation are those fields among $\psi^\alpha, \Lambda_\alpha$ that have the vanishing ghost degree. The respective classical action S^P_0 is obtained from S^P by putting all the fields of a nonvanishing ghost degree to zero. Gauge transformations for the fields are then read off from the complete BRST differential $s^P = \left(\cdot, S^P\right)$ by $\delta\phi^i = s^P\phi^i$, where in the Right-Hand Side we put all the fields of ghost degrees different from $0, 1$ to zero and replace degree-1 fields with gauge parameters.

It turns out that the parent formulation determined by S^P is equivalent to the starting point theory determined by S through the elimination of generalized auxiliary fields. It is then a BV master action for the parent theory of [28] in the case where the starting point theory is Lagrangian (recall also that $\gamma L_0 = 0$ and the gauge algebra is closed on our setting). Moreover, S^P is a proper solution to the master equation provided the starting point S is a proper one. In the rest of the paper we extend the construction to generic gauge theories, identify the structure of the parent formulation for diffeomorphism-invariant theories, prove the equivalence to the starting point theory, and illustrate the constructions by concrete examples.

2.3 Equivalence proof

According to the definition from [41] fields χ^i, χ^*_i are generalized auxiliary fields for the master action S if they are canonically conjugate in the antibracket and equations $\frac{\delta S}{\delta \chi^i}\Big|_{\chi^*_i=0} = 0$ can be algebraically solved for χ^i.

Proposition 2.2. *The BV formulation determined by $S^P, \left(\cdot,\cdot\right)^P$ and the starting point*

[1] Compared to the maximal order of space-time derivatives involved in the starting point Lagrangian L_0.

theory $S, (\cdot,\cdot)$ are equivalent via elimination of generalized auxiliary fields.

Proof. All the fields $\psi^A_{(\lambda)[\nu]}$ save for $\psi^A = \psi^A_{()[]}$ can be grouped into two sets w^a and v^b such that $\sigma^F w^a = v^a$. The set of fields and antifields can then be split as $\psi^A, \Lambda_A, w^a, v^a, w^*_a, v^*_a$. Let us show that v^a, w^a, v^*_a, w^*_a are generalized auxiliary fields. More precisely, as χ^i and χ^*_i we take respectively v^a, w^*_a and v^*_a, w^a.

Varying first with respect to w^*_a and putting v^*, w to zero, we find

$$\left[(d^F - \sigma^F + \bar{\gamma})w^a\right]|_{w=0} = 0 \quad \Leftrightarrow \quad v^a = \left[(d^F + \bar{\gamma})w^a\right]|_{w=0}\,. \tag{2.13}$$

It is almost clear from the last formula that it can be solved for v. The detailed proof uses the extra degrees (ghost degree and N_{∂_θ}) and was given in detail in [28]. In particular, one finds that all v^a vanish except for $\psi^A_{(\lambda)[]}$. If the theory is not truncated then $\psi^A_{(\lambda)[]} = \partial_{(\lambda)}\psi^A$. For the truncated theory this is only true for lower order derivatives [28]. However, if the truncation degree is high enough this does not affect the reduced action because L_0 involves y-derivatives of bounded order.

Varying then with respect to v^a and putting v^*, w to zero gives:

$$w^*_a = \frac{\delta^R}{\delta v^a}\left[w^*_b(d^F + \bar{\gamma})w^b + \Lambda_A\bar{\gamma}\psi^A + L_0\right]\Big|_{w=0}\,. \tag{2.14}$$

The second and the third terms cannot spoil the solvability with respect to w^*_a because they do not involve w^*_a. To see that this is also true for the first term, we use the following modification of the truncation degree: $N_{\partial_y} + N_{\partial_\theta} - (K+1)\mathrm{gh_T}$. In the linear order, we then find that $((d^F + \bar{\gamma})w^b)|_{w=0}$ can only involve variables v of the degree lower than that of w^b. It follows that $(\frac{\delta}{\delta v^a}(w^*_b(d^F + \bar{\gamma})w^b)|_{w=0}$ can only involve w^*-variables of degree higher then that of w^*_a. Because S^P is assumed truncated and hence does not involve fields of sufficiently high degree the equation can be solved order by order using the above degree and the homogeneity in the fields.

Finally, putting to zero all v^*_a, w^a as well as all v^a except $\psi^A_{(\mu)[]} = \partial_{(\mu)}\psi^A$ the master action S^P reduces to

$$\widetilde{S} = S_0[\psi^A] + \Lambda_A\gamma\psi^A\,, \tag{2.15}$$

which is exactly the starting point master action (2.3) if one identifies Λ_A with ψ^*_A. $\square$

2.4 Generalization

In order to allow for Lagrangians that are γ-closed only modulo a total derivative we need some more technique. In the setting of the starting point theory, we introduce the algebra of local forms $\widehat{\Omega}$ that are forms on x-space with values in local functions. As a usual technical assumption we in addition exclude field-independent forms from $\widehat{\Omega}$. Local forms can be seen as functions in the fields, their derivatives, the coordinates x^μ, and the fermionic variables θ^μ standing for basic differentials dx^μ. As is implied by the notation, the variables θ^μ are to be identified with the θ^μ of the previous sections.

In the usual local BRST cohomology considerations (see, e.g., [11]) it is quite useful to employ the extended BRST differential (recall that γ acts from the right)

$$\widetilde{\gamma} = -d_H + \gamma\,, \qquad d_H = \overleftarrow{\partial}_\mu \theta^\mu\,, \tag{2.16}$$

where d_H is often refereed to as total de Rham differential. For instance the ghost degree-g cohomology of γ in the space of local functionals is in fact isomorphic to the cohomology of $\widetilde{\gamma}$ in degree $g+n$ (the total degree such that θ carries unit degree is assumed) in the space of local forms without field-independent terms.

A particularly important representative of the local BRST cohomology is the Lagrangian density itself. It can be represented by a local form $\widehat{L}[\psi, x, \theta]$ of the total degree n such that $\widetilde{\gamma}\widehat{L} = 0$. The usual Lagrangian is just a coefficient in $\widehat{L}$ proportional to the volume form $\theta^1 \dots \theta^n$. More precisely $L_0[\psi, x] = \int d\theta^n \dots d\theta^1 \widehat{L}[\psi, x, \theta]$ and $\widetilde{\gamma}\widehat{L} = 0$ implies $\gamma L_0 = \partial_\mu j_1^\mu$, $\gamma j_1^\mu = \partial_\nu j_2^{\nu\mu}$, etc. with some $j_k^{\mu_1 \dots \mu_k}$, $\mathrm{gh}(j_k) = k$. Note that because of the above isomorphism any L_0 that is γ-closed modulo a total derivative can be represented by such a $\widetilde{\gamma}$-cocycle $\widehat{L}$. Obtaining $\widehat{L}$ can be also seen as solving the respective descent equation (see, e.g., [11]) with $\theta^1 \dots \theta^n L_0$ being the local form of maximal degree.

Representing the Lagrangian density through $\widehat{L}$ we easily generalize the above parent master action by

$$S^P = \int d^n x \left[\Lambda_\alpha (d^F - \sigma^F + \bar{\gamma})\psi^\alpha + \int d^n\theta \widehat{L}(\widetilde{\psi}^A_{(\lambda)}, x, \theta)\right]\,, \tag{2.17}$$

where by a slight abuse of notation we have denoted $\widetilde{\psi}^A_{(\lambda)} = \sum \frac{1}{k!}\theta^{\nu_k} \dots \theta^{\nu_1}\psi^A_{(\lambda)\nu_1 \dots \nu_k} \equiv \theta^{[\nu]}\psi^A_{(\lambda)[\nu]}$.

To see that S^P indeed satisfies the master equation modulo total derivatives, we first observe that

$$\int d\theta^n \dots d\theta^1 \widehat{L}(\widetilde{\psi}^A_{(\lambda)}, x, \theta) = \left[\partial^\theta_1 \dots \partial^\theta_n \widehat{L}(\psi^A_{(\lambda)}, x, \theta)\right]\Big|_{\theta=0}\,, \tag{2.18}$$

where $\partial^\theta_\mu = \frac{\overleftarrow{\partial}}{\partial\theta^\mu} - \frac{\partial^F}{\partial\theta^\mu}$ is a total right derivative with respect to θ^μ. It is then useful to employ the extended parent differential [28]:

$$(\widetilde{\gamma})^P = -(\frac{\overleftarrow{\partial}}{\partial x^\mu} + \frac{\partial^F}{\partial y^\mu})\theta^\mu + d^F - \sigma^F + \bar{\gamma}\,, \tag{2.19}$$

which is nilpotent and satisfies $(\widetilde{\gamma})^P|_{\theta=0} = \gamma^P$ and $[\partial^\theta_\mu, (\widetilde{\gamma})^P] = -\partial_\mu$.

Using then $[\partial^\theta_\mu, (\widetilde{\gamma})^P] = -[\partial^\theta_\mu, d_H]$ gives

$$\gamma^P \left[\partial^\theta_1 \dots \partial^\theta_n \widehat{L}(\psi^A_{(\lambda)}, x, \theta)\right]\Big|_{\theta=0} = (-1)^n \left[\partial^\theta_1 \dots \partial^\theta_n \, d_H \widehat{L}(\psi^A_{(\lambda)}, x, \theta)\right]\Big|_{\theta=0} = (-1)^n \int d^n\theta \; d_H \widehat{L}(\widetilde{\psi}^A_{(\lambda)}, x, \theta)\,, \quad (2.20)$$

so that the master equation is indeed satisfied modulo a total derivative. Finally one can check that the equivalence proof of Section **2.3** is not affected by the extra terms in the parent Lagrangian.

The structure of the parent formulation can be simplified by packing the fields $\Lambda^{(\mu)[\nu]}_A$ into superfields $\widetilde{\Lambda}^{(\mu)}_A(\theta)$ such that $\Lambda_\alpha \psi^\alpha = \Lambda^{(\mu)[\nu]}_A \psi^A_{(\mu)[\nu]} = \int d^n\theta \widetilde{\Lambda}^{(\mu)}_A \widetilde{\psi}^A_{(\mu)}$. It is then useful to employ the language of supergeometry. Namely, consider a supermanifold $\mathcal{M}$ with coordinates being $\psi^A_{(\mu)}$ and $\Lambda^{(\mu)}_A$, $\mathrm{gh}(\Lambda^{(\mu)}_A) = -\mathrm{gh}(\psi^A_{(\mu)}) + n - 1$ and equipped with the (odd) Poisson bracket defined by

$$\left\{\psi^A_{(\mu)}, \Lambda^{(\nu)}_B\right\} = \delta^A_B \delta^{(\nu)}_{(\mu)}\,. \quad (2.21)$$

The bracket carries ghost degree $1-n$ and the Grassmann parity $(1-n) \bmod 2$.

We consider the function

$$S_{\mathcal{M}}(\psi, \Lambda, x, \theta) = \Lambda^{(\mu)}_A \bar{\gamma} \psi^A_{(\mu)} + \widehat{L}(\psi^A_{(\mu)}, x, \theta)\,, \quad (2.22)$$

where as before $\widehat{L}(\psi^A_{(\mu)}, x, \theta)$ is obtained from $\widehat{L}[\psi]$ by replacing $\partial_{(\mu)}\psi^A$ with $\psi^A_{(\mu)}$. Note that $\mathrm{gh}(S_{\mathcal{M}}) = n$ and $|S_{\mathcal{M}}| = n \bmod 2$. Master action (2.17) can then be written as

$$S^P = \int d^n x d^n\theta \left[\widetilde{\Lambda}^{(\lambda)}_A d\widetilde{\psi}^A_{(\lambda)} - \widetilde{\Lambda}^{(\lambda)}_A \sigma^F \widetilde{\psi}^A_{(\lambda)} + S_M(\widetilde{\psi}, \widetilde{\Lambda}, x, \theta)\right]\,. \quad (2.23)$$

The space of field histories can be identified in this representation with the space of maps from the source supermanifold with coordinates x^μ, θ^μ into the target-space supermanifold with coordinates $\psi^A_{(\lambda)}, \Lambda^{(\lambda)}_A$. In particular, the antibracket (2.9) is induced on the space of maps from the target space bracket (2.21) (see e.g. [31, 39] for details on brackets related in this way). If $\widehat{L}, \gamma$ can be chosen x, θ-independent and the 2nd term can be removed by a field redefinition, then the above master action defines what is known as the AKSZ sigma model. As we are going to see next this is exactly what happens if the starting point theory is diffeomorphism invariant.

2.5 Diffeomorphism-invariant theories

We now specialize to the case where the starting point theory is diffeomorphism invariant and diffeomorphisms are in the generating set of gauge transformations so that γ contains a piece γ' such that $\gamma'\psi^A = (\partial_\mu \psi^A)\xi^\mu$, where ξ^μ are diffeomorphism ghosts and ψ^A all the fields including ξ^μ. We assume in addition that this is the only term in γ involving undifferentiated ξ^μ. Under this condition it is known [42] that by changing coordinates on the space of local forms as $\xi^\mu - \theta^\mu \to \xi^\mu$, the $-(\overleftarrow{\partial}_\mu - \frac{\overleftarrow{\partial}}{\partial x^\mu})\theta^\mu$ term in $\widetilde{\gamma}$ can be absorbed by γ so that $\widetilde{\gamma} = \frac{\overleftarrow{\partial}}{\partial x^\mu}\theta^\mu + \bar{\gamma}$ after the redefinition. It then follows that representatives of the $\widetilde{\gamma}$ cohomology can be assumed x, θ-independent as we do from now on. Note that in many cases $\widehat{L}$ can be taken in the form $\xi^1 \dots \xi^n L[\psi]$, where L is a Lagrangian density.

Turning to the parent formulation and following [28] we in addition redefine the θ-descendants of ξ^μ accordingly, i.e., $\xi^\mu_{()\nu} \to \xi^\mu_{()\nu} - \delta^\mu_\nu$ while keeping all the other fields unchanged. By this field redefinition, the term σ^F in γ^P is absorbed into $\bar{\gamma}$. The following statement follows from $\widetilde{\gamma}\widehat{L} = 0$ and the representation (2.23) of the parent master action

Proposition 2.3. *Let the starting point theory be diffeomorphism invariant. $S_{\mathcal{M}}$ defined by* (2.22) *is x, θ-independent and hence defines a function on $\mathcal{M}$. $S_{\mathcal{M}}$ satisfies the master equation*

$$\{S_{\mathcal{M}}, S_{\mathcal{M}}\} = 0\,. \tag{2.24}$$

Parent master action (2.17) *can be represented in the explicitly AKSZ form*

$$S^P = \int d^n x d^n \theta \, \left[\widetilde{\Lambda}^{(\mu)}_A d\widetilde{\psi}^A_\mu + S_{\mathcal{M}}(\widetilde{\psi}, \widetilde{\Lambda})\right] \tag{2.25}$$

where the tilde indicates that the variables are now fields depending on both x^μ and θ^ν.

We stress that in order for (2.25) to define a theory equivalent to (2.17), we need to restrict to field configurations where the $\xi^\mu_{()\nu}(x)$ invertible. Note also that just like in the non-Lagrangian case considered in [28] once the theory is rewritten in the form of an AKSZ sigma model one can use generic coordinates x^a, θ^a on the source space that are not at all related to the starting point coordinates x^μ. Field $\xi^\mu_{()a}(x)$ is then identified as the respective frame field.

To complete the discussion of the diffeomorphism invariance, we note that similarly to [28] any theory can be reformulated as an AKSZ sigma model by adding y^μ, ξ^μ as extra variables in the target space and replacing γ with $\widetilde{\gamma}$ where the role of x^μ, θ^μ is played by y^μ, ξ^μ. In this way one arrives at the parametrized parent formulation. In the Lagrangian setting under consideration now, the parametrized parent formulation should also involve antifields/momenta conjugate to y^μ, ξ^μ and their θ-descendants. For instance, in the well-known case of a 1-dimensional system (mechanics) these are the momenta conjugate to time variable and the reparametrization ghost momenta.

We finally comment on the interpretation of the (odd) symplectic manifold $\mathcal{M}$ equipped with $\{\,\cdot\,,\,\cdot\,\}$ and $S_{\mathcal{M}}$. In the 1d case the structure of (2.25) coincides with the AKSZ-type representation in [31] of the BV master action associated to a constrained Hamiltonian system with the trivial Hamiltonian. Moreover, $\mathcal{M}$ is an extended phase space of the respective Batalin–Fradkin–Vilkovisky (BFV) formulation [43, 44, 45] with $\{\cdot\,,\,\cdot\}$ being the extended Poisson bracket, $S_{\mathcal{M}}$ being the BRST charge, and (2.24) the BFV version of the master equation. Note that this interpretation is compatible with the ghost degree and Grassmann parity as $\mathrm{gh}(S_{\mathcal{M}}) = |S_{\mathcal{M}}| = 1$ and the bracket has zero degrees in this case. Of course, to relate $\mathcal{M}$ to the usual extended phase space, one first needs to eliminate many trivial pairs (see, e.g., the example in Section 3.2). In fact already master action (2.23) can be interpreted in terms of the Hamiltonian BFV formalism by relating the second term in (2.23) to a Hamiltonian (indeed it can be represented as a term linear in θ^μ) in agreement with [31]. In the general case it is natural to consider $\mathcal{M}$ equipped with the bracket and $S_{\mathcal{M}}$ as a multidimensional generalization of the BFV extended phase space.

3 Examples

3.1 Mechanics

Consider the mechanical system described by a Lagrangian $L(q, \partial q)$, where ∂ denotes total time derivative. If there is no gauge symmetry differential γ vanishes and parent action (2.10) truncated at degree 2 takes the familiar form [46]

$$S^P = S^P_0 = \int dt\, \left[p(\partial q - q_{(1)}) + p^{(1)}(\partial q_{(1)} - q_{(2)}) + L(q, q_{(1)})\right] , \tag{3.1}$$

where $q_{(l)} = (\frac{\partial^F}{\partial y})^l q$, $p = (\frac{\partial^F}{\partial \theta} q)^*$, and $p^{(1)} = (\frac{\partial^F}{\partial \theta} q_{(1)})^*$. The total set of variables is given by $q, q_{(1)}, q_{(2)}, p, p^{(1)}$, which have zero ghost degree, and their conjugate in the antibracket variables $q^*, q^*_{(l)}, l = 1, 2$ and $\frac{\partial^F}{\partial \theta} q$, $\frac{\partial^F}{\partial \theta} q_{(1)}$ of ghost degree -1. These last are to be interpreted as antifields. Note that the parent master action S^P coincides with the classical action S^P_0 because there is no gauge symmetry.

The variables $p, p^{(1)}$ and $q_{(1)}, q_{(2)}$ are clearly auxiliary fields and their elimination brings back the starting point Lagrangian with $q_{(1)}$ replaced by the "true" time derivative ∂q. This argument is essentially a specific realization of the general equivalence proof in Section **2.3**.

A general feature that can be seen already in this naive example is that a different reduction is also possible. To see this, we first eliminate $q_{(2)}, p^{(1)}$ as before and suppose for simplicity that there are no constraints so that equation $p = \frac{\partial}{\partial q_{(1)}} L$ can be solved for $q_{(1)}$. The variable $q_{(1)}$ is then an auxiliary field. Indeed, varying with respect to $q_{(1)}$ gives $p = \frac{\partial}{\partial q_{(1)}} L$. Solving this for $q_{(1)}$ gives

$$S^{\mathrm{red}}_0 = \int dt\, (p\partial q - (pq_{(1)}(q, p) - L(q, q_{(1)}(q, p)))\, , \tag{3.2}$$

which is easily recognized as a Hamiltonian action where p plays the role of momenta. We also note that the respective phase space can be seen as a reduction of the manifold $\mathcal{M}$ while the canonical Poisson bracket is simply the reduced version of the bracket (2.21).

This example has a straightforward generalization to the case of field theory without gauge symmetry. Taking for definiteness the scalar field with the Lagrangian $L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$ and reducing the resulting parent action as in the above example one arrives at

$$S_0^{red} = \int d^n x \left[\pi^\mu \partial_\mu \phi - (\frac{1}{2}\pi^\mu\pi_\mu + V(\phi))\right] . \tag{3.3}$$

This is a usual first-order action of the scalar field. We note that by separating space and time components, this action is seen to become a Hamiltonian action.

Although the construction is almost trivial in this simple example, it is much less obvious in the case of gauge theories. From the perspective of the above example, parent action (2.17) is a natural generalization of (3.1) to the case of gauge field theories. Moreover, this generalization maintains (general) covariance of the starting point formulation in a manifest way.

We also mention an interpretation of action (3.3) as a covariant Hamiltonian action of the De Donder–Weyl (DW) formalism (see, e.g., [47, 48]). For instance the second term is identified with the DW Hamiltonian while π^μ as the polymomenta. Moreover, the polysymplectic form of [47] can be related to the (odd) Poisson bracket (2.21) of the parent formulation. A similar interpretation can be given in the general case and will be discussed elsewhere.

3.2 Relativistic particle

The relativistic particle is defined by the Lagrangian

$$S[X,\lambda] = \frac{1}{2}\int d\tau \left[\lambda^{-1} g_{\mu\nu}(X)\partial X^\mu \partial X^\nu + \lambda m^2\right] = \int d\tau \mathcal{L} \,. \tag{3.4}$$

The BRST description is achieved by introducing the ghost ξ and the BRST differential

$$\gamma X^\mu = \xi \partial X^\mu \,, \quad \gamma\lambda = \partial(\xi\lambda) \,, \quad \gamma\xi = \partial\xi\xi \,, \tag{3.5}$$

Note that $\gamma\mathcal{L} = \partial(\xi\mathcal{L})$ so that $\widehat{L} = (\xi - \theta)\mathcal{L}$, which becomes θ-independent after the redefinition and can be used in (2.22).

Because of the diffeomorphism invariance, $\gamma\psi^A$ contains $\partial\psi^A\xi$ and the parent theory is an AKSZ-type sigma model with the target space being a supermanifold with the coordinates X^μ, ξ, λ, all their derivatives $X^\mu_{(l)}$, $\xi_{(l)}$, $\lambda_{(l)}$ considered as independent coordinates, and canonically conjugate momenta $p^{(l)}_\mu$, $\xi^{(l)}_*$, $\lambda^{(l)}_*$ (these are momenta not antifields because the bracket (2.21) has zero ghost degree and Grassmann parity). Here we use the notation such that (l) refers to the order of the y-derivative, e.g., $\lambda_{(l)} = (\frac{\partial^F}{\partial y})^l \lambda$. The source space is simply given by a time line with a coordinate τ extended by the Grassmann odd variable θ. The target space function $S_{\mathcal{M}}$ is given by

$$S_{\mathcal{M}} = p_\mu \xi X^\mu_{(1)} - \xi^* \xi \xi_{(1)} + \lambda^* \xi \lambda_{(1)} + \lambda^* \xi_{(1)} \lambda + \frac{1}{2} \xi (\lambda^{-1} g_{\mu\nu} X^\mu_{(1)} X^\nu_{(1)} + \lambda m^2) + \ldots \tag{3.6}$$

where dots refer to terms in $\Lambda^{(l)}_A \bar{\gamma} \psi^A_{(l)}$ with $l \geqslant 1$ and whose explicit form is in fact not needed.

It turns out that all the variables except X, p, ξ, ξ^* are trivial in the sense that all the fields they give rise to (i.e. their θ-derivatives) are generalized auxiliary fields. By inspecting the definition of generalized auxiliary fields it follows that it is enough to show that these variables are generalized auxiliary fields for $S_{\mathcal{M}}$ considered as a master action. In turn, this can be easily seen using a new coordinate system where X, λ are unchanged while ξ is replaced by $C = \lambda \xi$. The derivatives $X^\mu_{(l)}, C_{(l)}, \lambda_{(l)}$ and conjugate momenta $p^{(l)}_\mu, C^{(l)}_*, \lambda^{(l)}_*$ are then defined as before but starting from the new coordinates and hence are related to the original ones through a canonical transformation. In terms of the new coordinate system, $S_{\mathcal{M}}$ takes the form

$$S_{\mathcal{M}} = p_\mu \lambda^{-1} C X^\mu_{(1)} + \lambda^* C_{(1)} + \frac{1}{2} C (\lambda^{-2} g_{\mu\nu} X^\mu_{(1)} X^\nu_{(1)} + m^2) + \ldots \tag{3.7}$$

It is now obvious that $C_{(1)}, \lambda - 1$ as well as $C_{(n+1)}$, $\lambda_{(n)}$ for $n \geqslant 1$, and their conjugate momenta are all generalized auxiliary fields (we chose $\lambda - 1$ because λ is assumed invertible). Moreover, the variables $X^\mu_{(l)}$ and $p^{(l)}_\mu$ for $l \geqslant 2$ are also generalized auxiliary fields (note, however, that if one works with the truncated action the truncation is to be done as $X^\mu_{(l)} = p^{(l)}_\mu = 0$ for $n \geqslant 2$ and even).

After the elimination we are left with

$$S_{\mathcal{M}} = C (p_\mu X^\mu_{(1)} + \frac{1}{2} g_{\mu\nu} X^\mu_{(1)} X^\nu_{(1)} + m^2) \,. \tag{3.8}$$

In fact $X_{(1)}$ and $p^{(1)}$ are also generalized auxiliary fields because the equation $\dfrac{\partial S_{\mathcal{M}}}{\partial X^\mu_1}$ can be algebraically solved for X^μ_1 (C is to be considered invertible because it contains an invertible einbein as its θ descendant). The reduction then gives $\Omega = -\frac{1}{2} C (g^{\mu\nu} p_\mu p_\nu - m^2)$ which is a BRST charge of the particle model. It is easy to see that the Poisson bracket of the remaining variables is not affected by the reduction[2] and is given by

$$\{X^\mu, p_\nu\} = \delta^\mu_\nu \,, \qquad \{C, \mathcal{P}\} = 1 \,, \tag{3.9}$$

where we denoted C_* by $\mathcal{P}$ to agree with the usual conventions of the BFV formalism.

[2]Strictly speaking the elimination of generalized auxiliary fields χ^a, χ^*_a is the reduction to the second class surface defined by $\chi^*_a = 0$ and $\dfrac{\delta S}{\delta \chi^a} = 0$ so that the reduced bracket is the Dirac bracket (see [25] for more details).

In this way we have reduced the theory to the 1d AKSZ sigma model with the target space being the BFV phase space of the relativistic particle equipped with the BRST charge Ω and the extended Poisson bracket. This AKSZ model is known [31] to be just the BV formulation of the respective first-order Hamiltonian action.

The example we have just described is the Lagrangian/Hamiltonian version of the one in [28] (see also [13] for the respective BRST cohomology treatment). We stress that although the algebraic procedure that leads from the Lagrangian to Hamiltonian description of a particle is somewhat analogous to the usual Legendre transform it is in fact applied to the gauge theory and is operated in terms of BRST theory. In particular, it allows identifying constraints and constructing the corresponding BFV–BRST formulation without actually resorting to the Dirac–Bergmann algorithm and subsequently constructing the BRST charge.

The last observation in fact remains true in field theory as well. By explicitly extracting the "time" coordinate and treating the spatial coordinates implicitly the parent master action can be represented as a 1d (generalized) AKSZ sigma model of the type proposed in [31]. Its target space comes equipped with the respective BRST charge and the BRST-invariant Hamiltonian so that by eliminating the generalized auxiliary fields in the target space one arrives at the usual BFV description.

3.3 Yang–Mills-type theory

The set of fields for Yang–Mills-type theory are the components of a Lie algebra valued 1-form H_μ and a ghost C. The gauge part of the BRST differential is given by

$$\gamma H_\mu = \partial_\mu C + [H_\mu, C]\,, \qquad \gamma C = \frac{1}{2}[C,C]\,. \tag{3.10}$$

The dynamics is determined by a gauge invariant Lagrangian $L_0[H]$.

We explicitly identify the field content and the action of the parent formulation. At ghost number zero we have fields $(H_\mu)_{(\lambda)[\,]}(x)$ and $C_{(\lambda)|\mu}(x)$. It is useful to keep the y variables and to work in terms of the following generating functions:

$$A_\mu(x|y) = -C_{(\lambda)|\mu}(x)y^{(\lambda)}\,, \quad B_\mu(x|y) = (H_\mu)_{(\lambda)[\,]}(x)y^{(\lambda)}\,. \tag{3.11}$$

The parent action takes the form (for simplicity we keep only fields of zero ghost number)

$$S_0^P = \int d^n x\Big[\langle \pi^{\mu\nu}, \partial_{[\nu}A_{\mu]} - \frac{\partial}{\partial y^{[\nu}}A_{\mu]} + \frac{1}{2}[A_\nu, A_\mu]\rangle + \\ \langle \Pi^{\mu\nu}, \partial_\nu B_\mu - \frac{\partial}{\partial y^\nu}B_\mu - \frac{\partial}{\partial y^\mu}A_\nu - [B_\mu, A_\nu]\rangle + L_0[B]\Big]\,. \tag{3.12}$$

where we have introduced the notation

$$\pi^{\mu\nu}(x|p) = \pi^{(\lambda)\mu\nu}(x)p_{(\lambda)}\,, \qquad \Pi^{\mu\nu}(x|p) = \Pi^{(\lambda)\mu\nu}(x)p_{(\lambda)} \tag{3.13}$$

for the generating functions involving antifields conjugate to respectively $C_{(\lambda)\mu\nu}$ and $(H_\mu)_{(\lambda)\mu\nu}$ and for the inner product $\langle,\rangle$ comprising the natural pairing between the Lie algebra and its dual and the inner product (contraction of indices) between polynomials in y^μ and p_μ. The gauge transformation for all the fields including the Lagrange multipliers π, Π can be read off from the complete S^P for which the above S_0^P is the classical action. We note that action S_0^P was implicit in [16]. We also mention a somewhat related formulations in terms of bi-local fields [49, 50, 51].

Following the same logic as in the above examples, we eliminate contractible pairs for $-\sigma^F + \bar{\gamma}$ and their conjugate antifields. As in [28] it is useful to identify contractible pairs for $-\sigma^F + \bar{\gamma}$ as the θ-descendants of $\widetilde{\gamma}$-trivial pairs in the starting point jet space. All the jet space coordinates are known to enter $\widetilde{\gamma}$-trivial pairs except for $\widetilde{C} = C - \theta^\mu H_\mu$ replacing the undifferentiated ghost C, curvature $F^y_{\mu\nu} = \frac{\partial^F}{\partial y^\mu} H_\nu - \frac{\partial^F}{\partial y^\nu} H_\mu + [H_\mu, H_\nu]$ and the independent components of its covariant derivatives. Here we identified jet space coordinates (besides θ^μ, x^μ) with the y-derivatives of C, H_μ. After eliminating the trivial pairs the reduced differential is determined by the "Russian formula" [52]

$$\widetilde{\gamma}\widetilde{C} = \frac{1}{2}[\widetilde{C}, \widetilde{C}] - F^y\,, \qquad F^y = \frac{1}{2} F^y_{\mu\nu}\theta^\mu\theta^\nu\,, \tag{3.14}$$

and further relations defining the action of $\widetilde{\gamma}$ on independent components of the (covariant derivatives of) $F^y_{\mu\nu}$. Note that after the reduction $F^y_{\mu\nu}$ are independent coordinates.

It then follows that all the parent theory fields are generalized auxiliary except for the θ-descendants of $\widetilde{C}$, $F^y_{\mu\nu}$ and its covariant derivatives, and their associated antifields. Moreover, the action of the reduced $-\sigma^F + \bar{\gamma}$ can be read off from (3.14) and its analog for the curvatures (see [28] for more details). In particular, (3.14) implies

$$(-\sigma^F + \bar{\gamma})^{\mathrm{red}}\widetilde{C}_{()\mu\nu} = -[\widetilde{C}_{()\mu}, \widetilde{C}_{()\nu}] + F^y_{\mu\nu} + \ldots \tag{3.15}$$

where the dots stand for the terms involving fields of nonvanishing ghost degree.

Assuming that the Lagrangian depends on undifferentiated curvature only one finds that all the θ-descendants of other curvatures along with their conjugate antifields are also generalized auxiliary fields because the corresponding equations of motion merely express the higher curvatures through the x-derivatives of the lower ones. After eliminating all the above generalized auxiliary fields one stays with just θ-descendants of $\widetilde{C}$, undifferentiated curvature F^y and their conjugate antifields. The action for ghost-number-zero fields $\pi^{\mu\nu} = \frac{1}{2}(\widetilde{C}_{()\mu\nu})^*$, $A_\mu = -\widetilde{C}_{()\mu}$, $F^y_{\mu\nu}$ takes then the form

$$S_0^{\mathrm{red}} = \int d^n x \left[\langle \pi^{\mu\nu}, \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\nu, A_\mu] - F^y_{\nu\mu}\rangle + L_0(F^y)\right]\,. \tag{3.16}$$

By eliminating π, F through their equations of motion one gets the starting point Lagrangian formulation where F^y in $L_0(F^y)$ is replaced with the usual curvature $dA + \frac{1}{2}[A, A]$.

Another reduction of (3.16) depends on the particular form of L_0. Taking for definiteness $L_0(F) = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\langle F_{\mu\nu}, F_{\rho\sigma}\rangle$ where by slight abuse of notation $\langle\, ,\, \rangle$ denotes a nondegenerate invariant form on the gauge algebra, one observes that varying with respect to F^y allows expressing F^y through π as $F^y_{\mu\nu} = -\eta_{\mu\rho}\eta_{\nu\sigma}\pi^{\rho\sigma}$ where the identification of the gauge algebra and its dual through the invariant form is implied. It follows F^y is an auxiliary field and the reduced action takes the well-known form (see, e.g., [53])

$$S_0^{\text{red}} = \int d^n x \big(\langle \pi^{\mu\nu}, \partial_\nu A_\mu - \partial_\mu A_\nu + [A_\nu, A_\mu] \rangle + \pi^{\mu\nu}\pi_{\mu\nu} \big)\,. \tag{3.17}$$

We note that the formulation in (3.16) has an advantage over (3.17) because it allows for more general Lagrangians, not necessarily of the form $\langle F^{\mu\nu}, F_{\mu\nu}\rangle$. Further generalizations can be achieved using the parent Lagrangian (3.12).

3.4 Metric Gravity

In the BRST description of metric gravity, the fields are the inverse metric g^{ab} and a ghost field ξ^a that replaces the vector field parametrizing an infinitesimal diffeomorphism. The gauge part of the BRST differential is given by

$$\gamma g^{ab} = L_\xi g^{ab} = \xi^c \partial_c g^{ab} - g^{cb}\partial_c \xi^a - g^{ac}\partial_c \xi^b\,, \qquad \gamma \xi^c = (\partial_a \xi^c)\xi^a\,. \tag{3.18}$$

The dynamics is specified by the diffeomorphism-invariant Lagrangian $L[g]$ that is assumed to satisfy $\gamma L = \partial_a(\xi^a L)$ along with the standard regularity conditions.

For metric gravity, γX contains $(\partial_a X)\xi^a$ for any field X so that the general discussion of diffeomorphism-invariant theories applies. In particular, the Lagrangian representative $\widehat{L}$ can be chosen as $\widehat{L} = \xi^1 \ldots \xi^n L_0[g]$ and parent formulation can be represented as the AKSZ sigma model. Its target space has coordinates $\xi^a_{(b)}$, $g^{ab}_{(c)}$ along with their canonically conjugate antifields/momenta $\pi_a^{(b)}$ and $u_{ab}^{(c)}$.

It is useful to work in terms of generating functions. For this, we introduce formal variables p_b in addition to y^a and consider the algebra of polynomials in y, p equipped with the standard Poisson bracket $\left\{y^a, p^b\right\} = \delta^b_a$. The target space coordinates $g^{ab}_{c_1\ldots c_l}$ and $\xi^a_{c_1\ldots c_l}$ can then be encoded in

$$G = \frac{1}{2} g^{ab}_{(c)} y^{(c)} p_a p_b\,, \qquad \Xi = \xi^a_{(c)} y^{(c)} p_a\,, \tag{3.19}$$

and the action of γ on these coordinates can be compactly written as

$$\gamma \Xi = \frac{1}{2}\{\Xi, \Xi\}\,, \qquad \gamma G = \{G, \Xi\}\,. \tag{3.20}$$

The same variables can be used to encode antifields/momenta into the generating functions:

$$\Pi = \pi_a^{(b)} p_{(b)} y^a\,, \qquad U = \frac{1}{2} u_{ab}^{(c)} p_{(c)} y^a y^b\,. \tag{3.21}$$

In addition, we introduce the natural symmetric inner product $\langle , \rangle$ on the space of polynomials in y, p such that e.g. $\langle y^a, p_b\rangle = \delta^a_b$. In components it simply amounts to natural contraction between indices of the coefficients. The parent master action then becomes

$$\begin{gathered} S^P = \int d^n x d^n\theta \left[\langle \widetilde{U}, d^F \widetilde{G}\rangle + \langle \widetilde{\Pi}, d^F \widetilde{\Xi}\rangle + S_{\mathcal{M}}(\widetilde{G}, \widetilde{\Xi}, \widetilde{U}, \widetilde{\Pi})\right] \,, \\ S_{\mathcal{M}} = \langle \widetilde{U}, \left\{\widetilde{G}, \widetilde{\Xi}\right\}\rangle + \frac{1}{2}\langle \widetilde{\Pi}, \left\{\widetilde{\Xi}, \widetilde{\Xi}\right\}\rangle + \widetilde{\xi}^1 \dots \widetilde{\xi}^n L_0[\widetilde{G}] \,. \end{gathered} \tag{3.22}$$

where as before the tilde indicates that the fields are functions of x, θ and $\widetilde{\xi}$ enters $\widetilde{\Xi}$ as a y-independent term.

We now concentrate on the classical action S^P_0. Fields F, A of vansihing ghost degree enter the expansions of G, Ξ in θ as

$$\widetilde{G}(x, \theta|y, p) = F(x, y, p) + \dots \,, \quad \widetilde{\Xi}(x, \theta|y, p) = \Xi(x|y, p) + A_\mu(x|y, p)\theta^\mu + \dots \,. \tag{3.23}$$

As regards the antifields/momenta, the $n-1$-form P and $n-2$ form π components of respectively U and Π are of vanishing ghost degree and play the role of Lagrange multipliers. The classical action then can be then written as

$$S^P_0 = \int d^n x d^n\theta \left[\langle P, dF + \{F, A\}\rangle + \langle \pi, dA + \frac{1}{2}\{A, A\}\rangle + e^1 \dots e^n L_0[F]\right] \,, \tag{3.24}$$

where $e^a = e^a_\mu(x)\theta^\mu$ enters $A(x, \theta|y, p)$ as $A = \theta^\mu e^a_\mu(x) p_a + \dots$ and is to be identified as the frame field. The action (3.24) was implicitly in [16]. Mention also somewhat related descriptions from [54, 55].

We now perform the reduction of the parent formulation for gravity leading to its frame like form. We are going to implement the Lagrangian version of the analogous reduction considered in [28] (see also [16, 25]). Details on identification of trivial pairs for the BRST differential can be found in [12, 4, 42]. In particular, all the variables in Ξ and G except $\xi^a_{()}, \xi^a_b$, metric g^{ab}, and (independent components of the covariant derivatives of) the curvature are contractible pairs for $\bar{\gamma}$. All their θ-descendants as well as all the associated antifields are then the generalized auxiliary fields for the parent formulation. Moreover, under the usual assumption that metric (entering G as a $g^{ab}p_a p_b$) is close to a flat metric η^{ab}, the components of the difference $g^{ab} - \eta^{ab}$ together with the symmetric part of $\xi^a_c \eta^{cb}$ and their associated antifields give rise to generalized auxiliary fields and hence can also be eliminated.

The action of the reduced $\bar{\gamma}$ on the remaining coordinates ξ^a, ξ^a_b, $R^a_{b\,cd}$ and $R^b_{c_1 \dots c_k a_1 a_2 a_3}$, where the latter denote the covariant derivatives of the curvature $R^a_{b\,cd}$ is given by (see e.g. [28, 12, 42] for more details)

$$\bar{\gamma}\xi^a = \xi^a_c \xi^c \,, \qquad \bar{\gamma}\xi^a_b = \xi^a_c \xi^c_b - \frac{1}{2}\xi^c \xi^d R^a_{b\,cd} \,, \tag{3.25}$$

and

$$\bar{\gamma} R^b_{c_1\ldots c_k a_1 a_2 a_3} = \xi^{c_0} R^b_{c_0 c_1 \ldots c_k a_1 a_2 a_3} - \xi^b_d R^d_{c_1\ldots c_k a_1 a_2 a_3} + \\ + \xi^d_{c_1} R^b_{d c_k a_1 a_2 a_3} + \cdots + \xi^d_{a_3} R^b_{c_1\ldots c_k a_1 a_2 d}\,. \quad (3.26)$$

If L_0 depend on undifferentiated curvature only all the fields associated to the covariant derivatives of the curvature are generalized auxiliary. Indeed, it follows from (3.26) that the respective equations of motion express $R^b_{c_1\ldots c_k a_1 a_2 a_3}$ through $R^b_{c_1\ldots c_{k-1} a_1 a_2 a_3}$ so that θ-derivatives of $R^b_{c_1\ldots c_k a_1 a_2 a_3}$ with $k > 0$ and all the associated antifields can be eliminated. In this way one ends up with only θ derivatives of $\xi^a, \xi^a_b, R^a_{b,cd}$ and the associated antifields/momenta.

We then introduce the component fields entering $\widetilde{\xi}^a, \widetilde{\xi}^a_b, \widetilde{R}^a_{b,cd}$:

$$\widetilde{\xi}^a(x,\theta) = \xi^a - \theta^\mu e^a_\mu + \frac{1}{2}\theta^\nu\theta^\mu \xi^a_{\mu\nu} + \ldots\,, \\ \widetilde{\xi}^a_b(x,\theta) = \xi^a_b - \theta^\mu \omega^a_{\mu b} + \frac{1}{2}\theta^\nu\theta^\mu \xi^a_{b\mu\nu} + \ldots\,, \quad \widetilde{R}^a_{b,cd}(x,\theta) = R^a_{b,cd} + \ldots \quad (3.27)$$

where dots stand for terms of higher order in θ. In particular, fields $e^a_\mu, \omega^a_{\mu b}, R^a_{b,cd}$ carry vanishing ghost degree. Besides them antifields $\pi^{\mu\nu}_a = (\xi^a_{\mu\nu})^*$ and $\pi^{b\mu\nu}_a = (\xi^a_{b\mu\nu})^*$ also carry vanishing ghost degree and play the role of Lagrange multipliers. After the reduction action (3.24) takes the following form

$$S^{\mathrm{red}}_0[\pi^a, \pi^a_b, e^a, \omega^{ab}] = \int d^n x d^n\theta \Big[\pi_a(de^a + \omega^a_b e^b) + \\ + \pi^b_a(d\omega^a_b + \omega^a_c\omega^c_b - \frac{1}{2}e^c e^d R^a_{bcd}) + e^1\ldots e^n L_0(R)\Big]\,, \quad (3.28)$$

where antifields π_a and π^b_a are represented in a dual way as $n-2$-forms. Fields $\pi^{b\mu\nu}_a$ and R^a_{bcd} are clearly auxiliary ones. By eliminating them the second term is gone and we get

$$S^{\mathrm{red}-1}_0[\pi^a, e^a, \omega^{ab}] = \int d^n x \pi^{\mu\nu}_a(\partial_{[\nu} e^a_{\mu]} + \omega^{ac}_{[\nu} e^c_{\mu]}) + \int d^n x d^n\theta\ e^1\ldots e^n L_0[e,\omega]\,. \quad (3.29)$$

Just like in other examples, it is now easy to explicitly get back the starting point Lagrangian. Indeed, the fields π and ω are auxiliary because varying with respect to $\pi^{\mu\nu}_a$ gives the condition $de^a + \omega^a_b e^b = 0$ that is uniquely solved for ω^a_b in terms of e^a. At the same time variation with respect to $\omega^a_{\mu b}$ gives equation $\pi^{\mu\nu}_a e^b_\nu + (\pi \text{ -independent terms}) = 0$ which can be uniquely solved for $\pi^{\mu\nu}_a$. Substituting the solutions back to (3.29) one finds that only the term with L_0 expressed through e^a stays.

If the starting point L_0 is the precisely Einstein-Hilbert Lagrangian another reduction is also possible that leads to the usual first order action

$$S_1[e^a, \omega^{ab}] = \int d^n x d^n\theta\ \epsilon_{a_1\ldots a_{n-2} a_{n-1} a_n} e^{a_1}\ldots e^{a_{n-2}}(d\omega^{a_{n-1}a_n} + \omega^{a_{n-1}}_c \omega^{c a_n})\,, \quad (3.30)$$

depending on e^a, ω^a_b as independent fields. The difference with (3.29) is only in the first term in (3.29) and its extra dependence on $\pi^{\mu\nu}_a$. That (3.29) is equivalent to (3.30) via eliminating auxiliary fields is obvious if one eliminates $\pi^{\mu\nu}_a$ and ω^{ab}_μ in (3.29) as explained above and eliminates ω^{ab}_μ through its own equations of motion in (3.30).

In fact (3.30) can be obtained from (3.29) via a straightforward reduction. Indeed, let us change the field variables such that $\omega^{ab}_\mu = \alpha^{ab}_\mu(e) + \bar{\omega}^{ab}_\mu$ where $\alpha^{ab}_\mu[e]$ is a unique solution to $de^a + \alpha^a_c e^c = 0$ so that field $\bar{\omega}^{ab}_\mu$ is related to torsion in an invertible way. In terms of $\bar{\omega}^{ab}_\mu$ action (3.30) decomposes as $S_1[e, \alpha(e)] + S_2[e, \bar{\omega}]$ where S_2 is bilinear in undifferentiated $\bar{\omega}^{ab}_\mu$. Using this representation for the second term in (3.29) one observes that $\bar{\omega}^{ab}_\mu$ is an auxiliary field and can be expressed through $\pi^{\mu\nu}_a$ and e^a_μ. Using then an invertible field redefinition such that a new $\omega^{ab}_\mu[e, \pi]$ replaces $\pi^{\mu\nu}_a$ the reduced action (3.29) can be brought to the form (3.30).

4 Conclusions

In this paper, we have specialized the parent formulation of [28] to the Lagrangian level. More precisely, for a given Lagrangian gauge theory, we have constructed the first-order parent BV formulation by explicitly specifying the field–antifield space, the antibracket, and the BV master action. As a technical assumption, we restricted ourselves to the case of theories with a closed gauge algebra. But the parent formulation can also be defined in general. Indeed, S^P can be defined in exactly the same way, and the only difference is that in the general case, it satisfies the master equation only modulo the parent equations of motion. These last are determined by the classical action S^P_0, which is also well defined in general and can be obtained from S^P by putting all the fields of nonzero ghost degrees to zero. The complete master action can then be obtained via the usual BV procedure starting from S^P_0 and its gauge symmetries.

Although the construction of the parent formulation applies to an already specified gauge theory, our hope is to use this formulation to construct new models in the parent form (or related forms) from the very beginning. This strategy has proved fruitful [56, 57, 58] in the context of higher-spin gauge theories, where a version of the parent formulation at the level of the equations of motion [25, 28, 27] was successfully used. Among possible applications of the present results, Vasiliev's interacting higher-spin theory [20, 21, 22], where the Lagrangian formulation is currently unknown, seems to be the most attracting. We hope that the present approach gives the correct framework for addressing this issue. This is supported by a concise parent-like formulation of the nonlinear higher-spin theory at the off-shell level [59] (see also [16]). Another interesting perspective is to relate the parent action to that of the recently proposed double field theory [60, 61].

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