Limit distribution in the q-CLT for $q \geq 1$ can not have a compact support

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Abstract

In a recent paper Hilhorst (1) illustrated that the q-Fourier transform for $q > 1$ is not invertible in the space of density functions. Using an invariance principle he constructed a family of densities with the same q-Fourier transform and claimed that q-Gaussians are not mathematically proved to be attractors. We show here that none of the distributions constructed in Hilhorst's counterexamples can be a limit distribution in the q-CLT, except the one whose support covers the whole real axis, which is precisely the q-Gaussian distribution.

Keywords: q-central limit theorem, q-Fourier transform, q-Gaussian

1 Introduction

Using a specific invariance principle Hilhorst (1; 2) showed that the q-Fourier transform (q-FT) is not invertible in the space of densities. He constructed a family of densities containing the q-Gaussian and with the same q-FT. Any density of this family except the q-Gaussian has a compact support.

In the present note we establish that a limit distribution in the q-central limit theorem (q-CLT) proved in (3) (see also (4)) can not have a compact support. This eliminates all the distributions in Hilhorst's counterexamples as a valid limiting distribution in the q-CLT, leaving only one, which is the distribution corresponding to the q-Gaussian density. We also show that, for $q > 1$, any density which has the same q-FT as the q-Gaussian and whose support covers the real axis is asymptotically equivalent to the q-Gaussian.

The q-CLT deals with sequences of random variables of the form

$$Z_N = \frac{S_N - N\mu_q}{\alpha(q)N^{\frac{1}{2(2-q)}}}, \tag{1}$$

where $S_N = X_1 + \cdots + X_N$, the random variables $X_1, \ldots X_N$ being identically distributed and q-independent, $\mu_q = \int x[f(x)]^q dx$, and $\alpha(q) = [\nu_{2q-1}\sigma_{2q-1}^2]^{1/(2-q)}$, with

$$\nu_q = \int [f(x)]^q dx, \ \sigma_{2q-1}^2 = \int (x - \mu_q)^2 [f(x)]^{2q-1} dx.$$

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Without loss of generality we assume that $\mu_q = 0$. Three types of q-independence were discussed in paper (3). Namely, identically distributed random variables X_N are q-independent (see (3)) of[1]

$$\text{Type I:} \quad F_q[X_1 + \cdots + X_N](\xi) = F_q[X_1](\xi) \otimes_q \cdots \otimes_q F_q[X_N](\xi); \tag{2}$$

$$\text{Type II:} \quad F_q[X_1 + \cdots + X_N](\xi) = F_q[X_1](\xi) \otimes_{q_1} \cdots \otimes_{q_1} F_q[X_N](\xi); \tag{3}$$

$$\text{Type III:} \quad F_q[X_1 + \cdots + X_N](\xi) = F_{q_1}[X_1](\xi) \otimes_{q_1} \cdots \otimes_{q_1} F_{q_1}[X_N](\xi), \tag{4}$$

if these relationships hold for all $N \geq 2$ and $\xi \in (-\infty, \infty)$; $q_1 = \frac{1+q}{3-q}$. Here the operator F_q is the q-FT defined as

$$F_q[X_1](\xi) = \tilde{f}_q(\xi) := \int_{-\infty}^{\infty} \frac{f(x)\, dx}{[1 + i(1-q)x\xi f^{q-1}(x)]^{\frac{1}{q-1}}}, \tag{5}$$

with $q > 1$.

We recall some facts about the q-exponential and q-logarithmic functions. These functions are respectively defined as (see for instance (3))

$$\exp_q(x) = [1 + (1-q)x]_+^{\frac{1}{1-q}} \quad (\exp_1(x) = \exp(x)),$$

and

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q}, \ x > 0 \quad (\ln_1(x) = \ln(x)).$$

It is easy to see (see (3)) that for the q-exponential, the relations $\exp_q(x \oplus_q y) = \exp_q(x) \exp_q(y)$ and $\exp_q(x + y) = \exp_q(x) \otimes_q \exp_q(y)$ hold. In terms of q-log these relations can be equivalently rewritten as follows: $\ln_q(x \otimes_q y) = \ln_q x + \ln_q y$, and $\ln_q(xy) = \ln_q x \oplus_q \ln_q y$. It is not hard to verify that if $1 < q_1 < q_2$, then

$$\ln_{q_2}(x) \leq \ln_{q_1}(x) \ \text{ for all } x > 1, \tag{6}$$

$$\ln_{q_2}(x) \geq \frac{q_1 - 1}{q_2 - 1} \ln_{q_1}(x) \ \text{ for all } x > 0. \tag{7}$$

For $q > 1$ the q-exponential is defined for all $x < \frac{1}{q-1}$ and blows up at the point $x = \frac{1}{q-1}$. The q-exponential can also be extended to the complex plane and it is bounded on the imaginary axis: $|\exp_q(iy)| \leq 1$. Moreover, $|\exp_q(iy)| \to 0$ if $|y| \to \infty$. Using the q-exponential function, the q-FT of f can be represented in the form

$$\tilde{f}_q(\xi) = \int_{-\infty}^{\infty} f(x) \exp_q\left(ix\xi[f(x)]^{q-1}\right) dx. \tag{8}$$

We refer the reader to papers (3; 4; 5; 6; 7; 8; 9; 10; 11; 12) for various properties of the q-FT.

2   On the support of a limit distribution

For the sake of simplicity we consider a continuous and symmetric about zero density function f of a random variable X_1. Other cases can be considered in a similar manner with appropriate

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[1]See (19) for definitions and properties of the q-product $\otimes_q$ and the q-sum $\oplus_q$.

care. Denote $\lambda(x) = x[f(x)]^{q-1}$, where $1 \le q < 2$. Since f is symmetric, it suffices to consider $\lambda(x)$ only for positive x. Suppose λ attains its maximum value m at a point $x_m > 0$, i.e. $m = \max_{0 < x \le a}\{x[f(x)]^{q-1}\} = x_m[f(x_m)]^{q-1}$.

Proposition 2.1. *Let f be a continuous symmetric density function with $\operatorname{supp} f \subseteq [-a, a]$. Then the q-FT of f satisfies the following estimate*

$$|\tilde{f}_q(\eta - i\tau)| \le \exp_q(x_m M_q \tau), \tag{9}$$

where $\eta \in (-\infty, \infty)$, $\tau < \frac{1}{m(q-1)}$, $M_q = \max_{[0,a]}\{[f(x)]^{q-1}\}$, and x_m is the point where xf^{q-1}

attains its maximum m.

Proof. For f with $\operatorname{supp} f \subseteq [-a, a]$, equation (5) takes the form

$$\tilde{f}_q(\xi) = \int_{-a}^{a} \frac{f(x)dx}{[1 + i(1-q)x\xi f^{q-1}(x)]^{\frac{1}{q-1}}}. \tag{10}$$

Let $\xi = \eta + i\tau$ where $\eta = \Re(\xi)$ is the real part of ξ and $\tau = \Im(\xi)$ is its imaginary part. We assume that $\eta \in (-\infty, \infty)$ and $\tau > -\frac{1}{m(q-1)}$. Then for the denominator of the integrand in (10) one has

$$[1 + i(1-q)x(\eta - i\tau)f^{q-1}(x)]^{\frac{1}{q-1}} = [1 + i(1-q)\eta f^{q-1}(x) + (1-q)\tau x f^{q-1}(x)]^{\frac{1}{q-1}}$$

$$= [1 + (1-q)\tau x f^{q-1}(x)]^{\frac{1}{q-1}}[1 + i\frac{(1-q)\eta f^{q-1}(x)}{1 - (1-q)\tau x f^{q-1}(x)}]^{\frac{1}{q-1}}$$

$$= \left(\exp_q(\tau x f^{q-1}(x))\right)^{-1} \left(\exp_q(i\frac{(1-q)\eta f^{q-1}(x)}{1 - (1-q)\tau x f^{q-1}(x)})\right)^{-1}. \tag{11}$$

Using the inequality $|\exp(iy)| \le 1$ valid for all $y \in (-\infty, \infty)$ if $q > 1$, it follows from (11) that

$$\left|1 + i(1-q)x(\eta + i\tau)f^{q-1}(x)\right|^{\frac{1}{q-1}} \ge \left(\exp_q(\tau x f^{q-1}(x))\right)^{-1}. \tag{12}$$

Now, (10) together with (12) and $f(x)$ being a density function, yield (9). ∎

Remark 2.2. *Proposition (2.1) can be viewed as a generalization of the well known Paley-Wiener theorem. Indeed, if $q = 1$ then (9) takes the form*

$$|\tilde{f}(\eta - i\tau)| \le \exp(a\tau), \ \eta + i\tau \in \mathcal{C}, \tag{13}$$

which represents the Paley-Wiener theorem for continuous density functions.

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Inequality (13) can be used for estimation of the size of the support of f. Consider an example with $f(x) = (2a)^{-1}\mathcal{I}_{[-a,a]}(x)$, where $\mathcal{I}_{[-a,a]}(x)$ is the indicator function of the interval $[-a, a]$. The Fourier transform of this function is $\tilde{f}(\xi) = (a\xi)^{-1}\sin(a\xi)$, $M_q = M_1 = 1$, and $x_m = a$. Therefore, we have $|\tilde{f}(-i\tau)| \leq e^{a\tau}, \tau > 0$. This inequality yields

$$2a \geq 2\sup \frac{\ln|\tilde{f}(-i\tau)|}{\tau},$$

which gives an estimate from below for the size $d(f) = 2a$ of the support of f.

This idea can be used to estimate the size of the support of f using the q-FT and Proposition 2.1. Namely, inequality (9) with $\eta = 0$ implies

$$d(f) = 2a \geq 2x_m \geq \frac{2}{M_q}\sup_\tau \frac{\ln_q|\tilde{f}_q(-i\tau)|}{\tau}. \tag{14}$$

We notice that the integrand in the integral

$$\tilde{f}_q(-i\tau) = \int_{-a}^a \frac{f(x)dx}{[1 - (q-1)\tau x f^{q-1}(x)]^{\frac{1}{q-1}}}$$

is strictly grater than $f(x)$ if $\tau > 0$, implying $|\tilde{f}_q(-i\tau)| > 1$, since f is a density function. Therefore, the right hand side of (14) is positive and gives indeed an estimate of the size of the support of f from below.

Let $f_N(x) = f_{S_N}(x)$ be the density function of $S_N = X_1 + \cdots + X_N$, where $X_1, \ldots, X_N$ are q-independent random variables with the same density function $f = f_{X_1}$ whose support is $[-a, a]$. We show that the q-independence condition can not reduce the support of f_N to an interval independent of N. More precisely, $d(f_N)$ increases at the rate of N when $N \to \infty$.

Theorem 2.3. *Let $X_1, \ldots, X_N$ be q-independent of any type I-III random variables all having the same density function f with supp $f \subseteq [-a, a]$. Then, for the size of the density f_N of S_N, there exists a constant $K_q > 0$ such that the estimate*

$$d(f_N) \geq K_q N \sup_\tau \frac{\ln_q|\tilde{f}_q(-i\tau)|}{\tau} \tag{15}$$

holds.

Proof. Using formula (14) one has

$$d(f_N) \geq \frac{2}{M_{q,N}}\sup_\tau \frac{\ln_q|\widetilde{(f_N)}_q(-i\tau)|}{\tau}, \tag{16}$$

where $M_{q,N} = \max_{x \in [-Na, Na]} f_N^{q-1}(x)$. It is clear from probabilistic arguments that $M_{q,N} \leq M_q$ for all $N \geq 2$. Therefore, it follows from (16) that

$$d(f_N) \geq \frac{2}{M_q}\sup_\tau \frac{\ln_q|\widetilde{(f_N)}_q(-i\tau)|}{\tau}, \tag{17}$$

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Let X_N be q-independent of type I (see (2)). Making use of the inequality $|z - r| \geq |z| - r$, which holds true for any complex z and positive integer number r, one has

$$
\begin{aligned}
|\widetilde{(f_N)}_q(-i\tau)| &= |\tilde{f}_q(-i\tau) \otimes_q \cdots \otimes_q \tilde{f}_q(-i\tau)| \\
&= |[N \left(\tilde{f}_q(-i\tau) \right)^{1-q} - (N-1)]^{\frac{1}{1-q}}| \\
&\geq [N|\tilde{f}_q(-i\tau)|^{1-q} - (N-1)]^{\frac{1}{1-q}} \\
&= |\tilde{f}_q(-i\tau)| \otimes_q \cdots \otimes_q |\tilde{f}_q(-i\tau)|.
\end{aligned}
$$

Taking q-logarith of both sides in this inequality and using the property $\ln_q(g \otimes_q h) = \ln_q g + \ln_q h$, one obtains

$$
\ln_q |\widetilde{(f_N)}_q(-i\tau)| \geq N \ln_q |\tilde{f}_q(-i\tau)|. \tag{18}
$$

Now estimate (15) follows from inequalities (17) and (18).

For random variables X_N independent of type II, equation (18) takes the form

$$
\ln_{q_1} |\widetilde{(f_N)}_q(-i\tau)| \geq N \ln_{q_1} |\tilde{f}_q(-i\tau)|. \tag{19}
$$

Since $1 < q < q_1$ and $\frac{q-1}{q_1-1} = \frac{3-q}{2}$, making use of inequalities (6) and (7) and assuming that $|\widetilde{(f_N)}_q(i\tau)| \geq 1$, one has

$$
\ln_q |\widetilde{(f_N)}_q(-i\tau)| \geq \frac{(3-q)N}{2} \ln_q |\tilde{f}_q(-i\tau)|. \tag{20}
$$

Similarly, for variables independent of type III, we have

$$
\ln_q |\widetilde{(f_N)}_q(-i\tau)| \geq N \ln_{q_1} |\tilde{f}_{q_1}(-i\tau)|. \tag{21}
$$

Both (20) and (21) obviously imply estimate (15). ∎

Corollary 2.4. *Let $X_1, \ldots, X_N$ be q-independent of any type I-III random variables all having the same density function f with supp $f \subseteq [-a, a]$. If the sequence Z_N has a distributional limit random variable in some sense, then this random variable can not have a density with compact support. Moreover, due to the scaling present in Z_N, the support of the limit variable is the entire axis.*

The proof obviously follows immediately from (15) upon letting $N \to \infty$.

3   On Hilhorst's counterexamples

We want to compare Theorem 2.3 with Hilhorst's counterexamples in (1). He used the invariance principle to show that q-FT is not invertible. Let $f(x)$, $x \in (-\infty, \infty)$, be a symmetric density function, such that $\lambda(x) = x[f(x)]^{q-1}$ restricted to the semiaxis $[0, \infty)$ has a unique (local) maximum m at a point x_m. In other words $\lambda(x)$ has two monotonic pieces, $\lambda_-(x)$, $0 \leq x \leq x_m$, and

$\lambda_+(x)$, $x_m \leq x < \infty$. Let $x_\pm(\xi)$, $0 \leq \xi \leq m$, denote the inverses of $\lambda_\pm(x)$, respectively. Then the q-FT $(1 < q < 2)$ of f can be expressed in the form, see (1)

$$\tilde{f}_q(\xi) = \int_{-\infty}^{\infty} F(\xi') \exp(i\xi\xi')d\xi',$$

where

$$F(\xi) = \frac{q-2}{q-1}\xi^{\frac{1}{q-1}}\frac{d}{d\xi}[x_-^{\frac{q-1}{q-2}}(\xi) - x_+^{\frac{q-1}{q-2}}(\xi)], \ \xi \in [0, m]. \tag{22}$$

The invariance principle yields

$$F(\xi) = \frac{q-2}{q-1}\xi^{\frac{1}{q-1}}\frac{d}{d\xi}[X_-^{\frac{q-1}{q-2}}(\xi) - X_+^{\frac{q-1}{q-2}}(\xi)], \ \xi \in [0, m], \tag{23}$$

where

$$X_\pm^{\frac{q-1}{q-2}}(\xi) = x_\pm^{\frac{q-1}{q-2}}(\xi) + H(\xi), \tag{24}$$

with $H(\xi)$ being a function defined on $[0, m]$, and such that $X_\pm^{\frac{q-1}{q-2}}(\xi)$ are invertible. Denote by $\Lambda(x)$ the function defined by the two pieces of inverses of $X_\pm(\xi)$, namely

$$\Lambda_H(x) = \begin{cases} X_-^{-1}(x), & \text{if } 0 \leq x \leq x_{m,H}, \\ X_+^{-1}(x), & \text{if } x > x_{m,H}, \end{cases}$$

where $x_{m,H} = [(q-1)^{\frac{q-1}{2(2-q)}} + H(m)]^{-\frac{2-q}{q-1}}$. The function $\Lambda_H(x)$ is continuous, since $X_-^{-1}(x_{m,H}) = X_+^{-1}(x_{m,H})$. Then

$$f_H(x) = \left(\frac{\Lambda(x)}{x}\right)^{\frac{1}{q-1}} \tag{25}$$

defines a density function with the same q-FT as of f. The density f_H coincides with f if $H(\xi)$ is identically zero.

Now assume that $f(x)$ is a q-Gaussian,

$$f(x) = G_q(x) = \frac{C_q^{q-1}}{[1 + (q-1)x^2]^{\frac{1}{q-1}}}, \ 1 < q < 2,$$

where C_q is the normalization constant. Obviously, $G_q(x)$ is symmetric, and the function $\lambda_q(x) = x[G_q(x)]^{q-1}$ considered on the semiaxis $[0, \infty)$ has a unique maximum $m = \frac{C_q^{q-1}}{2\sqrt{q-1}}$ attained at the point $x_m = (q-1)^{-\frac{1}{2}}$. Moreover, the functions $x_\pm(\xi)$ in this case take the forms (see (1))

$$x_\pm(\xi) = \frac{C_q^{q-1} \pm [C_q^{2(q-1)} - 4(q-1)\xi^2]^{\frac{1}{2}}}{2\xi(q-1)}, \ 0 < \xi \leq m. \tag{26}$$

We denote the density $f_H(x)$ and the function $\Lambda_H(x)$ corresponding to the q-Gaussian by $G_{q,H}(x)$ and $\Lambda_{q,H}(x)$, respectively. Hilhorst, selecting $H(\xi) = A \geq 0$ constant, constructed a family of densities

$$G_{q,A}(x) = \frac{C_q(x^{\frac{q-2}{q-1}} - A)^{\frac{1}{q-2}}}{x^{\frac{1}{q-1}}[1 + (q-1)(x^{\frac{q-2}{q-1}} - A)^{2\frac{q-1}{q-2}}]^{\frac{1}{q-1}}}, \tag{27}$$

which have the same q-FT as the q-Gaussian for all A. The following statement shows that none of the densities $G_{q,A}(x)$ can be a limit distribution in the q-CLT, except the one, corresponding to $A = 0$, which coincides with the q-Gaussian, $G_{q,0}(x) = G_q(x)$.

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Proposition 3.1. *Let $H(0) > 0$. Then the support of $G_{q,H}(x)$ is compact, and*

$$supp\, G_{q,H} = \left[-[H(0)]^{\frac{q-1}{q-2}}, [H(0)]^{\frac{q-1}{q-2}} \right].$$

Proof. Since $\lim_{\xi \to 0} x_+(\xi) = +\infty$, the largest value of X_+ is equal to $\lim_{\xi \to 0} X_+(\xi) = [H(0)]^{\frac{q-1}{q-2}}$. Therefore, the inverse of X_+ is defined on the interval $[x_0, [H(0)]^{\frac{q-1}{q-2}}]$, where $x_0 > 0$ is some number obtained by **a** shifting of x_m depending on $H(m)$. On the other hand the smallest value of x_- is zero, taken at $\xi = 0$. Therefore, the inverse of X_- is defined on the interval $[0, x_0]$. Hence, by symmetry, $G_{q,H}$ has the support $\left[-[H(0)]^{\frac{q-1}{q-2}}, [H(0)]^{\frac{q-1}{q-2}} \right]$. ∎

Remark 3.2. *Note that $H(0)$ can not be negative. In fact, if $H(0) < 0$, then either $X_\pm$ is not invertible or, if it is invertible, its inverse does not identify a density function.*

Proposition 3.1 implies that if $H(0) > 0$ then, due to Corollary 2.4, $G_{q,H}(x)$ can not be the density function of the limit distribution in the q-CLT. Thus none of the densities in Hilhorst's counterexamples[2] except the q-Gaussian can serve as an attractor in the q-CLT.

Only one possibility is left, namely $H(0) = 0$. The next proposition establishes that, in this case, $G_{q,H}(x)$ is asymptotically equivalent to $G_q(x) \equiv G_{q,0}(x)$.

Proposition 3.3. *Let $H(0) = 0$. Then*

$$\lim_{|x| \to \infty} \frac{G_{q,H}(x)}{G_q(x)} = 1.$$

Proof. Since $H(0) = 0$, then obviously

$$\lim_{\xi \to 0} \frac{X_+(\xi)}{x_+(\xi)} = \lim_{\xi \to 0} \left(1 + \frac{H(\xi)}{x_+(\xi)} \right) = 1.$$

Therefore, for inverses one has

$$\lim_{x \to +\infty} \frac{X_+^{-1}(x)}{x_+^{-1}(x)} = 1.$$

This implies

$$\lim_{x \to +\infty} \frac{G_{q,H}(x)}{G_q(x)} = \lim_{x \to +\infty} \left(\frac{\frac{X_+^{-1}(x)}{x}}{\frac{x_+^{-1}(x)}{x}} \right)^{\frac{1}{q-1}} = 1. \qquad \blacksquare$$

Remark 3.4. *Propositions 3.1 and 3.3 establish that $G_{q,H}$ can identify a limiting distribution in the q-CLT only if $H(0) = 0$. However, in this case, independently from other values of $H(\xi)$, the density $G_{q,H}(x)$ is asymptotically equivalent to the q-Gaussian, i.e. $G_{q,H}(x) \sim G_q(x)$ as $|x| \to \infty$.*

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[2] See Examples 2 and 3 in (1). Example 4 is not relevant to the q-CLT, since in this case, $(2q - 1)$-variance of the 2-Gaussian does not exist, and consequently the q-CLT is not applicable.

We return to this question in the Conclusion, where we discuss whether $G_{q,H}$ can, for H not identically zero, be an attractor in the q-CLT.

The statement of the following proposition can be proved exactly as Proposition 3.3, replacing $X_+(\xi)$, $x_+(\xi)$ by functions $X_-(\xi)$, $x_-(\xi)$, respectively.

Proposition 3.5. *Let $H(0) = 0$. Then*

$$\lim_{x \to 0} \frac{G_{q,H}(x)}{G_q(x)} = 1.$$

4   Other relations of $G_{q,H}$ with the q-Gaussian

In light of Propositions 3.1 and 3.3, we will assume below that $H(0) = 0$ and clarify other conditions for $H(\xi)$. As above we use notations $\Lambda_{q,H}(x) = x[G_{q,H}(x)]^{q-1}$ and $\lambda_q(x) = x[G_q(x)]^{q-1}$.

Proposition 4.1. *Let $H(m) = 0$. Then the function $\Lambda_{q,H}(x)$ attains its unique maximum at the point x_m, and $\Lambda_{q,H}(x_m) = \lambda_q(x_m) = m$.*

Proof. If $H(m) = 0$, then it follows from (24) immediately, that $X_\pm(m) = x_\pm(m)$, which implies $\Lambda_{q,H}(x_m) = \lambda_q(x_m) = m$. ∎

The statement below clarifies the range of the values $H(\xi)$.

Proposition 4.2. *The function*

$$f_{q,H}(x) = \left(\frac{\Lambda_{q,H}(x)}{x} \right)^{\frac{1}{q-1}}$$

defines a density function if $H(\xi)$ for all $\xi \in (0, m]$ satisfies the following condition

$$- \frac{1}{[x_+(\xi)]^{\frac{q-1}{2-q}}} < H(\xi) < \left(\frac{C_q^{q-1}}{\xi} \right)^{\frac{q-1}{2-q}} - \frac{1}{[x_-(\xi)]^{\frac{q-1}{2-q}}}. \tag{28}$$

Proof. Let $0 < \xi \leq m$. The conditions $X_-(\xi) > C_q^{1-q}\xi$ and $X_+(\xi) < \infty$ together with (24) imply estimate (28). ∎

Remark 4.3. *Proposition 4.2 implies that the range of $H(m)$ is restricted to the interval*

$$-(q-1)^{\frac{q-1}{2(2-q)}} < H(m) < (q-1)^{\frac{q-1}{2(2-q)}}.$$

Proposition 4.4. *Let $H(m) \neq 0$. Then the function $\Lambda_{q,H}(x)$ attains its unique maximum at the point $x_{m,H} = \left[(q-1)^{\frac{q-1}{2(2-q)}} + H(m) \right]^{-\frac{2-q}{q-1}}$, and $\Lambda_{q,H}(x_{m,H}) = \lambda_q(x_m) = m$.*

Proof. Since $x_{m,H} = X_-(m) = X_+(m)$, the statement of this proposition can easily be derived upon computing $X_-(m)$. ∎

Proposition 4.5. *Let $q \in (3/2, 2)$. The inequality $X'_+(\xi) < 0$ holds near zero if and only if $H \geq 0$ near zero.*

Proof. It is not hard to verify that $x_+(\xi) = \frac{A_q}{\xi} + O(\xi)$ and $x'_+(\xi) = -\frac{A_q}{\xi^2} + O(1)$, as $\xi \to 0$, where $A_q = C_q^{q-1}/(q-1)$. Differentiating both sides of equation (24) for X_+, one has

$$X'_+(\xi) = \left(\frac{X_+(\xi)}{x_+(\xi)} \right)^{\frac{1}{2-q}} x'_+(\xi) - H'(\xi) \frac{(2-q)(X_+(\xi))^{\frac{1}{2-q}}}{q-1}. \tag{29}$$

Due to Proposition (3.5) $X_+(\xi) \sim x_+(\xi)$ as $\xi \to 0$. Therefore,

$$X'_+(\xi) \sim x'_+(\xi) - H'(\xi) \frac{(2-q)(x_+(\xi))^{\frac{1}{2-q}}}{q-1}$$

$$\sim -\frac{A_q}{\xi^2} - H'(\xi) \frac{B_q}{\xi^{\frac{1}{2-q}}}, \quad \xi \to 0, \tag{30}$$

where $B_q = \frac{(2-q)A_q^{\frac{1}{2-q}}}{q-1}$. If $H(\xi) \geq 0$, then obviously, $X'_+(\xi) < 0$ near zero. Now assume that $H(\xi) < 0$ near zero (that is in an interval $(0, \varepsilon)$ with some $\varepsilon > 0$). Then the second term in (30) grows faster then the first term near zero if $q > 3/2$, implying $X'_+(\xi) \geq 0$ near zero. ∎

Proposition 4.6. *Let x_1^a and x_2^a be two numbers such that $0 < x_1^a < x_{m,H} < x_2^a$, and*

$$\Lambda_{q,H}(x_1^a) = \Lambda_{q,H}(x_2^a) = a,$$

then $x_1^a x_2^a = const$ for all values of $a \in (0, m]$ if and only if $H(\xi)$ is identically zero.

Proof. Let $H(\xi) \equiv 0$. Then

$$\Lambda_{q,H}(x) = \lambda_q(x) = \frac{C_q^{q-1} x}{1 + (q-1)x^2}.$$

In this case the conclusion of the proposition can be established by direct calculation. Indeed, $x_1^a x_2^a = (q-1)^{-1}$, which is independent of the values of a. Now assume that $H(\xi) \neq 0$. Then, using (24) it is readily seen that

$$X_+(a)X_-(a) = \left([x_+(a)x_-(a)]^{\frac{q-1}{q-2}} + \mu(a) \right)^{\frac{q-2}{q-1}}$$

where $\mu(a) = H(a)\left(x_+^{\frac{q-1}{q-2}}(a) + x_+^{\frac{q-1}{q-2}}(a) + H(a)\right)$. Since $x_-(a)$ and $x_+(a)$ equal respectively x_1^a and x_2^a corresponding to the case $H(\xi) \equiv 0$, it follows that

$$X_+(a)X_-(a) = (const + \mu(a))^{\frac{q-2}{q-1}}. \tag{31}$$

Now inverting $X_\pm(x)$ we obtain that the product $x_1^a x_2^a$ is dependent of a.

Equation (31) also implies the necessity of the condition $H(\xi) \equiv 0$ for $x_1^a x_2^a$ to be independent on a. $\blacksquare$

5 Conclusion

Concluding, we would like to note some key points related to the limiting distribution in the q-CLT, the role of the q-FT in this as well as other relevant theorems, and also briefly address other concerns raised by Hilhorst in his paper (1).

1. In the proof of the q-CLT (see (3)), the q-FT is used only to establish the existence of a limiting distribution. If we assume that there is another (non unique) limiting distribution, then, due to Propositions 3.3 and 3.5, this distribution shares the same value at the origin and the same asymptotic behavior at infinity as the q-Gaussian. Therefore, such a density may be seen as a (slight) deformation of the q-Gaussian.

   However, can a distribution defined by $G_{q,H}$, if H is not identically zero, be a limit distribution in the q-CLT? Our belief is that this can not happen. Two strong arguments in favor of this belief are the following ones:

   (i) Proposition 4.5 states that $H(\xi)$ can not be negative near the origin for $3/2 < q < 2$. Due to the plausible nature of attractors, it is very unlikely that $H(\xi)$ has non-smooth points and drastic changes including sign changes. If $H(\xi)$ is not smooth at some points then $G_{q,H}$ will have singular points. A change of sign adds a new inflection point in the graph of the density of a limiting distribution. Therefore, if H is not negative in some small interval then it is not negative on the whole interval $[0, m]$. All these facts essentially restrict the set of functions $H(\xi)$ used in the invariance principle (possibly reducing this set to $\{0\}$), for which $G_{q,H}$ would be an attractor.

   (ii) Due to the strong dependence between q-independent random variables, a possible *symmetry-like* relationship between probabilities for small values and those for large values of the scaled sums Z_N is expected. Since the function $\lambda_q(x) = x[G_q(x)]^{q-1}$ is used extensively, this function may possess such a "symmetry", if it exists. In fact if x_1^a and x_2^a are two numbers such that $0 < x_1^a < x_m < x_2^a$, and $\lambda_q(x_1^a) = \lambda_q(x_2^a) = a$, then the product $x_1^a x_2^a = \frac{1}{q-1}$ is constant for all values of $a \in (0, m]$. We call this property *the hyperbolicity property* (since $x_2^a = \frac{1}{(q-1)x_1^a}$ is a hyperbola in (x_1^a, x_2^a)-plane). This heuristic argument leads to the following conjecture:

   Conjecture: The limit distribution of the sequence Z_N in equation (1) for q-independent random variables X_N possesses the hyperbolicity property.

   Proposition 4.6 shows that the only density with this property in the class of functions $G_{q,H}$, $H(0) = 0$, is the q-Gaussian. If the above conjecture is true, then the uniqueness of a limit distribution in the q-CLT will immediately follow from Proposition 4.6.

2. Regarding the paper (4), we recall that it is devoted to the asymptotical analysis of limiting distributions of (q, α)-stable distributions, explicitly addressing the "asymptotic equivalence." This paper, like (3), essentially uses the q-FT technique to establish the existence of a limiting distribution. To this end, we recall that even the usual FT (i.e., the 1-FT) can not be used for the rigorous proof of the uniqueness of solution in many situations, even though it is invertible. Therefore, in the classical theory the Fourier series or Fourier transform techniques are used for the existence of a solution. The use of this technique for the uniqueness is restricted to classes of functions which are representable as a Fourier series or Fourier transformable. For the uniqueness usually other methods are involved, like the "maximum principle", "energy integral", etc. Concluding this remark, we would like to note that in the references (7; 4; 8) mentioned by Hilhorst in his paper (1), the q-FT is used only for existence purposes.

3. Another question raised by Hilhorst in his papers (1; 2) is the lack of examples of q-independent random variables and their applications. Mathematically, a non-vacuous definition of some notion is a recipe for producing examples. The notion of q-independence generalizes the notion of usual independence, hence, containing as a trivial particular case the usual independence. Non-trivial examples can be produced at will using the definition of q-independence. As an example, for practical applications, paper (13) proves that sequences of independent random variables mixed with the help of a chi-square distribution are asymptotically q-independent. Paper (14) shows that such sequences can be considered as variance mixtures of normals, a wide class of distributions with applications in Beck-Cohen superstatistics (15). It is not surprising that sequences of variance mixtures of normals have limit distributions with q-Gaussian densities (see (14)).

4. Last but not least. It is definitively clear that no experimental, observational or computational results will ever determine an analytical function unless we have strong (physical) reasons to severely restrict its class. Independently from the q-CLT, a wide spectrum of experimental and computational distributions have been interpreted in the literature as q-Gaussians. Hence, various strong analytical reasons do exist which make q-Gaussians quite special. These include:

    (i) Under simple width constraints, q-Gaussians extremize the nonadditive entropy S_q, whose uniqueness (under natural axioms) and physical relevance has been repeatedly exhibited in the literature from many standpoints (see (16; 17; 18; 19; 20));

    (ii) q-Gaussians have been shown to exactly solve, for all values of space and time, the so-called *Porous Medium Equation* (21; 22), a very basic nonlinear Fokker-Planck equation (which satisfies the H-theorem precisely for the entropy S_q (23; 24), and which can be deduced from a quite simple non-Markovian Langevin equation) (25);

    (iii) Scale-invariant probabilistic models have been analytically shown to yield q-Gaussian limiting distributions for large systems, in a way totally analogous to how the de Moivre-Laplace theorem yields Gaussians, with the latter result being recovered as the $q = 1$ case of these models (26; 27);

    (iv) q-Gaussians consistently are attractors of q-CLTs that do not use the q-FT in their proofs (13; 14).

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