

M-theory, the signature theorem, and geometric invariants

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Abstract

The equations of motion and the Bianchi identity of the C-field in M-theory are encoded in terms of the signature operator. We then reformulate the topological part of the action in M-theory using the signature, which leads to connections to the geometry of the underlying manifold, including positive scalar curvature. This results in a variation on the miraculous cancellation formula of Alvarez-Gaumé and Witten in twelve dimensions and leads naturally to the Kreck-Stolz s -invariant in eleven dimensions. Hence M-theory detects diffeomorphism type of eleven-dimensional (and seven-dimensional) manifolds, and in the restriction to parallelizable manifolds classifies topological eleven-spheres. Furthermore, requiring the phase of the partition function to be anomaly-free imposes restrictions on allowed values of the s -invariant. Relating to string theory in ten dimensions amounts to viewing the bounding theory as a disk bundle, for which we study the corresponding phase in this formulation.

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1 Introduction

In this paper we show that M-theory encodes geometric invariants via the signature. The index theorem is a powerful tool in characterizing anomalies of physical theories. The simplest example of an index theorem in even dimensions is the index of the de Rham operator resulting in the Gauss-Bonnet theorem, which gives the Euler characteristic as a topological invariant. Due to the presence of spinors in a supersymmetric theory, it is very common to use the Dirac operator, whose index leads to the \hat{A} -genus as a topological invariant. In odd dimensions, there is a geometric/analytical correction term, namely the eta-invariant in the Atiyah-Patodi-Singer (APS) index theorem [2]. In this paper we consider instead, in the context of M-theory, the signature operator on differential forms, whose index is the signature [14]. In the presence of an odd-dimensional boundary, the APS index theorem for the signature expresses the signature of a Riemannian manifold with boundary in terms of the integral of the Hirzebruch L-polynomial and the eta-invariant of the boundary [3].

The C-field in eleven-dimensional M-theory with field strength G_4 has a dual field G_7 , which is just the Hodge dual at the level of differential forms. Due to the structure of the equation of motion of the C-field, G_7 can also be viewed as a potential with a field strength G_8 [7] [22] [23] [24] [25]. We show how the signature operator in eleven dimensions encodes the dynamics of the C-field and its dual. We also consider the index of this operator. The harmonic part of the C-field C_3 is already studied in [28]. In a complementary way, we consider here the harmonic part of the field strength G_4 . This is done in section 2, where we provide observations which serve as preparation for the main discussion in section 3.

Witten [36] wrote the topological part of the action in M-theory on a Spin eleven-manifold Y^{11} , namely the combination of the Chern-Simons term and the one-loop term, using index theory. This is done by lifting to the ‘bounding theory’ on a Spin twelve-manifold Z^{12} and involves an index of the Dirac operator coupled to an E_8 bundle as well as the index of the Rarita-Schwinger operator, that is the Dirac operator coupled to the virtual vector bundle $TZ^{12} - 4\mathcal{O}$. The subtraction of four copies of the trivial line bundle $-4\mathcal{O}$ from the tangent bundle comes from the consideration of ghosts in eleven dimensions. In section 3.1 we give an alternative description of the topological part of the action, using the Hirzebruch signature theorem, and hence the Hirzebruch L-polynomial [14]. This leads to a variant of the miraculous cancellation formula of Alvarez-Gaumé and Witten [1] which we might call “quantum” in the sense that ghosts coming from the path integral – a quantum effect – are accounted for.

The phase of the partition function in eleven dimensions (as opposed to twelve) involves the eta-invariants of the E_8 Dirac operator and of the Rarita-Schwinger operator. In section 3.2 we show that the above-mentioned reformulation in terms of the signature and the L-polynomial, when the E_8 bundle is trivial,

leads essentially to the s -invariant of Kreck and Stolz [19], defined in the rational numbers for positive scalar curvature metrics on our eleven-manifolds. Furthermore, absence of anomalies from the phase imposes a condition on the allowed values of the s -invariant. Issues of positive scalar curvature in M-theory in relation to the partition function are studied extensively in [29]. The s -invariant requires the vanishing of the rational Pontrjagin classes p_i . For p_1 this is weaker than requiring a String structure, the obstruction to which is $\frac{1}{2}p_1 \in H^4\mathbb{Z}$, because of possible 2-torsion. Similarly for p_2 this is weaker than requiring a Fivebrane structure [30] [31], the obstruction to which is $\frac{1}{6}p_2 \in H^8\mathbb{Z}$, because of possible 2- and 3-torsion.

The restriction of the s -invariant to parallelizable manifolds is given by the Eells-Kuiper invariant [10]. Since this invariant classifies topological spheres, we get in section 3.2 that M-theory classifies topological eleven-spheres. On the other hand, the extensions to the case when the E_8 bundle is no longer trivial suggests a possible generalization of the Kreck-Stolz invariant and which is defined for manifolds of positive scalar curvature together with a degree four cohomology class. To make our statements about M-theory will rely on the corresponding constructions in [19].

We also relate the geometric/analytical invariants in eleven dimensions to type IIA string theory via dimensional reduction on the circle S^1 in section 4.1. We consider the adiabatic limit of the eta-invariant of the signature operator, as opposed to that of the (twisted) Dirac operator considered previously in [21] [26] [29], and building on [8]. The bounding theory is then taken on a disk bundle Z^{12} over the ten-dimensional manifold of type IIA string theory. The proof in [3] of the index theorem with boundary assumes that the Riemannian manifold has a product metric near the boundary. For general manifolds, there is a correction form [12], which should be used for disk bundles. The signature of a disk bundle is given in terms of the integral of a characteristic class on the base manifold and a limiting eta-invariant [32]. We discuss this in section 4.2.

2 The signature (operator) in twelve and eleven dimensions

2.1 The signature (operator) in twelve dimensions

The signature operator on closed twelve-manifolds. Let Z^{12} be an oriented Riemannian twelve-manifold. The de Rham operator d and its adjoint d^* act on differential forms. The operator $d + d^*$ acts on the space Ω_Z^* of all differential forms and anticommutes with the involution τ defined by $\tau\phi = -i^{p(p-1)} *_{12} \phi$ for $\phi \in \Omega_Z^p$ a p -form on Z^{12} . Denoting by Ω_Z^+ and Ω_Z^- the \pm -eigenspaces of τ , we have that $d + d^*$ interchanges Ω_Z^+ and Ω_Z^- and hence defines by restriction the signature operator $\sigma : \Omega_Z^+ \rightarrow \Omega_Z^-$.

For Z^{12} closed, Hodge theory gives the equality [14]

$$\text{sign}(Z^{12}) = \text{index}(\sigma) = \int_{Z^{12}} L, \quad (2.1)$$

where L is the Hirzebruch L-polynomial and the right-hand side is the signature of the quadratic form on $H^6(Z^{12}; \mathbb{R})$ given by the cup product. There is a bilinear form on $H^6(Z^{12}) \otimes H^6(Z^{12}) \xrightarrow{\cup} \mathbb{R}$, where the cup product \cup is symmetric and nondegenerate, and the signature of Z^{12} is $\sigma(Z^{12}) = \sigma(\cup)$ with the following relevant properties

1. *Product:* $\sigma(M^4 \times N^8) = \sigma(M^4)\sigma(N^8)$. This will be useful in compactifications to four dimensions and to relating the corresponding secondary invariants in eleven dimensions to those in seven dimensions.
2. *Bordism invariance:* If $Z^{12} = \partial W^{13}$ then $\sigma(Z^{12}) = 0$. In this case the integral of the L-genus is zero.

Example: Calabi-Yau compactification. The signature can be used to derive consistency conditions on realistic compactifications [16]. Consider the bounding theory on Z^{12} taken to be a product $M^4 \times N^8$, where M^4 is a four-dimensional manifold and N^8 is a Calabi-Yau four-fold $N^8 = X_{\mathbb{C}}^4$. If M^4 is flat then the

signature $\sigma(N^4) = 0$ so that, by the product property above, the signature of Z^{12} is zero. If M^4 is not flat then $\sigma(M^4)$ can be nonzero and N^8 can be taken so that $\sigma(Z^{12})$ does not vanish. The middle cohomology of $X_{\mathbb{C}}^4$ splits as $H^4(X_{\mathbb{C}}^4) = B_+(X_{\mathbb{C}}^4) \oplus B_-(X_{\mathbb{C}}^4)$ into a selfdual ($*\omega = \omega$) subspace $B_+(X_{\mathbb{C}}^4)$ and anti-selfdual ($*\omega = -\omega$) subspace $B_-(X_{\mathbb{C}}^4)$, whose dimensions are determined by the Hirzebruch signature as

$$\begin{aligned}\sigma(X_{\mathbb{C}}^4) &= \dim B_+(X_{\mathbb{C}}^4) - \dim B_-(X_{\mathbb{C}}^4) \\ &= \int_{X_{\mathbb{C}}^4} L_2 = \frac{1}{45} \int_{X_{\mathbb{C}}^4} (7p_2 - p_1^2) = \frac{\chi(X_{\mathbb{C}}^4)}{3} + 32.\end{aligned}$$

The symmetric inner product $(\omega_1, \omega_2) = \int_{X_{\mathbb{C}}^4} \omega_1 \wedge *\omega_2$ is positive definite on $H^4(X_{\mathbb{C}}^4)$, and $H^4(X_{\mathbb{C}}^4; \mathbb{Z})$ is unimodular by Poincaré duality. The symmetric quadratic form $Q(\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2$ is positive definite on $B_+(X_{\mathbb{C}}^4)$ and negative definite on $B_-(X_{\mathbb{C}}^4)$. The reduction of the M-theory action is performed in [16] where the one-loop degree eight polynomial I_8 is taken to have components along the Calabi-Yau space, which leads to a quantization on the values of the Euler characteristic

$$I = - \int_{X_{\mathbb{C}}^4} I_8 = \int_{X_{\mathbb{C}}^4} (4p_2 - p_1^2)/192 = \frac{\chi(X_{\mathbb{C}}^4)}{24} \in \mathbb{Z}. \quad (2.2)$$

The C-field in M-theory satisfies $[G_4] - \frac{1}{4}p_1 \in H^4(X_{\mathbb{C}}^4; \mathbb{Z})$ [36]. If G_4 is zero then $\frac{1}{4}p_1$ has to be an integral class. This implies by Wu's formula that $x^2 \equiv 0 \pmod{2}$ for any $x \in H^4(X_{\mathbb{C}}^4; \mathbb{Z})$. From (2.2), this means that $H^4(X_{\mathbb{C}}^4; \mathbb{Z})$ is an even self-dual lattice with signature $\sigma(X_{\mathbb{C}}^4)$. The requirement that $\chi = 0 \pmod{24}$ is consistent with the fact that every even self-dual lattice should have $\sigma = 0 \pmod{8}$. If $\frac{1}{4}p_1$ is half-integral then $[G_4]$ has to be half-integral, and a potentially non-integral contribution to the one-point function can be cancelled also for Calabi-Yau's for which $\chi \neq 0 \pmod{24}$ [16].

2.2 The signature (operator) in eleven dimensions and the C-field

Now suppose Z^{12} has a boundary Y^{11} and is isometric to a product near the boundary.² Then near Y^{11} , σ is of the form $\sigma = (\frac{\partial}{\partial z} + \mathcal{S})$ with \mathcal{S} a self-adjoint operator on Y^{11} . The restriction of Ω_Z^+ to Y^{11} can be identified with the space Ω_Y^* of all differential forms on Y^{11} . On Y^{11} , the signature operator is defined as [3]

$$\mathcal{S}\phi = (-1)^p(\epsilon * d - d*)\phi, \quad (2.3)$$

where $*$ is the Hodge star operator defined by the metric g_Y and with $\epsilon = 1$ for ϕ a $2p$ -form and $\epsilon = -1$ for ϕ a $(2p-1)$ -form. The operator \mathcal{S} commutes with the parity of differential forms on Y^{11} and commutes with the operator which is essentially the Hodge duality operator $\phi \mapsto (-1)^p * \phi$. When p is even then \mathcal{S} commutes precisely with $*$, and when p is odd then \mathcal{S} commutes with $*$ up to a sign. This splits the operator \mathcal{S} , according to form degree, into even and odd parts $\mathcal{S} = \mathcal{S}^{\text{ev}} \oplus \mathcal{S}^{\text{odd}}$.

The even and odd signature operators. Let (Y^{11}, g_Y) be a compact oriented eleven-dimensional Riemannian manifold. The odd signature operator $\mathcal{S}_p^{\text{odd}}$ is defined on $\bigoplus_{p=1}^6 \Omega^{2p-1}$, the differential forms of odd degree, by

$$\mathcal{S}_p^{\text{odd}} = (-1)^{p+1}(*d + d*). \quad (2.4)$$

Similarly, the even signature operator $\mathcal{S}_p^{\text{ev}}$ is defined on $\bigoplus_{p=0}^5 \Omega^{2p}$, the differential forms of even degree, by

$$\mathcal{S}_p^{\text{ev}} = (-1)^p(*d - d*), \quad (2.5)$$

The operator \mathcal{S}^{ev} isomorphic to the operator \mathcal{S}^{odd} [5].

²The general case will be considered in section 4.1.

The even signature operator on the fields G_4 and G_8 . We now consider the field G_4 and its Hodge dual G_7 in eleven dimensions out of which we build a field G_8 . The equation of motion for G_4 ,

$$d * G_4 = \frac{1}{2} G_4 \wedge G_4 - I_8(g_Y), \quad (2.6)$$

involves dG_7 , which we call G_8 (this is called Θ in [7]). The even signature operator $\mathcal{S}(g_Y) = \oplus_p \mathcal{S}_p : \Omega^{2*}(Y^{11}) \rightarrow \Omega^{2*}(Y^{11})$ on (Y^{11}, g_Y) acts on even degree forms as

$$\begin{aligned} \mathcal{S}_p : \Omega^{2p}(Y^{11}) &\rightarrow \Omega^{10-2p}(Y^{11}) \oplus \Omega^{12-2p}(Y^{11}) \\ G_{2p} &\mapsto (-1)^p (*d - d*)G_{2p}. \end{aligned} \quad (2.7)$$

For $p = 2$ this gives the action on the field strength G_4 of the C-field

$$\begin{aligned} \mathcal{S}_2 : \Omega^4(Y^{11}) &\rightarrow \Omega^6(Y^{11}) \oplus \Omega^8(Y^{11}) \\ G_4 &\mapsto *dG_4 - d * G_4, \end{aligned} \quad (2.8)$$

which, upon use of the Bianchi identity $dG_4 = 0$, gives $G_4 \mapsto -d * G_4$. This can further be expanded using the equation of motion (2.6), resulting in

$$\mathcal{S}_2 : G_4 \mapsto I_8(g_Y) - \frac{1}{2} G_4 \wedge G_4. \quad (2.9)$$

Next, for $p = 4$, the operator (2.7) acting on the field strength G_8 gives

$$\mathcal{S}_4 : G_8 \mapsto *dG_8 - d * G_8. \quad (2.10)$$

As mentioned above, we take G_8 to be the right hand side of the equation of motion of the C-field, i.e. $G_8 = d * G_4 - \frac{1}{2} G_4 \wedge G_4$. Consequently, $dG_8 = 0$ and $d * G_8 = \Delta G_4 - \frac{1}{2} d * (G_4 \wedge G_4)$, so that

$$\mathcal{S}_4 : G_8 \mapsto d * G_8 = -\Delta G_4 + \frac{1}{2} d * (G_4 \wedge G_4). \quad (2.11)$$

Here $\Delta = (d + d^*)^2$ is the Hodge Laplacian, which on G_4 and G_8 is simply dd^* since both of these fields are closed.

The index of \mathcal{S}_2 . Consider the kernel of the operator (2.8). From (2.9), this space is

$$\text{Ker}(\mathcal{S}_2) = \{G_4 \mid d * G_4 = 0\} \quad (2.12)$$

so that it is given by requiring the eight-form on the right-hand side of the equation of motion (2.6) to be zero

$$\frac{1}{2} G_4 \wedge G_4 - I_8(g_Y) = 0. \quad (2.13)$$

This expression can be easily arranged to hold by requiring each of the two terms to vanish separately. The quadratic term would be zero if G_4 is taken to have specific components which run over less than eight values. The one-loop term is zero for manifolds (M, g_M) for which the Pontrjagin forms satisfy $p_2(g_M) - \frac{1}{4} p_1(g_M)^2 = 0$. At the level of cohomology this is satisfied for eight-manifolds with nowhere vanishing spinors (see [15]) or with a $\text{PU}(3)$ structure (see [33]). The cokernel of the operator (2.8) is the kernel of the adjoint operator \mathcal{S}_2^\dagger so that $\text{Coker}(\mathcal{S}_2) = \text{Ker}(\mathcal{S}_2^\dagger)$ is given by $G_4 = 0$, that is by flat C-fields. The dimension of the space of C-fields satisfying the equations of motion, when $G_4 \wedge G_4 - I_8(g_Y) = 0$, is given by $\dim(\text{Ker}(\mathcal{S}_2))$. The dimension of the space of flat C-fields is then given by $\dim(\text{Ker}(\mathcal{S}_2^\dagger))$. The index of the operator \mathcal{S}_2 is

$$\begin{aligned} \text{Index}(\mathcal{S}_2) &= \dim(\text{Ker}(\mathcal{S}_2)) - \dim(\text{Coker}(\mathcal{S}_2)) \\ &= \{\text{“on-shell” C-fields}\} - \{\text{flat C-fields}\}. \end{aligned}$$

The index of \mathcal{S}_4 . Now consider the kernel of the operator (2.10) acting on the dual field G_8 , $\text{Ker}(\mathcal{S}_4) = \{G_8 \mid d * G_8 = 0\}$. From (2.11), this is given in terms of G_4 by

$$\text{Ker}(\mathcal{S}_4) = \{G_8 \mid \Delta G_4 - \frac{1}{2}d * (G_4 \wedge G_4) = 0\} . \quad (2.14)$$

The differential equation giving the condition in (2.14) does not seem to have a general solution. Instead, we will give a characterization in certain special cases. When $G_4 \wedge G_4 = 0$, as discussed above for $\text{Ker}(\mathcal{S}_2)$, the condition in (2.14) is simply that G_4 is harmonic. However, when $*(G_4 \wedge G_4)$ is a nonzero three-form, say proportional to the C-field itself, with $dC_3 = G_4$, then the condition is $(\Delta - m)G_4 = 0$, that is G_4 is annihilated by the ‘‘massive’’ Laplacian. Here m is the parameter of proportionality, that is $*(G_4 \wedge G_4) = 2mC_3$. This is reminiscent of the phenomenon of odd-dimensional self-duality which appears in supergravity theories in odd dimensions, where a field strength is (Hodge) dual to a potential. The cokernel of the operator \mathcal{S}_4 is the kernel of the adjoint operator \mathcal{S}_4^\dagger , where G_8 being zero implies that G_7 is constant, which is the same as $*G_4$ being a constant. The index can be found similarly to the case of \mathcal{S}_2 (within the above specialization).

The odd signature operator on the potential fields C_3 and G_7 . We now consider the signature operator acting on the odd forms in M-theory. These are the C-field C_3 and the field G_7 , the Hodge dual of G_4 . For the first field we have

$$\mathcal{S}_2^{\text{odd}} : C_3 \mapsto -(d * + * d)C_3 = -G_7 , \quad (2.15)$$

so that the kernel is given by flat dual fields. For the second field we have

$$\mathcal{S}_4^{\text{odd}} : G_7 \mapsto -(d * + * d)G_7 = - * d * G_4 , \quad (2.16)$$

so that the kernel is given by co-closed G_4 . The cokernels are given by zero C_3 and by flat C-fields, i.e. those for which $G_4 = 0$. Therefore, the indices are given, respectively, by

$$\text{Index}(\mathcal{S}_2^{\text{odd}}) = \{\text{dual flat fields}\} - \{\text{zero C - fields}\} \quad (2.17)$$

$$\text{Index}(\mathcal{S}_4^{\text{odd}}) = \{\text{coclosed } G_4\} - \{\text{flat C - fields}\} . \quad (2.18)$$

Alternative form. Alternatively, instead of the operator $d * \pm * d$, we could use the operator $d \pm * d *$, which is manifestly (anti) self-adjoint. Then, for example, on G_4 we would get $G_4 \mapsto * (\frac{1}{2}G_4 \wedge G_4 - I_8(g))$.

The eta-invariant. Since \mathcal{S} is self-adjoint and its square $\mathcal{S}^2 = \Delta$ is the Hodge Laplacian, it is diagonalizable with real eigenvalues λ_n . The eta-invariant $\eta(\mathcal{S}(Y^{11}))$, which is a measure of asymmetry of the spectrum of the operator, is defined as the value at $s = 0$ of [3]

$$\eta(s) = \sum_{\lambda_n \neq 0} \frac{\text{sign} \lambda_n}{|\lambda_n|^s} . \quad (2.19)$$

The fact that the operator \mathcal{S}^{ev} isomorphic to the operator \mathcal{S}^{odd} implies, in particular, that the eta-invariant and number of zero modes corresponding to \mathcal{S} can be written in terms of those of \mathcal{S}^{ev} as

$$\eta(\mathcal{S}) = 2\eta(\mathcal{S}^{\text{ev}}) , \quad h(\mathcal{S}) = 2h(\mathcal{S}^{\text{ev}}) , \quad (2.20)$$

and similarly for \mathcal{S}^{odd} . Therefore, in dealing with the eta-invariant, one can formulate expressions using the ‘total’ signature operator, the odd signature operator, or the even signature operator, with the simple prescribed way of transforming from one formulation to the other.

Having discussed the effect of the signature operator on the fields, next we turn to the corresponding effect on the action and partition function of the theory.

3 The invariants from M-theory

In this section we start by recalling the APS index theorem for the signature operator and then provide our main arguments in this context in section 3.1 and section 3.2.

The APS index theorem for the signature operator. Let (Z^{12}, g_Z) be a compact oriented Riemannian manifold with boundary (Y^{11}, g_Y) , and assume that near the boundary the twelve-manifold is isometric to a product. Then the APS index theorem relates a topological invariant on one side to a sum of a differential geometric and a spectral invariant on the other side [3]

$$\text{sign}(Z^{12}, Y^{11}) = \int_{Z^{12}} L(p(g_Z)) - \eta(\mathcal{S}(g_Y)) , \quad (3.1)$$

where

(i) $\text{sign}(Z^{12})$ is the signature of the nondegenerate quadratic form defined by the cup product on the image of $H^6(Z^{12}, Y^{11})$ in $H^6(Z^{12})$. Looking at relative cohomology is appropriate since there are no six-form field strengths in M-theory in eleven dimensions.

(ii) $L(p(g_Z)) = L_{12}(p_1, p_2, p_3)$, where L_{12} is the 3rd Hirzebruch L-polynomial (of degree twelve) and the p_i are the Pontrjagin forms of the curvature built out of the Riemannian metric g_Z .

(iii) $\eta(\mathcal{S}(g_Y))$ is the eta-function for the self-adjoint operator \mathcal{S}_p on even forms on Y^{11} given by $G_{2p} \mapsto (-1)^p(*d - d*)G_{2p}$. The multiplicity of the zero eigenvalue of \mathcal{S}_p is $h = \dim(\text{Ker}(\mathcal{S}_p))$. For the values $p = 2, 4$, we considered this in the previous section.

Notice that if $\text{sign}(Z^{12}, Y^{11}) = 0$ then the topological quantity is given in terms of the geometric/analytical quantity $\int_{Z^{12}} L(p(g_Z)) = \eta(\mathcal{S}(g_Y))$. This can happen, for instance, when Z^{12} itself is a boundary, as indicated in the bordism property mentioned in section 2.1.

3.1 A variation on the miraculous cancellation formula in twelve dimensions

The topological part of the action in M-theory, namely the combination of the Chern-Simons term and the one-loop term

$$I = I_{CS} + I_{1\text{-loop}} = \frac{1}{6} \int_{Z^{12}} G_4 \wedge G_4 \wedge G_4 - \int_{Z^{12}} I_8 \wedge G_4 , \quad (3.2)$$

is formulated in [36] in terms of index theory. This involves I^{E_8} , the index of the Dirac operator coupled to an E_8 bundle, as well as I^{RS} , the index of the Rarita-Schwinger operator, that is the Dirac operator coupled to the virtual vector bundle $TZ^{12} - 4\mathcal{O}$. The subtraction of four copies of the trivial line bundle, $-4\mathcal{O}$, comes from the inclusion of effect of ghosts required to fix the gauge invariances of the Rarita-Schwinger operator in eleven dimensions. The exponent in the phase of the partition function is [36]

$$\frac{1}{2\pi} I = \frac{1}{2} I^{E_8} + \frac{1}{4} I^{RS} . \quad (3.3)$$

The factor of $1/2$ on the right hand side is due to a Majorana-Weyl (MW) condition. The factor of $1/4$ is due to a MW condition and the fact that the characteristic class should to be related to half of the gravitino-dilatino anomaly (in comparing to heterotic string theory).

We now provide our alternative description, using the the Hirzebruch signature theorem, and hence the Hirzebruch L-polynomial. We start with the Rarita-Schwinger index in twelve dimensions. This is given by

$$\begin{aligned} I_{12}^{RS} &= \widehat{A}(Z^{12}) \text{ch}(TZ_{\mathbb{C}}^{12} - 4\mathcal{O}) \\ &= \frac{1}{2^{10} \cdot 3^3 \cdot 5 \cdot 7} (-31p_1^3 + 44p_1p_2 - 16p_3) \cdot 8 + \frac{1}{2^7 \cdot 3^2 \cdot 5} (7p_1^2 - 4p_2) \cdot p_1 \\ &\quad - \frac{1}{2^3 \cdot 3} (p_1) \cdot \frac{1}{2^2 \cdot 3} (p_1^2 - 2p_2) + 1 \cdot \frac{1}{2^3 \cdot 3^2 \cdot 5} (p_1^3 - 3p_1p_2 + 3p_3) \\ &= \frac{1}{2^3 \cdot 3^3 \cdot 5 \cdot 7} (2p_1^3 - 13p_1p_2 + 62p_3) \end{aligned}$$

On the other hand, the degree-twelve part of the Hirzebruch L-polynomial is given by (see e.g. [14])

$$L_{12} = \frac{1}{3^3 \cdot 5 \cdot 7} (62p_3 - 13p_1p_2 + 2p_1^3) . \quad (3.4)$$

Now we get a formula which is a variation on the miraculous anomaly cancellation formula of Alvarez-Gaumé and Witten [1]. This relies on the curious degree twelve expression ³

$$8I_{12}^{RS} = L_{12} . \quad (3.5)$$

To some extent, our formula can be viewed as a quantum counterpart of the classical miraculous cancellation formula. By “quantum” we mean in the sense of accounting for ghosts coming from the path integral are accounted for. However, a fully quantum version would involve setting up effective actions as in [11], which will be discussed separately elsewhere. Note that for a twelve-manifold M with tangent bundle TM the miraculous cancellation formula is [1]

$$L(M) = 8\widehat{A}(M, TM) - 32\widehat{A}(M) , \quad (3.6)$$

where $\widehat{A}(M, TM) = \widehat{A}(M)\text{ch}(TM)$, with $\text{ch}(TM) = \sum_j e^{x_j} + e^{-x_j} = \sum_j 2\cosh x_j$. It is easily seen that $L(M) = 8\widehat{A}(M)[\text{ch}(TM) - 4]$. We now rewrite (3.3), the exponent in the phase of the partition function, arriving at the alternative expression which trades the Rarita-Schwinger index with the Hirzebruch L-polynomial via (3.5)

$$\frac{1}{2\pi}I = \frac{1}{2}I^{E_8} + \frac{1}{32}L_{12} . \quad (3.7)$$

To motivate what might be gained by doing this, let us consider the case when the E_8 bundle is trivial. In this case the top degree component of (3.7) reduces to

$$\frac{1}{2\pi}I = 124 \left(\widehat{A}_{12} + \frac{1}{2^7 \cdot 31} L_{12} \right) . \quad (3.8)$$

Absence of anomalies requires that the right hand side be an integer (as it is a phase in the effective action), so that we arrive at the condition

$$\left\langle \widehat{A}_{12} + \frac{1}{2^7 \cdot 31} L_{12}, [Z^{12}] \right\rangle \in \mathbb{Z}/124 . \quad (3.9)$$

Proposition 1 (i). *The miraculous cancellation formula in twelve dimensions can be written as $8I_{12}^{RS} = L_{12}$, where I_{12}^{RS} is the Rarita-Schwinger index and L_{12} is the Hirzebruch L-polynomial in degree 12.*

(ii). *The topological action in M-theory is $\frac{1}{2}I^{E_8} + \frac{1}{32}L_{12}$, where I^{E_8} is the index of the Dirac operator coupled to an E_8 bundle.*

(iii). *The phase is not anomalous if $\left\langle \widehat{A}_{12} + \frac{1}{2^7 \cdot 31} L_{12}, [Z^{12}] \right\rangle \in \mathbb{Z}/124$.*

In the following section we show that this naturally leads to the Kreck-Stolz s -invariant.

3.2 The Kreck-Stolz s -invariant and scalar curvature in eleven dimensions

Let Z^{12} be a twelve-dimensional compact Spin manifold with boundary $\partial Z^{12} = Y^{11}$. Let g_Z be a Riemannian metric on Z^{12} which coincides with a product metric on $Y^{11} \times I$ in a collar neighborhood of the boundary and let g_Y be its restriction to the boundary. Let $D^+(Z^{12}, g_Z)$ be the (chiral) Dirac operator with respect to the metric g_Z from the positive to negative spinors on Z^{12} . This becomes a Fredholm operator if we impose the APS boundary condition [3], i.e. if we restrict to spinors on Z^{12} whose restriction to ∂Z^{12} is in the kernel of P , the spectral projection corresponding to nonnegative eigenvalues of the (total) Dirac

³We will label the \widehat{A} -genus and the L-genus by their form-degree rather than by that divided by four.

operator $D(Y^{11}, g_Y)$ on Y^{11} . Denote by $\text{index}(D^+(Z^{12}, g_Z))$ the index of this Fredholm operator. If $g_Z(t)$ is a continuous family of metrics on Z^{12} , then this index is independent of t , which can be seen as follows (cf. [3] [19]). The corresponding family of spectral projections $P(t)$ is not continuous for those parameter values t where an eigenvalue of $D(Y^{11}, g_Y)$ crosses the origin. If $g_Y(t)$ has positive scalar curvature metric then the Weitzenböck formula gives [20] that $\text{Ker}(D(Y^{11}, g_Y(t))) = 0$. Hence $D^+(Z^{12}, g_Z(t))$ is a continuous family of Fredholm operators and thus $\text{index}(D^+(Z^{12}, g_Z(t)))$ is independent of t . Note the following:

1. If g_Z has a positive scalar curvature metric then, from [4], $\text{index}(D^+(Z^{12}, g_Z))$ vanishes.
2. If g_Z is a metric on Z^{12} whose restriction to the boundary g_Y has positive scalar curvature then $\text{index}(D^+(Z^{12}, g_Z))$ depends only on the connected component of g_Y in $\mathcal{R}^+(Y^{11})$, the space of positive scalar curvature metrics on Y^{11} [19].

The APS index theorem for the Dirac operator is [3]

$$\text{index}(D^+(Z^{12}, g_Z)) = \int_{Z^{12}} \widehat{A}(p_i(g_Z)) - \frac{1}{2} (h(Y^{11}) + \eta(D(Y^{11}, g_Y))) . \quad (3.10)$$

Here

- $p_i(g_Z)$ are the Pontrjagin forms of Z^{12} with respect to the Levi-Civita connection ∇_Z^L determined by g_Z ,
- $D(Y^{11}, g_Y)$ is the Dirac operator on $Y^{11} = \partial Z^{12}$,
- $h(Y^{11})$ is the dimension of the kernel of $D(Y^{11}, g_Y)$ which consists of harmonic spinors on the boundary,
- and $\eta(D(Y^{11}, g_Y))$ is the η -invariant, which measure the asymmetry of the spectrum of the self-adjoint operator $D(Y^{11}, g_Y)$.

Additivity and the space of harmonic spinors on Y^{11} . Consider two twelve-manifolds (Z^{12}, g_Z) and (Z'^{12}, g'_Z) . If we glue these two manifolds along isometric boundary component Y^{11} , then the extension of the Chern-Simons term $I_{CS} = \frac{1}{6} \int_{Y^{11}} G_4 \wedge G_4 \wedge C_3$ from Y^{11} to Z^{12} leading to $I_{CS} = \frac{1}{6} \int_{Z^{12}} G_4 \wedge G_4 \wedge G_4$, is independent of the choice of the bounding twelve-manifold [36]. However, the index formula shows that the index of the Dirac operator behaves additively, provided that there are no harmonic spinors on Y^{11} , i.e. $h(Y^{11}) = 0$. This happens, for example, if the scalar curvature of that piece of the eleven-dimensional boundary is positive. Such situations are considered extensively in [29].

The Kreck-Stolz invariant $s(Y^{11}, g_Y)$. We will arrive at an invariant $s(Y^{11}, g_Y) \in \mathbb{Q}$, defined in [19], as an absolute version of the Gromov-Lawson invariant [13], which in our case would be for a pair of positive scalar curvature metrics g_1 and g_2 on Y^{11} . We will first describe this invariant, following [19], and then show how M-theory leads to it naturally. This invariant is obtained by rewriting (3.10) as a sum of two terms, one depending only on the geometry of Y^{11} and another depending only on the topology of Z^{12} . This can be done provided that the real Pontrjagin classes of Y^{11} vanish. The construction relies on treating the decomposable vs. nondecomposable summands in $\int_{Z^{12}} \widehat{A}(p_i(g_Z))$ separately [19].

Decomposable summands: Let α_4 and β_8 be differential forms of positive degree on Z^{12} whose restrictions to the boundary Y^{11} are coboundaries, i.e. there are forms c_3 and c_7 on Y^{11} such that $dc_3 = \alpha_4|_{Y^{11}}$ and $dc_7 = \beta_8|_{Y^{11}}$. Then the wedge products are related as

$$\int_{Z^{12}} \alpha_4 \wedge \beta_8 = \int_{Y^{11}} \alpha_4 \wedge c_7 + \langle j^{-1}[\alpha_4] \cup j^{-1}[\beta_8], [Z^{12}, Y^{11}] \rangle , \quad (3.11)$$

where $j^{-1}[\alpha_4] \in H^4(Z^{12}, Y^{11}; \mathbb{R})$ in any preimage of the de Rham cohomology class $[\alpha_4] \in H^4(Z^{12}; \mathbb{R})$ under the natural map $j : H^4(Z^{12}, Y^{11}; \mathbb{R}) \rightarrow H^4(Z^{12}; \mathbb{R})$ and similarly for $[\beta_8]$, and $\langle \cdot, [Z^{12}, Y^{11}] \rangle$ is the Kronecker product with the fundamental class. Note that the integral on the right hand side of (3.11) is independent of the choice of c_7 and the Kronecker product is independent of the preimages $j^{-1}[\alpha_4]$ and $j^{-1}[\beta_8]$. Taking α_4 and β_8 to be rational multiples of the Pontrjagin forms $p_1(g_Z)$ and $p_2(g_Z)$, respectively, gives that the decomposable summands in $\widehat{A}(p_i(g_Z))$ can be written as a sum of two terms.

Nondecomposable summands: This summand in $\widehat{A}(p_i(g_Z))$ is a nontrivial multiple of the top Pontrjagin form $p_3(g_Z)$. Since the Hirzebruch L-polynomial also involves this form (with another multiple) then one can arrange for a combination of \widehat{A} and L which cancels p_3 , namely [14] the following combination $\widehat{A}_3 + \frac{1}{27 \cdot 31} L_3$. Let $j^{-1}p_i(Z^{12})$ be any preimage under the natural map $j : H^{4i}(Z^{12}, Y^{11}; \mathbb{R}) \rightarrow H^{4i}(Z^{12}; \mathbb{R})$. This exists because we are assuming $p_i(Y^{11}) = 0 \in H^{4i}(Y^{11}; \mathbb{R})$. Then

$$\begin{aligned} \text{index}(D^+(Z^{12}, g_Z)) &= \int_{Y^{11}} d^{-1} \left(\widehat{A} + \frac{1}{27 \cdot 31} L \right) (p_i(Y^{11}, g_Y)) - \frac{1}{2} (h(Y^{11}) + \eta(D(Y^{11}, g_Z|_Y))) \\ &\quad - \frac{1}{27 \cdot 31} \eta(\mathcal{S}(Y^{11}, g_Z|_Y)) - t(Z^{12}), \end{aligned}$$

where the topological term is

$$t(Z^{12}) = - \left\langle \left(\widehat{A} + \frac{1}{27 \cdot 31} L \right) (j^{-1}p_i(Z^{12})), [Z^{12}, Y^{11}] \right\rangle + \frac{1}{27 \cdot 31} \text{sign}(Z^{12}). \quad (3.12)$$

In particular, if all the real Pontrjagin classes of Y^{11} vanish then we can apply the formula to $Z^{12} = Y^{11} \times I$, in which case $t(Z^{12})$ vanishes.

Given a closed eleven-dimensional Spin manifold Y^{11} with vanishing real Pontrjagin classes and positive scalar curvature metric g_Y on Y^{11} we define, following [19],

$$s(Y^{11}, g_Y) := -\frac{1}{2} \eta(D(Y^{11}, g_Y)) - \frac{1}{27 \cdot 31} \eta(\mathcal{S}(Y^{11}, g_Y)) + \int_{Y^{11}} d^{-1} \left(\widehat{A}_{12} + \frac{1}{27 \cdot 31} L_{12} \right) (p_i(Y^{11}, g_Y)). \quad (3.13)$$

Properties of $s(Y^{11}, g_Y)$. Let Y^{11} and Y'^{11} be eleven-dimensional closed Spin manifolds with vanishing real Pontrjagin classes and positive scalar curvature metrics g_Y and g'_Y , respectively. Then, specializing [19],

1. If $f : s(Y^{11}, g_Y) \rightarrow s(Y'^{11}, g'_Y)$ is a Spin preserving isometry, then $s(Y^{11}, g_Y) = s(Y'^{11}, g'_Y)$.
2. $s(Y^{11}, g_Y)$ depends only on the connected component of g_Y in $\mathcal{R}_{\text{scal}}^+(Y^{11})$, the moduli space of positive scalar curvature metrics on Spin eleven-manifolds.
3. If Y^{11} bounds a Spin manifold Z^{12} and g_Z is a metric on Z^{12} extending g_Y , which is a product metric near the boundary, then

$$s(Y^{11}, g_Y) = \text{index}(D^+(Z^{12}, g_Z)) + t(Z^{12}). \quad (3.14)$$

4. $s(Y^{11}, g_Y)$ depends on the choice of Spin structure on Y^{11} .

Consider the expression (3.8) on a twelve-manifold with boundary. Using the APS index theorem, both for the signature operator (3.1) and for the Dirac operator (3.10), we get that the phase of the M-theory partition function in eleven dimensions is given by expression (3.13). Given the identification of the phase in the M-theory partition function essentially with the Kreck-Stolz s -invariant, the anomaly cancellation condition (3.9) in twelve dimensions can now be recast as saying that in eleven dimensions $s(Y^{11}, g_Y) \in \mathbb{Z}/124$. We therefore have

Theorem 2 *Consider M-theory on Spin (Y^{11}, g_Y) , where g_Y is a metric of positive scalar curvature, and let the E_8 bundle on Y^{11} be trivial. Then*

- (i). *The phase of the M-theory partition function is anomaly free provided $s(Y^{11}, g_Y) \in \mathbb{Z}/124$.*
- (ii). *M-theory on a Spin manifold with positive scalar curvature metric (Y^{11}, g_Y) detects diffeomorphism types.*
- (iii). *The topological part of the action is invariant under Spin isometries.*
- (iv). *The topological part of the action depends only on the connected component of the metric in the moduli space of positive scalar curvature metrics.*
- (v). *The topological part of the action depends on the choice of Spin structure.*

The last part of the theorem is discussed extensively in [29] from another point of view. See also section 4.1.

Conditions and examples of s -invariants satisfying anomaly cancellation. We would like to check that the condition (3.9) or, more precisely the condition in part (i) of Theorem 2, is satisfied for some relevant Spin eleven-manifolds. Since the eta-invariant is additive under direct sum, we could consider decomposable manifolds and restrict to seven-manifolds, as internal spaces of compactifications to four dimensions. An important class of such Spin Einstein manifolds which solve the supergravity equations of motion is the Witten spaces $M_{k,l}$ [34], which are principal S^1 bundles over $\mathbb{C}P^2 \times \mathbb{C}P^1$ classified by $lx + ky \in H^2(\mathbb{C}P^2 \times \mathbb{C}P^1; \mathbb{Z})$, where x and y are the generators of $H^2(\mathbb{C}P^2; \mathbb{Z})$ and $H^2(\mathbb{C}P^1; \mathbb{Z})$, respectively [18]. Let k and l be relatively prime integers with k even. Then the s -invariant of $M_{k,l}$ with Einstein metric $g_{k,l}$ is [19]

$$s(M_{k,l}, g_{k,l}) = -\frac{3}{27 \cdot 7} \frac{k(l^2 + 3)(l^2 - 1)}{l^2}, \quad (3.15)$$

from which we observe that for the values $(k, l) = (14, 3)$ we get $124s(M_{14,3}, g_{14,3}) = 744$. With this integer value for s , there are no anomalies in the phase. Another example we consider is the family of Aloff-Wallach spaces of positive sectional curvature, for which the s -invariant is [19]

$$S(N_{k,l}, g_{k,l}) = \frac{1}{2^5 \cdot 7} kl(k+l). \quad (3.16)$$

We see that for the values $(k, l) = (8, 7)$ we find that $s(N_{8,7}, g_{8,7}) = 15/4$, so that the phase is $2\pi i$ times the integer $3 \cdot 5 \cdot 31$. Other values can also be found by solving the above Diophantine equations (but we do not need that here).

The Eells-Kuiper invariant and stably parallelizable eleven-manifolds. We now consider a special class of eleven-manifolds, for which the s -invariant reduces to a more classical invariant. Let Y^{11} be a stably parallelizable compact Spin eleven-manifold without boundary. That is, $TY^{11} \oplus \mathcal{O}$ is trivial. Let $Y^{11} = \partial Z^{12}$ with Z^{12} a Spin twelve-manifold. The Eells-Kuiper invariant is defined as [10]

$$ek(Y^{11}) = \left\langle \left(\widehat{A}_{12}(p_1, p_2, p_3) - \frac{1}{2^7 \cdot 31} L_{12}(p_1, p_2, p_3) \right), [Z^{12}, Y^{11}] \right\rangle + \frac{1}{2^7 \cdot 31} \text{sign}(Z^{12}) \in \mathbb{Q}/\mathbb{Z}, \quad (3.17)$$

where p_i are the relative Pontrjagin classes of (Z^{12}, Y^{11}) corresponding to some framing of the stable tangent bundle of Y^{11} . Let ω be a connection on the stable tangent bundle of Y^{11} . Extend ω over Z^{12} as a product on a collar neighborhood. Following [9], the Chern-Simons invariants are given by

$$\begin{aligned} CS(L, Y^{11}, \omega) &= \text{sign}(Z^{12}) - \int_{Z^{12}} L_{12}(p_i(\omega)) \in \mathbb{R}, \\ CS(\widehat{A}, Y^{11}, \omega) &= - \int_{Z^{12}} \widehat{A}_{12}(p_i(\omega)) \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

If ω is the trivial connection corresponding to some framing then $p_i(\omega)$ represent the relative Pontrjagin classes of that framing and

$$ek(Y^{11}) = -CS(\widehat{A}, Y^{11}, \omega) + \frac{1}{2^7 \cdot 31} CS(L, Y^{11}, \omega) \in \mathbb{Q}/\mathbb{Z}; \quad (3.18)$$

Suppose Y^{11} is endowed with a Riemannian metric g_Y . Define the signature operator on even forms on Y^{11} as above, and extend g_Y to a metric on Z^{12} as a product near the boundary. Let ∇_{g_Y} be the Levi-Civita connection associated with g_Y . The APS index theorem [3] gives the eta invariant for this metric

$$\begin{aligned} \eta(S(Y^{11})) &= CS(L, Y^{11}, \nabla_{g_Y}) \\ \frac{1}{2} (h - \eta(D(Y^{11}))) &= CS(\widehat{A}, Y^{11}, \nabla_{g_Y}) \in \mathbb{R}/\mathbb{Z}. \end{aligned}$$

Let f be any section of the stable tangent bundle and $TP(\omega)$ are the canonical forms satisfying $dTP(\omega) = P(\omega)$, which is another way of writing d^{-1} . Then, from [9],

$$ek(Y^{11}) = \frac{1}{2} (\eta(D(Y^{11})) - h) + \frac{1}{27 \cdot 31} \eta(\mathcal{S}(Y^{11})) - \int f^* \left(T\hat{A}(\nabla_{g_Y} + \nabla_0) - \frac{1}{27 \cdot 31} TL(\nabla_{g_Y} - \nabla_0) \right) \in \mathbb{Q}/\mathbb{Z}, \quad (3.19)$$

where ∇_0 is a trivial connection on TY^{11} . This is analogous to similar discussions on framing in [28]. In this case of stably parallelizable manifolds, the phase of the partition function is given by the Eells-Kuiper invariant. Since $ek(Y^{11})$ classifies topological eleven-spheres, then

Observation 3 *The topological action in M-theory classifies topological eleven-spheres.*

This is related to the global gravitational anomalies of [35] although M-theory is not chiral.

A generalization of the Kreck-Stolz invariant? We define a new expression which, in addition to dependence on the metric and Spin structure, depends also on a degree four cohomology class a . Recall that in the M-theory expression (3.9), which led to the Kreck-Stolz s -invariant, we assumed that the E_8 bundle in eleven and twelve dimensions is trivial, that is its degree four characteristic class a is zero. Note that $BE_8 \sim K(\mathbb{Z}, 4)$ in our range of dimensions so that a can take on any value. However, the action in M-theory involves an E_8 bundle which is in general not trivial. Therefore, we would like to consider the effect of including this class, together with the geometry. Assuming as in [19] that the real Pontrjagin classes vanish, implies in particular that the first Pontrjagin class appearing in the flux quantization condition of [36], $G_4 - \frac{1}{4}p_1 = a \in H^4(Y^{11}; \mathbb{Z})$, is absent so that allowing a degree four class is essentially the same as allowing a C-field through its field strength G_4 , at least rationally away from torsion. The inclusion of the nontrivial class a leads to a contribution of the corresponding Pontrjagin character of the E_8 bundle E , $\text{Ph}(E) = 248 + 60a + 6a^2 + \frac{1}{3}a^3$. Thus, we have

$$\text{Definition 4 } s(Y^{11}, g_Y, a) = s(Y^{11}, g_Y) + d^{-1} \left(\frac{1}{3}a^3 + 6a^2\hat{A}_4 + 60a\hat{A}_2 \right).$$

We propose this as the geometric invariant when a nontrivial C-field is present. While we have written this invariant in eleven dimensions (as relevant for M-theory), the extension to other dimensions is obvious from our use of the index theorem. It would be interesting to work this out explicitly.

4 Comparison to type IIA string theory in ten dimensions

In this section we relate the expressions we considered above in section 3, for M-theory in eleven and (extension to) twelve dimensions, to corresponding ones in string theory in ten dimensions. This comparison requires Y^{11} to be a circle bundle.

4.1 The s -invariant for Y^{11} a circle bundle

Consider Y^{11} to be the principal circle bundle $S^1 \rightarrow Y^{11} \xrightarrow{\pi} X^{10}$ with positive scalar curvature metric g_Y , as considered in [29]. Corresponding to the circle bundle is a complex line bundle \mathcal{L} with first Chern class $c = c_1(\mathcal{L})$. The tangent bundle splits as $TY^{11} \cong \pi^*(TX^{10}) \oplus T_F Y^{11}$, where the tangent bundle along the fibers $T_F Y^{11}$ is trivial, with a trivialization provided by the vector field generating the S^1 -action on Y^{11} . Since we are assuming $p_i(Y^{11}) = 0 \in H^{4i}(Y^{11}; \mathbb{R})$, $i = 1, 2$, the splitting of the tangent bundle implies that $\pi^*(p_i(X^{10})) = p_i(Y^{11}) = 0$. Then the Gysin exact sequence

$$\dots \longrightarrow H^{4i-2}(X^{10}; \mathbb{R}) \xrightarrow{\cup c} H^{4i}(X^{10}; \mathbb{R}) \xrightarrow{\pi^*} H^{4i}(Y^{11}; \mathbb{R}) \longrightarrow \dots \quad (4.1)$$

relates the fields on Y^{11} to the fields on X^{10} (see [21]) and in our case shows that $p_i(X^{10})$ is divisible by c . That is, there are elements (in the notation of [19]) $\bar{p}_i \in H^{4i-2}(X^{10}; \mathbb{R})$ such that $p_i(X^{10}) = \bar{p}_i c$, $i = 1, 2$. This gives that the Pontrjagin classes of X^{10} are zero when the Chern class of the line bundle is zero; otherwise they are in general not zero.

The bilinear form. As above, consider a line bundle \mathcal{L} with Euler class $e = c(\mathcal{L}) \in H^2(X^{10})$ over a base manifold X^{10} . There is a natural symmetric bilinear form on the degree four cohomology

$$B_c : H^4(X^{10}; \mathbb{Z}) \times H^4(X^{10}; \mathbb{Z}) \rightarrow \mathbb{R} \quad (4.2)$$

defined by $B_c(a, b) := \langle a \cup b \cup c(\mathcal{L}), [X^{10}] \rangle$, where $[X^{10}]$ is the fundamental homology class of $[X^{10}]$. This bilinear form is part of the expression for the phase in the Spin^c case (see [29]). The Thom isomorphism theorem implies that

$$\text{sign}(\mathbb{D}(\mathcal{L})) = \text{signature of } B_c, \quad (4.3)$$

where $\pi_D : \mathbb{D}(\mathcal{L}) = Y^{11} \times_{S^1} \mathbb{D}^2 \rightarrow X^{10}$ is the disk bundle associated to the S^1 -action on Y^{11} with orbit manifold $X^{10} = Y^{11}/S^1$. In the general case when the Pontrjagin classes of Y^{11} are not required to vanish, the obstruction to expressing the signature of the disk bundle $\mathbb{D}(\mathcal{L})$ as an evaluation of a characteristic class on X^{10} is the limiting eta-invariant [32].

Dependence of the $s(Y^{11}, g_Y)$ on the Spin structure on Y^{11} . There are two cases to consider:

1. X^{10} is *Spin*: In this case $w_2(X^{10}) = 0 \in H^2(X^{10}; \mathbb{Z}_2)$. By the decomposition of the tangent bundle of Y^{11} we see that a Spin structure on X^{10} induces a Spin structure on Y^{11} , which we denote by ξ .
2. X^{10} is *Spin^c*: In this case $w_2(X^{10}) = c \pmod{2}$. Then $TX^{10} \oplus \mathcal{L}$ admits a Spin structure. The choice of such a Spin structure gives a Spin structure on the disk bundle $\mathbb{D}(\mathcal{L})$ whose restriction to the sphere bundle $S\mathcal{L} = Y^{11}$ is a Spin structure ξ' .

In the Spin case, i.e. when $w_2(X^{10}) = 0$ and $c = 0 \pmod{2}$, we have that ξ and ξ' are different Spin structures on Y^{11} , since the restriction of ξ to a fiber S^1 is the nontrivial Spin structure, which does not extend over the 2-disk \mathbb{D}^2 , whereas the restriction of ξ' extends by construction. This is again discussed more fully in [29].

The s -invariant for the case when X^{10} is Spin^c. The disk bundle $\mathbb{D}(\mathcal{L})$ is a twelve-manifold with boundary Y^{11} and induced Spin structure ξ' on that boundary. Then the index $\text{index}(D^+(\mathbb{D}(\mathcal{L}), g_{\mathbb{D}(\mathcal{L})})) = 0$ for any metric $g_{\mathbb{D}(\mathcal{L})}$ on $\mathbb{D}(\mathcal{L})$ which restricts to g_Y on the boundary and is a product metric in a collar neighborhood of the boundary. Now for a line bundle \mathcal{L} of Chern class c , the genera are given by [14]

$$\widehat{A}(\mathcal{L}) = \frac{c}{2\sinh(c/2)}, \quad L(\mathcal{L}) = \frac{c}{\tanh(c)}, \quad (4.4)$$

so that the s -invariant, using [19], is

$$s(Y^{11}, \xi', g_Y) = \left\langle \widehat{A}(TX^{10}) \frac{1}{2\sinh(c/2)} + \frac{1}{2^7 \cdot 31} L(TX^{10}) \frac{1}{\tanh(c)}, [X^{10}] \right\rangle + \frac{1}{2^7 \cdot 31} \text{sign}(B_c) \quad (4.5)$$

The s -invariant for the case when X^{10} is Spin. The general discussion is more difficult since there is no obvious Spin twelve-manifold Z^{12} bounding Y^{11} with the Spin structure ξ . It is also very difficult to compute the index $\text{index}(D^+(Z^{12}, g_Z))$. However, as argued more generally in [19], when g_Z has a metric of positive scalar curvature then the index is zero, in which case the s -invariant is given by

$$s(Y^{11}, \xi, g_Y) = \left\langle \widehat{A}(TX^{10}) \frac{1}{2\sinh(c/2)} + \frac{1}{2^7 \cdot 31} L(TX^{10}) \frac{1}{\tanh(c)}, [X^{10}] \right\rangle + \frac{1}{2^7 \cdot 31} \text{sign}(B_c). \quad (4.6)$$

The s -invariant can be related to the eta-invariants in the adiabatic limit as follows [6]. Let $\pi : E \rightarrow X^{10}$ be an oriented 2-dimensional real vector bundle over X^{10} and g_E a fiber metric on E with a compatible connection ∇^E . Let Y^{11} be the unit sphere bundle of E with the induced metric g_Y , so that Y^{11} is a circle

bundle over X^{10} with an induced Spin structure ξ . For $\epsilon > 0$ consider the metric $g_\epsilon = g_\epsilon^Y = g_E \oplus \pi^*(\frac{1}{\epsilon}g_X)$. Taking the adiabatic limit, $\epsilon \rightarrow 0$, gives

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2} \eta(D(Y^{11}, g_\epsilon^Y)) &= - \left\langle \widehat{A}(TX^{10}) \left(\frac{1}{e} - \frac{1}{2 \tanh(e/2)} \right), [X^{10}] \right\rangle, \\ \lim_{\epsilon \rightarrow 0} \eta(\mathcal{S}(Y^{11}, g_\epsilon^Y)) &= - \left\langle L(TX^{10}) \left(\frac{1}{\tanh(e)} - \frac{1}{e} \right), [X^{10}] \right\rangle - \text{sign}(B_e). \end{aligned}$$

Combining the two gives the expression (4.6) of the s -invariant. Trading the L-genus with the Rarita-Schwinger index gives back the expressions derived in [21] [26] [29], where the dimensional reduction to type IIA string theory is first interpreted via the adiabatic limit.

4.2 Disk bundles and the secondary correction term

We consider the general case when the twelve-manifold no longer has a product metric near the boundary, which is a departure from the set-up of APS [3]. Assume then that (Y^{11}, g_Y) bounds a (general) twelve-dimensional Riemannian manifold (Z^{12}, g_Z) . Let $N^{12} = Y^{11} \times [0, 1]$ with product metric g_0 , and extend g_Z smoothly to a metric g_1 on $Z^{12} \cup N^{12}$ in such a way that g_1 is a product metric near $Y^{11} \times \{1\}$. Then, from [12], the signature of Z^{12} is given by

$$\text{sign}(Z^{12}) = \int_{Z^{12}} L_{12}(p_i(g_Z)) + \int_{Y^{11}} TL_{12}(g_0, g_1) - \eta(Y^{11}), \quad (4.7)$$

where the boundary correction term TL_k is the secondary characteristic L_k -class corresponding to the Levi-Civita connections of g_0 and g_1 . Let h be a fiber metric on the line bundle \mathcal{L} and $\nabla^{\mathcal{L}}$ connection on \mathcal{L} compatible with h , so that $g_{\mathcal{L}} = h + \pi^*(g)$ is an induced natural Riemannian metric on the total space \mathcal{L} . Let $S_r(\mathcal{L})$ be the circle bundle of radius r corresponding to the bundle \mathcal{L} . Consider two concentric disk bundles $\mathbb{D}_\rho(\mathcal{L})$ and $\mathbb{D}_r(\mathcal{L})$ where $\rho < r$. Let $g_0(\rho, r)$ be the product metric on the annulus $N^{12} = S_r(\mathcal{L}) \times [\rho, r]$. Extend the metric $g_{\mathcal{L}}$ on $\mathbb{D}_\rho(\mathcal{L})$ to a metric $g_1(\rho, r)$ on $\mathbb{D}_r(\mathcal{L})$ in such a way that $g_1(\rho, r) = g_0(\rho, r)$ near the boundary $S_r(\mathcal{L})$. Let ∇^r be the Levi-Civita connection for the product metric g_0 on $\mathbb{D}_r(\mathcal{L}) - \{0\}$ and let $\alpha(r)$ be the corresponding connection form. Let ∇^ρ be the Levi-Civita connection of $g_{\mathcal{L}}|_{\mathbb{D}_\rho(\mathcal{L})}$ and $\beta(\rho)$ the corresponding connection form. Let $\theta = \beta(r) - \alpha(\rho)$ and consider $\Omega_t = \Omega_t(\rho, r)$, the curvature of the connection $(1-t)\nabla^r + t\nabla^\rho$. The secondary characteristic L_{12} -class is defined in [32] by

$$TL_{12}(g_0, g_1) := 6 \int_0^1 L_{12}(\theta, \Omega_t, \dots, \Omega_t) dt. \quad (4.8)$$

Then the signature of the disk bundle is

$$\text{sign}(\mathbb{D}_r(\mathcal{L})) = \int_{\mathbb{D}_r(\mathcal{L})} L_{12}(g_1) + \int_{S_r(\mathcal{L})} TL_{12}(g_0, g_1) - \eta(S_r(\mathcal{L})), \quad (4.9)$$

The first term on the right hand side goes to zero as $\rho \rightarrow 0$, so that as in [32] [17]

$$\text{sign}(\mathbb{D}_r(\mathcal{L})) = \int_{S(\mathcal{L})} \lim_{\rho \rightarrow 0} TL_{12}(g_0, g_1) - \eta(S_r(\mathcal{L})). \quad (4.10)$$

Let V be a real vector bundle of rank 2 over a compact oriented Riemannian ten-manifold. Then, using [32], the signature of the disk bundle $\mathbb{D}(V)$ is

$$\text{sign}(\mathbb{D}(V)) = \int_{X^{10}} L_{12}(V, X^{10}) - \lim_{r \rightarrow 0} \eta(S_r(V)), \quad (4.11)$$

where $\eta(S_r(V))$ is the eta-invariant of the circle bundle of radius r and $L_{12}(V, X^{10})$ is the characteristic polynomial of degree ten which is expressed explicitly in terms of the coefficients of the Hirzebruch L_{12} polynomial, the Euler class $e(V)$ and the Pontrjagin classes $p_i(X^{10})$

$$L_{12}(V, X^{10}) = \frac{1}{3^3 \cdot 5 \cdot 7} [8e(V)^5 - 14e(V)^3 p_1(X^{10}) + 49e(V) p_2(X^{10}) - 7e(V) p_1(X^{10})^2] . \quad (4.12)$$

Expression (4.11) shows that $\eta(S_r(V))$ is in general not topological, and the limiting eta-invariant needs to be included in the expression of the phase when considering disk bundles. The geometric correction (4.12) occur because we are considering nontrivial circle bundles. Otherwise, when $e(V) = 0$ we have $L_{12}(V, X^{10}) = 0$. Also, the expression (4.12) would simplify depending on the values of $p_1(X^{10})$ and $p_2(X^{10})$, which unlike the ones for Y^{11} , we are not assuming to vanish. In the case of disk bundles, expression (4.12) is the result in ten dimensions of the corresponding expression for the L-genus in twelve dimensions, used in our main discussions in section 3.

Example: Hopf bundle over $\mathbb{C}P^5$. Consider type IIA string theory on the complex projective space $\mathbb{C}P^5$. For the canonical line bundle γ over $\mathbb{C}P^5$, the characteristic polynomial is, from [32],

$$L_{12}(\gamma, \mathbb{C}P^5) = \frac{1}{3^3 \cdot 5 \cdot 7} [8c_1(\gamma)^5 + 14c_1(\gamma)^3 c_2(\mathbb{C}P^5) + 49c_1(\gamma) c_4(\mathbb{C}P^5) - 7c_1(\gamma) c_2(\mathbb{C}P^5)^2] . \quad (4.13)$$

Integrating gives $\int_{\mathbb{C}P^5} L_{12}(\gamma, \mathbb{C}P^5) = -\frac{2^4 \cdot 53}{3^3 \cdot 5 \cdot 7}$. Now with $\text{sign}(\mathbb{D}(\gamma)) = \text{sign}(\mathbb{C}P^5) = 0$, expression (4.11) gives the value for the limiting eta-invariant $\lim_{r \rightarrow 0} \eta(S_r(\gamma)) = -\frac{2^4 \cdot 53}{3^3 \cdot 5 \cdot 7}$.

We hope to make further use of the appearance of the signature and geometric invariants in M-theory in the near future.

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