A SIMPLER PROOF OF THE BOROS–FÜREDI–BÁRÁNY–PACH–GROMOV THEOREM

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Abstract. A short and almost elementary proof of the Boros–Füredi–Bárány–Pach–Gromov theorem on the multiplicity of covering by simplices in $\mathbb{R}^d$ is given.

Let us give a proof of the Boros–Füredi–Bárány–Pach–Gromov theorem [1, 4, 7, 6] that is actually the "decoded" and refined proof from [6] (see also [5, Section 2] for a similar proof in the two-dimensional case). Unlike the proof in [6], the only topological notion that is used here is the degree of a piece-wise smooth map.

Consider a set of $d+1$ probabilistic absolutely continuous measures $\mu_0, \mu_1, \dots, \mu_d$ on $\mathbb{R}^d$. Define a *random simplex* of dimension k as a simplex spanned by $k+1$ points $x_{d-k}, \dots, x_d \in \mathbb{R}^d$, where the point x_i is distributed according to the measure μ_i.

Theorem 1. *Under the above assumptions there exists a point $c \in \mathbb{R}^d$ such that the probability for a random d-simplex to contain c is*

$$\geq p_d = \frac{1}{(d+1)!}.$$

Note that in [6] a stronger result is proved: the maps $\Delta^N \to Y$ of a simplex with measure to a smooth manifold were considered. Here we give the statement of Theorem 1 that is closer to the original theorems in [1, 4, 7].

Proof of Theorem 1. Assume the contrary. Take some small $\varepsilon > 0$. Consider a fine enough triangulation Y of $\mathbb{R}^d$ so that for any $0 < k \leq d$ and any k-face σ of Y the probability of a random $(d-k)$-simplex $x_k x_{k+1} \dots x_d$ to intersect σ is $< \varepsilon$. Here and below we always assume that μ_i is the distribution of x_i. Such a triangulation exists because the measures μ_i are absolutely continuous. The absolute continuity is essentially needed here.

Consider a $(d+1)$-dimensional simplicial complex $Y * 0$ (the cone over Y with apex 0). We assume that $\mathbb{R}^d$ is contained in its one-point compactification $S^d = \mathbb{R}^d \cup \{0\}$ (note that 0 is used in a non-standard way). We also assume that $Y \cup \{0\}$ is a finite triangulation of S^d. Now we are going to build a (piece-wise smooth) map $f : (Y * 0)^{(d)} \to S^d$ (from the d-skeleton) which is "economical" with respect to the measures μ_i (this phrase will be clarified below), and coincides with the identification $Y = \mathbb{R}^d$ on $Y \subset (Y * 0)^{(d)}$.

Proceed by induction:

- Map 0 to $0 \in S^d$;
- For any vertex $v \in Y$ map $[v0]$ to an open ray starting from v (and ending at $0 \in S^d$) so that the probability for a random $(d-1)$-simplex $x_1 \dots x_d$ to meet $f([v0])$ is $< p_d$. This is possible because a simplex $x_0 x_1 \dots x_d$ contains v iff the $(d-1)$-simplex $x_1 \dots x_d$ intersects the ray from v opposite to $x_0 - v$. Since the

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probability for a random d-simplex to contain v is $< p_d$, for some of such rays the corresponding probability is also $< p_d$.

- Step to the k-skeleton of $Y * 0$ as follows. Let $\sigma = v_1 \dots v_k 0$ be a k-simplex of $Y * 0$. The map f is already defined for $\partial\sigma$. We know that the probability for a random $(d-k+1)$-simplex $x_{k+1} \dots x_d$ to meet some $f(v_1 \dots \hat{v}_i \dots v_k 0)$ $(i = 1, \dots, k)$ is $< (k-1)!p_d$, and the probability to meet $f(v_1 \dots v_k)$ is $< \varepsilon$. If ε is chosen small enough we see that a random $(d-k+1)$-simplex $x_{k+1} \dots x_d$ intersects $f(\partial\sigma)$ with probability $< k!p_d$. There exist a point x_k outside $f(\partial\sigma)$ such that the probability for $x_k x_{k+1} \dots x_d$ (with random last $d-k$ points) to meet $f(\partial\sigma)$ is $< k!p_d$. Let us define the map f on the simplex σ treated as a join $\partial\sigma * c$ so that c is mapped to $0 \in S^d$, and every segment $[vc]$ $(v \in \partial\sigma)$ is mapped to the infinite ray from $f(v)$ in the direction opposite to $x_k - v$. More explicitly: map $[vc]$ to $[f(v), x_k]$ first; then apply the inversion with center x_k and radius $|x_k - f(v)|$ that maps $[f(v), x_k]$ to $[f(v), 0]$; if $f(v) = 0$ then map $[vc]$ to the point $0 \in S^d$. Now the probability for a random $(d-k)$-simplex to intersect $f(\sigma)$ is $< k!p_d$.

Finally for any d-simplex σ of Y we have that the boundary of the cone $\sigma * 0$ is mapped so that

$$\mu_d(f(\partial(\sigma * 0))) < (d+1)!p_d = 1,$$

if we again use small enough ε. Therefore $f(\partial(\sigma * 0)) \neq S^d$ and the restriction $f|_{\partial(\sigma*0)}$ has zero degree. By summing up the degrees (the d-faces of $(\partial\sigma) * 0$ go pairwise and cancel, because Y is a triangulation) we see that the map $f|Y$ has even degree but it is the identity map, which is a contradiction. □

This theorem can be sharpened (following [6]) if two of the measures coincide.

Theorem 2. *If some two measures coincide then the bound in Theorem 1 can be improved to*

$$p'_d = \frac{2d}{(d+1)!(d+1)}.$$

Proof. Assume $\mu_{d-1} = \mu_d$. We proceed in the same way building $f : (Y * 0)^{(d)} \to \mathbb{R}^d$, but we slightly change the construction on the last step.

On the last step we have a $(d-1)$-simplex σ of Y, and f is already defined for $\partial(\sigma * 0)$ so that the probability for a random segment $[x_{d-1}x_d]$ to intersect $D = f(\partial(\sigma * 0))$ is $< d!p'_d = \frac{2d}{(d+1)^2}$.

We are going to extend f to $\sigma * 0$ so that its image $f(\sigma * 0)$ mod 2 has measure $< \frac{1}{d+1}$. Here the image mod 2 is the set of points in $\mathbb{R}^d$ that are covered by $f(\sigma * 0)$ odd number of times. We have noted in the proof of Theorem 1 that we essentially use the covering parity at the final degree reasoning.

It can be easily seen that D "partitions" $\mathbb{R}^d$ into two parts A and B characterized by the following property: any generic piece-wise linear path from A to B meets D odd number of times, and any generic piece-wise linear path with both ends in A (or both in B) meets D even number of times. The sets A and B are the only possibilities of image of $f(\sigma * 0)$ mod 2, because the covering parity of $f|_{\sigma*0}$ changes only at crossing with $f(\partial(\sigma*0)) = D$.

If $\mu_d(A) = x$ and $\mu_d(B) = 1 - x$ then the probability for a random segment $[x_{d-1}x_d]$ (recall that $\mu_{d-1} = \mu_d$) to meet D is at lest $2x(1-x)$, that is

$$\frac{2d}{(d+1)^2} > 2x(1-x).$$

It follows easily that in this case either x or $1-x$ is $< \frac{1}{d+1}$ and we can define f as required again. □

Remark. Unlike the approach here, the previous papers [1, 4, 7, 6] mostly considered discrete measures concentrated on finite point sets in $\mathbb{R}^d$. In this case Theorems 1 and 2 hold, because we may approximate a discrete measure by an absolutely continuous measure, distributed on a set of δ-balls with centers at the original concentration points. After going to the limit $\delta \to 0$ we may also assume by the standard compactness reasoning that the centers c_δ also tend to some point c. Then a simple argument shows that c is the required point for the original discrete measure.

Remark. Imre Bárány has noted that Theorem 1 implies the colorful Tverberg theorem[0] [2, 8, 3] with a bad bound $T(r, d)$ of order

$$T(r, d) \sim \frac{r}{1 - (1 - p_d)^{1/(d+1)}} \sim r(d + 1)!(d + 1).$$

Of course, this bound is much worse that the known other bounds (the optimal bounds are in [3] and have order r), but unlike the previous known proofs this proof uses very little topology.

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[0] Given a family of $(d + 1)T(r, d)$ points in $\mathbb{R}^d$ colored into $d + 1$ colors each containing $T(r, d)$ points, there exist r disjoint "rainbow" $(d + 1)$-tuples of points such that the corresponding r convex hulls of the $(d + 1)$-tuples have a common point.