On sub-ideal causal smoothing filters

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Abstract

Smoothing filters are studied for continuous time processes. The consideration is restricted by the linear causal filters represented as convolution integrals over the historical data. This filters are used in dynamic smoothing when the future values of the process are not available. The paper suggests a family of sub-ideal smoothing filters with almost exponential dumping of the energy on the higher frequencies.

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1 Introduction

The paper studies dynamic smoothing filters of continuous time processes. The consideration is restricted by the linear causal filters represented as convolution integrals over the historical data. This filters are used in dynamic smoothing, when future values of the process are not available.

In frequency domain, smoothing means reduction of the energy on higher frequencies. In particular, an ideal low-pass filter is a smoothing filter. This filter is not causal, i.e., it requires the future value of the process.

One of possible application of dynamic smoothing is forecasting. A typical approach is to smooth the original process and forecast the resulting smooth process. This approach explores the fact that sufficient rate of decay of energy on higher frequencies ensures some opportunities for prediction and interpolation of the processes. The classical result is Nyquist-Shannon-Kotelnikov interpolation theorem that implies that if a process $y(t)$ is band-limited then it is predictable (see, e.g., [1]-[3], [8]-[12], [16]-[18]). Therefore, an ideal low-pass filter cannot be causal. Moreover, a filter with exponential dumping of the energy on the high frequencies cannot be causal, similarly to the low-pass filter. It is not surprising since a general process cannot be transformed to a predictable process by a causal filter, and a process with exponential decay of energy on high frequencies is weakly predictable on a finite time horizon [6].

We suggest a family of causal smoothing filters with "almost" exponential rate of dumping the energy on the higher frequencies. This leads to a sub-ideal filter in the sense that the effectiveness of the damping of higher frequencies cannot be increased; any faster decay of the transfer function would lead to the loss of causality. In addition, the effectiveness of smoothing of a given reference set of non-causal filters (2.1) can be approximated by this family of causal filters.

2 Problem setting

Let $x(t)$ be a continuous time process, $t \in \mathbf{R}$. The output of a linear filter is the process

$$y(t) = \int_{-\infty}^{\infty} k(t-s)x(s)ds,$$

where $k(\cdot) : \mathbf{R} \to \mathbf{R}$ is a given kernel.

If $k(t) = 0$ for $t < 0$, then the output of the corresponding filter is

$$y(t) = \int_{-\infty}^{t} k(t-s)x(s)ds.$$

In this case, the filter and the kernel are said to be causal. The output of a causal filter at time t can be calculated using only past historical values $x(s)|_{s \le t}$ of the currently observable continuous input process.

The goal is to approximate $x(t)$ by a smooth filtered process $y(t)$ via selection of an appropriate smoothing causal kernel $k(\cdot)$.

We are looking for families of the causal smoothing kernels $k(\cdot)$ with the following properties:

(A) The outputs $y(t)$ approximate processes $x(t)$; the arbitrarily close approximation can be achieved by selection of an appropriate kernel from the family.

(B) For processes $x(t) \in L_2(\mathbf{R})$, the outputs $y(t)$ are infinitely differentiable functions. The values of the transfer function of the filter are as small as possible on the higher frequencies, to achieve the most effective damping of the energy on the higher frequencies of x.

(C) The effectiveness of the damping of higher frequencies achieved for this family cannot be exceeded; any faster decay of the value of the transfer function would lead to the loss of causality.

(D) The family of these causal kernels ensures effectiveness of the smoothing that approximate the effectiveness of some reasonable reference family of non-causal smoothing filters.

Note that it is not a trivial task to satisfy Conditions (C)-(D). For instance, consider a family of low-pass filters with increasing pass interval $[-\Delta,\Delta]$, where $\Delta>1$. Clearly, the corresponding smoothed processes approximate the original process as $\Delta\to+\infty$, i.e., Condition (A) is satisfied. However, the distance of the set of these ideal low-pass filters from the set of all causal filters is positive (see [1]).

For $x\in L_2(\mathbf{R})$, we denote by $X=\mathcal{F}x$ the function defined on $i\mathbf{R}$ as the Fourier transform of x;

$$X(i\omega)=(\mathcal{F}x)(i\omega)=\int_{-\infty}^{\infty}e^{-i\omega t}x(t)dt,\quad \omega\in\mathbf{R}.$$

Here $i=\sqrt{-1}$. For $x\in L_2(\mathbf{R})$, then X is defined as an element of $L_2(\mathbf{R})$ (more precisely, $X(i\cdot)\in L_2(\mathbf{R})$).

Consider a reference family of smoothing filters with transfer functions (i.e., Fourier transforms of the kernels)

$$M_\mu(i\omega)=e^{-\mu|\omega|},\quad \mu\in(0,1]. \tag{2.1}$$

These kernel are non-causal; at time t, the output processes of these filters are weakly predictable with finite horizon $[t,t+\mu)$ [6].

Let us show that requirements (A),(B),(C), and (D), will be satisfied for the family of filters with causal kernels $\{k_\nu(\cdot)\}\subset L_2(\mathbf{R})$ such that $k_\nu(t)=0$ for $t<0$, where $\nu=1,2,...,\ \nu\to+\infty$, and with the corresponding Fourier transforms $K_\nu(i\omega)$, with the following properties:

(a) *Approximation property:*

(a1) For any $\Omega>0$, $K_\nu(i\omega)\to1$ as $\nu\to+\infty$ uniformly in $\omega\in[-\Omega,\Omega]$.

(a2) For any $x(\cdot)\in L_2(\mathbf{R})$,

$$\|y_\nu(\cdot)-x(\cdot)\|_{L_2(\mathbf{R})}\to0\quad\text{as}\quad\nu\to\infty,$$

where y_ν is the smoothed process

$$y_\nu(t)=\int_{-\infty}^{t}k_\nu(t-s)x(s)ds.$$

(b) *Smoothing property:* For every $\nu > 0$, there exists $\rho > 0$ such that for any $n \geq 1$,

$$\int_{-\infty}^{\infty} e^{|\omega|^\rho} |K_\nu(i\omega)|^n d\omega < +\infty.$$

(c) *Sub-ideal smoothing:* For any $\delta > 1$, there exists $\nu > 0$ such that

$$\int_{-\infty}^{\infty} \frac{|\log |K_\nu(i\omega)||^\delta}{1+\omega^2} d\omega = +\infty. \tag{2.2}$$

(d) *Approximation of effectiveness of non-causal filters (2.1):* For any $\varepsilon > 0$ and $\delta > 0$, there exists $\nu = \nu(\delta) > 0$ such that

$$\| |K_\nu(i\omega)| - |M_\delta(i\omega)| \|_{L_r(\mathbf{R})} \leq \varepsilon.$$

Let us show that that Conditions (a)-(d) ensures that Condition (A)-(D) are satisfied, in a certain sense.

Clearly, Condition (a) ensures that Condition (A) is satisfied in a certain sense.

Let us show that Condition (b) ensures that Condition (B) is satisfied. It can be seen from the following: for any $k > 0$ and $\nu > 0$,

$$\int_{-\infty}^{\infty} (1 + |\omega|^k)^4 |K_\nu(i\omega)|^4 d\omega < +\infty.$$

Let $x(\cdot) \in L_2(\mathbf{R})$, $X = \mathcal{F}X$, and $Y_\nu(i\omega) = K_\nu(i\omega)X(i\omega)$. By Hölder inequality, it follows that

$$\int_{-\infty}^{\infty} (1 + |\omega|^k)^2 |Y_\nu(i\omega)|^2 d\omega \leq \left(\int_{-\infty}^{\infty} (1 + |\omega|^k)^4 |K_\nu(i\omega)|^4 d\omega \right)^{1/2} \|X\|_{L_2(\mathbf{R})}^{1/2} < +\infty.$$

Hence $y_\nu(t)$ has derivatives in $L_2(\mathbf{R})$ of any order, and, therefore, is infinitely differentiable in the classical sense.

Let us show that Condition (c) ensures that Condition (C) is satisfied. Let $\delta > 1$ be fixed, and let $m = m(\delta)$ be such that (2.2) holds. Let us show that the kernel $\widetilde{k} = \mathcal{F}^{-1}\widetilde{K}$ cannot be causal for some "better" transfer function $\widetilde{K}(i\omega)$ such that

$$|\widetilde{K}(i\omega)| = o(|K_\nu(i\omega)|) \quad \text{as} \quad \omega \to +\infty. \tag{2.3}$$

More precisely, we will show that $\widehat{k}_\nu$ cannot be causal with a stronger condition that there exist $\delta > 1$ and $\Omega_1 > 0$ such that

$$|\log |\widetilde{K}(i\omega)|| \geq |\log |\widetilde{K}(i\omega)||^\delta, \quad \omega \in [-\Omega_1, \Omega_2]. \tag{2.4}$$

In particular, this condition implies that $\log |K_\nu(i\omega)| / \log |\widetilde{K}(i\omega)| \to 0$ as $\omega \to +\infty$.

The desired fact that $\widetilde{k}$ cannot be causal can be seen from the following: by Paley and Wiener Theorem [14], the Fourier transform $K(i\omega)$ of a causal kernel $k(\cdot) \in L_2(\mathbf{R})$ have to be such that

$$\int_{-\infty}^{\infty} \frac{|\log|K(i\omega)||}{1+\omega^2} d\omega < +\infty.$$

(see, e.g., [13], p. 35). Since $\nu = \nu(\delta)$ is such that (2.2) holds, it follows from (2.4) that $\widehat{k}$ cannot be causal. Therefore, Condition (c) ensures that Condition (C) is satisfied.

Finally, Condition (d) ensures that Condition (D) is satisfied, since the effectiveness of smoothing is defined by the rate of decay of the energy on the higher frequencies.

3 A family of smoothing kernels

Let us consider a set of kernels

$$K_{\alpha,\beta,q}(p) \triangleq e^{-\alpha(p+\beta)^q}, \quad p \in \mathbf{C}^+. \tag{3.1}$$

Here $\alpha > 0$, $\beta > 0$, and $q \in (\frac{1}{2}, 1)$, are rational numbers. We mean the branch of $(p+\beta)^q$ such that its argument is $q\text{Arg}\,(p+\beta)$, where $\text{Arg}\,z \in (-\pi, \pi]$ denotes the principal value of the argument of $z \in \mathbf{C}$. This set was introduced in [4] as am auxiliary tool for solution of a parabolic equation in frequency domain.

Theorem 1 *Conditions (a)-(d) are satisfied for the countable family of kernels (3.1) with rational numbers α, $\beta = 1/m$, $q = 1 - 1/n$, where $m \to +\infty$ and $n \to +\infty$, and where m and n are integers. We assume that this family of kernels is such that it can be counted as a sequence $\{K_\nu\}_{\nu=1}^{\infty}$ such that $\alpha \to 0$ and $n \to +\infty$ as $\nu \to +\infty$.*

Note that we have to include kernels with all positive rational numbers α to ensure that Condition (d) is satisfied.

Proof of Theorem 1. Let $\mathbf{C}^+ \triangleq \{z \in \mathbf{C} : \ \text{Re}\, z > 0\}$. Let H^r be the Hardy space of holomorphic on $\mathbf{C}^+$ functions $h(p)$ with finite norm $\|h\|_{H^r} = \sup_{s>0} \|h(s+i\omega)\|_{L_r(\mathbf{R})}$, $r \in [1, +\infty]$ (see, e.g., [7]).

The functions $K_\nu(p)$ are holomorphic in $\mathbf{C}^+$, and

$$\ln|K_\nu(p)| = -\text{Re}\,(\alpha(p+\beta)^q) = -\alpha|p+\beta|^q \cos[q\text{Arg}\,(p+\beta)]. \tag{3.2}$$

In addition, there exists $M = M(\beta, q) > 0$ such that $\cos[q\text{Arg}\,(p+\beta)] > M$ for all $p \in \mathbf{C}^+$. It follows that

$$|K_\nu(p)| \le e^{-\alpha M |p+\beta|^q} < 1, \quad p \in \mathbf{C}^+. \tag{3.3}$$

Hence $K_\nu \in H^r$ for all $r \in [1, +\infty]$. By Paley-Wiener Theorem, $k_\nu = \mathcal{F}^{-1} K_\nu(i\omega)$ are causal kernels, i.e., $k_\nu(t) = 0$ for $t < 0$ (see, e.g., [19], p.163).

Let $x \in L_2(\mathbf{R})$, $X = \mathcal{F}x$, and $Y_\nu = K_\nu X$.

Let us show that Condition (a) holds. Since $\alpha \to 0$ as $\nu \to +\infty$, it follows that $K_\nu(i\omega) \to 1$ as $\nu \to +\infty$ for any ω. It follows that Condition (a1) holds. By Condition (a1), $Y_\nu(i\omega) \to X(i\omega)$ as $\nu \to +\infty$ for all $\omega \in \mathbf{R}$. In addition, $|K_\nu(i\omega)| \leq 1$. Hence $|Y_\nu(i\omega) - X(i\omega)| \leq 2|X(i\omega)|$. We have that $\|X(i\omega)\|_{L_2(\mathbf{R})} = \|x\|_{L_2(\mathbf{R})} < +\infty$. By Lebesgue Dominance Theorem, it follows that

$$\|Y_\nu(i\omega) - X(i\omega)\|_{L_2(\mathbf{R})} \to 0 \quad \text{as} \quad \nu \to +\infty.$$

Therefore, Condition (a) holds.

Let us show that Condition (b) holds. By (3.3), it follows that

$$|K(i\omega)| \leq e^{-\alpha M|\omega|^q}, \quad \omega \in \mathbf{R}. \tag{3.4}$$

Therefore, Condition (b) holds with $\delta < q$.

To see that Condition (c) holds, it suffices to observe that (2.2) holds if $\delta q > 1$, i.e., $q > 1/\delta$.

Let us show that Condition (d) holds. We assume that the family of kernels of $K_{\alpha,\beta,q}(\cdot)$ is counted as a sequence $\{K_\lambda\}_{\lambda=1}^{\infty}$ such that $\alpha \to \mu$, $\beta \to 0$, $q \to 1$ as $\lambda \to +\infty$. By (3.2), $|K_\lambda(i\omega)| \to |M_\mu(i\omega)|$ as $q \to 1 - 0$ for all $\omega \in \mathbf{R}$. In addition, $|K_\lambda(i\omega)| + |M_\alpha(i\omega)| \leq 2e^{-\min(\mu,\alpha)}$. By Lebesgue Dominance Theorem, it follows that

$$\|K_\lambda(i\omega) - M_\mu(i\omega)\|_{L_2(\mathbf{R})} \to 0 \quad \text{as} \quad \nu \to +\infty.$$

This completes the proof of Theorem 1. □

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