A GAUGE THEORETIC APPROACH TO ELASTICITY WITH MICROROTATIONS

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Abstract. We formulate elasticity theory with microrotations using the framework of gauge theories, which has been developed and successfully applied in various areas of gravitation and cosmology. Following this approach, we demonstrate the existence of particle-like solutions. Mathematically this is due to the fact the our equations of motion are of Sine-Gordon type and thus have soliton type solutions. Similar to Skyrmions and Kinks in classical field theory, we can show explicitly that these solutions have a topological origin.

1. Introduction

1.1. **Elasticity and gauge theories.** In classical elasticity one assumes the independence of displacements and (micro)rotations. This independence has been dropped when the Cosserat brothers developed an extended framework of elasticity [6], often called Cosserat elasticity. This idea has led to a rich variety of models which are known by various different names like oriented or multipolar medium, asymmetric elasticity, micropolar elasticity, see e.g. [10, 11, 12, 26, 38, 39, 15, 13, 33, 34]. This field is also related to other areas of continuum mechanics, like the theory of granular media, ferromagnetic materials, cracked media and liquid crystals.

In Cosserat elasticity one considers a medium which can experience displacements and microrotations. This theory has two limiting cases, one with no microrotations which is classical elasticity, and another case where one assumes that the medium only experiences rotations and no displacements. Models of this type, though somewhat counter intuitive, have in fact a long history which can be traced back to MacCullagh in 1839, see [41]. In recent work [2, 3] such models have been investigated in the fully nonlinear setting and plane wave type solutions were explicitly constructed. The existence of such solutions is a highly non-trivial fact. In a linearised setting, similar solutions were investigated in [27, 28, 29].

Recent years have seen a revival of elasticity, in particular in the theory of dislocations which has been analysed from a gauge theoretic point of view. Of particular interest is the fact that these dislocations behave very much like particles. This gauge theoretic approach has been developed and applied for instance in [23, 24]. From a more formal point of view, one of the open questions in this field is whether or not it is possible to find soliton type solutions which could justify the particle interpretation from a mathematical point of view. The main result of this paper is that we are able to find soliton type solutions in a particular model of elasticity motivated by gauge theories of gravity.

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1.2. **Geometry and elasticity.** We consider an elastic medium which occupies the whole of $\mathbb{R}^3$, and we can identify materials points with points in space. We use the Latin alphabet to label the coordinate indices, $a, b, c, \dots = 1, 2, 3$, say, we write x^i for the spatial (holonomic) coordinates of $\mathcal{M}$, a 3-dimensional simply connected manifold which we identify with the material points of the continuum. This manifold is embedded into the 3-dimensional Euclidean space $\mathbb{R}^3$. Greek indices refer to the (co)frame indices (also tangent space or anholonomic indices), we use $\alpha, \beta, \dots = 1, 2, 3$ to label the frame covectors (1 forms) $\vartheta^\alpha = \vartheta^\alpha_b dx^b$ which we can refer to as the distortion in elasticity. One can think of ϑ^α as a set of three mutually orthogonal and normal covectors which form a basis at any point $p \in T^*\mathcal{M}$. Each such basis vector has components $(\vartheta^\alpha)_a$ and thus we can also view the object ϑ^α_a as an orthogonal matrix. We denote the frame by $e_\alpha = e^b_\alpha\, \partial/\partial x^b$, such that $e_\alpha \rfloor \vartheta^\beta = e^b_\alpha \vartheta^\beta_b = \delta^\beta_\alpha$. The e_α form the basis vectors of the tangent manifold $T\mathcal{M}$, and $\rfloor$ denotes the inner product. The metric is defined by $g_{\alpha\beta} = e_\alpha \otimes e_\beta$. We can think of the metric of the deformed medium as the Cauchy-Green tensor in elasticity.

When considering deformations of an elastic material, one places certain compatibility conditions on the induced stress. These so called Saint-Venant compatibility conditions are a form of integrability conditions so that the stress can be expressed and the symmetric derivative of the displacement. In a more geometrical language, by deforming the medium, we do not want to induce any curvature into the material. This means that one assumes the Riemann curvature tensor of the deformed medium to vanish identically. It is well known that these two views are equivalent, see for instance [1, 37]. One of the main ideas of this work is to carry this requirement over to more general, non-Riemannian geometries. In these geometries the notion of curvature is supplemented by a new geometrical quantity called torsion which was first introduced by Cartan in 1922. Therefore, we will analyse elastic materials using the language of non-Riemannian geometries with torsion and we will stick to the well established requirement of having vanishing total curvature. In the following subsection we will collect the most important basic facts of such geometries.

1.3. **Geometries with vanishing Riemann curvature tensor.** Let us describe each material point of our elastic material by a coframe ϑ^α and a connection $\Gamma_\alpha{}^\beta$. The coframe specifies the orientation of the orthonormal basis vectors at this point, while the connection determines how an arbitrary vector is parallelly transported near this point. The vanishing of the Riemann curvature tensor (a matrix valued 2-form) takes the form

$$R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta + \Gamma_\sigma{}^\beta \wedge \Gamma_\alpha{}^\sigma \equiv 0, \tag{1.1}$$

where d denotes the exterior derivative and $\wedge$ is the exterior (alternating) product. In geometries where the Riemann curvature identically vanishes the connection is often referred to as a Weitzenböck connection. We follow the Einstein summation convention whereby we sum over twice repeated indices. The condition $R_\alpha{}^\beta \equiv 0$ is known in the literature by various names like teleparallel, fernparallel of distant parallel geometry. Geometrically this condition means that the notion of parallelism is no longer a local statement, but can be defined globally, this means on the entire manifold. A straight line connecting two material points defines an absolute notion of parallelism. The only nonvanishing geometrical object in such geometries is

Cartan's torsion (a vector valued 2-form) defined by

$$T^\alpha = d\vartheta^\alpha + \Gamma_\beta{}^\alpha \wedge \vartheta^\beta. \tag{1.2}$$

It is important to note that there is a gauge freedom in choosing the variables $\{\vartheta^\alpha, \Gamma_\alpha{}^\beta\}$. All coframes are related to each other by means of local rotations. Therefore, the same elastic medium can be described equivalently by another pair $\{\vartheta'^\alpha, \Gamma'_\alpha{}^\beta\}$ which is related to the former by means of the gauge transformation

$$\vartheta'^\alpha = L^\alpha{}_\sigma\, \vartheta^\sigma, \tag{1.3}$$

$$\Gamma'_\beta{}^\alpha = L^\alpha{}_\sigma\, \Gamma_\rho{}^\sigma\, (L^{-1})^\rho{}_\beta + L^\alpha{}_\sigma\, d(L^{-1})^\sigma{}_\beta, \tag{1.4}$$

where $L^\alpha{}_\beta$ is an arbitrary 3×3 rotation matrix, this means $L \in \mathrm{SO}(3)$. The condition $R_\alpha{}^\beta \equiv 0$ is invariant under this gauge transformations, whereas torsion T^α transforms covariantly, $T'^\alpha = L^\alpha{}_\sigma\, T^\sigma$.

Let ε denote the volume 3-form, then the dual forms are defined by

$$\varepsilon_\alpha := *\vartheta_\alpha = e_\alpha \rfloor \varepsilon, \tag{1.5}$$

$$\varepsilon_{\alpha\beta} := *(\vartheta_\alpha \wedge \vartheta_\beta) = e_\beta \rfloor \varepsilon_\alpha, \tag{1.6}$$

$$\varepsilon_{\alpha\beta\gamma} := *(\vartheta_\alpha \wedge \vartheta_\beta \wedge \vartheta_\gamma) = e_\gamma \rfloor \varepsilon_{\alpha\beta}. \tag{1.7}$$

The operator $*$ is called the Hodge operator.The dual forms satisfy the following useful identities

$$\vartheta^\beta \wedge \varepsilon_\alpha = \delta^\beta_\alpha \varepsilon, \tag{1.8}$$

$$\vartheta^\beta \wedge \varepsilon_{\mu\nu} = \delta^\beta_\nu \varepsilon_\mu - \delta^\beta_\mu \varepsilon_\nu, \tag{1.9}$$

$$\vartheta^\beta \wedge \varepsilon_{\alpha\mu\nu} = \delta^\beta_\alpha \varepsilon_{\mu\nu} + \delta^\beta_\mu \varepsilon_{\nu\alpha} + \delta^\beta_\nu \varepsilon_{\alpha\mu}. \tag{1.10}$$

The object $\varepsilon_{\alpha\beta\gamma}$ is the totally antisymmetric Levi-Civita symbol with $\varepsilon_{123} = 1$.

1.4. **Elastic invariants.** Since T^α is the only nontrivial geometric object, it is therefore the only non-trivial object characterising the deformations of the medium. It is natural to decompose it into its irreducible pieces which will serve as our building blocks of elastic invariants. However, torsion is represented as a rank 3 tensor which is skew-symmetric in one pair of indices. Since we work in $\mathbb{R}^3$, it seems more natural to consider the Hodge dual of torsion which can be regarded as a 3×3 matrix with no *a priori* symmetries. Therefore, let us define $\mathcal{T}^\alpha := *T^\alpha$.

The three irreducible pieces of torsion are defined by

$${}^{(2)}T^\alpha = \frac{1}{2}\, \vartheta^\alpha \wedge e_\beta \rfloor T^\beta, \tag{1.11}$$

$${}^{(3)}T^\alpha = \frac{1}{3}\, *(\vartheta_\beta \wedge T^\beta)\, \varepsilon^\alpha, \tag{1.12}$$

$${}^{(1)}T^\alpha = T^\alpha - {}^{(2)}T^\alpha - {}^{(3)}T^\alpha. \tag{1.13}$$

Making use of the local transformations (1.3)-(1.4), we can eliminate either of the variables in the pair $\{\vartheta^\alpha, \Gamma_\alpha{}^\beta\}$. In particular, in [2], the teleparallel connection is gauged away $\Gamma_\alpha{}^\beta = 0$, and the coframe ϑ^α is left as the only variable. Alternatively in [3] the coframe is gauged to the standard constant values $\vartheta^\alpha_i = \delta^\alpha_i$ and the only variable is the connection, in which case once can write

$$\Gamma_\beta{}^\alpha = \Lambda^\alpha{}_\sigma\, d(\Lambda^{-1})^\sigma{}_\beta. \tag{1.14}$$

Here $\Lambda \in \mathrm{SO}(3)$ parametrises a teleparallel connection. We will refer to this choice of variables as the connection gauge since the only dynamical variable is the connection.

2. The potential energy

2.1. **Basic equations of the static model.** The most natural potential energy is based on the sum of the three quadratic elastic invariants constructed from the nonvanishing torsion

$$V = \sum_{i=1}^{3} c_i \, {}^{(i)}T^\alpha \wedge *T_\alpha, \tag{2.1}$$

where c_i, $i = 1, 2, 3$, are positive constants to which will refer as elastic moduli. However when our medium occupies the whole of $\mathbb{R}^3$, these quadratic terms are not fully independent because of the condition $R_\alpha{}^\beta \equiv 0$. This condition, as demonstrated in Appendix A allows to express the square of the first irreducible part in terms of the trace square and the axial trace square

$${}^{(1)}T^\alpha \wedge *T_\alpha = 2 \, {}^{(2)}T^\alpha \wedge *T_\alpha - \frac{1}{2} \, {}^{(3)}T^\alpha \wedge *T_\alpha + d \left(2\vartheta^\alpha \wedge *T_\alpha \right). \tag{2.2}$$

In tensor language, using (1.11), (1.12), and (1.8), (1.9), we have

$${}^{(2)}T^\alpha \wedge *T_\alpha = \frac{1}{2} \, T^\nu{}_{\alpha\nu} T_\mu{}^{\alpha\mu} \, \varepsilon, \tag{2.3}$$

$${}^{(3)}T^\alpha \wedge *T_\alpha = \frac{1}{12} \left(T_{\mu\rho\sigma} \varepsilon^{\mu\rho\sigma} \right)^2 \varepsilon. \tag{2.4}$$

It is straightforward to establish relation between the irreducible parts of the torsion 2-form T^α and its dual 1-form $\mathcal{T}^\alpha$. We have $\mathcal{T}^\alpha = \mathcal{T}^{\alpha\beta} \, \vartheta_\beta$, denote the matrix $\mathcal{T}^{\alpha\beta}$ by $\mathbf{T}$, and we find explicitly the decomposition of the second rank tensor

$$\mathcal{T}^{\alpha\beta} = {}^{(1)}\mathcal{T}^{\alpha\beta} + {}^{(2)}\mathcal{T}^{\alpha\beta} + {}^{(3)}\mathcal{T}^{\alpha\beta}, \tag{2.5}$$

$$\mathbf{T} = {}^{(1)}\mathbf{T} + {}^{(2)}\mathbf{T} + {}^{(3)}\mathbf{T}, \tag{2.6}$$

into the trace, antisymmetric and traceless symmetric parts:

$${}^{(3)}\overset{*}{T}{}^{\alpha\beta} = \frac{1}{6} \, T_{\mu\rho\sigma} \varepsilon^{\mu\rho\sigma} \, g^{\alpha\beta}, \qquad {}^{(3)}\mathbf{T} = \frac{1}{3} \left(\mathrm{tr}\mathbf{T} \right) \mathbf{I}, \tag{2.7}$$

$${}^{(2)}\overset{*}{T}{}^{\alpha\beta} = \frac{1}{2} \, \varepsilon^{\alpha\beta\mu} \, T^\nu{}_{\mu\nu}, \qquad {}^{(2)}\mathbf{T}^{\mathrm{T}} = - \, {}^{(2)}\mathbf{T} \tag{2.8}$$

$${}^{(1)}\overset{*}{T}{}^{\alpha\beta} = \frac{1}{2} \, T^\alpha{}_{\mu\nu} \varepsilon^{\mu\nu\beta}, \qquad {}^{(1)}\mathbf{T}^{\mathrm{T}} = {}^{(1)}\mathbf{T}, \quad \mathrm{tr} \, {}^{(1)}\mathbf{T} = 0. \tag{2.9}$$

2.2. **The choice of the gauge.** In [2], the coframe gauge is used when the connection is zero (completely gauged out). The components of the coframe are the only dynamical variables. Geometrically, they represent an arbitrary orthogonal 3×3 matrix. Since the connection is trivial in this gauge, the torsion 2-form reduces to $T^\alpha = d\vartheta^\alpha$. Using the components of the coframe and frame, $\vartheta^\alpha = h^\alpha_i dx^i$, $e_\alpha = h^i_\alpha \partial_i$, we have for the holonomic torsion

$$T^i = h^i_\alpha T^\alpha = h^i_\alpha d(h^\alpha_j) \wedge dx^j = h^i_\alpha \partial_k h^\alpha_j \, dx^k \wedge dx^j. \tag{2.10}$$

The dual 1-form then reads

$$\overset{*}{T}{}^i = \overset{*}{T}{}^i{}_j \, dx^j = h^i_\alpha \partial_k h^\alpha_l \, \varepsilon^{kl}{}_j dx^j. \tag{2.11}$$

A complementary approach is developed in [3], where the only dynamical variable is the connection, whereas the coframe is fixed to its trivial constant value $\vartheta^\alpha = \delta^\alpha_i \, dx^i$. The torsion (1.2) reduces in this gauge to $T^\alpha = \Gamma_\beta{}^\alpha \wedge \vartheta^\beta$, and since in the Weitzenböck space the connection is given by (1.14), we have

$$T^i = \delta^i_\alpha \, T^\alpha = \delta^i_\alpha \Lambda^\alpha{}_\sigma \, d(\Lambda^{-1})^\sigma{}_\beta \, \delta^\beta_j \, dx^j. \tag{2.12}$$

Denoting $u^i_\sigma := \delta^i_\alpha \Lambda^\alpha{}_\sigma$, we thus have

$$T^i = u^i_\alpha d(u^\alpha_j) \wedge dx^j = u^i_\alpha \partial_k u^\alpha_j \, dx^k \wedge dx^j. \tag{2.13}$$

This is completely equivalent (2.10), with the difference that the orthogonal matrix h^i_α, representing the coframe, is replaced with the orthogonal matrix $\mathbf{u} := u^i_\alpha$, representing the connection.

In [3], there are two technical deviations as compared to the model developed in [2]. The first deviation is the different choice of the variables. Noticing that $(\mathbf{u}^{\mathrm{T}}\partial\mathbf{u})^i{}_{kj} = u^i_\alpha \partial_k u^\alpha_j$ is skew-symmetric in i, j (being an element of the Lie algebra of the orthogonal group), in [3] one chooses as a basic variable a new 3×3 matrix

$$\mathbf{A} = \star(\mathbf{u}^{\mathrm{T}}\partial\mathbf{u}), \tag{2.14}$$

which has no a priori symmetries. This $\star$ denotes the dualization operator which relates skew symmetric matrices to vectors, similar to the Hodge $*$ operator. Writing out the indices explicitly, this matrix can be written in the following way

$$A_{lk} = \frac{1}{2} \varepsilon_{li}{}^j \, u^i_\alpha \partial_k u^\alpha_j. \tag{2.15}$$

The inverse reads

$$u^i_\alpha \partial_k u^\alpha_j = A_{lk} \, \varepsilon^{li}{}_j. \tag{2.16}$$

This quantity is known in the polar elasticity theory under various different names: the wryness tensor, the second Cosserat deformation tensor, the third order right micropolar curvature tensor, or torsion-curvature tensor (note that this is somewhat a misnomer since it neither directly relates to torsion nor curvature in the sense of differential geometry). Substituting this into (2.13), we can find a relation between the the second rank tensor (which we can view as a matrix in $\mathbb{R}^3$) to the matrix $\mathbf{A}$, so that

$$\overset{*}{\mathbf{T}} = \mathbf{A} - \mathrm{tr}(\mathbf{A}) \, \mathbf{I}. \tag{2.17}$$

The second deviation of [3] from [2] is a different structure of the Lagrangian. Namely, the Lagrangian of a rotationally elastic medium is assumed in [3] to be a function of orthogonal matrix only.

2.3. **Kinetic energy and total Lagrangian.** The Lagrangian 4-form $V = L \, dt \wedge \varepsilon$ is constructed by taking $L = L(u, \partial_0 u, \partial_i u)$ as a quadratic function of the irreducible parts of A_{lk}. In addition, the kinetic term is chosen to be

$$L^{\mathrm{kin}} = \frac{1}{2}\mathrm{tr}(\dot{u}^{\mathrm{T}}\dot{u}) = \frac{1}{2}(\partial_t u^i_\alpha)(\partial_t u^\alpha_i). \tag{2.18}$$

This form of the kinetic energy is well motivated, since when linearised it yields angular kinetic energy or rotation energy.

Introducing, analogously to (2.15), the velocity of the deformations of the material continuum

$$A_{lt} = \frac{1}{2}\varepsilon_{li}{}^{j}\, u^i_\alpha \partial_t u^\alpha_j, \tag{2.19}$$

one can recast (2.18) as

$$L^{\rm kin} = \frac{1}{2}(\partial_t u^i_\alpha)(\partial_t u^\alpha_i) = (A_{lt})^2. \tag{2.20}$$

Taking into account the identity (2.2), the potential energy contains just two quadratic invariants. As a result, the general Lagrangian reads

$$L = L^{\rm kin}(A_{lt}) - L^{\rm pot}(A_{lk}), \qquad L^{\rm pot}(A_{lk}) = \lambda_1\,(A^k{}_k)^2 + \lambda_2\,(A_{[lk]})^2. \tag{2.21}$$

One can use different parametrizarions of the orthogonal matrices. For example, in [2], the spinor parametrization was used. Here we find it more convenient to describe an arbitrary orthogonal matrix with the help of the three real functions β^α. From these independent constituents, an orthogonal matrix is constructed as follows

$$u^i_\alpha = \delta^i_\alpha + 2\beta_\alpha\beta^i - 2\delta^i_\alpha\beta^2 + 2\alpha\beta^\gamma\varepsilon^i{}_{\alpha\gamma}. \tag{2.22}$$

Here $\alpha^2 + \beta^2 = 1$ (with $\beta^2 = \beta_\alpha\beta^\alpha$), and Greek indices are freely converted into Latin ones (and vice versa) using the Kronecker deltas. The inverse matrix reads

$$u^\alpha_i = \delta^\alpha_i + 2\beta^\alpha\beta_i - 2\delta^\alpha_i\beta^2 - 2\alpha\beta^\gamma\varepsilon^\alpha{}_{i\gamma}. \tag{2.23}$$

Substituting (2.22) and (2.23) into (2.15) and (2.19), we find

$$A_{lk} = 2\left(\varepsilon_{lij}\beta^i\partial_k\beta^j + \beta_l\partial_k\alpha - \alpha\partial_k\beta_l\right), \tag{2.24}$$

$$A_{lt} = 2\left(\varepsilon_{lij}\beta^i\partial_t\beta^j + \beta_l\partial_t\alpha - \alpha\partial_t\beta_l\right). \tag{2.25}$$

For small β, the above formulas can be linearised, so that $A_{lk} \approx -2\partial_k\beta_l$, $A_{lt} \approx -2\partial_t\beta_l$, and the linearised model is described by the Lagrangian

$$L \approx 4\dot{\beta}^2 - 4\lambda_1({\rm div}\beta)^2 - 2\lambda_2({\rm curl}\beta)^2. \tag{2.26}$$

2.4. **The nonlinear equations of motion.** The complete nonlinear equations are more nontrivial. Denote the derivatives

$$H^{lk} = \frac{\partial L^{\rm pot}}{\partial A_{lk}}, \qquad H^{lt} = \frac{\partial L^{\rm kin}}{\partial A_{lt}}. \tag{2.27}$$

Then the field equations read

$$(\partial_t H^{it} - \partial_k H^{ik})P_{ij} + 2(H^{it}Q_{tij} - H^{ik}Q_{kij}) = 0, \tag{2.28}$$

where we introduced

$$P_{ij} = \varepsilon_{ijl}\beta^l + \frac{1}{\alpha}(\delta_{ij} - \delta_{ij}\beta^2 + \beta_i\beta_j), \tag{2.29}$$

$$Q_{tij} = \left[\varepsilon_{ijl} - \frac{1}{\alpha}(\delta_{ij}\beta_l - \delta_{il}\beta_j)\right]\partial_t\beta^l, \tag{2.30}$$

$$Q_{kij} = \left[\varepsilon_{ijl} - \frac{1}{\alpha}(\delta_{ij}\beta_l - \delta_{il}\beta_j)\right]\partial_k\beta^l. \tag{2.31}$$

The equations (2.28) are valid for *any* Lagrangian $L = L^{\rm kin}(A_{lt}) - L^{\rm pot}(A_{lk})$ with an arbitrary dependence on the variables A_{lt}, A_{lk}. However, for the specific choice of the quadratic Lagrangian (2.20) and (2.21), we have explicitly

$$H^{it} = 2A^{it}, \qquad H^{ik} = 2\lambda_1\, A^l{}_l\delta^{ik} + 2\lambda_2\, A^{[ik]}. \tag{2.32}$$

The tensor (2.29) is invertible and the inverse reads

$$(P^{-1})^{jk} = \alpha\,\delta^{jk} - \varepsilon^{jkn}\beta_n. \tag{2.33}$$

One can easily check that $P_{ij}(P^{-1})^{jk} = \delta^k_i$. Multiplying (2.28) with this inverse, we obtain another convenient form of the fields equations:

$$\partial_t H^{it} - \partial_k H^{ik} + 2(H^{jt}G_{tj}{}^i - H^{jk}G_{kj}{}^i) = 0, \tag{2.34}$$

where we have introduced

$$G_{tj}{}^i = Q_{tjl}(P^{-1})^{li} = \varepsilon_j{}^{il}\partial_t(\alpha\beta_l) + \beta^i\partial_t\beta_j - \beta_j\partial_t\beta^i, \tag{2.35}$$

$$G_{kj}{}^i = Q_{kjl}(P^{-1})^{li} = \varepsilon_j{}^{il}\partial_k(\alpha\beta_l) + \beta^i\partial_k\beta_j - \beta_j\partial_k\beta^i. \tag{2.36}$$

Note that both objects are antisymmetric in i, j.

The resulting system of nonlinear differential equations (obtained after substituting (2.32), (2.29)-(2.31) and (2.24), (2.25) into (2.28)) is quite nontrivial. It is possible, however, to find simple solutions under the additional assumptions.

3. Spherically-symmetric solutions – the soliton

Let us now look for the configurations with a 3-dimensional spherical symmetry. The corresponding ansatz reads

$$\beta^\alpha = \frac{x^\alpha}{r}\cos w, \qquad \alpha = \sin w, \tag{3.1}$$

where the scalar function $w = w(t, r)$ depends on time and on the radial variable $r = \sqrt{x_i x^i}$.

Then (2.24) and (2.25) yield

$$A_{lk} = 2\left[\varepsilon_{lik}\frac{x^i}{r^2}\cos^2 w + \frac{x_l x_k}{r^2}\,w' - \sin w\cos w\left(\frac{\delta_{lk}}{r} - \frac{x_l x_k}{r^3}\right)\right], \tag{3.2}$$

$$A_{lt} = 2\frac{x_l}{r}\,\dot w. \tag{3.3}$$

Hereafter the dot and the prime denote the derivatives with respect to time and radius, respectively.

Accordingly, the trace and the skew-symmetric parts of (3.2) are

$$A^k{}_k = 2w' - 4\,\frac{\sin w\cos w}{r}, \tag{3.4}$$

$$A_{[lk]} = -2\varepsilon_{lkn}\frac{x^n}{r^2}\cos^2 w. \tag{3.5}$$

As a result, we find

$$H^{it} = \frac{4x^i}{r}\,\dot w, \tag{3.6}$$

$$H^{ik} = 2\lambda_1\,\delta^{ik}\left(2w' - 4\,\frac{\sin w\cos w}{r}\right) - 4\lambda_2\,\varepsilon^{ikn}\frac{x_n}{r^2}\cos^2 w. \tag{3.7}$$

It is straightforward to see that

$$\partial_t H^{it} = \frac{4x^i}{r}\,\ddot{w}, \qquad H^{jt}G_{tj}{}^{i} = 0, \tag{3.8}$$

$$G_{kj}{}^{i} = \varepsilon_{kj}{}^{i}\,\frac{\sin w\cos w}{r} + \varepsilon_{j}{}^{il}\,\frac{x_l x_k}{r^2}\left(\frac{\sin w\cos w}{r}\right)' + (x^i\delta_{kj} - x_j\delta^i_k)\frac{\cos^2 w}{r^2}. \tag{3.9}$$

Using these results, we find that the field equations (2.34) reduce to

$$\frac{4x^i}{r}\left[\ddot{w} - \lambda_1\left(w'' + \frac{2}{r}\,w'\right) - \frac{1}{r^2}\,U(w)\right] = 0, \tag{3.10}$$

where

$$U(w) = \sin(2w)[(\lambda_2 - \lambda_1) + (\lambda_2 - 2\lambda_1)\cos(2w)]. \tag{3.11}$$

Since the two terms in the round brackets are the Laplacian in spherical coordinates, this equation of motion can also be written in the neat form $\ddot{w} - \lambda_1\Delta w = U(w)/r^2$. Let us introduce a new function $\varphi := wr$, and next let us rescale $r \mapsto \sqrt{\lambda_1}t$, $\varphi \mapsto \sqrt{\lambda_1}\varphi$, then (3.10) becomes

$$\ddot{\varphi} - \varphi'' + \frac{U(\varphi/r)}{\lambda_1^{3/2}\,r} = 0. \tag{3.12}$$

The resulting equation is closely related to the spherical 'sine-Gordon' equation, see for instance [7, 31] and references therein. The main difference between previously studied spherical sine-Gordon equations and our equation (3.12) is that the non-linearity carries an extra factor of $1/r$. This is similar to angular momentum when studying the spherical Schrödinger equation or Newton's equation. This additional factor has some interesting implications. At large distances from the centre this term becomes negligible and asymptotically we recover the wave equations. Thus, we have established the existence of soliton solutions in this model with a localised solution near the centre.

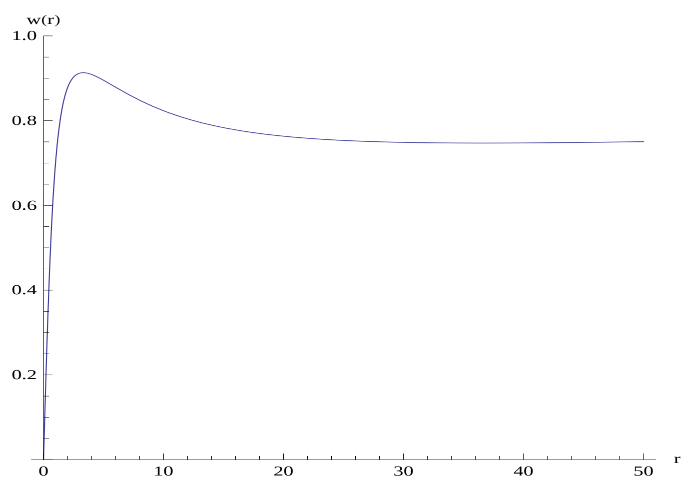

FIGURE 1. Soliton solution of the rotational elasticity.

In the static case, the equations of motion reduce to the single ODE

$$\lambda_1 (r^2 w')' + U(w) = 0. \tag{3.13}$$

For $\lambda_2 = 2\lambda_1$ this further reduces to $(r^2 w')' + \sin(2w) = 0$, whereas when $\lambda_2 = \lambda_1$ one is left with $(r^2 w')' - \frac{1}{2}\sin(4w) = 0$. The qualitative analysis of the equation (3.13) reveals the existence of static solutions that vanish at $r = 0$ and approach asymptotically $\pi/4$ at infinity for $r \to \infty$.

Numeric integration is straightforward. The form of the solution depends on the coupling constants and on the initial value of $w'(0)$. However, the qualitative behaviour remains the same. As a specific example, Fig. 1 presents the soliton for $\lambda_2 = \lambda_1$ when $w'(0) = 1$.

In the static case we can introduce the new function f, defined by

$$w = \frac{1}{2}\arctan\{\sinh(f(\log(r)))\} \tag{3.14}$$

which transforms (3.10) into an autonomous second order differential equation

$$f_{\beta\beta} + f_\beta(1 - \tanh(f) f_\beta) - 2\sinh(f) - 4\tanh(f) + 2\frac{\lambda_2}{\lambda_1}(\sinh(f) + \tanh(f)) = 0, \tag{3.15}$$

where $\beta = \log(r)$. This equation can now be analysed using standard techniques from ordinary differential equations or dynamical systems, and we find that this static system has three equilibrium points

$$f = 0, \qquad \sinh f = \pm\frac{\sqrt{\lambda_1}\sqrt{3\lambda_1 - 2\lambda_2}}{\lambda_1 - \lambda_2}, \tag{3.16}$$

with eigenvalues $(0, -1)$ in all three cases which in turn yields interesting (in)stability properties.

4. Discussion

The rotational elasticity model has many features similar to the model suggested by Skyrme [36]. Although the Lagrangians are different, the dynamics looks qualitatively the same. In particular, the remarkable feature of the rotational elasticity is the existence of solitons which are close relatives to the Skyrmions. The stability of solutions obtained is guaranteed by the topological nature of such configurations.

Recently it was noticed [32] that in the framework of the gauge gravity approach (that underlies the model under consideration, see Sec. 1.3) one can define an identically conserved current 3-form that gives rise to the topological charge that naturally classifies the field configurations. Specialising to the case of the Weitzenböck geometry with the flat curvature (1.1), this topological current reduces to

$$J^{\rm top} = \Gamma_\alpha{}^\beta \wedge d\Gamma_\beta{}^\alpha + \frac{2}{3}\,\Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha. \tag{4.1}$$

This 3-form is identically conserved, $dJ^{\rm top} \equiv 0$ in view of (1.1). As a result, we can construct the topological charge

$$\mathcal{Q} = \frac{1}{96\pi^2}\int \Gamma_\alpha{}^\beta \wedge \Gamma_\beta{}^\gamma \wedge \Gamma_\gamma{}^\alpha. \tag{4.2}$$

The integral is taken over the whole 3-space and the result is a constant for configurations that we studied in the previous section when w vanishes at the origin and approaches constant value at infinity. By direct computation we can verify that $Q = 1$ for the soliton described above.

A multi-soliton generalisation is straightforward. As a first step, we replace the ansatz (3.1) by shifting the origin from $x^a = 0$ to an arbitrary point $x^a_{(I)} = 0$:

$$\beta^\alpha = \frac{x^\alpha - x^\alpha_{(I)}}{|x - x_{(I)}|} \cos w, \qquad \alpha = \sin w, \tag{4.3}$$

where the function $w = w(|x - x_{(I)}|)$ depends on the Euclidean distance $|x - x_{(I)}|$. Then the N-soliton generalisation is obtained by taking instead of (2.22) the orthogonal matrix that is a product of N factors of the form (4.3)

$$u^i_\alpha = \left(u_{(1)} \cdot u_{(2)} \cdot \cdots \cdot u_{(N)}\right)^i_\alpha. \tag{4.4}$$

The dot denotes the usual matrix product. Such a generalisation would, for instance, allow us to study the interaction of solitons.

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References

[1] Yu. A. Amenzade, *Theory of Elasticity* (Mir Publishers, Moscow, 1979).

[2] C.G. Böhmer, R. Downes, and D. Vassiliev, *Rotational elasticity*, (2010) 20 pp.; arXiv:1008.3833.

[3] C.G. Böhmer, *Modelling rotational elasticity with orthogonal matrices*, (2010) 10 pp.; arXiv:1008.4005.

[4] G. Capriz, *Continua with microstructure*, Springer Tracts in Natural Phylosophy **35** (springer, Berlin, 1989).

[5] O. Chervova and D. Vassiliev, *The stationary Weyl equation and Cosserat elasticity*, J. Phys. A: Math. Theor. **43** (2010) 335203 (14 pages); arXiv:1001.4726.

[6] E. Cosserat and F. Cosserat, *Théorie des Corps Déformables* (Paris: Hermann, 1909); English translation by D.H. Delphenich available at http://www.uni-due.de/∼hm0014/Cosserat_files/Cosserat09_eng.pdf.

[7] G. H. Derrick, *Comments on Nonlinear Wave Equations as Models for Elementary Particles*, J. Math. Phys. **5** (1964) 1252-1254.

[8] J. Dyszlewicz, *Micropolar Theory of Elasticity*, Lecture Notes in Applied and Computational Mechanics, v. 15 (Springer, Berlin, 2004).

[9] J.L. Ericksen and C. Truesdell, *Exact theory of stress and strain in rods and shells*, Arch. Rat. Mech. Anal. **1** (1957) 295-323.

[10] J. L. Ericksen, *Hydrostatic theory of liquid crystals*, Arch. Rational Mech. Anal. **9** (1962) 379394.

[11] J. L. Ericksen, *Kinematics of macromolecules*, Arch. Rational Mech. Anal. **9** (1962) 18.

[12] J. L. Ericksen, *Twisting of liquid crystals*,J. Fluid Mech. **27** (1967) 5964.

[13] A. C. Eringen and E. S. Suhubi, *Nonlinear theory of simple microelastic solids I*, Int. J. Eng. Sci. **2** (1964) 189204; *Nonlinear theory of simple microelastic solids II*, Int. J. Eng. Sci. **2** (1964) 389404.

[14] A.C. Eringen, editor, *Continuum Physics, Vol. IV, Polar and nonlocal field theories* (Academic Press, New York, 1976).

[15] A. E. Green, *Multipolar continuum mechanics*, Arch. Rational Mech. Anal. **17** (1964) 113147.

[16] G. Grioli, *Elasticità asimmetrica*, Ann. Mat. pura e Appl. **50** (1960) 389-417.

[17] G. Grioli, *Contributo per una formulazione di tipo integrale della meccanica dei continui di Cosserat*, Ann. Mat. pura e Appl. **111** (1976) 175-183.

[18] G. Grioli, *Microstructures as a refinement of Cauchy theory. Problems of physical concreteness*, Continuum Mechanics and Thermodynamics **15** (2003) 441-450.

[19] F. Gronwald and F.W. Hehl, *Stress and hyperstress as fundamental concepts in continuum mechanics and in relativistic field theory*, In: G. Ferrarese, ed., "Advances in Modern Continuum Dynamics", International Conference in Memory of Antonio Signorini, Isola d'Elba, June 1991, pp. 1-32. Pitagora Editrice, Bologna (1993); arXiv:gr-qc/9701054.

[20] F.W. Hehl and Yu.N. Obukhov, *Élie Cartan's torsion in geometry and in field theory, an essay*, Ann. Fond. Louis Broglie **32** (2007) 157-194.
[21] F.W. Hehl, J.D. McCrea, E.W. Mielke, and Y. Ne'eman, *Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance*, Phys. Repts **258** (1995) 1-171.
[22] E. Kröner, *Mechanics of Generalized Continua*, Proc. IUTAM Symp. Fredenstadt-Stuttgart, 1967 (Springer, Berlin, 1967).
[23] M. Lazar and F.W. Hehl, *Cartan's spiral staircase in physics and, in particular, in the gauge theory of dislocations*, Found. Phys. **40** (2010) 1298-1325; arXiv:0911.2121.
[24] M. Lazar, *On the fundamentals of the three-dimensional translation gauge theory of dislocations*, Mathematics and Mechanics of Solids **15** (2010) published online Sept. 2010; arXiv:1003.3549.
[25] G.A. Maugin. *On the structure of the theory of polar elasticity*, Phil. Trans. R. Soc. Lond. **A356** (1998) 1367-1395.
[26] R.D. Mindlin, *Micro-structure in linear elasticity*, Arch. Rat. Mech. Anal. **16** (1964) 51-78.
[27] D. Natroshvili, L. Giorgashvili and I. G. Stratis, *Mathematical problems of the theory of elasticity of chiral materials*, Applied Mathematics, Informatics and Mechanics **8** (2003) 47103.
[28] D. Natroshvili, L. Giorgashvili and I. G. Stratis, *Representation formulae of general solutions in the theory of hemitropic elasticity*, Q. Jl Mech. Appl. Math. **59** (2006) 451-474.
[29] D. Natroshvili, R. Gachechiladze, A. Gachechiladze and I. G. Stratis, *Transmission problems in the theory of elastic hemitropic materials*, Applicable Analysis **86** (2007) 14631508.
[30] W. Nowacki, *Theory of Asymmetric Elasticity*, 2nd ed. (Pergamon, Oxford, 1985).
[31] O. H. Olsen and M. R.Samuelsen, *Rotationally symmetric breather-like solutions to the sine-Gordon equation*, Physics Letters **A77** (1980) 95-99.
[32] A. Randono and T.L. Hughes, *Torsional monopoles and torqued geometries in gravity and condensed matter* (2010) 4 pp.; arXiv:1010.1031.
[33] H. Schaefer, *Das Cosserat Kontinuum*, ZAMM **47** (1967) 485498.
[34] H. Schaefer, *Analysis der Motorfelder im Cosserat-Kontinuum*, ZAMM **47** (1967) 319328.
[35] H. Schaefer, *Die Motorfelder des dreidimensionalen Cosserat-Kontinuums im Kalkül der Differentialformen*, Lecture course No. 19, International Centre for Mechanical Sciences (ICMS, Udine, 1970) 1-60.
[36] T.H.R. Skyrme, *A non-linear field theory*, Proc. Roy. Soc. London **A260** (1961) 127-138.
[37] I. S. Sokolnikoff, *Mathematical Theory of Elasticity* (McGraw-Hill, New York, 1956).
[38] R.A. Toupin, *Elastic materials with couple-stresses*, Arch. Rat. Mech. Anal. **11** (1962) 385-414.
[39] R.A. Toupin, *Theories of elasticity with couple-stress*, Arch. Rat. Mech. Anal. **17** (1964) 85-112.
[40] C. Truesdell and R. Toupin, *The Classical Field Theories. Encyclopedia of Physics*, Vol. III/1. (Berlin-Göttingen-Heidelberg: Springer, 1960).
[41] E. Whittaker, *A History of the Theories of Aether and Electricity* (Thomas Nelson and Sons, London, United Kingdom, 1951).

Appendix A. Some identities

$\overset{*}{T}{}^{\alpha} := *T^{\alpha}$, where $*$ is the Hodge operator.

Note that there is an identity (see Eq. (5.9.18) of [21]) that is valid in all dimensions:

(A.1)

$$\tilde{R}^{\alpha\beta} \wedge \varepsilon_{\alpha\beta} = R^{\alpha\beta} \wedge \varepsilon_{\alpha\beta} + \left(- {}^{(1)}T^{\alpha} + 2\, {}^{(2)}T^{\alpha} - \frac{1}{2}\, {}^{(3)}T^{\alpha} \right) \wedge *T_{\alpha} + d\left(2\vartheta^{\alpha} \wedge *T_{\alpha}\right).$$

The tilde denotes the Riemannian geometric objects, i.e. those constructed from the Christoffel symbols of the corresponding Riemannian metric of the manifold.

Taking into account that the Riemann-Cartan curvature vanishes due to the teleparallel constraint (1.1) and that by assumption the elastic medium is embedded in a flat Euclidean space with $\tilde{R}^{\alpha\beta} = 0$, the identity (A.1) allows to express the square of the first (tensor) irreducible part in terms of the trace and axial trace squares:

$$ {}^{(1)}T^{\alpha} \wedge *T_{\alpha} = 2\, {}^{(2)}T^{\alpha} \wedge *T_{\alpha} - \frac{1}{2}\, {}^{(3)}T^{\alpha} \wedge *T_{\alpha} + d\,(2\vartheta^{\alpha} \wedge *T_{\alpha})\,. \tag{A.2} $$

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