Blow-up phenomena and global existence for a periodic two-component Hunter-Saxton system

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Abstract

This paper is concerned with blow-up phenomena and global existence for a periodic two-component Hunter-Saxton system. We first derive precise blow-up scenarios for strong solutions to the system. Then, we present several new blow-up results of strong solutions and a new global existence result to the system. Our obtained results for the system are sharp and improve considerably earlier results.

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1 Introduction

In this paper, we study the Cauchy problem of the following periodic two-component Hunter-Saxton system:

$$\begin{cases} u_{txx}+2u_xu_{xx}+uu_{xxx}-k\rho\rho_x=0, & t>0,\ x\in\mathbb{R}, \\ \rho_t+(\rho u)_x=0, & t>0,\ x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}, \\ \rho(0,x)=\rho_0(x), & x\in\mathbb{R}, \\ u(t,x+1)=u(t,x), & t\geq 0, x\in\mathbb{R}, \\ \rho(t,x+1)=\rho(t,x), & t\geq 0, x\in\mathbb{R}, \end{cases} \tag{1.1}$$

with $k\in\{-1,1\}$, which appears originally in [18] and is the short-wave limit of the two-component Camassa-Holm system [4, 8]. The system (1.1) is a special case of Green-Naghdi system, which models the non-dissipative dark matter [19].

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For $\rho \equiv 0$, the system (1.1) reduces to the Hunter-Saxton equation [11], which describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field. The single-component model also arises in a different physical context as the high-frequency limit [7, 12] of the Camassa-Holm equation for shallow water waves [2, 13], a re-expression of the geodesic flow on the diffeomorphism group of the circle [5] with a bi-Hamiltonian structure [9] which is completely integrable [6]. The Hunter-Saxton equation also has a bi-Hamiltonian structure [13, 18] and is completely integrable [1, 12]. Moreover, the Hunter-Saxton equation has a geometric interpretation which was intensively studied in [16].

The initial value problem for the Hunter-Saxton equation on the line (nonperiodic case) was studied by Hunter and Saxton in [11]. Using the method of characteristics, they showed that smooth solutions exist locally and break down in finite time, see [11]. The occurrence of blow-up can be interpreted physically as the phenomenon by which waves that propagate away from the perturbation knock the director field out of its unperturbed state [11]. The initial value problem for the Hunter-Saxton equation on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ was discussed in [22]. The author proved the local existence of strong solutions to the periodic Hunter-Saxton equation, showed that all strong solutions except space-independent solutions blow up in finite time by using Kato semigroup method [14]. Moreover, the behavior of the solutions exhibits different features.

For $\rho \not\equiv 0$, peakon solutions and the Cauchy problem of the system (1.1) with $k \in \{1, -1\}$ have been discussed in [4] and [20] respectively. Recently, [21] deals with a generalization of the two-component Hunter-Saxton system set both on the unit torus and on the real line. Moreover, we find another paper [10] just before submitting the present paper. Some results in [10] and the present paper are similar. The aim of this paper is to study further blow-up phenomena and global existence of the system (1.1). Blow-up scenarios, several new blow-up results and a new global existence result of strong solutions to the system (1.1) are presented. The obtained results are sharp and improve considerably the recent results in [20].

The paper is organized as follows. In Section 2, we recall the local existence of the initial value problem associated with the system (1.1) and give a more precise explanation. In Section 3, we derive two precise blow-up scenarios. In Section 4, we present several explosion criteria of strong solutions to the system (1.1) with rather general initial data. In Section 5, we give a new global existence result of strong solutions to the system (1.1).

Notation Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations of function spaces if there is no ambiguity. We let $[A, B]$ denote the commutator of linear operator A and B. For convenience, we let $(\cdot|\cdot)_{s\times r}$ and $(\cdot|\cdot)_s$ denote the inner products of $H^s \times H^r$, $s, r \in \mathbb{R}_+$ and H^s, $s \in \mathbb{R}_+$, respectively.

2 Local existence

We provide now the framework in which we shall reformulate the system (1.1). Integrating both sides of the first equation of the system (1.1) with respect to x, we obtain

$$u_{tx} + uu_{xx} + \frac{1}{2}u_x^2 - \frac{k}{2}\rho^2 = a(t),$$

where

$$a(t) = -\frac{1}{2}\int_{\mathbb{S}}(k\rho^2 + u_x^2)dx$$

and

$$\frac{d}{dt}a(t) = 0,$$

cf. [20]. For convenience, write $a := a(0)$. Therefore,

$$u_{tx} + uu_{xx} = \frac{k}{2}\rho^2 - \frac{1}{2}u_x^2 + a. \tag{2.1}$$

Integrating (2.1) once more in x, we obtain

$$u_t + uu_x = \partial_x^{-1}(\frac{k}{2}\rho^2 + \frac{1}{2}u_x^2 + a) + h(t), \tag{2.2}$$

where $\partial_x^{-1} f(x) := \int_0^x f(y)dy$ and $h(t) : [0, \infty) \to \mathbb{R}$ is an arbitrary continuous function.

Thus we get an equivalent form of the system (1.1)

$$\begin{cases} u_t + uu_x = \partial_x^{-1}(\frac{k}{2}\rho^2 + \frac{1}{2}u_x^2 + a) + h(t), & t > 0,\, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0,\, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ \rho(t, x+1) = \rho(t,x), & t \geq 0, x \in \mathbb{R}, \\ u(t, x+1) = u(t,x), & t \geq 0, x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where $k \in \{-1, 1\}$, $\partial_x^{-1} f(x) := \int_0^x f(y)dy$ and $h(t) : [0, \infty) \to \mathbb{R}$ is an arbitrary continuous function.

Next, we will establish the local well-posedness for the Cauchy problem of the system (2.3) in $H^s \times H^{s-1}$, $s \geq 2$, with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ (the circle of unit length) by applying Kato's theory.

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X and let $Q : Y \to X$ be a topological isomorphism. $\| \cdot \|_X$ and $\| \cdot \|_Y$ denote the norm of Banach space X and Y, respectively. Let $L(Y, X)$ denote the space of all bounded linear operators from Y to X ($L(X)$, if $X = Y$.).

Theorem 2.1 *[20] Given $h(t) \in C([0, \infty); \mathbb{R})$ and $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$, then there exists a maximal $T = T(a, h(t), \| z_0 \|_{H^s \times H^{s-1}}) > 0$, and a unique solution $z = (u, \rho)$ to (2.3) such that*

$$z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2})$$

is continuous.

As a consequence of Theorem 2.1 and the relation between the solution of the system (1.1) and the solution of the system (2.3) we have the following:

Theorem 2.2 *Given $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, s \geq 2$. Then there exists locally a family of solutions to (1.1).*

Note that the solution of the system (2.3) for any fixed $h(t)$ is unique. However, the solution of the system (1.1) given by Theorem 2.2 is not unique by the arbitrariness of $h(t)$. In the following sections, we discuss the corresponding unique solution to the system (2.3) with a fixed $h(t)$.

3 The precise blow-up scenario

In this section, we present the precise blow-up scenarios for strong solutions to the system (1.1).

We first recall the following lemmas.

Lemma 3.1 *[15] If $r>0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\| fg \|_{H^r} \leq c(\| f \|_{L^\infty} \| g \|_{H^r} + \| f \|_{H^r} \| g \|_{L^\infty}),$$

where c is a constant depending only on r.

Lemma 3.2 *[15] If $r>0$, then*

$$\| [\Lambda^r, f]g \|_{L^2} \leq c(\| \partial_x f \|_{L^\infty} \| \Lambda^{r-1} g \|_{L^2} + \| \Lambda^r f \|_{L^2} \| g \|_{L^\infty}),$$

where c is a constant depending only on r.

Lemma 3.3 *[3] Let $t_0>0$ and $v \in C^1([0,t_0); H^2(\mathbb{R}))$. Then for every $t \in [0,t_0)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}} \{v_x(t,x)\} = v_x(t,\xi(t)),$$

and the function m is almost everywhere differentiable on $(0,t_0)$ with

$$\frac{d}{dt} m(t) = v_{tx}(t,\xi(t)) \qquad \text{a.e. on } (0,t_0).$$

Remark 3.1 *If $v \in C^1([0,t_0); H^s(\mathbb{R}))$, $s>\frac{3}{2}$, then Lemma 3.3 also holds true. Meanwhile, Lemma 3.3 works analogously for*

$$M(t) := \sup_{x\in\mathbb{R}} \{v_x(t,x)\}.$$

Lemma 3.4 *Assume $k=1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s>\frac{5}{2}$, be given and assume that T is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with the initial data z_0. Then*

$$\|\rho_x(t,\cdot)\|_{L^\infty} \leq K \exp\left\{-2\int_0^t u_x(s,\xi(s))ds\right\},$$

where $(s,\xi(s))$ is a maximal point of $u_{xx}^2+\rho_x^2$ in $[0,T)\times\mathbb{S}$ and $K = \|u_{0,xx}\|_{L^\infty} + \|\rho_{0,x}\|_{L^\infty}$.

Proof Multiplying the first equation in (1.1) by u_{xx}, we get

$$\frac{1}{2}(u_{xx}^2)_t + 2u_x u_{xx}^2 + u\frac{1}{2}(u_{xx}^2)_x - \rho\rho_x u_{xx} = 0. \tag{3.1}$$

Differentiating the second equation in (1.1) in x and multiplying the obtained equation by ρ_x, we get

$$\frac{1}{2}(\rho_x^2)_t + 2u_x\rho_x^2 + u\frac{1}{2}(\rho_x^2)_x + \rho\rho_x u_{xx} = 0. \tag{3.2}$$

Adding the above two equations, we have

$$\frac{1}{2}(u_{xx}^2 + \rho_x^2)_t + 2u_x(u_{xx}^2 + \rho_x^2) + \frac{1}{2}(u_{xx}^2 + \rho_x^2)_x = 0. \tag{3.3}$$

By $z \in C([0,T); H^s \times H^{s-1})$, $s > \frac{5}{2}$, we know $u_{xx} \in H^{s-2}$, $\rho_x \in H^{s-2}$. Moreover, since H^{s-2} is a Banach algebra for $s > \frac{5}{2}$, $u_{xx}^2 + \rho_x^2 \in H^{s-2}$, $s > \frac{5}{2}$. Let $M(t) = \sup_{x\in\mathbb{S}}(u_{xx}^2 + \rho_x^2)(t,x)$. It follows from Remark 3.1 that there is a point $(t, \xi(t)) \in [0,T) \times \mathbb{S}$ such that $M(t) = (u_{xx}^2 + \rho_x^2)(t, \xi(t))$. Evaluating (3.3) on $(t, \xi(t))$ we get

$$\frac{dM(t)}{dt} = -4u_x(t, \xi(t))M(t).$$

Then, we obtain

$$M(t) = M(0)\exp\left\{\int_0^t -4u_x(s, \xi(s))ds\right\}.$$

Note that

$$M(0) = \sup_{x\in\mathbb{S}}(u_{0,xx}^2 + \rho_{0,x}^2) \le \|u_{0,xx}\|_{L^\infty}^2 + \|\rho_{0,x}\|_{L^\infty}^2.$$

Thus, we get

$$\|\rho_x\|_{L^\infty} \le K\exp\left\{-2\int_0^t u_x(s, \xi(s))ds\right\}.$$

Next we prove the following useful result on global existence of solutions to (1.1).

Theorem 3.1 *Assume $k = 1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s > \frac{5}{2}$, be given and assume that T is the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) with the initial data z_0. If there exists $M > 0$ such that*

$$\|u_x(t,\cdot)\|_{L^\infty} + \|\rho(t,\cdot)\|_{L^\infty} \le M, \quad t \in [0,T),$$

then the $H^s \times H^{s-1}$-norm of $z(t,\cdot)$ does not blow up on [0,T).

Proof Let $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (2.3) with the initial data $z_0 \in H^s \times H^{s-1}$, $s > \frac{5}{2}$, and let T be the maximal existence time of the corresponding solution z, which is guaranteed by Theorem 2.1. Throughout this proof, $c > 0$ stands for a generic constant depending only on s.

By $\|u_x(t,\cdot)\|_{L^\infty} \le M$ and Lemma 3.4, we get

$$\|\rho_x\|_{L^\infty} \le K\exp\left\{2\int_0^t u_x(s, \xi(s))ds\right\} \le Ke^{2Mt} := c(t). \tag{3.4}$$

Applying the operator Λ^s to the first equation in (2.3), multiplying by $\Lambda^s u$, and integrating over $\mathbb{S}$, we obtain

$$\frac{d}{dt}\|u\|_{H^s}^2 = -2(uu_x, u)_s + 2(u, \partial_x^{-1}(\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a) + h(t))_s. \tag{3.5}$$

Let us estimate the first term of the right-hand side of (3.5).

$$\begin{aligned}
|(uu_x, u)_s| &= |(\Lambda^s(u\partial_x u), \Lambda^s u)_0| \\
&= |([\Lambda^s, u]\partial_x u, \Lambda^s u)_0 + (u\Lambda^s\partial_x u, \Lambda^s u)_0| \\
&\leq \|[\Lambda^s, u]\partial_x u\|_{L^2}\|\Lambda^s u\|_{L^2} + \frac{1}{2}|(u_x\Lambda^s u, \Lambda^s u)_0| \\
&\leq (c\|u_x\|_{L^\infty} + \frac{1}{2}\|u_x\|_{L^\infty})\|u\|_{H^s}^2 \\
&\leq c\|u_x\|_{L^\infty}\|u\|_{H^s}^2,
\end{aligned}$$

where we used Lemma 3.2 with $r = s$. Then, we estimate the second term of the right-hand side of (3.5) in the following way:

$$\begin{aligned}
&|(\partial_x^{-1}(\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a) + h(t), u)_s| \\
\leq\ & \|\partial_x^{-1}(\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a) + h(t)\|_{H^s}\|u\|_{H^s} \\
\leq\ & (\|\partial_x^{-1}(\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a)\|_{L^2} + \|\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a\|_{H^{s-1}} + \|h(t)\|_{H^s})\|u\|_{H^s} \\
\leq\ & (\|\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a\|_{L^2} + \|\frac{1}{2}\rho^2 + \frac{1}{2}u_x^2 + a\|_{H^{s-1}} + \|h(t)\|_{H^s})\|u\|_{H^s} \\
\leq\ & c(\|\rho^2\|_{H^{s-1}} + \|u_x^2\|_{H^{s-1}} + 2\|a\|_{H^{s-1}} + \|h(t)\|_{H^s})\|u\|_{H^s} \\
\leq\ & c(\|\rho\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|u_x\|_{L^\infty}\|u_x\|_{H^{s-1}} + |a| + |h(t)|)\|u\|_{H^s} \\
\leq\ & c(\|\rho\|_{L^\infty} + \|u_x\|_{L^\infty} + 1)(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1),
\end{aligned}$$

where we used Lemma 3.1 with $r = s - 1$. Combining the above two inequalities with (3.5), we get

$$\frac{d}{dt}\|u\|_{H^s}^2 \leq c(\|\rho\|_{L^\infty} + \|u_x\|_{L^\infty} + 1)(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1). \tag{3.6}$$

In order to derive a similar estimate for the second component ρ, we apply the operator Λ^{s-1} to the second equation in (2.3), multiply by $\Lambda^{s-1}\rho$, and integrate over $\mathbb{S}$, to obtain

$$\frac{d}{dt}\|\rho\|_{H^{s-1}}^2 = -2(u\rho_x, \rho)_{s-1} - 2(u_x\rho, \rho)_{s-1}. \tag{3.7}$$

Let us estimate the first term of the right hand side of (3.7)

$$\begin{aligned}
&|(u\rho_x, \rho)_{s-1}| \\
=\ & |(\Lambda^{s-1}(u\partial_x\rho), \Lambda^{s-1}\rho)_0| \\
=\ & |([\Lambda^{s-1}, u]\partial_x\rho, \Lambda^{s-1}\rho)_0 + (u\Lambda^{s-1}\partial_x\rho, \Lambda^{s-1}\rho)_0| \\
\leq\ & \|[\Lambda^{s-1}, u]\partial_x\rho\|_{L^2}\|\Lambda^{s-1}\rho\|_{L^2} + \frac{1}{2}|(u_x\Lambda^{s-1}\rho, \Lambda^{s-1}\rho)_0| \\
\leq\ & c(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|\rho_x\|_{L^\infty}\|u\|_{H^{s-1}})\|\rho\|_{H^{s-1}} + \frac{1}{2}\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}}^2 \\
\leq\ & c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2),
\end{aligned}$$

here we applied Lemma 3.2 with $r = s-1$. Then we estimate the second term of the right hand side of (3.7). Based on Lemma 3.1 with $r = s-1$, we get

$$\begin{aligned}|(u_x\rho, \rho)_{s-1}| &\le \|u_x\rho\|_{H^{s-1}}\|\rho\|_{H^{s-1}}\\ &\le c(\|u_x\|_{L^\infty}\|\rho\|_{H^{s-1}} + \|\rho\|_{L^\infty}\|u_x\|_{H^{s-1}})\|\rho\|_{H^{s-1}}\\ &\le c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|\rho\|^2_{H^{s-1}} + \|u\|^2_{H^s}).\end{aligned}$$

Combining the above two inequalities with (3.7), we get

$$\frac{d}{dt}\|\rho\|^2_{H^{s-1}} \le c(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}} + 1). \tag{3.8}$$

By (3.6) and (3.8), we have

$$\begin{aligned}&\frac{d}{dt}(\|\rho\|^2_{H^{s-1}} + \|u\|^2_{H^s} + 1)\\ &\le c(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty} + 1)(\|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}} + 1).\end{aligned}$$

An application of (3.4), Gronwall's inequality and the assumption of the theorem yield

$$(\|\rho\|^2_{H^{s-1}} + \|u\|^2_{H^s} + 1) \le \exp(c(M + c(t) + 1))(\|\rho_0\|^2_{H^{s-1}} + \|u_0\|^2_{H^s} + 1).$$

This completes the proof of the theorem.

Given $z_0 \in H^s \times H^{s-1}$ with $s \ge 2$. Theorem 2.1 ensures the existence of a maximal $T > 0$ and a solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) such that

$$z = z(\cdot, z_0) \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2}).$$

Consider now the following initial value problem

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}, \end{cases} \tag{3.9}$$

where u denotes the first component of the solution z to (2.3). Then we have the following two useful lemmas.

Applying classical results in the theory of ordinary differential equations, one can obtain the following result on q which is crucial in the proof of blow-up scenarios.

Lemma 3.5 *[8, 17] Let $u \in C([0,T); H^s)\bigcap C^1([0,T); H^{s-1}), s \ge 2$. Then Eq.(3.9) has a unique solution $q \in C^1([0,T)\times\mathbb{R};\mathbb{R})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = exp\left(\int_0^t u_x(s, q(s,x))ds\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

Following the similar proof in [8], we obtain the next result:

Lemma 3.6 *Assume $k \in \{-1,1\}$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s \geq 2$ and let $T > 0$ be the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1). Then we have*

$$\rho(t, q(t,x))q_x(t,x) = \rho_0(x), \quad \forall\ (t,x) \in [0,T) \times \mathbb{S}. \tag{3.10}$$

Moreover, if there exists $M_1 > 0$ such that $u_x \geq -M_1$ for all $(t,x) \in [0,T) \times \mathbb{S}$, then

$$\|\rho(t,\cdot)\|_{L^\infty} = \|\rho(t, q(t,\cdot))\|_{L^\infty} \leq e^{M_1 T}\|\rho_0(\cdot)\|_{L^\infty}, \quad \forall\ t \in [0,T).$$

Furthermore, if $\rho_0 \in L^1$, then

$$\int_{\mathbb{S}} |\rho(t,x)|dx = \int_{\mathbb{S}} |\rho_0(x)|dx, \quad \forall\ t \in [0,T).$$

Our next result describes the precise blow-up scenarios for sufficiently regular solutions to (1.1).

Theorem 3.2 *Assume $k = 1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s > \frac{5}{2}$ be given and let T be the maximal existence time of the corresponding solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) with the initial data z_0. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t \to T} \inf_{x \in \mathbb{S}} u_x(t,x) = -\infty.$$

Proof By Theorem 2.1 and Sobolev's imbedding theorem it is clear that if

$$\liminf_{t \to T} \inf_{x \in \mathbb{S}} u_x(t,x) = -\infty,$$

then $T < \infty$.

Let $T < \infty$. Assume that there exists $M_1 > 0$ such that

$$u_x(t,x) \geq -M_1, \quad \forall\ (t,x) \in [0,T) \times \mathbb{S}.$$

By Lemma 3.6, we have

$$\|\rho(t,\cdot)\|_{L^\infty} \leq e^{M_1 T}\|\rho_0\|_{L^\infty}, \quad \forall\ t \in [0,T).$$

Take $K_0 = K^2 e^{4M_1 T}$. By the first equation in (2.3), a direct computation implies the following inequality

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{S}} u(t,x)^2 dx \qquad\qquad (3.11)\\
&= 2\int_{\mathbb{S}} u\partial_x^{-1}(\frac{1}{2}u_x^2 + \frac{k}{2}\rho^2 + a)dx + 2h(t)\int_{\mathbb{S}} u dx\\
&\leq \int_{\mathbb{S}} u^2 dx + \frac{1}{4}\int_{\mathbb{S}} \left(\int_0^x (u_y^2 + k\rho^2 + 2a)dy\right)^2 dx + |h(t)|\left(1 + \int_{\mathbb{S}} u(t,x)^2 dx\right)\\
&\leq |h(t)| + (1 + |h(t)|)\int_{\mathbb{S}} u(t,x)^2 dx + \frac{1}{4}\left(\int_0^1 (u_x^2 + \rho^2 + 2|a|)dx\right)^2\\
&= |h(t)| + (1 + |h(t)|)\int_{\mathbb{S}} u(t,x)^2 dx + \frac{1}{4}\left[2|a| + \int_0^1 (u_{0,x}^2 + \rho_0^2)dx\right]^2
\end{aligned}$$

for $t \in (0, T)$.

Multiplying (2.1) by u_x and integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^2 dx &= 2\int_{\mathbb{S}} u_x(-uu_{xx} + \frac{k}{2}\rho^2 - \frac{1}{2}u_x^2 + a)dx \\ &= \int_{\mathbb{S}} -2uu_xu_{xx}dx + k\int_{\mathbb{S}} u_x\rho^2 dx - \int_{\mathbb{S}} u_x^3 dx + 2a\int_{\mathbb{S}} u_x dx \\ &= k\int_{\mathbb{S}} u_x\rho^2 dx \\ &\le \|\rho\|_{L^\infty}^2 + \|\rho\|_{L^\infty}^2 \int_{\mathbb{S}} u_x^2 dx.\end{aligned} \tag{3.12}$$

Multiplying the first equation in (1.1) by $m = u_{xx}$ and integrating by parts, we find

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} m^2 dx = &-4\int_{\mathbb{S}} u_x m^2 dx - 2\int_{\mathbb{S}} umm_x dx + 2k\int_{\mathbb{S}} m\rho\rho_x dx \\ = &-3\int_{\mathbb{S}} u_x m^2 dx + 2k\int_{\mathbb{S}} m\rho\rho_x dx \\ \le &\ 3M_1\int_{\mathbb{S}} m^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} m^2 + \rho_x^2 dx \\ \le &\ (3M_1 + \|\rho\|_{L^\infty})\int_{\mathbb{S}} m^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} \rho_x^2 dx.\end{aligned} \tag{3.13}$$

Differentiating the first equation in (1.1) with respect to x, multiplying the obtained equation by $m_x = u_{xxx}$, integrating by parts and using Lemma 3.4, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}} m_x^2 dx \\ = &-4\int_{\mathbb{S}} m^2 m_x dx - 6\int_{\mathbb{S}} u_x m_x^2 - 2\int_{\mathbb{S}} um_{xx}m_x + 2k\int_{\mathbb{S}} \rho_x^2 m_x + 2k\int_{\mathbb{S}} \rho\rho_{xx}m_x dx \\ = &-5\int_{\mathbb{S}} u_x m_x^2 dx + 2k\int_{\mathbb{S}} \rho_x^2 m_x dx + 2k\int_{\mathbb{S}} \rho\rho_{xx}m_x dx \\ \le &\ 5M_1\int_{\mathbb{S}} m_x^2 dx + 2\|\rho_x\|_{L^\infty}^2 \int_{\mathbb{S}} |m_x| dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} (\rho_{xx}^2 + m_x^2) dx \\ \le &\ 5M_1\int_{\mathbb{S}} m_x^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} (\rho_{xx}^2 + m_x^2) dx + 2\|\rho_x\|_{L^\infty}^2 + 2\|\rho_x\|_{L^\infty}^2 \int_{\mathbb{S}} m_x^2 dx \\ \le &\ (5M_1 + \|\rho\|_{L^\infty} + 2K_0)\int_{\mathbb{S}} m_x^2 dx + \|\rho\|_{L^\infty}\int_{\mathbb{S}} \rho_{xx}^2 dx + 2K_0.\end{aligned} \tag{3.14}$$

Multiplying the second equation in (1.1) by ρ and integrating by parts, we have

$$\frac{d}{dt}\int_{\mathbb{S}} \rho^2 dx = -\int_{\mathbb{S}} u_x\rho^2 dx \le M_1\int_{\mathbb{S}} \rho^2 dx. \tag{3.15}$$

Differentiating the second equation in (1.1) with respect to x, multiplying the obtained equation by ρ_x and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}}\rho_x^2dx =& -3\int_{\mathbb{S}}u_x\rho_x^2dx-2\int_{\mathbb{S}}m\rho\rho_xdx \\ \leq& 3M_1\int_{\mathbb{S}}\rho_x^2dx+\|\rho\|_{L^\infty}\int_{\mathbb{S}}(m^2+\rho_x^2)dx \\ \leq& (3M_1+\|\rho\|_{L^\infty})\int_{\mathbb{S}}\rho_x^2dx+\|\rho\|_{L^\infty}\int_{\mathbb{S}}m^2dx.\end{aligned} \tag{3.16}$$

Differentiating the second equation in (1.1) with respect to x twice, multiplying the obtained equation by ρ_{xx}, integrating by parts and using Lemma 3.4, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}}\rho_{xx}^2dx \\ =& -5\int_{\mathbb{S}}u_x\rho_{xx}^2dx+\int_{\mathbb{S}}u_{xxx}(3\rho_x^2-2\rho\rho_{xx})dx \\ \leq& 5M_1\int_{\mathbb{S}}\rho_{xx}^2dx+\int_{\mathbb{S}}m_x(3\rho_x^2-2\rho\rho_{xx})dx \\ \leq& 5M_1\int_{\mathbb{S}}\rho_{xx}^2dx+3\|\rho_x\|_{L^\infty}^2\int_{\mathbb{S}}|m_x|dx+\|\rho\|_{L^\infty}\int_{\mathbb{S}}2m_x\rho_{xx}dx \\ \leq& (5M_1+\|\rho\|_{L^\infty})\int_{\mathbb{S}}\rho_{xx}^2dx+(3\|\rho_x\|_{L^\infty}^2+\|\rho\|_{L^\infty})\int_{\mathbb{S}}m_x^2+3\|\rho_x\|_{L^\infty}^2 \\ \leq& (5M_1+\|\rho\|_{L^\infty})\int_{\mathbb{S}}\rho_{xx}^2dx+(3K_0+\|\rho\|_{L^\infty})\int_{\mathbb{S}}m_x^2dx+3K_0.\end{aligned} \tag{3.17}$$

Summing (3.10)-(3.16), we have

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{S}}(u^2+u_x^2+m^2+m_x^2+\rho^2+\rho_x^2+\rho_{xx}^2)dx \\ \leq& K_1\int_{\mathbb{S}}(u^2+u_x^2+m^2+m_x^2+\rho^2+\rho_x^2+\rho_{xx}^2)dx+K_2,\end{aligned}$$

where

$$K_1=1+\max_{t\in[0,T]}|h(t)|+8e^{M_1T}\|\rho_0\|_{L^\infty}+(e^{M_1T}\|\rho_0\|_{L^\infty})^2+17M_1+5K_0,$$

$$K_2=\max_{t\in[0,T]}|h(t)|+\frac{1}{4}\left[2|a|+\int_0^1(u_{0,x}^2+\rho_0^2)dx\right]^2+(e^{M_1T}\|\rho_0\|_{L^\infty})^2+5K_0.$$

By means of Gronwall's inequality and the above inequality, we deduce that

$$\begin{aligned}&\|u(t,\cdot)\|_{H^3}^2+\|\rho(t,\cdot)\|_{H^2}^2 \\ \leq& e^{K_1t}(\|u_0\|_{H^3}^2+\|\rho_0\|_{H^2}^2+\frac{K_2}{K_1}), \quad \forall\ t\in[0,T).\end{aligned}$$

The above inequality, Sobolev's imbedding theorem and Theorem 3.1 ensure that the solution z does not blow-up in finite time. This completes the proof of the theorem.

Note that when $k=-1$, we cannot get Lemma 3.4. However, following the similar proof of Theorems 3.1-3.2 we obtain the following two results:

Theorem 3.3 *Assume $k=-1$. Let $z_0=\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s\times H^{s-1}$, $s\geq 2$, be given and assume that T is the maximal existence time of the corresponding solution $z=\begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) with the initial data z_0. If there exists $M>0$ such that*

$$\|u_x(t,\cdot)\|_{L^\infty}+\|\rho(t,\cdot)\|_{L^\infty}+\|\rho_x(t,\cdot)\|_{L^\infty}\leq M, \quad t\in[0,T),$$

then the $H^s\times H^{s-1}$-norm of $z(t,\cdot)$ does not blow up on [0,T).

Theorem 3.4 *Assume $k=-1$. Let $z_0=\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s\times H^{s-1}$, $s>\frac{5}{2}$, be given and let T be the maximal existence time of the corresponding solution $z=\begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) with the initial data z_0. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{S}} u_x(t,x)=-\infty \quad or \quad \limsup_{t\to T}\{\|\rho_x\|_{L^\infty}\}=+\infty.$$

For initial data $z_0=\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2\times H^1$, we have the following precise blow-up scenario.

Theorem 3.5 *Assume $k\in\{-1,1\}$. Let $z_0=\begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2\times H^1$, and let T be the maximal existence time of the corresponding solution $z=\begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) with the initial data z_0. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\to T}\inf_{x\in\mathbb{S}} u_x(t,x)=-\infty.$$

Proof Let $z=\begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (2.3) with the initial data $z_0\in H^2\times H^1$, and let T be the maximal existence time of the solution z, which is guaranteed by Theorem 2.1.

Let $T<\infty$. Assume that there exists $M_1>0$ such that

$$u_x(t,x)\geq -M_1, \quad \forall\ (t,x)\in[0,T)\times\mathbb{S}.$$

By Lemma 3.6, we have

$$\|\rho(t,\cdot)\|_{L^\infty}\leq e^{M_1T}\|\rho_0\|_{L^\infty}, \quad \forall\ t\in[0,T).$$

Combining (3.11)-(3.13) and (3.15)-(3.16), we obtain

$$\frac{d}{dt}\int_{\mathbb{S}} u^2+u_x^2+m^2+\rho^2+\rho_x^2dx\leq K_3\int_{\mathbb{S}} u^2+u_x^2+m^2+\rho^2+\rho_x^2dx+K_4,$$

where

$$K_3=1+\max_{t\in[0,T]}|h(t)|+(e^{M_1T}\|\rho_0\|_{L^\infty})^2+7M_1+4e^{M_1T}\|\rho_0\|_{L^\infty},$$

$$K_4 = \max_{t\in[0,T]} |h(t)| + \frac{1}{4}\left[2|a| + \int_0^1 (u_{0,x}^2 + \rho_0^2)dx\right]^2 + (e^{M_1 T}\|\rho_0\|_{L^\infty})^2.$$

By means of Gronwall's inequality and the above inequality, we get

$$\|u(t,\cdot)\|_{H^2}^2 + \|\rho(t,\cdot)\|_{H^1}^2 \le e^{K_3 t}(\|u_0\|_{H^2}^2 + \|\rho_0\|_{H^1}^2 + \frac{K_4}{K_3}).$$

The above inequality ensures that the solution z does not blow-up in finite time.

On the other hand, by Sobolev's imbedding theorem, we see that if

$$\liminf_{t\to T} \inf_{x\in\mathbb{S}} u_x(t,x) = -\infty,$$

then the solution will blow up in finite time. This completes the proof of the theorem.

Remark 3.2 *Note that Theorem 3.2 and Theorem 3.5 show that*

$$T(a, h(t), \|z_0\|_{H^s\times H^{s-1}}) = T(a, h(t), \|z_0\|_{H^{s'}\times H^{s'-1}}) = T(a, h(t), \|z_0\|_{H^2\times H^1})$$

with $k = 1$ for each $s, s' > \frac{5}{2}$. Furthermore, the maximal existence time T of the family of solutions to (1.1) given in Theorem 2.2 can be chosen independent of s. Moreover, Theorem 3.5 implies that

$$T(a, h(t), \|z_0\|_{H^s\times H^{s-1}}) \le T(a, h(t), \|z_0\|_{H^2\times H^1})$$

with $k \in \{-1, 1\}$ for each $s \ge 2$.

Remark 3.3 *Note that Theorem 3.4 shows that*

$$T(a, h(t), \|z_0\|_{H^s\times H^{s-1}}) = T(a, h(t), \|z_0\|_{H^{s'}\times H^{s'-1}})$$

with $k = -1$ for each $s, s' > \frac{5}{2}$. Moreover, the maximal existence time T of the family of solutions to (1.1) given in Theorem 2.2 can be chosen independent of s.

4 Blow-up

In this section, we discuss the blow-up phenomena of the system (1.1) and prove that there exist strong solutions to (1.1) which do not exist globally in time.

Theorem 4.1 *Assume $k = 1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \ge 2$, and T be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with the initial data z_0. If $\rho_0 \not\equiv 0$ or $u_0 \not\equiv c$ for any $c \in \mathbb{R}$, and there exists a point $x_0 \in \mathbb{S}$, such that $\rho_0(x_0) = 0$, then the corresponding solutions to (1.1) blow up in finite time.*

Proof We use the integrated representation (2.1). Let $m(t) = u_x(t, q(t, x_0))$, $\gamma(t) = \rho(t, q(t, x_0))$, where $q(t, x)$ is the solution of Eq.(3.9). By Eq.(3.9) we can obtain

$$\frac{dm}{dt} = (u_{tx} + uu_{xx})(t, q(t, x_0)).$$

Evaluating (2.1) at $(t, q(t, x_0))$ we get

$$\frac{d}{dt}m(t) = \frac{1}{2}\gamma(t)^2 - \frac{1}{2}m(t)^2 + a.$$

Since $\gamma(0) = 0$, we infer from Lemmas 3.5-3.6 that $\gamma(t) = 0$ for all $t \in [0, T)$. Note that $a = -\frac{1}{2}\int_{\mathbb{S}}(\rho_0^2 + u_{0,x}^2)dx < 0$ since $\rho_0 \not\equiv 0$ or $u_0 \not\equiv c$. Then we have $\frac{d}{dt}m(t) \le a < 0$. Thus, it follows that $m(t_0) < 0$ for some $t_0 \in (0, T)$. Solving the following inequality yields

$$\frac{d}{dt}m(t) \le -\frac{1}{2}m(t)^2.$$

Therefore

$$0 > \frac{1}{m(t)} \ge \frac{1}{m(t_0)} + \frac{1}{2}(t - t_0).$$

The above inequality implies that $T < t_0 - \frac{2}{m(t_0)}$ and $\lim_{t\to T} m(t) = -\infty$. In view of Theorem 3.5 and Remark 3.2, this completes the proof of the theorem.

Corollary 4.1 *Assume $k = 1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \ge 2$, and T be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with the initial data z_0. If ρ_0 is odd, either $\rho_0 \not\equiv 0$ or $u_0 \not\equiv 0$ is odd, then the corresponding solutions to (1.1) blow up in finite time.*

Proof Since ρ_0 is odd, $\rho_0(0) = 0$. $u_0 \not\equiv 0$ is odd implies $u_0 \not\equiv c$ for any $c \in \mathbb{R}$. From Theorem 4.1 we can get the desired result.

Theorem 4.2 *Assume $k = -1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s > \frac{5}{2}$, and T be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with the initial data z_0. The corresponding solutions to (1.1) blow up in finite time if one of the following conditions holds: (1) $a < 0$, (2) $a > 0$ and there exists some $x_0 \in \mathbb{S}$ such that $u_0'(x_0) < -\sqrt{2a}$, (3) $a = 0$ and there exists some $x_0 \in \mathbb{S}$ such that $u_0'(x_0) \le 0$, $\rho_0(x_0) \ne 0$.*

Proof Applying Remark 3.3 and a simply density argument, it is clear that we may consider the case $s = 3$. Define now

$$m(t) := \min_{x\in\mathbb{S}}\{u_x(t, x)\}, \quad t \in [0, T)$$

and let $\xi(t) \in \mathbb{S}$ be a point where this minimum is attained by Lemma 3.3. It follows that

$$m(t) = u_x(t, \xi(t)).$$

Clearly $u_{xx}(t, \xi(t)) = 0$ since $u(t, \cdot) \in H^3(\mathbb{S}) \subset C^2(\mathbb{S})$. Using the integrated representation (2.1) and evaluating it at $(t, \xi(t))$, we obtain

$$\frac{d}{dt}m(t) \le -\frac{1}{2}m(t)^2 + a.$$

Let (1) hold. Note that $\frac{dm(t)}{dt} \leq a$. It then follows that there is a point $x_0 \in \mathbb{S}$ such that $m(t_0) < 0$. Solving the following inequality

$$\frac{d}{dt}m(t) \leq -\frac{1}{2}m(t)^2,$$

we obtain

$$0 > \frac{1}{m(t)} \geq \frac{1}{m(t_0)} + \frac{1}{2}(t - t_0).$$

This implies that $T < t_0 - \frac{2}{m(t_0)}$ and $\lim\limits_{t\to T} m(t) = -\infty$.

Let (2) hold. Note that if $m(0) = u_0'(\xi(0)) \leq u_0'(x_0) < -\sqrt{2a}$, then $m(t) < -\sqrt{2a}$ for all $t \in [0, T)$. From the above inequality we obtain

$$\frac{m(0) + \sqrt{2a}}{m(0) - \sqrt{2K}} e^{\sqrt{2a}\ t} - 1 \leq \frac{2\sqrt{2a}}{m(t) - \sqrt{2a}} \leq 0.$$

Since $0 < \frac{m(0)+\sqrt{2a}}{m(0)-\sqrt{2a}} < 1$, there exists

$$0 < T \leq \frac{1}{\sqrt{2a}} \ln \frac{m(0) - \sqrt{2a}}{m(0) + \sqrt{2a}},$$

such that $\lim_{t\to T} m(t) = -\infty$. Theorem 3.5 and Remark 3.2 imply that the corresponding solution to (2.3) blows up in finite time if condition (1) or condition (2) holds.

Let (3) hold. We use the integrated representation (2.1). Let $h(t) = u_x(t, q(t, x_0))$, $\gamma(t) = \rho(t, q(t, x_0))$, where $q(t, x)$ is the solution of Eq.(3.9). By Eq.(3.9) we can obtain

$$\frac{dh}{dt} = (u_{tx} + uu_{xx})(t, q(t, x_0)).$$

Evaluating (2.1) at $(t, q(t, x_0))$ we get

$$\frac{d}{dt}h(t) = -\frac{1}{2}\gamma(t)^2 - \frac{1}{2}h(t)^2.$$

By $\gamma(0) \neq 0$, we infer from Lemmas 3.5-3.6 that $\gamma(t) \neq 0$ for all $t \in [0, T)$. Since $h(0) \leq 0$ and $\frac{d}{dt}h(t) < 0$, it follows that $h(t_0) < 0$ for some $t_0 \in [0, T)$. Solving the following inequality

$$\frac{d}{dt}h(t) \leq -\frac{1}{2}h(t)^2,$$

we obtain

$$0 > \frac{1}{h(t)} \geq \frac{1}{h(t_0)} + \frac{1}{2}(t - t_0).$$

This implies that $T < t_0 - \frac{2}{h(t_0)}$ and $\lim\limits_{t\to T} h(t) = -\infty$. In view of Theorem 3.5 and Remark 3.2, this completes the proof of the theorem.

Corollary 4.2 *Assume $k = -1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s > \frac{5}{2}$, and T be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (2.3) with the initial data z_0. The corresponding solution to (2.3) blows up in finite time if one of the following conditions holds: (1) $a > 0$ and $u_0'(0) < -\sqrt{2a}$, (2) $a = 0$ and $u_0'(0) \leq 0$, $\rho_0(0) \neq 0$.*

5 Global Existence

In this section, we will present a global existence result.

Theorem 5.1 *Assume $k = 1$. Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, where $s = 2$ or $s \geq 3$ and T be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with the initial data z_0. If $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$, then the corresponding solutions z exist globally in time.*

Proof By Lemma 3.5, we know that $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t, x) = exp\left(\int_0^t u_x(s, q(s, x))ds\right) > 0, \quad \forall\ (t, x) \in [0, T) \times \mathbb{R}.$$

Moreover,

$$\inf_{y\in\mathbb{S}} u_x(t, y) = \inf_{x\in\mathbb{R}} u_x(t, q(t, x)),\ \forall\ t \in [0, T). \tag{5.1}$$

Set $M(t, x) = u_x(t, q(t, x))$ and $\alpha(t, x) = \rho(t, q(t, x))$ for $t \in [0, T)$ and $x \in \mathbb{R}$. By (1.1) and Eq.(3.9), we have

$$\frac{\partial M}{\partial t} = (u_{tx} + uu_{xx})(t, q(t, x)) \ \text{ and } \ \frac{\partial \alpha}{\partial t} = -\alpha M. \tag{5.2}$$

Evaluating (2.1) at $(t, q(t, x))$ we get

$$\partial_t M(t, x) = -\frac{1}{2}M(t, x)^2 + \frac{1}{2}\alpha(t, x)^2 + a. \tag{5.3}$$

By Lemmas 3.5-3.6, we know that $\alpha(t, x)$ has the same sign with $\alpha(0, x) = \rho_0(x)$ for every $x \in \mathbb{R}$. Moreover, there is a constant $\beta > 0$ such that $\inf_{x\in\mathbb{R}} |\alpha(0, x)| = \inf_{x\in\mathbb{S}} |\rho_0(x)| \geq \beta > 0$ since $\rho_0(x) \neq 0$ for all $x \in \mathbb{S}$ and $\mathbb{S}$ is a compact set. Thus,

$$\alpha(t, x)\alpha(0, x) > 0, \quad \forall x \in \mathbb{R}.$$

Next, we consider the following Lyapunov function first introduced in [4].

$$w(t, x) = \alpha(t, x)\alpha(0, x) + \frac{\alpha(0, x)}{\alpha(t, x)}(1 + M^2), \quad (t, x) \in [0, T) \times \mathbb{R}. \tag{5.4}$$

By Sobolev's imbedding theorem, we have

$$\begin{aligned} 0 < w(0, x) &= \alpha(0, x)^2 + 1 + M(0, x)^2 \\ &= \rho_0(x)^2 + 1 + u_{0,x}(x)^2 \\ &\leq 1 + \max_{x\in\mathbb{S}}(\rho_0(x)^2 + u_{0,x}(x)^2) := C_1. \end{aligned} \tag{5.5}$$

Differentiating (5.4) with respect to t and using (5.2)-(5.3), we obtain

$$
\begin{aligned}
\frac{\partial w}{\partial t}(t,x) &= \frac{\alpha(0,x)}{\alpha(t,x)} M(t,x)\,(2a+1) \\
&\leq |1+2a| \frac{\alpha(0,x)}{\alpha(t,x)}(1+M^2) \\
&\leq |1+2a| w(t,x).
\end{aligned}
$$

By Gronwall's inequality, the above inequality and (5.5), we have

$$
w(t,x) \leq w(0,x) e^{|1+2a|t} \leq C_1 e^{|1+2a|t}
$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. On the other hand,

$$
w(t,x) \geq 2\sqrt{\alpha^2(0,x)(1+M^2)} \geq 2\beta |M(t,x)|, \quad \forall \ (t,x) \in [0,T) \times \mathbb{R}.
$$

Thus,

$$
M(t,x) \geq -\frac{1}{2\beta} w(t,x) \geq -\frac{1}{2\beta} C_1 e^{|1+2a|t}
$$

for all $(t,x) \in [0,T) \times \mathbb{R}$. Then by (5.1) and the above inequality, we have

$$
\liminf_{t \to T} \inf_{y \in \mathbb{S}} u_x(t,y) = \liminf_{t \to T} \inf_{x \in \mathbb{R}} u_x(t,q(t,x)) \geq -\frac{1}{2\beta} C_1 e^{|1+2a|t}.
$$

This completes the proof by using Theorem 3.2 and Remark 3.2.

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