DENOMINATOR IDENTITY FOR AFFINE LIE SUPERALGEBRAS WITH ZERO DUAL COXETER NUMBER

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Abstract. We prove a denominator identity for non-twisted affine Lie superalgebras with zero dual Coxeter number.

0. Introduction

0.1. Let $\mathfrak{g}$ be a complex finite-dimensional contragredient Lie superalgebra. These algebras were classified by V. Kac in [K1] and the list (excluding Lie algebras) consists of four series: $A(m|n), B(m|n), C(m), D(m|n)$ and the exceptional algebras $D(2,1,a), F(4), G(3)$. The finite-dimensional contragredient Lie superalgebras with zero Killing form (or, equivalently, with dual Coxeter number equal to zero) are $A(n|n), D(n|n+1)$ and $D(2,1,a)$.

Denote by Δ_{+0} (resp., Δ_{+1}) the set of positive even (resp., odd) roots of $\mathfrak{g}$. The Weyl denominator R and the affine Weyl denominator $\hat{R}$ are given by the following formulas

$$R=\frac{R_0}{R_1},\quad \hat{R}=\frac{\hat{R}_0}{\hat{R}_1},$$

where

$$R_0:=\prod_{\alpha\in\Delta_{+0}}(1-e^{-\alpha}),\quad \hat{R}_0:=R_0\cdot\prod_{k=1}^{\infty}(1-q^k)^{\operatorname{rank}\mathfrak{g}}\prod_{\alpha\in\Delta_0}(1-q^ke^{-\alpha}),$$
$$R_1:=\prod_{\alpha\in\Delta_{+1}}(1+e^{-\alpha}),\quad \hat{R}_1:=R_1\cdot\prod_{k=1}^{\infty}\prod_{\alpha\in\Delta_1}(1+q^ke^{-\alpha}).$$

Let $\hat{\mathfrak{g}}$ be the non-twisted affinization of $\mathfrak{g}$, $\hat{\mathfrak{h}}$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$ and $\hat{\Delta}_+$ be the set of positive roots of $\hat{\mathfrak{g}}$. The affine Weyl denominator is the Weyl denominator of $\hat{\mathfrak{g}}$. Let $\hat{\rho}\in\hat{\mathfrak{h}}$ be such that $2(\hat{\rho},\alpha)=(\alpha,\alpha)$ for each simple root $\alpha\in\hat{\Delta}_+$.

If $\mathfrak{g}$ has a non-zero Killing form, the affine denominator identity, stated in [KW] and proven in [KW],[G2], takes the form

$$\hat{R}e^{\hat{\rho}}=\sum_{w\in T'}w(Re^{\hat{\rho}}), \tag{1}$$

where T' is the affine translation group corresponding to the *"largest"* root subsystem of Δ_0 (see Section 1.2.1 below). The affine denominator identity for strange Lie superalgebras $Q(n)$, which are not contragredient, was stated in [KW] and proven in [Z].

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Suppose $\mathfrak{g}$ has zero dual Coxeter number, that is $\mathfrak{g}$ is $A(n|n), D(n|n+1)$ or $D(2,1,a)$. In this case, $\hat{\rho} = \rho = \frac{1}{2}(\sum_{\alpha\in\Delta_{+0}} \alpha - \sum_{\alpha\in\Delta_{+1}} \alpha)$. In this paper we will prove the following formulas

$$(2) \qquad \begin{array}{ll} \hat{R}e^{\hat{\rho}} \cdot f(q, e^{\mathfrak{str}}) = \sum_{w\in T'} w(Re^{\hat{\rho}}) & \text{for } A(n|n), \\ \hat{R}e^{\hat{\rho}} \cdot f(q) = \sum_{w\in T'} w(Re^{\hat{\rho}}) & \text{for } D(n+1|n), D(2,1,a), \end{array}$$

where T' is the affine translation group corresponding to the *"smallest"* root subsystem of Δ_0 (see 0.2 below) and $f(q, e^{\mathfrak{str}}), f(q)$ are given by the formulas (3) below. The affine denominator identity for $\mathfrak{gl}(2|2)$ was stated by V. Kac and M. Wakimoto in [KW] and proven in [G3] (the proof in [G3] is different from the proof presented below).

In order to write down $f(q)$, we introduce the following infinite products after [DK]: for a parameter q and a formal variable x we set

$$(1+x)_q^\infty := \prod_{k=0}^{\infty}(1+q^k x), \quad \text{and} \quad (1-x)_q^\infty := \prod_{k=0}^{\infty}(1-q^k x).$$

These infinite products converge for any $x \in \mathbb{C}$ if the parameter q is a real number $0 < q < 1$. In particular, they are well defined for $0 < x = q < 1$ and $(1\pm q)_q^\infty := \prod_{n=1}^{\infty}(1\pm q^n)$.

For $A(n|n) = \mathfrak{gl}(n|n)$ denote by $\mathfrak{str}$ the restriction of the supertrace to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (thus $\mathfrak{str} \in \mathfrak{h}^*$). One has

$$(3) \qquad \begin{array}{ll} f(q, e^{\mathfrak{str}}) = \frac{(1-q(-1)^n e^{\mathfrak{str}})_q^\infty \cdot (1-q(-1)^n e^{-\mathfrak{str}})_q^\infty}{((1-q)_q^\infty)^2} & \text{for } \mathfrak{gl}(n|n), \\ f(q) = \left((1-q)_q^\infty\right)^{-1} & \text{for } D(n+1|n). \end{array}$$

As it was pointed by P. Etingof, the terms $f(q, e^{\mathfrak{str}}), f(q)$ can be interpreted using "degenerate" cases $n = 1$; for example, for $\mathfrak{gl}(1|1)$ we obtain the formula

$$\hat{R}e^{\hat{\rho}} = \frac{((1-q)_q^\infty)^2}{(1+qe^{\mathfrak{str}})_q^\infty \cdot (1+qe^{-\mathfrak{str}})_q^\infty} Re^{\hat{\rho}},$$

which is trivial since $\mathfrak{gl}(1|1)$ has the only positive root $\beta = \mathfrak{str}$, which is odd.

Since $\mathfrak{sl}(n|n) = \{a \in \mathfrak{gl}(n|n) |\ \mathfrak{str}(a) = 0\}$ and $\operatorname{rank} \mathfrak{sl}(n|n) = 2n-1 = \operatorname{rank} \mathfrak{gl}(n|n) - 1$, one has

$$f(q) = \left\{ \begin{array}{ll} (1-q)_q^\infty & \text{for } \mathfrak{sl}(2n|2n), \\ \frac{((1+q)_q^\infty)^2}{(1-q)_q^\infty} & \text{for } \mathfrak{sl}(2n+1|2n+1). \end{array} \right.$$

The root datum of $D(2,1,a)$ is the same as the root datum of $D(2|1)$ so the affine denominator identity for $D(2,1,a)$ is the same as the affine denominator identity for $D(2|1)$.

As it is shown in [KW], the evaluation of the affine denominator identity for $\mathfrak{gl}(2|2)$ (i.e., (2) for $A(1|1)$) gives the following Jacobi identity [J]:

$$\Box(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{(j+1)k} k^3 q^{jk}, \tag{4}$$

where $\Box(q) = \sum_{j\in\mathbb{Z}} q^{j^2}$ and thus the coefficient of q^m in the power series expansion of $\Box(q)^8$ is the number of representation of a given integer as a sum of 8 squares (taking into the account the order of summands).

0.2. In order to define T' for $A(n|n), D(n+1|n)$ we present the set of even roots in the form $\Delta_0 = \Delta' \coprod \Delta''$, where

$$\begin{array}{ll} \Delta' \cong \Delta'' = A_{n-1} & \text{for } A(n-1|n-1) = \mathfrak{gl}(n|n), \\ \Delta' = C_n,\ \Delta'' = D_{n+1} & \text{for } D(n+1|n). \end{array}$$

Let W' be the Weyl group of Δ' and $\hat{W}'$ be the corresponding affine Weyl group. Then $\hat{W}' = W' \ltimes T'$, where T' is a translation group, see [K2], Chapter VI. Notice that for $D(n+1|n)$ the rank of root system Δ' is smaller than the rank of Δ''; by contrast, for Lie superalgebras with non-zero Killing form, the lattice T' in (1) corresponds to the root system Δ', whose rank is not smaller than the rank of Δ'' (one has $\Delta_0 = \Delta' \coprod \Delta''$ as before). It is not possible to change T' to T'' in Identity (1) and in Identity (2) for $D(n+1|n)$, since the sum $\sum_{w\in T''} w(Re^{\rho})$ is not well defined if $\Delta' \not\supseteq \Delta''$ (see Remark 2.1.4).

We prove Identity (2) and outline a similar proof for Identity (1). The key point is Proposition 2.3.2, where it is shown that for any complex finite-dimensional contragredient Lie superalgebra, the expansion of $Y := \hat{R}^{-1} e^{-\hat{\rho}} \sum_{w\in T'} w(Re^{\hat{\rho}})$ contains only $\hat{W}$-invariant elements. This implies that $Y = f(q)$ for $\mathfrak{g} \neq \mathfrak{gl}(n|n)$ and $Y = f(q, e^{-\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. We determine $f(q)$ for $D(n+1|n)$ and $f(q, e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$ using suitable evaluations. For other finite-dimensional contragredient simple Lie superalgebras the equality $f(q) = 1$ can be obtained in two steps: first, using the Casimir operator and the fact that the dual Coxeter number is non-zero, we show that $f(q)$ is scalar; then one deduces that this scalar is equal to 1 from the denominator identity for $\mathfrak{g}$ (this is done in [G2]).

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1. Preliminary

One readily sees (for instance, [G2], 1.5) that $Re^{\hat{\rho}}$ and $\hat{R}e^{\hat{\rho}}$ do not depend on the choice of set of positive roots Δ_+ so it is enough to establish the identity for one choice of Δ_+. Similarly, it is enough to establish the identity for one choice of A_{n-1} for $\mathfrak{gl}(n|n)$. In Section 1.1 we describe our choice of the set of of positive roots for $\mathfrak{gl}(n|n), D(n+1|n)$.

In Section 1.2 we introduce notation for affine Lie superalgebra $\hat{\mathfrak{g}}$. In Section 1.3 we introduce the algebra $\mathcal{R}$ of formal power series in which we expand R and $\hat{R}$.

1.1. **Root systems.** Let $\mathfrak{g}$ be $\mathfrak{gl}(n|n)$ or $D(n|n+1)$ and let $\mathfrak{h}$ be its Cartan subalgebra. We fix the following sets of simple roots:

$$\Pi=\{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\dots,\varepsilon_n-\delta_n\} \text{ for } \mathfrak{gl}(n|n),$$
$$\Pi=\{\varepsilon_1-\delta_1,\delta_1-\varepsilon_2,\varepsilon_2-\delta_2,\dots,\varepsilon_n-\delta_n,\delta_n\pm\varepsilon_{n+1}\} \text{ for } D(n+1|n).$$

We fix a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ and denote by $(-,-)$ the induced non-degenerate symmetric bilinear form on $\mathfrak{h}^*$; we normalize the form in such a way that $-(\varepsilon_i,\varepsilon_j)=(\delta_i,\delta_j)=\delta_{ij}$; notice that $\{\varepsilon_i,\delta_i|1\leq i\leq n\}$ (resp., $\{\varepsilon_j,\delta_i|1\leq i\leq n,1\leq j\leq n+1\}$ is an orthogonal basis of $\mathfrak{h}^*$ for $\mathfrak{gl}(n|n)$ (resp., for $D(n+1|n)$).

For this choice one has

$$\Delta_{0+}=\{\varepsilon_i-\varepsilon_j\}_{1\leq i<j\leq n}\coprod\{\delta_i-\delta_j\}_{1\leq i<j\leq n} \text{ for } \mathfrak{gl}(n|n),$$
$$\Delta_{1+}=\{\varepsilon_i-\delta_j\}_{1\leq i\leq j\leq n}\cup\{\delta_i-\varepsilon_j\}_{1\leq i<j\leq n} \text{ for } \mathfrak{gl}(n|n),$$
$$\Delta_{0+}=\{\varepsilon_i\pm\varepsilon_j\}_{1\leq i<j\leq n+1}\coprod\{\delta_s\pm\delta_t\}_{1\leq s<t\leq n}\cup\{2\delta_s\}_{1\leq s\leq n} \text{ for } D(n+1|n),$$
$$\Delta_{1+}=\{\varepsilon_i-\delta_s\}_{1\leq i\leq s\leq n}\cup\{\delta_s-\varepsilon_j\}_{1\leq s<j\leq n+1}\cup\{\delta_i+\varepsilon_j\}_{1\leq i\leq n;1\leq j\leq n+1} \text{ for } D(n+1|n).$$

For $D(n+1|n)$ one has $\rho=0$ for $D(n+1|n)$. For $\mathfrak{gl}(n|n)$ one has $\mathfrak{str}=\sum_{i=1}^n(\varepsilon_i-\delta_i)$ and $\rho=-\frac{1}{2}\mathfrak{str}$.

Recall that $\mathfrak{sl}(n|n)=\{a\in\mathfrak{gl}(n|n)|\ \mathfrak{str}(a)=0\}$ and so $\mathfrak{h}^*$ for $\mathfrak{sl}(n|n)$ is the quotient of $\mathfrak{h}^*$ for $\mathfrak{gl}(n|n)$ by $\mathbb{C}\mathfrak{str}$.

By above, Δ_0 is the union of two irreducible root systems, and we write $\Delta_0=\Delta''\coprod\Delta'$, where Δ'' lies in the span of ε_is and Δ' lies in the span of δ_is (this notation is compatible with notations in Section 0.2).

1.2. **Non-twisted affinization.** Let $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be any complex finite-dimensional contragredient Lie superalgebra with a fixed triangular decomposition, and let Δ_+ be its set of positive roots. Let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$ and let $\hat{\mathfrak{h}}$ be its Cartan subalgebra, see [K2], Chapter VI. Recall that $\mathfrak{g}=[\mathfrak{g},\mathfrak{g}]\oplus\mathbb{C}D$ for some $D\in\hat{\mathfrak{h}}$. Let $\hat{\Delta}=\hat{\Delta}_0\coprod\hat{\Delta}_1$ be the set of roots of $\hat{\mathfrak{g}}$. We set

$$\hat{\Delta}^+=\Delta_+\cup(\cup_{k=1}^{\infty}\{\alpha+k\delta|\ \alpha\in\Delta\})\cup(\cup_{k=1}^{\infty}\{k\delta\}),$$

where δ is the minimal imaginary root. Let W (resp., $\hat{W}$) be the Weyl group of Δ_0 (resp., $\hat{\Delta}_0$). One has $(\hat{\mathfrak{h}}^*)^{\hat{W}}=\mathbb{C}\delta$ for $\mathfrak{g}\neq\mathfrak{gl}(n|n)$ and $(\hat{\mathfrak{h}}^*)^{\hat{W}}=\mathbb{C}\delta\oplus\mathbb{C}\mathfrak{str}$ for $\mathfrak{g}=\mathfrak{gl}(n|n)$.

We extend the non-degenerate symmetric invariant bilinear form from $\mathfrak{g}$ to $\hat{\mathfrak{g}}$ and denote by $(-,-)$ the induced non-degenerate symmetric bilinear form on $\hat{\mathfrak{h}}^*$ (the above-mentioned form on $\mathfrak{h}^*$ is induced by this form on $\hat{\mathfrak{h}}^*$). For $A\subset\hat{\mathfrak{h}}^*$ we set $A^{\perp}=\{\mu\in\hat{\mathfrak{h}}^*|\ \forall\nu\in A\ (\mu,\nu)=0\}$.

1.2.1. In Section 1.1 we introduced the root systems Δ', Δ'' for $\mathfrak{g} = \mathfrak{gl}(n|n), D(n+1|n)$. For $\mathfrak{g} \neq \mathfrak{gl}(n|n), D(n+1|n), D(2,1,a)$ the Killing form κ is non-zero; in this case, we introduce Δ', Δ'' by the formulas: $\Delta' := \{\alpha|\kappa(\alpha,\alpha) > 0\}$, $\Delta'' := \{\alpha|\kappa(\alpha,\alpha) < 0\}$. One has $\Delta_0 = \Delta' \coprod \Delta''$ and $\Delta'' = \emptyset$ if Δ_0 is irreducible. Let W' (resp., W'') be the Weyl group of Δ' (resp., Δ''). One has $W = W' \times W''$.

1.2.2. Now that we have introduced the decomposition $\Delta_0 = \Delta' \coprod \Delta''$ for any complex finite-dimensional contragredient Lie superalgebra, we denote by $\hat{W}'$ the Weyl group of the affine root system $\hat{\Delta}'$. Recall that $\hat{W}' = W' \ltimes T'$, where T' is a translation group (see [K2], Chapter VI).

1.2.3. For $N \subset \hat{\mathfrak{h}}^*$ we use the notation $\mathbb{Z}N$ for the set $\sum_{\mu\in N} \mathbb{Z}\mu$. Set

$$Q^+ := \sum_{\mu\in\Delta_+} \mathbb{Z}_{\geq 0}\mu, \quad Q := \mathbb{Z}\Delta, \quad \hat{Q}^{\pm} := \pm \sum_{\mu\in\hat{\Delta}_+} \mathbb{Z}_{\geq 0}\mu, \quad \hat{Q} := \mathbb{Z}\hat{\Delta}_+.$$

We introduce the standard partial order on $\hat{\mathfrak{h}}^*$: $\mu \leq \nu$ if $(\nu - \mu) \in \hat{Q}^+$.

1.3. **Algebra $\mathcal{R}$.** We are going to use notation of [G2], 1.4, which we recall below. Retain notation of Section 1.2.

1.3.1. Call a $\hat{Q}^+$-*cone* a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

For a formal sum of the form $Y := \sum_{\nu\in\hat{\mathfrak{h}}^*} b_\nu e^\nu$, $b_\nu \in \mathbb{Q}$ define the *support* of Y by $\mathrm{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^* |\ b_\nu \neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu\in\hat{Q}^+} b_\nu e^{\lambda-\nu}$, where $\lambda \in \hat{\mathfrak{h}}^*$, $b_\nu \in \mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y = \sum_{\nu\in\hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of $\hat{Q}^+$-cones.

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

1.3.2. *Action of the Weyl group.* For $w \in \hat{W}$ set $w(\sum_{\nu\in\hat{\mathfrak{h}}^*} b_\nu e^\nu) := \sum_{\nu\in\hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. By above, $wY \in \mathcal{R}$ iff $w(\mathrm{supp}\, Y)$ is a subset of a finite union of $\hat{Q}^+$-cones. For each subgroup $\tilde{W}$ of $\hat{W}$ we set $\mathcal{R}_{\tilde{W}} := \{Y \in \mathcal{R}|\ wY \in \mathcal{R}$ for each $w \in \tilde{W}\}$; notice that $\mathcal{R}_{\tilde{W}}$ is a subalgebra of $\mathcal{R}$.

1.3.3. *Infinite products.* An infinite product of the form $Y = \prod_{\nu\in X}(1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q}$, $r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \hat{\Delta}_+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu\in X}(1 + a_\nu e^{-w\nu})^{r(\nu)},$$

is again an infinite product of the above form, since the set $w\hat{\Delta}_+\setminus\hat{\Delta}_+$ is finite (see for example [G2], Lemma 1.2.8). Hence $\mathcal{Y}$ is a $\hat{W}$-invariant multiplicative subset of $\mathcal{R}_{\hat{W}}$.

The elements of $\mathcal{Y}$ are invertible in $\mathcal{R}$: using the geometric series we can expand Y^{-1} (for example, $(1-e^{\alpha})^{-1}=-e^{-\alpha}(1-e^{-\alpha})^{-1}=-\sum_{i=1}^{\infty}e^{-i\alpha}$).

1.3.4. *The subalgebra $\mathcal{R}'$.* Denote by $\mathcal{R}'$ the localization of $\mathcal{R}_{\hat{W}}$ by $\mathcal{Y}$. By above, $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$. Observe that $\mathcal{R}'\not\subset\mathcal{R}_{\hat{W}}$: for example, $(1-e^{-\alpha})^{-1}\in\mathcal{R}'$, but $(1-e^{-\alpha})^{-1}=\sum_{j=0}^{\infty}e^{-j\alpha}\not\in\mathcal{R}_{\hat{W}}$. We extend the action of $\hat{W}$ from $\mathcal{R}_{\hat{W}}$ to $\mathcal{R}'$ by setting $w(Y^{-1}Y'):=(wY)^{-1}(wY')$ for $Y\in\mathcal{Y},\ Y'\in\mathcal{R}_{\hat{W}}$.

Notice that an infinite product of the form $Y=\prod_{\nu\in X}(1+a_\nu e^{-\nu})^{r(\nu)}$, where a_ν, X are as above and $r(\nu)\in\mathbb{Z}$, lies in $\mathcal{R}'$ and $wY=\prod_{\nu\in X}(1+a_\nu e^{-w\nu})^{r(\nu)}$. The support $\text{supp}(Y)$ has a unique maximal element (with respect to the standard partial order) and this element is given by the formula

$$\max\text{supp}(Y)=-\sum_{\nu\in X\setminus\hat{\Delta}_+:a_\nu\neq 0}r_\nu\nu.$$

1.3.5. Let $\tilde{W}$ be a subgroup of $\hat{W}$. For $Y\in\mathcal{R}'$ we say that Y *is $\tilde{W}$-invariant (resp., $\tilde{W}$-anti-invariant)* if $wY=Y$ (resp., $wY=\text{sgn}(w)Y$) for each $w\in\tilde{W}$.

Let $Y=\sum a_\mu e^\mu\in\mathcal{R}_{\tilde{W}}$ be $\tilde{W}$-anti-invariant. Then $a_{w\mu}=(-1)^{\text{sgn}(w)}a_\mu$ for each μ and $w\in\tilde{W}$. In particular, $\tilde{W}\,\text{supp}(Y)=\text{supp}(Y)$, and, moreover, for each $\mu\in\text{supp}(Y)$ one has $\text{Stab}_{\tilde{W}}\,\mu\subset\{w\in\tilde{W}|\ \text{sgn}(w)=1\}$. The condition $Y\in\mathcal{R}_{\tilde{W}}$ is essential: for example, for $\tilde{W}=\{\text{id},s_\alpha\}$, the expressions $Y:=e^\alpha-e^{-\alpha}$, $Y^{-1}=e^{-\alpha}(1-e^{-2\alpha})^{-1}$ are $\tilde{W}$-anti-invariant, $\text{supp}(Y)=\{\pm\alpha\}$ is s_α-invariant, but $\text{supp}(Y^{-1})=\{-\alpha,-3\alpha,\dots\}$ is not s_α-invariant.

For $Y\in\mathcal{R}_{\tilde{W}}$ such that each $\tilde{W}$-orbit in $\hat{\mathfrak{h}}^*$ has a finite intersection with $\text{supp}(Y)$, introduce the sum

$$\mathcal{F}_{\tilde{W}}(Y):=\sum_{w\in\tilde{W}}\text{sgn}(w)wY.$$

This sum is well defined, but does not always belong to $\mathcal{R}$. For $Y=\sum a_\mu e^\mu$ one has $\mathcal{F}_{\tilde{W}}(Y)=\sum b_\mu e^\mu$, where $b_\mu=\sum_{w\in\tilde{W}}\text{sgn}(w)a_{w\mu}$; in particular, $b_\mu=\text{sgn}(w)b_{w\mu}$ for each $w\in\tilde{W}$. One has

$$Y\in\mathcal{R}_{\tilde{W}}\ \&\ \mathcal{F}_{\tilde{W}}(Y)\in\mathcal{R}\ \Longrightarrow\ \begin{cases}\text{supp}(\mathcal{F}_{\tilde{W}}(Y))\text{ is }\tilde{W}\text{-stable},\\ \mathcal{F}_{\tilde{W}}(Y)\in\mathcal{R}_{\tilde{W}};\\ \mathcal{F}_{\tilde{W}}(Y)\text{ is }\tilde{W}\text{-anti-invariant.}\end{cases}$$

We call a vector $\lambda\in\hat{\mathfrak{h}}^*$ *$\tilde{W}$-regular* if $\text{Stab}_{\tilde{W}}\,\lambda=\{\text{id}\}$, and we say that the orbit $\tilde{W}\lambda$ is $\tilde{W}$-regular if λ is $\tilde{W}$-regular (so the orbit consists of $\tilde{W}$-regular points). If $\tilde{W}$ is an affine Weyl group, then for any $\lambda\in\hat{\mathfrak{h}}^*$ the stabilizer $\operatorname{Stab}_{\tilde{W}}\lambda$ is either trivial or contains a reflection. Thus for $\tilde{W}=\hat{W}',\ \hat{W}''$ one has

$$Y\in\mathcal{R}_{\tilde{W}}\ \&\ \mathcal{F}_{\tilde{W}}(Y)\in\mathcal{R}\implies \operatorname{supp}(\mathcal{F}_{\tilde{W}}(Y))\ \text{ is a union of } \tilde{W}\text{-regular orbits.}$$

For $Y\in\mathcal{R}'$ the sum $\sum_{w\in\tilde{W}}\operatorname{sgn}(w)wY$ is not always $\tilde{W}$-anti-invariant: for example, for $\tilde{W}=\{\mathrm{id},s_\alpha\}$ one has $\sum_{w\in\tilde{W}}\operatorname{sgn}(w)w((1-e^{-\alpha})^{-1})=(1-e^{-\alpha})^{-1}-(1-e^{\alpha})^{-1}=1+2e^{-\alpha}+2e^{-2\alpha}+\ldots$, which is not $\tilde{W}$-anti-invariant.

2. Proof

As it is pointed out in Section 1, it is enough to establish the denominator identity for a particular choice of Δ_+ and we do this for the choice described in Section 1.1. Recall that the group T' was introduced in Section 1.2.2. The steps of the proof are the following.

1) In Section 2.1 we check that for $\mathfrak{g}=\mathfrak{gl}(n|n), D(n+1|n)$, the sum $\mathcal{F}_{T'}(Re^{\hat{\rho}})$ is well-defined and belongs to $\mathcal{R}$.

2) In Section 2.2 we prove the inclusions

$$\operatorname{supp}(\mathcal{F}_{T'}(Re^{\hat{\rho}})),\operatorname{supp}(\hat{R}e^{\hat{\rho}})\subset U, \tag{5}$$

where

$$U:=\{\mu\in\hat{\rho}-\hat{Q}^+|\ (\mu,\mu)=(\hat{\rho},\hat{\rho})\} \tag{6}$$

for $\mathfrak{g}=\mathfrak{gl}(n|n)$ and $D(n+1|n)$.

For simple contragredient Lie superalgebras with non-zero Killing form steps (1), (2) are performed in [G2], 2.4.

3) In Section 2.3 we show that for any finite-dimensional simple contragredient Lie superalgebra $\mathfrak{g}$ the inclusions (5) imply that $\operatorname{supp}\big(\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})\big)\subset\hat{Q}^{\hat{W}}$. As a result, $\hat{R}^{-1}e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^{\hat{\rho}})$ takes the form $f(q)$ (resp., $f(q,e^{\mathfrak{str}})$) for $\mathfrak{g}\neq\mathfrak{gl}(n|n)$ (resp., for $\mathfrak{gl}(n|n)$).

4) In Section 2.4 we compute $f(q)$ (resp., $f(q,e^{\mathfrak{str}})$) for $D(n+1|n)$ (resp., for $\mathfrak{gl}(n|n)$). This completes the proof of Identity (2).

In Section 2.5 we briefly repeat the arguments of [G2] showing that $f(q)=1$ for $\mathfrak{g}\neq\mathfrak{gl}(n|n), D(n+1|n), D(2,1,a)$. This completes the proof of Identity (1).

2.1. **Step 1.** In this subsection we show that for $\mathfrak{g}=\mathfrak{gl}(n|n), D(n+1|n)$, the sum $\mathcal{F}_{T'}(Re^{\hat{\rho}})$ is a well-defined element of $\mathcal{R}$. Since $\hat{\rho}=\rho$ is $\hat{W}$-invariant, it is enough to verify that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$.

Recall that $T'=\mathbb{Z}\{t_{\delta_i-\delta_{i+1}}\}_{i=1}^{n-1}$ for $\mathfrak{gl}(n|n)$ and $T'=\mathbb{Z}\{t_{\delta_i}\}_{i=1}^{n}$ for $D(n+1|n)$, where

$$t_\mu(\alpha)=\alpha-(\alpha,\mu)\delta\ \text{ for any } \alpha\in\hat{Q}. \tag{7}$$

2.1.1. By Section 1.3.4 one has

$$\max\operatorname{supp}\big(w(R)\big)=\sum_{\alpha\in\Delta_{0+}:w\beta<0} w\alpha-\sum_{\beta\in\Delta_{1+}:w\beta<0} w\beta.$$

For $w\in T'$ write $w=t_\mu$, where $\mu\in\mathbb{Z}\{\delta_i-\delta_{i+1}\}_{1\leq i<n}$ for $\mathfrak{gl}(n|n)$ and $\mu\in\mathbb{Z}\{\delta_i\}_{i=1}^n$ for $D(n+1|n)$. From (7) we get

$$\{\beta\in\Delta_{i+}|w\beta<0\}=\{\beta\in\Delta_{i+}|(\beta,\mu)>0\}\ \text{ for } i=0,1.$$

We obtain $\max\operatorname{supp}\big(t_\mu(R)\big)=-v(\mu)+(v(\mu),\mu)\delta$, where

$$v(\mu):=\sum_{\beta\in\Delta_{0+}:(\beta,\mu)>0}\beta-\sum_{\beta\in\Delta_{1+}:(\beta,\mu)>0}\beta.$$

In order to prove that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$ we verify that

$$(i)\ \ \forall\mu\ (v(\mu),\mu)\leq 0;\quad (ii)\ \ \forall N>0\ \ \{\mu|\ (v(\mu),\mu)\geq -N\}\ \text{ is finite.} \tag{8}$$

The condition (ii) ensures that the sum $\mathcal{F}_{T'}(R)=\sum_\mu t_\mu(R)$ is well-defined and the condition (i) means that for each μ one has

$$\max\operatorname{supp}(t_\mu(R))=-v(\mu)\leq\sum_{\beta\in\Delta_{1+}}\beta$$

so $\operatorname{supp}\big(\mathcal{F}_{T'}(R)\big)\subset\sum_{\beta\in\Delta_{1+}}\beta-\hat{Q}^+$ and thus $\mathcal{F}_{T'}(R)\in\mathcal{R}$.

2.1.2. *Case* $\mathfrak{gl}(n|n)$. Recall that $w\in T'$ has the form $w=t_\mu,\ \mu=\sum_{i=1}^n k_i\delta_i$, where the k_is are integers and $\sum_{i=1}^n k_i=0$. One has

$$\begin{array}{l}\{\alpha\in\Delta_{+0}|\ (\alpha,\mu)>0\}:=\{\delta_i-\delta_j|\ i<j,k_i>k_j\},\\ \{\alpha\in\Delta_{+1}|\ (\alpha,\mu)>0\}:=\{\varepsilon_i-\delta_j|\ k_j<0,i\leq j\}\cup\{\delta_i-\varepsilon_j|\ k_i>0,i<j\},\end{array}$$

where $1\leq i,j\leq n$.

Write $v(\mu)=v'+v''$, where $v'=\sum_{i=1}^n a_i\delta_i$ and v'' lies in the span of ε_is. By above, for $k_i>0$ one has $a_i\leq(n-i)-(n-i)=0$ and for $k_j<0$ one has $a_j\geq-(j-1)+j=1$. Therefore $(v(\mu),\mu)=\sum_{i=1}^n a_ik_i\leq\sum_{k_i<0}k_i\leq 0$ and the set $\{\mu|\ (v(\mu),\mu)\geq -N\}$ is a subset of the set $\{\mu|\ \sum_{k_i<0}k_i\geq -N\}$, which is finite for any N, because k_is are integers and $\sum_{i=1}^n k_i=0$. This establishes conditions (8).

2.1.3. *Case* $D(n+1|n)$. Recall that $w\in T'$ has the form $w=t_\mu,\ \mu=\sum k_i\delta_i$, where the k_is are integers. One has

$$\begin{array}{l}\{\alpha\in\Delta_{+0}|\ (\alpha,\mu)>0\}:=\{\delta_i-\delta_j|\ i<j,k_i>k_j\}\cup\{\delta_i+\delta_j|\ i\neq j,\ k_i+k_j>0\}\cup\{2\delta_i|\ k_i>0\},\\ \{\alpha\in\Delta_{+1}|\ (\alpha,\mu)>0\}:=\{\varepsilon_s-\delta_j|\ k_j<0,s\leq j\}\cup\{\delta_i-\varepsilon_s|\ k_i>0,i<s\}\cup\{\delta_i+\varepsilon_s|\ k_i>0\},\end{array}$$

where $1\leq i,j\leq n$ and $1\leq s\leq n+1$.

Write $v(\mu)=v'+v''$, where $v'=\sum_{i=1}^n a_i\delta_i$ and v'' lies in the span of ε_is. By above, for $k_i>0$ one has $a_i\leq (2n+1-i)-(2n+2-i)=-1$ and for $k_j<0$ one has $a_j\geq -(j-1)+j=1$. Therefore

$$(v(\mu),\mu)=\sum_{i=1}^n a_ik_i\leq -\sum_{k_i>0}k_i+\sum_{k_j<0}k_j=-\sum_{1=1}^n|k_i|\leq 0$$

so the set $\{\mu|\ (v(\mu),\mu)\geq -N\}$ is a subset of the set $\{\mu|\ \sum_{i=1}^n|k_i|\leq N\}$, which is finite for any N. This establishes conditions (8).

2.1.4. *Remark.* For $\mathfrak{gl}(n|n)$ one can interchange Δ' and Δ'' so the sum $\mathcal{F}_{T''}(R)$ is well-defined. One readily sees that $\mathcal{F}_{T''}(R)$ is not well-defined for $D(n+1|n)$. For instance, for $n>1$, for each $k>0$ one has $v(-2k\varepsilon_1)=0$ so $\max\operatorname{supp}\big(t_{-2k\varepsilon_1}(R)\big)=0$ and the sum $\sum_{k=1}^\infty t_{-2k\varepsilon_1}(R)$ is not well-defined; hence $\mathcal{F}_{T''}(R)$ is not well-defined as well.

2.2. **Step 2.** By Section 1.3.3, $\hat{R}$ is an invertible element of $\mathcal{R}'$. From representation theory we know that since $\hat{\mathfrak{g}}$ admits a Casimir element [K2], Chapter II, the character of the trivial $\hat{\mathfrak{g}}$-module is a linear combination of the characters of Verma $\hat{\mathfrak{g}}$-modules $M(\lambda)$, where $\lambda\in-\hat{Q}$ are such that $(\lambda+\hat\rho,\lambda+\hat\rho)=(\hat\rho,\hat\rho)$. Since the character of $M(\lambda)$ is equal to $\hat{R}^{-1}e^\lambda$, we obtain

$$1=\sum_{\substack{\lambda\in\hat{Q}^-,\\(\lambda+\hat\rho,\lambda+\hat\rho)=(\hat\rho,\hat\rho)}} a_\lambda\hat{R}^{-1}e^\lambda,$$

where $a_\lambda\in\mathbb{Z}$. This can be rewritten as

$$\hat{R}e^{\hat\rho}=\sum_{\substack{\lambda\in\hat\rho-\hat{Q}^+,\\(\lambda,\lambda)=(\hat\rho,\hat\rho)}} a_\lambda e^\lambda,$$

that is $\operatorname{supp}(\hat{R})\subset U$, see (6) for notation.

It remains to verify the inclusion $\operatorname{supp}\big(\mathcal{F}_{T'}(Re^{\hat\rho})\big)\subset U$. The denominator identity for $\mathfrak{g}$ (see [KW],[G1]) takes the form

$$Re^\rho=\mathcal{F}_{W''}\Big(\frac{e^\rho}{\prod_{\beta\in S}(1+e^{-\beta})}\Big),$$

where $S:=\{\varepsilon_i-\delta_i\}_{i=1}^n$ (the identity for $\mathfrak{gl}(n|n)$ immediately follows from the identity for $\mathfrak{sl}(n|n)$). Since $\rho=\hat\rho$ is $\hat{W}$-invariant, this implies

$$t_\mu(Re^{\hat\rho})=e^{\hat\rho}\sum_{w\in W''}\operatorname{sgn}(w)\prod_{\beta\in S}(1+e^{-t_\mu w\beta})^{-1}.$$

For each $t_\mu\in T'$ and $w\in W''$ one has

$$\operatorname{supp}\Big(\prod_{\beta\in S}(1+e^{-t_\mu w\beta})^{-1}\Big)\subset V,\ \text{where } V:=\mathbb{Z}\{t_\mu w\beta|\ \beta\in S\}\cap\hat{Q}^-.$$

Since $(t_\mu w\beta, t_\mu w\beta') = (\beta,\beta') = (t_\mu w\beta,\hat\rho) = (\hat\rho,\beta) = 0$ for any $\beta,\beta'\in S$, one has $(V,V) = (V,\hat\rho) = 0$. Therefore $V+\hat\rho\subset U$ so $\operatorname{supp}\bigl(t_\mu(Re^{\hat\rho})\bigr)\subset U$ for each μ. This establishes the required inclusion $\operatorname{supp}\bigl(\mathcal{F}_{T'}(Re^{\hat\rho})\bigr)\subset U$ and completes the proof of (5).

2.3. **Step 3.** Let us deduce the inclusion $\operatorname{supp}(\hat R^{-1}e^{\hat\rho}\cdot\mathcal{F}_{T'}(Re^{\hat\rho}))\subset(\hat Q^-)^{\hat W}$ from (5).

2.3.1. **Lemma.** *For any simple finite-dimensional contragredient Lie superalgebra $\mathfrak{g}$ the term $\mathcal{F}_{T'}(Re^{\hat\rho})$ is a $\hat W'$-anti-invariant element of $\mathcal{R}_{\hat W'}$.*

Proof. In the light of Section 1.3.5, it is enough to present $\mathcal{F}_{T'}(Re^{\hat\rho})$ in the form $\mathcal{F}_{\hat W'}(Y)$ for some $Y\in\mathcal{R}_{\hat W}$. Let R_0', R_0'' be the Weyl denominators for Δ',Δ'' respectively (i.e., $R_0'=\prod_{\alpha\in\Delta'_+}(1-e^{-\alpha})$). Below we will prove the formula

$$\mathcal{F}_{T'}(Re^{\hat\rho}) = \mathcal{F}_{\hat W'}\Bigl(\frac{R_0''e^{\hat\rho}}{R_1}\Bigr). \tag{9}$$

By Section 1.3.3, $R_1^{-1}R_0''e^{\hat\rho}\in\mathcal{R}_{\hat W}$, so the formula establishes the required assertion.

Let us show that the right-hand side of (9) is well-defined. Since R_0'' is $\hat W'$-invariant, it is enough to verify that $\mathcal{F}_{\hat W'}\bigl(e^{\hat\rho}R_1^{-1}\bigr)$ is a well-defined element of $\mathcal{R}$. For $\mathfrak{g}\neq\mathfrak{gl}(n|n), D(n+1|n)$ this is proven in [G2], 2.4.1 (i). Consider the case $\mathfrak{g}=\mathfrak{gl}(n|n), D(n+1|n)$. Since $\hat\rho$ is $\hat W$-invariant, it is enough to check that $\mathcal{F}_{\hat W'}(R_1^{-1})$ is a well-defined element of $\mathcal{R}$. By Section 1.3.4, for each $w\in\hat W'$ one has

$$\max\operatorname{supp}\bigl(w(R_1^{-1})\bigr) = \sum_{\beta\in\Delta_{1+}:w\beta<0} w\beta.$$

In particular, $\operatorname{supp}\bigl(w(R_1^{-1})\bigr)\subset\hat Q^-$, so, if the sum $\mathcal{F}_{\hat W'}(R_1^{-1})=\sum_{w\in\hat W'}\operatorname{sgn} w\cdot w(R_1^{-1})$ is well-defined, it lies in $\mathcal{R}$. In order to see that this sum is well-defined let us check that for each $\nu\in\hat Q^-$ the set

$$X(\nu) := \{w\in\hat W'|\ \sum_{\beta\in\Delta_{1+}:w\beta<0} w\beta\geq\nu\}$$

is finite. One has

$$X(\nu)\subset\{w\in\hat W'|\ \forall\beta\in\Delta_{1+}\ w\beta\geq\nu\}.$$

Write $\nu=-k\delta+\nu'$, where $k\geq 0,\ \nu'\in Q$, and write $w\in X(\nu)$ in the the form $w=t_\mu y$, where $t_\mu\in T', y\in W'$. Since $w\beta=y\beta-(y\beta,\mu)\delta$ for $\beta\in\Delta_{1+}$, one has $(y\beta,\mu)\geq -k$ for each $\beta\in\Delta_{1+}$. Since $\{\varepsilon_i-\delta_i,\delta_i-\varepsilon_{i+1}\}\subset\Delta_{1+}$, this gives $|(\mu,y\delta_i)|\leq k$ for $i=1,\dots,n$. Combining the facts that W' is a subgroup of signed permutation of $\{\delta_j\}_{j=1}^n$ and that (μ,δ_i) is integral for each i, we conclude that $X(\nu)$ is finite. Thus $\mathcal{F}_{\hat W'}\bigl(\frac{R_0''}{R_1}\bigr)$ is a well-defined element of $\mathcal{R}$.

Now let us prove the formula (9). Recall that $\rho=\rho_0'+\rho_0''-\rho_1$, where

$$\rho_0':=\sum_{\alpha\in\Delta_{0+}'}\alpha/2,\quad \rho_0'':=\sum_{\alpha\in\Delta_{0+}''}\alpha/2,\quad \rho_1:=\sum_{\beta\in\Delta_{1+}}\beta/2.$$

The Weyl denominator identity for Δ_0'' takes the form

$$R_0'e^{\rho_0'}=\mathcal{F}_{W'}(e^{\rho_0'}).$$

Since $R_1e^{\rho_1}=\prod_{\beta\in\Delta_{1+}}(e^{\beta/2}+e^{-\beta/2})$ is W-invariant and $R_0''e^{\rho_0''}$ is W'-invariant, we get

$$Re^{\rho}=\frac{R_0''e^{\rho_0''}}{R_1e^{\rho_1}}\cdot\mathcal{F}_{W'}(e^{\rho_0'})=\mathcal{F}_{W'}\big(\frac{e^{\rho_0'}R_0''e^{\rho_0''}}{R_1e^{\rho_1}}\big)=\mathcal{F}_{W'}\big(\frac{R_0''e^{\rho}}{R_1}\big).$$

Using the W-invaraince of $\hat{\rho}-\rho$, we obtain

$$\mathcal{F}_{T'}\big(Re^{\hat{\rho}}\big)=\mathcal{F}_{T'}\big(\mathcal{F}_{W'}\big(\frac{R_0''e^{\hat{\rho}}}{R_1}\big)\big)=\mathcal{F}_{\hat{W}'}\big(\frac{R_0''e^{\hat{\rho}}}{R_1}\big)$$

as required. This completes the proof. □

2.3.2. **Proposition.** *Let $\mathfrak{g}$ be a simple finite-dimensional contragredient Lie superalgebra. One has*

$$\mathrm{supp}(\hat{R}^{-1}e^{\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}}))\subset(\hat{Q}^-)^{\hat{W}}=\hat{Q}^-\cap\hat{Q}^{\perp}.$$

Proof. By Section 2.1.1, $\mathcal{F}_{T'}(Re^{\hat{\rho}})\in\mathcal{R}$; by Section 1.3.3, $\hat{R}^{-1}\in\mathcal{R}$ so

$$Y:=\hat{R}^{-1}e^{-\hat{\rho}}\cdot\mathcal{F}_{T'}(Re^{\hat{\rho}})\in\mathcal{R}.$$

The affine root system $\hat{\Delta}'$ is a subsystem of $\hat{\Delta}_0$. Set $\hat{\Delta}'_+=\hat{\Delta}'\cap\hat{\Delta}_+$ and let $\hat{\Pi}'$ be the corresponding set of simple roots. Fix $\hat{\rho}'\in\hat{\mathfrak{h}}^*$ such that $2(\hat{\rho}',\alpha)=(\alpha,\alpha)$ for each $\alpha\in\hat{\Pi}'$.

It is easy to see that $\hat{R}_0e^{\hat{\rho}'},\hat{R}e^{\hat{\rho}}$ are $\hat{W}'$-anti-invariant elements of $\mathcal{R}'$ (see, for instance, [G2], 1.5.1). Thus $\hat{R}_1e^{\hat{\rho}'-\hat{\rho}}=\hat{R}_0e^{\hat{\rho}'}\cdot(\hat{R}e^{\hat{\rho}})^{-1}$ is a $\hat{W}'$-invariant element of $\mathcal{R}'$. By Section 1.3.3, $\hat{R}_1\in\mathcal{R}_{\hat{W}}$ so $\hat{R}_1e^{\hat{\rho}'-\hat{\rho}}$ is a $\hat{W}'$-invariant element of $\mathcal{R}_{\hat{W}}$. Using Lemma 2.3.1, we get

(10) $\qquad \hat{R}_0e^{\hat{\rho}'}Y=\hat{R}_1e^{\hat{\rho}'-\hat{\rho}}\mathcal{F}_{T'}(R)$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$.

Write $Y=Y_1+Y_2$, where $\mathrm{supp}(Y_1)=\mathrm{supp}(Y)\cap\hat{Q}^{\perp}$ and $\mathrm{supp}(Y_2)=\mathrm{supp}(Y)\setminus\hat{Q}^{\perp}$. Note that $Y_1,Y_2\in\mathcal{R}$. Assume that $Y_2\neq 0$. Let μ be a maximal element in $\mathrm{supp}(Y_2)$. One has $\mathrm{supp}(\hat{R}^{-1})\subset\hat{Q}^-$ and $\mathrm{supp}\big(\mathcal{F}_{T'}(R)e^{\hat{\rho}}\big)\subset\hat{\rho}-\hat{Q}^+$, by Section 1.3.4 and (5) respectively. Thus $\mathrm{supp}(Y)\subset\hat{Q}^-$ and so $\mu\in\hat{Q}^-$.

Since $\mathrm{supp}(Y_1)\subset\hat{Q}^{\perp}$, Y_1 is a $\hat{W}$-invariant element of $\mathcal{R}_{\hat{W}}$ so $\hat{R}_0e^{\hat{\rho}'}Y_1$ is a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$. In the light of (10), the product $\hat{R}_0e^{\hat{\rho}'}Y_2$ is also a $\hat{W}'$-anti-invariant element of $\mathcal{R}_{\hat{W}'}$. Clearly, $\hat{\rho}'+\mu$ is a maximal element in the support of $\hat{R}_0e^{\hat{\rho}'}Y_2$.

By Section 1.3.5, this support is the union of $\hat{W}'$-regular orbits (recall that regularity means that each element has the trivial stabilizer in $\hat{W}'$), so $\hat{\rho}'+\mu$ is a maximal element in a regular $\hat{W}'$-orbit and thus $\frac{2(\hat{\rho}'+\mu,\alpha)}{(\alpha,\alpha)}\not\in\mathbb{Z}_{\leq 0}$ for each $\alpha\in\hat{\Pi}'$. Since $\mu\in\hat{Q}^-$ one has $\frac{2(\mu,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}$ for each $\alpha\in\hat{\Pi}'$. Taking into account that $\frac{2(\hat{\rho}',\alpha)}{(\alpha,\alpha)}=1$ for each $\alpha\in\hat{\Pi}'$, we obtain

$$\forall\alpha\in\hat{\Pi}'\ \ \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}_{\geq 0}. \tag{11}$$

Recall that $\delta=\sum_{\alpha\in\hat{\Pi}'}k_\alpha\alpha$ for some $k_\alpha\in\mathbb{Z}_{>0}$ (see [K2], Chapter VI). Since $\mu\in\hat{Q}^-$ one has $(\mu,\delta)=0$. Combining with (11), we get $(\mu,\alpha)=0$ for each $\alpha\in\hat{\Pi}'$ so $\mu\in(\hat{\Delta}')^{\perp}$.

One has

$$(\hat{\Delta}')^{\perp}\cap\hat{Q}=(\hat{Q}^{\perp}\cap\hat{Q})\oplus V,$$

where the restriction of $(-,-)$ to $\mathbb{Q}V$ is negatively definite; more precisely, one has

$\mathfrak{g}$	$\mathfrak{gl}(n\vert n)$	$\mathfrak{gl}(m\vert n),\ m\neq n$	$C(n)$	other cases
$\hat{Q}^{\perp}\cap\hat{Q}$	$\mathbb{Z}\{\delta,\mathfrak{str}\}$	$\mathbb{Z}\delta$	$\mathbb{Z}\delta$	$\mathbb{Z}\delta$
V	$\mathbb{Z}\Delta''$	$\mathbb{Z}\Delta''\oplus\mathbb{C}\xi$	$\mathbb{Z}\Delta''\oplus\mathbb{C}\xi$	$\mathbb{Z}\Delta''$

For $\mathfrak{g}=\mathfrak{gl}(m|n),\ m\neq n$ and $\mathfrak{g}=C(n)$ the element ξ is given in [G2], 3.2; one has $(\Delta'',\xi)=0,\ (\xi,\xi)<0$. Since $V\subset\hat{Q}$, one has $(V,\hat{Q}^{\perp})=0$. Now combining the formulas $\mu\in(\hat{Q}^{\perp}\cap\hat{Q})\oplus V,\ (\mu,\mu)=0$ with the fact that $(\nu,\nu)<0$ for each non-zero $\nu\in V$, we obtain $\mu\in\hat{Q}^{\perp}\cap\hat{Q}=\hat{Q}^{\hat{W}}$, which contradicts to the construction of Y_2. Hence $Y_2=0$ as required. ☐

2.3.3. Using the table in the proof of Proposition 2.3.2, we obtain the following corollary.

Corollary. *For $\mathfrak{g}\neq\mathfrak{gl}(n|n)$ one has $f(q)\cdot\hat{R}e^{\hat{\rho}}=\mathcal{F}_{T'}(Re^{\hat{\rho}})$ for some $f(q)=\sum_{k=0}^{\infty}a_kq^k$ ($a_k\in\mathbb{Z}$). For $\mathfrak{g}=\mathfrak{gl}(n|n)$ one has $f(q,e^{\mathfrak{str}})\cdot\hat{R}e^{\hat{\rho}}=\mathcal{F}_{T'}(Re^{\hat{\rho}})$ for some $f(q,e^{\mathfrak{str}})=\sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty}a_{k,m}q^ke^{m\cdot\mathfrak{str}}$ ($a_{k,m}\in\mathbb{Z}$).*

2.4. **Step 4 for $\mathfrak{g}=\mathfrak{gl}(n|n),D(n+1|n)$.** In this subsection we complete the proof of the denominator identities (2) by proving the formulas (3). We prove them by taking a suitable evaluation of $\hat{R}^{-1}\sum_{t\in T'}t(R)$. By Corollary 2.3.3, $\hat{R}^{-1}\sum_{t\in T'}t(R)$ is equal to $f(q)$ for $D(n+1|n)$ and to $f(q,e^{\mathfrak{str}})$ for $\mathfrak{gl}(n|n)$. Now we consider q as a real parameter between 0 and 1. We choose the evaluation in such a way that the evaluation of $\hat{R}^{-1}\sum_{t\in T'}t(R)$ is equal to the evaluation of $\hat{R}^{-1}R$. As a result, $f(q)$ (resp., $f(q,e^{\mathfrak{str}})$) is equal to the evaluation of $\hat{R}^{-1}R$, which can be easily computed.

2.4.1. *Case $D(n+1|n)$.* Take a complex parameter x and consider the following evaluation: $e^{-\varepsilon_i}:=x^{a_i},\ e^{-\delta_j}:=-x^{b_j}$, where $a_i,\ (i=1,\dots,n+1),\ b_j,(j=1,\dots,n)$ are integers such that $a_i\pm b_j\neq 0,a_i\pm a_j\neq 0,b_i\pm b_j\neq 0,b_i\neq 0$ for all indexes i,j. We denote the evaluation of R (resp., $\hat{R}$) by $R(x)$ (resp., $\hat{R}(x)$). The functions $R(x), \hat{R}(x)$ are meromorphic. One has

$$R(x) = \frac{\prod_{1\leq i<j\leq n+1}(1-x^{a_i\pm a_j})\cdot\prod_{1\leq i<j\leq n}(1-x^{b_i\pm b_j})\cdot\prod_{1\leq i\leq n}(1-x^{2b_i})}{\prod_{1\leq i\leq j\leq n}(1-x^{a_i\pm b_j})\prod_{1\leq j<i\leq n+1}(1-x^{a_i\pm b_j})}.$$

One readily sees that $R(x)$ has a pole at $x=1$ of order $|\Delta_{1+}|-|\Delta_{0+}|=n$.

One has

$$\left.\frac{\hat{R}(x)}{R(x)}\right|_{x=1} = \frac{((1-q)_q^\infty)^{\dim\mathfrak{g}_0}}{((1-q)_q^\infty)^{\dim\mathfrak{g}_1}} = ((1-q)_q^\infty)^{\dim\mathfrak{g}_0-\dim\mathfrak{g}_1} = (1-q)_q^\infty.$$

In particular, $\hat{R}(x)$ also has a pole of order n at $x=1$.

The evaluation of $(t_{\sum k_i\delta_i}(R))(x)$ is

$$\frac{\prod_{1\leq i<j\leq n+1}(1-x^{a_i\pm a_j})\cdot\prod_{1\leq i\leq n}(1-q^{-2k_i}x^{2b_i})\cdot\prod_{1\leq i<j\leq n}(1-q^{-k_i\mp k_j}x^{b_i\pm b_j})}{\prod_{1\leq i\leq j\leq n}(1-q^{\mp k_j}x^{a_i\pm b_j})\prod_{1\leq j<i\leq n+1}(1-q^{\mp k_j}x^{-a_i\pm b_j})}$$

which is a meromorphic function. Let s be the number of zeros among $k_1,\dots,k_n$. Then at $x=1$ the order of zero of the numerator is at least is $n(n+1)+s^2$, and the order of zero of the denominator is $2(n+1)s$. Therefore at $x=1$ the function $(t_{\sum k_i\delta_i}(R))(x)$ has the pole of order at most $2(n+1)s-n(n+1)-s^2=n+1-(n+1-s)^2$; in particular, $(t_{\sum k_i\delta_i}(R))(x)$ has the pole of order at most n and it is equal to n iff $n=s$ that is $\sum k_i\delta_i=0$ and $(t_{\sum k_i\delta_i}(R))(x)=R(x)$.

We conclude that $(\hat{R}(x))^{-1}\cdot\sum_{t\in T':t\neq\mathrm{id}}(t(R))(x)$ is holomorphic at $x=1$ and its value is equal to zero, and that $(\hat{R}(x))^{-1}\cdot\sum_{t\in T'}(t(R))(x)$ is holomorphic at $x=1$ and its value is equal to $\frac{R(x)}{\hat{R}(x)}|_{x=1}$. In the light of Corollary 2.3.3 we obtain

$$f(q) = \left.\frac{R(x)}{\hat{R}(x)}\right|_{x=1} = ((1-q)_q^\infty)^{-1}.$$

2.4.2. *Case* $\mathfrak{gl}(n|n)$. Fix $y>1$. Take a complex parameter x and consider the following evaluation

$$e^{-\varepsilon_1} := y,\ e^{-\varepsilon_i} := x^i,\ \text{for } i=2,\dots,n\ e^{-\delta_i} := -x^{-i} \text{ for } i=1,\dots,n.$$

The functions $R(x), \hat{R}(x)$ are meromorphic. One has

$$R(x) = \frac{\prod_{1<i\leq n}(1-yx^{-i})\cdot\prod_{1<i<j\leq n}(1-x^{i-j})\cdot\prod_{1\leq i<j\leq n}(1-x^{j-i})}{\prod_{1\leq i\leq n}(1-yx^{i})\cdot\prod_{1<i\leq j\leq n}(1-x^{i+j})\cdot\prod_{1\leq j<i\leq n}(1-x^{-i-j})}.$$

Therefore the function $R(x)$ has a pole of order $n-1$ at $x=1$.

One has

$$\frac{\hat{R}(x)}{R(x)}\Big|_{x=1} = \frac{((1-q)_q^\infty)^{\dim\mathfrak{g}_0-2(n-1)}\cdot((1-qy)_q^\infty)^{n-1}\cdot((1-qy^{-1})_q^\infty)^{n-1}}{((1-q)_q^\infty)^{\dim\mathfrak{g}_1-2n}\cdot((1-qy)_q^\infty)^{n}\cdot((1-qy^{-1})_q^\infty)^{n}}.$$

Thus $\hat{R}(x)$ also has a pole of order $n-1$ at $x=1$. Since $\dim\mathfrak{g}_0=\dim\mathfrak{g}_1$ and $e^{\mathfrak{str}}=(-1)^n y^{-1}$ for $x=1$ we obtain

$$\frac{\hat{R}(x)}{R(x)}\Big|_{x=1} = \frac{((1-q)_q^\infty)^2}{(1-q(-1)^n e^{\mathfrak{str}})_q^\infty\cdot(1-q(-1)^n e^{-\mathfrak{str}})_q^\infty}.$$

One has

$$(t_{\sum k_i\delta_i}(R))(x,y)=\frac{\prod_{1<i\leq n}(1-yx^{-i})\cdot\prod_{1<i<j\leq n}(1-x^{i-j})\cdot\prod_{1\leq i<j\leq n}(1-q^{k_j-k_i}x^{j-i})}{\prod_{1\leq i\leq n}(1-q^{k_i}yx^{i})\cdot\prod_{1<i\leq j\leq n}(1-q^{k_j}x^{i+j})\cdot\prod_{1\leq j<i\leq n}(1-q^{-k_j}x^{-i-j})},$$

which is a meromorphic function.

Let s be the number of zeros among $k_1,\dots,k_n$. Then at $x=1$ the order of zero of the numerator is at least $\frac{(n-1)(n-2)+s(s-1)}{2}$, and the order of zero of the denominator is $(n-1)s$. Therefore at $x=1$ the function $(t_{\sum k_i\delta_i}(R))(x,y)$ has the pole of order at most $(n-1)s-\frac{(n-1)(n-2)+s(s-1)}{2}=\frac{3n-s-2-(n-s)^2}{2}$, so the order is at most $n-1$ and it is equal to $n-1$ iff $s=n-1,n$. Notice that $s\neq n-1$, since $\sum k_i=0$. Therefore the pole has order $n-1$ iff $\sum k_i\delta_i=0$.

We conclude that the function $(\hat{R}(x))^{-1}(\mathcal{F}_{T'}(R))(x)$ is holomorphic at $x=1$ and its value is equal to $\frac{R(x)}{\hat{R}(x)}|_{x=1}$. Using Corollary 2.3.3 we obtain

$$f(q,e^{\mathfrak{str}})=\frac{R(x)}{\hat{R}(x)}\Big|_{x=1}=\frac{(1-q(-1)^n e^{\mathfrak{str}})_q^\infty\cdot(1-q(-1)^n e^{-\mathfrak{str}})_q^\infty}{((1-q)_q^\infty)^2}.$$

2.5. **Step 4 for $\mathfrak{g}\neq\mathfrak{gl}(n|n), D(n+1|n), D(2,1,a)$.** In this case the dual Coxeter number is non-zero. Recall that $q=e^{-\delta}$. Write $f(q)=\sum_{k=0}^\infty a_k e^{-k\delta}$. Since $f(q)\cdot\hat{R}e^{\hat{\rho}}=\mathcal{F}_{T'}(Re^{\hat{\rho}})$, we have

$$\sum_{k=1}^\infty a_k e^{-\delta}\cdot\hat{R}e^{\hat{\rho}}=\mathcal{F}_{T'}(Re^{\hat{\rho}})-a_0\hat{R}e^{\hat{\rho}}.$$

By (5), for any ν in the support of the right-hand side, one has $(\nu,\nu)=(\hat{\rho},\hat{\rho})$, and for any ν in the support of the left-hand side one has $(\nu,\nu)=(\hat{\rho},\hat{\rho})-2k(\delta,\hat{\rho})$ for some $k>0$. Since $(\hat{\rho},\delta)$ is equal to the dual Coxeter number, which is non-zero, we conclude that the intersection of supports is empty. Hence $f(q)=a_0$. Since the coefficient of $e^{\hat{\rho}}$ in $\hat{R}e^{\hat{\rho}}$ is equal to one, a_0 is equal to the coefficient of $e^{\hat{\rho}}$ in $\mathcal{F}_{T'}(Re^{\hat{\rho}})$. As it is shown in [G2], this coefficient is equal to one so $f(q)=1$ as required.

3. Other forms of denominator identity

Recall that denominator identity for a basic Lie superalgebra can be written in the form

$$Re^{\rho}=\mathcal{F}_{W^{\sharp}}\Big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big), \tag{12}$$

where $W^{\sharp}:=W'$ for $\mathfrak{g}\not=D(n+1|n),D(2,1,a)$ and $W^{\sharp}:=W''$ for $\mathfrak{g}=D(n+1|n),D(2,1,a)$, and $S\subset\Pi$ is the maximal isotropic system (see [KW],[G1]). If the dual Coxeter number of $\mathfrak{g}$ is non-zero the affine denominator identity for $\mathfrak{g}$ can be written in the form

$$\hat{R}e^{\hat{\rho}}=\mathcal{F}_{\hat{W}^{\sharp}}\Big(\frac{e^{\hat{\rho}}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)$$

see [KW],[G2]. In this section we will show that for $\mathfrak{gl}(n|n)$ the denominator identity can be written in a similar form:

$$\hat{R}e^{\rho}=f(q,e^{\mathrm{str}})\cdot\mathcal{F}_{\hat{W}'}\Big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big), \tag{13}$$

and that the denominator identities for $D(n+1|n)$ can not be written in a similar form, since the expressions $\mathcal{F}_{\hat{W}''}\big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)$, $\mathcal{F}_{\hat{W}'}\big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)$ are not well defined.

3.1. **Case** $D(n+1|n)$.

Let us show that the expressions $\mathcal{F}_{\hat{W}''}\big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)$, $\mathcal{F}_{\hat{W}'}\big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)$ are not well-defined for $D(n+1|n)$. Fix Π as in Section 1.1 and recall that $\rho=0$.

We repeat the reasonings of Section 2.1.1. One has

$$\sum_{\beta\in V(w)}w\beta\in\mathrm{supp}\Big(\frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})}\Big)\subset\sum_{\beta\in V_S(w)}w\beta-\hat{Q}^+\subset\hat{Q}^-,$$

where

$$V_S(w)=\{\beta\in S|w\beta<0\}.$$

Therefore $1\in\mathrm{supp}\big(\frac{1}{\prod_{\beta\in S}(1+e^{-w\beta})}\big)$ iff $wS\subset\Delta_+$.

Take $S=\{\varepsilon_i-\delta_i\}$; then $t_{\mu}S\subset\Delta_+$ if $(\varepsilon_i-\delta_i,\mu)<0$ for all i which holds for all $\mu\in\sum\mathbb{Z}_{<0}\varepsilon_i$ and all $\mu\in\sum\mathbb{Z}_{>0}\delta_i$. Hence the sums $\mathcal{F}_{\hat{W}''}\big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)$, $\mathcal{F}_{\hat{W}'}\big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\big)$ contain infinitely many summands equal to 1 and thus they are not well-defined.

3.2. **Case** $\mathfrak{gl}(n|n)$.

Fix Π as in Section 1.1; then $S=\{\varepsilon_i-\delta_i\}$.

In order to deduce the formula (13) from (12) and (2) it is enough to verify that the expression

$$\mathcal{F}_{\hat{W}'}\Big(\frac{e^{\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)=e^{\rho}\mathcal{F}_{\hat{W}'}\Big(\frac{1}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)$$

is well-defined (since $\rho = \mathfrak{str}/2$ is $\hat{W}$-invariant). As in Section 2.1.1, it amounts to show that

$$X_S(\nu) := \{w \in \hat{W}' | \sum_{\beta \in V_S(w)} w\beta \geq -\nu\}$$

is finite for any $\nu \in \hat{Q}^+$ (where $V_S(w)$ is defined as in Section 3.1). As in Section 2.1.1, writing $\nu = k\delta + \nu_+$, where $\nu_+ \in \mathbb{Z}\Delta$, we get

$$X_S(\nu) \subset \{t_\mu y | \mu \in T', y \in W' \text{ s.t. } (yS, \mu) \geq -k\}.$$

Since y permutes δ_is, $t_\mu y \in X_S(\nu)$ forces $(\delta_i, \mu) \geq -k$ for all i. Taking into account that μ lies in the $\mathbb{Z}$-span of δ_i and $(\mu, \sum_{i=1}^n \delta_i) = 0$, we conclude that $X_S(\nu)$ is finite. This establishes (13).

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