BOUNDARY VALUE PROBLEM FOR A CLASSICAL SEMILINEAR PARABOLIC EQUATION

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Abstract. In this paper, we study the boundary value problem of the classical semilinear parabolic equations

$$u_t - \Delta u = |u|^{p-1}u, \quad in \ \ \Omega \times (0,T)$$

and $u = 0$ on the boundary $\partial\Omega \times [0,T)$ and $u = \phi$ at $t = 0$, where $\Omega \subset R^n$ is a compact C^1 domain, $1 < p \leq p_S$ is a fixed constant, and $\phi \in C_0^2(\Omega)$ is a given smooth function. Introducing new idea, we show that there are two sets $\tilde{W}$ and $\tilde{Z}$ such that for $\phi \in W$, there is a global positive solution $u(t) \in \tilde{W}$ with h^1 omega limit $\{0\}$ and for $\phi \in \tilde{Z}$, the solution blows up at finite time.

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1. Introduction

In this paper, we study the Dirichlet boundary value problem of the classical semilinear parabolic equation

$$u_t - \Delta u = |u|^{p-1}u, \quad in \ \ \Omega \times (0,T) \tag{1}$$

with $u = 0$ on the boundary $\partial\Omega \times [0,T)$ and $u = \phi$ at $t = 0$, where $T > 0$, $\Omega \subset R^n$ is a compact C^1 domain, $p > 1$ is a fixed constant, and $\phi \in C_0^2(\Omega)$ is a given smooth function. Assume that $p \leq p_S = \frac{n+2}{n-2}$ for $n \geq 3$ and $p < \infty$ for $n = 1, 2$. By the standard theory we know that there is a local time positive solution to (1). With the help of Nehari functional, one may find the threshold of the initial datum such that the solution either exists globally or blows up in finite time. More interesting results about (1) can be found in the recent work [1]. Since the equation (1) is a model problem, it deserves to have more understanding. Introducing new idea, we show in this paper that there are two new sets $\tilde{W}$ and $\tilde{Z}$ such that for $\phi \in \tilde{W}$, there is a global positive solution in $\tilde{W}$ with the H^1 omega limit 0 and for $\phi \in \tilde{Z}$, the solution blows up at finite time. We may extend the method used in this paper to treat Neumann boundary value problem of semilinear parabolic equation with negative power in [4]. To define the invariant set $\tilde{Z}$, we shall use the fact that the cones

$$C_+ = \{u \in C_0^1(\Omega); u \geq 0, \quad u \neq 0\}$$

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and

$$C_- = \{u \in C_0^1(\Omega); u \leq 0, \quad u \neq 0\}$$

are invariant sets of (1). This fact can be proved by applying the maximum principle.

We now recall the standard way to construct the invariant sets for (1). Formally, (1) has a Lyapunov functional; namely,

$$J(u) = \int \frac{1}{2}|\nabla u|^2 - \frac{1}{1+p}u^{p+1}.$$

Here and after, we use $\int$ to denote the integration over Ω. In fact, we may consider (1) as the negative L2-gradient flow of the functional $J(\cdot)$. That is, abstractly, (1) can be written as

$$u_t = -J'(u).$$

Hence, we have

$$\frac{d}{dt}J(u(t)) =< J'(u), u_t >= -|u_t|_2^2 = -|J'(u)|_{L^2}^2.$$

Let $f(u) = u^p$ and its primitive

$$F(u) = \frac{u^{p+1}}{p+1}.$$

Introduce the working space

$$\Sigma = \{u \in H_0^1; u \neq 0, \int F(u) < \infty\}.$$

The condition $\int F(u) < \infty$ is always true by using the Sobolev inequality.

Define on Σ, the functional

$$M(u) = \frac{1}{2}\int_\Omega |u|^2$$

and the Nehari functional

$$I(u) = \int |\nabla u|^2 - uf(u) = \int |\nabla u|^2 - |u|^{p+1}.$$

Note that these two functionals are well-defined on Σ.

Along the flow (1) we can see that

$$\frac{d}{dt}M(u) = \int uu_t = -I(u). \tag{2}$$

Let

$$d = \inf\{J(u); u \in \Sigma; I(u) = 0\}.$$

Define

$$W = \{u \in \Sigma; J(u) < d, I(u) > 0\} \bigcup \{0\}$$

and

$$Z = \{u \in \Sigma; J(u) < d, I(u) < 0\}.$$

The classical result says that W and Z are invariant sets of (1); furthermore, for $1 < p < p_S$ and for any initial data $\phi \in W$, the solution exists globally;

for $1 < p \leq p_S$ and for any initial data $\phi \in Z$, the solution blows up at finite time. One may see [5] for more results and references.

We now introduce new functionals. For $\lambda \in \mathbf{R}_+$, define

$$E_\lambda(u) = J(u) + \lambda M(u).$$

Then along the flow (1), we have

$$\frac{d}{dt}E_\lambda(u) = -|J'(u)|_2^2 - \lambda I(u). \tag{3}$$

From this, it is clear that for $\lambda \geq 0$, we have

$$\frac{d}{dt}E_\lambda(u) > 0$$

except $|J'(u)| = 0$.

Introduce

$$d_\lambda = \inf\{E_\lambda(u); u \in \Sigma; I(u) = 0\}.$$

As in the case for the quantity d, we can give it the mountain-pass characterization.

Assume it is finite at this moment. Define

$$W_\lambda = \{u \in \Sigma; E_\lambda(u) < d_\lambda, I(u) > 0\} \bigcup \{0\}.$$

For convenient we set $W_0 = W$. Arguing as in W, one can see that W_λ with $\lambda > 0$ is non-empty.

Then by (3) and the standard argument we know that for $\lambda \geq 0$, W_λ is a invariant set of the flow (1).

One of our main results for (1) is to show the the following conclusion.

Theorem 1. *Fix any power $1 < p < p_S$, we have for $\lambda > 0$ that*

(1). d_λ is finite, and $d_\lambda > d$ for $\lambda > 0$;

(2). for $\phi \in W_\lambda$ with $\lambda \geq 0$, the flow exists globally and its omega limit is $\{0\}$. Hence

$$\tilde{W} := \bigcup_{\lambda \geq 0} W_\lambda$$

is invariant set of (1).

We remark that since $d_\lambda > d$, we know that the set W_λ is different from the set W.

To find the set for blow-up solutions to (1), we need to use the comparison argument. We shall restrict the initial data being positive. Let $\delta \geq 0$. Consider the boundary value problem of the following semilinear parabolic equation

$$v_t - \Delta v + \delta v = v^p, \quad u > 0, \quad in \quad \Omega \times (0, T) \tag{4}$$

with $u = 0$ on the boundary $\partial\Omega \times [0, T)$ and $u = \phi$ at $t = 0$, where $T := T_{max}(\phi) > 0$ is the maximal existence time of the solution $v(t)$. Define on $\Sigma_+ = \Sigma \bigcap C_+$,

$$J_\delta(v) = J(v) + \delta M(v),$$

$$I_\delta(v) = I(v) + 2\delta M(v),$$

and on the set where $\{I_\delta(v) = 0\}$

$$E^\delta(v) = J_\delta(v) = (\frac{1}{2} - \frac{1}{p+1}) \int |u|^{p+1}.$$

Define

$$d_\delta = \inf\{E^\delta(v); v \in \Sigma_+, I_\delta(u) = 0\}.$$

For $\epsilon > 0$,

$$d_{\delta,\epsilon} = \inf\{J_\delta(u); u \in \Sigma_+; I_\delta(u) = \epsilon\}$$

and

$$Z_\delta = \{u \in \Sigma_+; J_\delta(u) < d_\delta, I_\delta(u) < 0\}.$$

Clearly, Z_δ is non-empty and it is a invariant set of the flow (4). We remark that one may make similar construction on $\Sigma_- = \Sigma \bigcap C_-$.

Theorem 2. *Fix $1 < p \leq p_S$. (1). For $\phi \in Z_\delta$ and $\phi \geq v_\delta$, the flow $(v(t))$ to (4) blows up in finite time.*

(2). Let $u(t)$ be the flow to (1) with the initial data ϕ as (1) above. Then $u(t) \geq v(t)$ and $u(t)$ blows up at some $t < \infty$.

As a consequence of Theorem 2, we have

Corollary 3. *Set $\tilde{Z} = \bigcup_{\delta \geq 0} Z_\delta$. Then for any $\phi \in \tilde{Z}$, the solution for (1) blows up at finite time.*

The results above will be proved in next section.

2. Global solution and finite time blow-up solution

We now prove Theorem 1.

(1). The finiteness of d_λ can be obtained in the similar way as in [5]. Since $1 < p < p_S$, we know that d_λ can also be achieved by some function u_λ(see [2] [3], or [6]). By this we know that d_λ is different from d for $\lambda > 0$. Hence, we have $d_\lambda > d$ for $\lambda > 0$.

(2). Since $I(\phi) > 0$, we have $I(u(t)) > 0$ for all $t \in [0, T)$. For otherwise, for some $t > 0$, $I(u(t)) = 0$. Using the definition of d_λ, we have $E_\lambda(u(t)) \geq d_\lambda$. This is a contradiction to the fact that

$$\frac{d}{dt} E_\lambda(u(t)) < 0, \quad and \quad E_\lambda(u(t)) < E_\lambda(\phi) < d_\lambda.$$

Using (2), we know that $M(u(t)) < M(\phi)$. With the help of the condition $E_\lambda(u(t)) < d$ and $1 < p < p_S$, we know that $u(t) \in H^1$ is uniformly bounded and bounding constant depends only on d, p, $|\Omega|$, and $M(\phi)$.

The H^1 omega limit at $t = \infty$ can be determined below. It is a classical fact ([5]) that the H^1 omega limit set $\omega(\phi)$ consists of classical equilibria. If $v \in \omega(\phi)$, we have $I(v) = 0$. If v is nontrivial, we have

$$E_\lambda(v) \geq d_\lambda.$$

Impossible. Hence $v = 0$, that is, $\omega(\phi) = \{0\}$.

This completes the proof of Theorem 1.

The remaining part of this section we give the proof of Theorem 2.

Proof. (Proof of Theorem 2). Introduce

$$A=\inf\{\frac{|\nabla u|_2^2+\delta|u|_2^2}{|u|_{p+1}^2};\quad u\in H_0^1(\Omega),\quad u\neq 0\}.$$

Then it is easy to see that $d_\delta=\frac{p-1}{2(p+1)}A^{(p+1)/(p-1)}$ (see [2] [3], or [6]). Assume that $0\neq v\in H_0^1(\Omega)$ such that $I_\delta(v)=-\epsilon$. Then

$$E^\delta(v)=\frac{p-1}{2(p+1)}\int(|\nabla v|^2+\delta v^2)-\frac{\epsilon}{p+1}. \tag{5}$$

Using the definition of A we have

$$\int(|\nabla v|^2+\delta v^2)\leq\int|v|^{p+1}\leq A^{-\frac{p+1}{2}}(\int(|\nabla v|^2+\delta v^2))^{\frac{p+1}{2}}.$$

Hence,

$$\int(|\nabla v|^2+\delta v^2)\geq A^{(p+1)/(p-1)}.$$

Combining this with (5) we have

$$d_{\delta,\epsilon}\geq d_\delta-\frac{\epsilon}{p+1}. \tag{6}$$

We now prove (1) in the statement of theorem 2.

(1). Take $\epsilon>0$ such that

$$\epsilon<\min(-I_\delta(\phi),d_\delta-J_\delta(\phi)).$$

Then using (3) and (6) we know that

$$J_\delta(v(t))\leq J_\delta(\phi)<d_\epsilon$$

for $t\in[0,T)$. Since $I_\delta(\phi)<-\epsilon$, by using the definition of $d_{\delta,\epsilon}$ and the continuity, we know that

$$I_\delta(v(t))<-\epsilon.$$

Note that

$$I_\delta(v)=2J_\delta(v)-(1-\frac{2}{p+1})\int|v|^{p+1}.$$

Assume that $T=T_{max}>0$ be the maximal time of the flow $(v(t))$. Assume that $T=\infty$. On one hand, using similar formula to (2) we have

$$\frac{1}{2}\frac{d}{dt}\int v^2=-I_\delta(v)\geq\epsilon>0,$$

and then

$$\int v^2\geq\int\phi^2+2\epsilon t\to\infty.$$

That is, $M(v(t))\to\infty$ as $t\to\infty$.

On the other hand,

$$\frac{1}{2}\frac{d}{dt}\int v^2 = -I_\delta(v) \geq -2d_\epsilon + (1-\frac{2}{p+1})\int |v|^{p+1}.$$

Then we have

$$\frac{d}{dt}M(v(t)) \geq -2d_\epsilon + C(p,|\Omega|)M(v(t))^{\frac{p+1}{2}}$$

for some uniform constant $C(p,|\Omega|) > 0$. Then using $M(v(t)) \to \infty$, we know that there exists $T_1 > 0$ such that for any $t > T_1$,

$$\frac{d}{dt}M(v(t)) \geq \frac{1}{2}C(p,|\Omega|)M(v(t))^{\frac{p+1}{2}}.$$

However, this implies that $T < \infty$. A contradiction. Hence $T < \infty$ and $M(v(t)) \to \infty$ as $t \to T$.

We shall prove (2) in the statement of theorem 2 by using the comparison lemma. (2). Let $T_{max} < \infty$ be the blow-up time of the flow $(v(t))$. Recall that $v(t) > 0$ for $t \in (0, T_{max})$. Let $w(t) = u(t) - v(t)$, $t < T_{\max}$. Then $w(t)$ is bounded in any finite time before the blowing up time of the solution $u(t)$. Note that

$$w_t - \Delta w = p\xi^{p+1}w + \delta v. \tag{7}$$

Recall that $w(0) = 0$ and $w(t)|_{\partial\Omega} = 0$. Let $w_-(t)$ be the negative part of $w(t)$. Multiplying both sides of (7) by $w_-(t)$ and integrating over Ω by $w_-(t)$, we get

$$\frac{d}{dt}\int |w_-(t)|^2 = -\int |\nabla w_-(t)|^2 + p\int \xi^{-p-1}|w_-(t)|^2 + \delta\int vw_-(t).$$

We remark that the last term is non-positive. Then we have

$$\frac{d}{dt}\int |w_-(t)|^2 \leq C\int |w_-(t)|^2.$$

By the Gronwall inequality we know that $\int |w_-(t)|^2 = 0$ for any $t > 0$. Hence we have $u(t) \geq v(t)$ and then

$$\int u(t)^2 \geq \int v(t)^2 \to \infty$$

as $t \to T_{max} < \infty$.

□

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